

The Optimal Group Size in Microcredit Contracts

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Abstract

We develop a model of a repeated microcredit lending and study how group size affects the optimal group lending contracts with joint liability, what should be the punishment for defaulting borrowers, and what is the effect of project correlation on the lending outcome. The story is that one benevolent lender gives microcredit to a group of borrowers to be invested in n projects that could be either disjoint or correlated. The outcome of each risky project is not observable by the lender. Therefore in case some of the borrowers default on their loan repayments, the lender is not able to identify strategic default. We characterize the optimal contract and determine the optimal size of the group of borrowers endogenously. We discuss that joint liability has positive effects on the repayment rate and borrowers' welfare, although joint liability contracts are feasible under a smaller set of parameter values than individual liability contract. Our analysis also suggests that group size should increase with project risk. Furthermore, we argue that less severe punishment doesn't have any effect on borrowers' welfare and repayment amount, and only the maximum loan that could be offered to borrowers should be lower. This negative effect could be compensated by enlarging the group size. We also discuss that when projects are correlated, joint liability contracts are increasingly more feasible when group becomes larger.

[*Keywords*: Microfinance, Group lending, Individual lending, Strategic default]

1 Introduction

Small-scale businesses are considered a major source of employment, specially in developing countries where these businesses employ more than half of the economically active population (de Mel et al., 2008). One of the most efficient tools for the development of microenterprises is accessing a source of finance (Berge et al., 2014). However, these enterprises are usually excluded from conventional financial services, either because they are too small, they are unable to offer formal guarantees or they live too far from the financial networks (Prior and Argandoña, 2009). In Peru in 2008, for example, there were 3,080,000 microenterprises that employed 76% of the country's labor force and accounted for 42% of the country's GDP, but they were never regarded as relevant by the formal banking system (Chu, 2015).

Microcredits may equip financial institutions with the right tool to rehearse their specific social responsibilities in developing countries of 'integrating people in the active population and combating the cause of social and financial exclusion' (Lacalle-Calderón and Rico-Garrido, 2006) and reducing poverty (Littlefield et al., 2003). Microcredits are small loans with little or no collateral offered to microentrepreneurs who are usually excluded from conventional financial services. This type of lending had some success in lending to the poor and has been publicly seen as one of the recent improvements in financial institutions to support development, although it is also claimed that microcredits only reach the moderately poor people and not the poorest of the poor (Scully, 2004; Marr, 2003). Microcredit lending has a long history of practice,¹ but it took more attention after the successful experiment of Grameen bank in Bangladesh in the mid 1970s conducted by Dr. Muhammad Yunus who received Nobel Peace Prize for his efforts in poverty reduction in 2006. The victorious trial of the Grameen bank's innovative business model has led to a rapid reproduction of a similar type of lending all over the world.² Beside being of interest to socially responsible investors, charitable institutions and governments, microcredits have also attracted enormous academic attention as a potential key to economic and social returns (for a survey see Milana and Ashta, 2012; Banerjee, 2013).

¹Credit cooperatives were active in nineteenth-century in Germany (Guinnane, 2001).

²Famous examples of successful MFIs include: Equitas Microfinance in India (Narayanan and Rangan, 2010), Mibanco in Peru (Chu and Gustavo, 2009), Tibet Poverty Alleviation Fund (TPAF) in Tibet Autonomous Region (Stuart et al., 2008), Kashf Foundation in Pakistan (Mahmood, 2007), Accion International a US organization who is active in Latin America (Quelch and Laidler, 2003).

Consider a benevolent lender (she) who wants to give loans to a group of n borrowers (he) with *joint liability (JL)*, that is members accept to be jointly liable for each member's loan. The loans will be invested in n projects that can be either disjoint or correlated. We assume that the realized output of each project is known to the group members but unknown to the lender. This is a plausible assumption, as groups of borrowers are self-selected groups, and have better information about each other than the lender has. Moreover, they have means for observing each other's output.

In this study, we try to answer three main questions. First, what kind of contracts should be offered to borrowers so that the lender could receive her money back while borrowers' lifetime profit is maximized? More specifically, what penalty function should be employed against defaulting borrowers so that they are encouraged to repay but not punished too harshly? Should they be excluded from the lending game forever or we could give them another chance? In this paper, we reconsider the optimal design of uncollateralized lending contracts with joint liability. We compare joint liability contracts, which deprive the strategically defaulting member from future loan forever, with joint liability contracts that deprive them for only T periods and let them rejoin the game afterwards. We examine positive and negative effects of this less severe punishment on the lending outcome.

The second question we are exploring is: how large should the group size n be in order to maximize the borrower's benefit while to leave the lender break even? On the one side, a larger group size can have a positive effect on the repayment rate, as having more people liable for repaying defaulted payments assures a higher rate of repayment. On the other side, a larger group size can be a threat towards members who repay their loans successfully, since they should pledge repaying for all their defaulting peers, and it may happen that every body else in the group is defaulting on his repayment. We propose an method to find the optimal group size that allows JL to provide larger loans to borrowers. The third question that leads our study is: how project correlation can affect the lending outcome? Assuming that chance of success in project is an increasing function of the group size maybe a natural assumption to make, as jointly liable group members are likely to help each other to succeed. However, the marginal desirability of forming larger groups should be decreasing, as very large groups have to handle higher tensions.

We model the lending situation described above as an infinitely repeated game. Inspired by Tedeschi (2006), we define two phases in our model, *lending phase* where the game starts and

continues until the group defaults on repayment, and *punishment phase* where no new loans are extended to the borrowers. We stay aligned with Bhole and Ogden’s (2010) simple group lending model that is designed for groups of two borrowers, and extend their established results on groups of n borrowers. In our model, borrowers receive contracts individually that determine the amount of loan L and repayment R . They invest their loans on n projects that are identical in terms of mean return and chance of success, however they can be disjoint or correlated. Projects will be either successful with high return or unsuccessful with low return. After the outcomes of projects are realized, each borrower decides to repay his loan or not. Only those who had successful project can repay. We make a difference between *strategic default* and *non-strategic default*. Borrowers default strategically, when they refuse to repay while they had high outcome from their projects; and they default non-strategically, when they don’t repay because they had low or no outcome from their project due to bad luck. The lender cannot identify strategic default, but members of the group can. If someone defaults non-strategically, his group members will repay his loan. The lender deprives a defaulting group from loans in future in order to decrease incentives for strategic default, however non-strategic defaults resulted from bad luck are unavoidable and repayment is not completely insured. We deviate from Bhole and Ogden (2010), where we assume that group members play grim-trigger between themselves. As a result if someone defaults strategically, other members of the group will not repay his loan as well as their own loans, and no further loans are granted to the group members. Later we relax the grim-trigger assumption and define *Flexible Joint liability (FJL)*, in which if someone defaults strategically, other members still repay his loan but punish him by excluding him from receiving loans for the next T periods.

The results of our study offer three sets of comparisons, joint liability vs. *individual liability (IL)* lending, unlimited length vs. limited length punishment phase, and disjoint vs. correlated projects. The first comparison shows that JL lending is feasible under a small set of parameter settings than IL lending, but it has higher performance in terms of borrowers’ welfare and repayment rate. JL lending can also outperform IL lending in terms of maximum loan that can be offered to borrowers if the group size is not too large. The intuition behind it is that groups can offer higher repayment insurance than individuals, thus is it less risky to offer them larger loans. But if the group size becomes too large, the risk of project failure and having more defaulting members also goes up. This risk is even higher when borrowers invest on riskier projects. We suggest an algorithm to

calculate the optimal group size given the amount of high and low returns of projects, the discount factor of borrowers for future loans, and the chance of success in projects.

The second comparison proves that FJL lending does not have a significant effect on borrowers' repayment amount and their welfare and it stays the same as JL lending. However the maximum loan that can be offered to borrowers under FJL lending should be lower compared to JL lending. More specifically, the the maximum loan that can be offered to borrowers under FJL lending is increasing in the length of punishment phase and is the highest when the length of the punishment phase goes to infinity, that is the maximum loan that can be offered to borrowers is highest under JL lending. We also show that, in FJL lending, when the length of punishment phase is not too long, larger groups should be formed. Intuitively, when the punishment is not very severe, members are more prone to default strategically, therefore a larger group is needed to increase the repayment insurance. Finally, the third comparison provides evidence that if projects are correlated, feasibility of FJL lending and consequently also JL lending contracts will be increasing in the group size, regardless of the degree of correlation and borrowers' discount factor.

2 Related Literature

Our study, in general, is in line with the recent and growing theoretical literature in microfinance that explores conditions, if any, under which group lending can enhance lending results and alleviate informational asymmetry and enforcement problems, specially in developing countries (see for example Ghatak, 2000; Armendáriz de Aghion and Gollier, 2000). Within this literature, we are related to researches that are concerned with the optimal design of contract that should be offered to borrowers and type of punishment that should be employed against defaulting members (see e.g. Bhole and Ogden, 2010; Tedeschi, 2006). Literature mostly conclude that when borrowers are unable to impose strong social sanctions on each other, lender will be better off by offering individual liability contracts rather than joint liability contracts (see e.g. Besley and Coate, 1995; Armendáriz de Aghion, 1999). In this paper, we discuss that joint liability lending can actually have a positive effect on borrowers' welfare and repayment amount compared to individual liability, although joint liability contracts are feasible under a smaller parameter setting. We also argue particularly that JL contracts, when feasible, outperforms IL contracts in terms of the maximum

loan that can be offered to the borrower. Furthermore, we discuss that less severe punishment does not change borrower's lifetime benefit and repayment amount, thus borrower's problem stay intact whether the joint liability contract comes with a very sever punishment (JL) or a more flexible punishment (FJL).

Literature investigating group lending with joint liability, pays little attention to group size as one of the potential influential factors in the relative success of group lending. Theoretical studies mostly analyze lending models of groups of two borrowers while experimental and empirical studies suggest the importance of group size (Abbink et al., 2006; Galak et al., 2011). We argue that group size is an important factor in increasing borrower's welfare and repayment rate in microcredit lending, and we find the optimal group size endogenously.

Yet the existing literature that consider group size as an important factor, conclude differently. Conning (2005) and Ahlin (2015) argue in favor of larger groups. Similar to us, they suggest that group size cannot grow too large, however they base their arguments on different reasons from us. Conning (2005) reasons that it becomes more and more costly to contain free-riding as group size increases. Ahlin (2015) discusses that presence of local borrower information is necessary for large groups to have any impact. On the contrary, there are Bourjade and Schindele (2012) and Baland et al. (2013) who suggest that small group size is better. Bourjade and Schindele (2012) explain that there is a trade-off between raising profits through increased group size and providing incentives for borrowers with less social ties. They conclude that argue that if group members have social ties, a rational lender should choose a group of limited size. Baland et al. (2013) characterize loan contracts as a function of borrower's initial wealth, and prove that a smaller group size in JL lending can raise efficiency.

Literature on group lending with joint liability mostly assumes that borrowers run independent projects (see e.g. Armendáriz de Aghion 1999). However, jointly liable groups are more likely to contribute to each other's success. Moreover they usually live in the same region and experience similar risks. There are studies claiming that joint liability contracts are less feasible when project outcomes are positively correlated. Ghatak, (2000), for example, argues that when projects are likely to succeed or fail simultaneously, the joint liability part of the contract comes less often to practice. We differ from this literature by analyzing the effect of project correlation on feasibility of joint liability contracts. We prove that when there is any correlation between projects, we could

increase the feasibility of joint liability contracts by enlarging the borrower’s group. Katzur and Lensink (2012) also analyse joint liability contracts with correlated project outcomes, however they have a different setting from us and focus on social ties.

3 Model

Consider an infinitely repeated principal-agent model, with a benevolent principal (*lender*) and n agents (*borrowers*) playing grim-trigger with each other. We consider a two-phase model in which the lender and borrowers start in a “lending phase”, if one loan is successfully repaid by the group, another loan is given. In any period, if borrowers default, the lender and borrowers engage in a “punishment phase”, where no new loans are extended to borrowers. Each period of our game has three steps:

- $s = 0$ Each borrower receives a contract (L, R) individually, specifying the amount of loan L and the repayment R .
- $s = 1$ Each borrower invests L on his project that will be either successful with chance of $\alpha \in [0, 1]$ and yields a high return Y^H , or not successful with chance of $1 - \alpha$, and yields a low return Y^L , such that $0 \leq Y^L < Y^H$. Project returns are public information to all members of the group of borrowers.
- $s = 2$ Borrowers decide simultaneously to either repay their share R or not, and the lender announces the remaining payment. In this case, if i members default on their loan, other partners will be asked to additionally pay an amount $\frac{iR}{n-i}$ to the lender for their defaulting peers. If the total repayment is equal to nR or more, the group receives future financing. Otherwise, the entire group will be excluded from financing next period by the lender.

These three stages will be repeatedly played until the lender realizes that borrowers are not entitled for financing next period, and each period of not receiving loan, borrowers’ utility will be zero. At the first step, we assume that projects do not differ in their riskiness (i.e. α is the same for all borrowers). We also assume that high or low returns of projects are borrowers’ private information, and each borrower always invests on the same project.

Two types of defaults are possible: *strategic default* in which borrower does not repay although he had high outcome Y^H , and *nonstrategic default* as a result of obtaining low outcome Y^L resulted from bad luck or negative economic shock. We assume that the lender is unable to observe whether a borrower's default is strategic or nonstrategic. However borrowers are able to observe strategic defaults of their peers costlessly.

Borrowers play a repeated game among themselves. They start out cooperating, and they continue repaying if they can, and if all others did repay or some defaulted non-strategically, they also repay the remaining share of their peers. At the first step, we assume that borrowers play grim-trigger against each other. Meaning that if at some period, some members default strategically on their repayments, other members stop repaying themselves and stop repaying the defaulting players' shares. They will therefore not get a loan in the next period. In our second step, we relax the grim-trigger assumption by assuming that if at some period, some members default strategically, other group members still repay their own loans as well as the defaulting members' share, but they punish the strategically defaulting members by excluding them from the lending game for T periods.

The benevolent lender maximizes the payoff of each borrower contingent on: first, each borrower must be willing to accept a loan (repayment amount must be affordable for him); second, each borrower must have correct incentives to repay for himself and his share for each defaulting peer, when he is able to pay (in the worst case that all other members default, he must be still willing to repay for the entire group); and third, the lender must break even, meaning that she must maintain a sustainable lending operation over the entire loan portfolio by charging the appropriate repayment. The exact terms of the maximization problem depend on whether we are examining individual lending or group lending and will be discussed in the following sections.

4 Joint Liability (JL) Contracts

In this section, we formalize the model assuming that if some borrowers default strategically, their group members punish them by playing grim-trigger strategy. Meaning that successful members refuse to pay for defaulting members as well as for themselves. In this case, they won't receive a loan next round. The probability that total loan is paid and lending can continue to the next round

is

$$\alpha^n \binom{n}{n} + \alpha^{n-1} (1-\alpha) \binom{n}{n-1} + \dots + \alpha^1 (1-\alpha)^{n-1} \binom{n}{1} = [1 - (1-\alpha)^n]$$

The expected repayment for a borrower who plays the repayment strategy in each period t can be stated as follows:

$$\sum_{i=0}^{n-1} \binom{n-1}{i} \alpha^{n-i} (1-\alpha)^i \left(R + \frac{i}{n-i} R \right) = [1 - (1-\alpha)^n] R$$

Thus the expected lifetime utility of a borrower, who plays a repayment strategy at any period onwards in which he gets financing, is determined as

$$V_{JL}^R = \mathbb{E}(Y) - [1 - (1-\alpha)^n] R + [1 - (1-\alpha)^n] \delta V_{JL}^R$$

where $0 \leq \delta \leq 1$ is borrower's discount factor that determines his valuation of tomorrow's utility of financing. The expected lifetime utility of a borrower can be rewritten as

$$V_{JL}^R = \frac{\mathbb{E}(Y) - R [1 - (1-\alpha)^n]}{1 - \delta [1 - (1-\alpha)^n]} \quad (1)$$

Now the lender's optimization problem for any $0 \leq Y^L < Y^H$, $0 \leq \alpha \leq 1$, and $0 \leq \delta \leq 1$ can be stated as $\underset{L,R}{\text{maximize}} V_{JL}^R$ subject to:

1. The stipulated repayment amount for a successful borrower cannot exceed his output (even in worse case that everyone else has failed) and it must be affordable,

$$nR \leq Y^H \quad (2)$$

2. "Each borrower is repaying when his project is successful" is a subgame perfect equilibrium, if the one shot deviation constraint is satisfied. Thus for a successful borrower, the payoff of strategically defaulting can not be larger than the payoff of repaying and being refinanced,

$$Y^H \leq Y^H - nR + \delta V_{JL}^R$$

The above condition can be simplified to

$$nR \leq \delta V^R \quad (3)$$

that also implies $R < \delta V^R$, which guarantees that a successful borrower pays repayment R when all his partners are successful.

3. The lender must be able to sustain the lending game over periods and at least break even.

So the expected repayment amount of the group has to be at least as large as $n(L + \epsilon)$,

$$\alpha^n \binom{n}{n} nR + \alpha^{n-1} (1 - \alpha) \binom{n}{n-1} nR + \dots + \alpha^1 (1 - \alpha)^{n-1} \binom{n}{1} nR \geq n(L + \epsilon)$$

that can be simplified to

$$R \geq \frac{L + \epsilon}{[1 - (1 - \alpha)^n]} \quad (4)$$

If there are some (L, R) that satisfy constraints (2), (3) and (4), then JL lending will be feasible, and these constraints will define its feasibility region. Note that individual lending can be considered as a special type of group lending with $n = 1$.

Proposition 1. *There are $\tilde{\delta}_{JL}(n, \alpha, Y^H, Y^L)$, $\tilde{L}_{JL}(n, \alpha, Y^H)$ and $\hat{L}_{JL}(n, \alpha, \delta, Y^H, Y^L)$ such that:*

a) *If $\delta \geq \tilde{\delta}_{JL}(n, \alpha, Y^H, Y^L)$, then JL lending is feasible iff $L \leq \tilde{L}_{JL}(n, \alpha, Y^H)$.*

b) *If $\delta \leq \tilde{\delta}_{JL}(n, \alpha, Y^H, Y^L)$, then JL lending is feasible iff $L \leq \hat{L}_{JL}(n, \alpha, \delta, Y^H, Y^L)$.*

Moreover, whenever JL lending is feasible: for any $\alpha \neq 0$, the lender demands an optimal repayment $R = \frac{L + \epsilon}{[1 - (1 - \alpha)^n]}$ from each borrower; and the expected lifetime utility for each borrower will amount to $V_{JL}^R = \frac{\mathbb{E}(Y) - (L + \epsilon)}{1 - \delta[1 - (1 - \alpha)^n]}$.

Proof. In case of JL lending, the optimal contract (L, R) is a solution to the following problem:

$$\begin{aligned} \underset{L, R}{\text{maximize}} \quad V_{JL}^R &= \frac{\mathbb{E}(Y) - R[1 - (1 - \alpha)^n]}{1 - \delta[1 - (1 - \alpha)^n]} \\ \text{s.t.} \quad nR &\leq \delta V_{JL}^R \\ nR &\leq Y^H \\ R &\geq \frac{L + \epsilon}{1 - (1 - \alpha)^n} \end{aligned}$$

As V_{JL}^R is decreasing in R , the lender would like to set R as low as possible. The third constraint gives the minimum R required for breaking even, $R = \frac{L+\varepsilon}{[1-(1-\alpha)^n]}$, for any $\alpha \neq 0$. As long as R is limited to the upper limits expressed in constraints 1 and 2, lending is feasible, otherwise is not feasible. Replacing $R = \frac{L+\varepsilon}{[1-(1-\alpha)^n]}$ in the objective function and constraints, we will have

$$\begin{aligned} \underset{L,R}{\text{maximize}} \quad V_{JL}^R &= \frac{\mathbb{E}(Y) - (L + \varepsilon)}{1 - \delta [1 - (1 - \alpha)^n]} \\ \text{s.t.} \quad L &\leq \frac{\delta \mathbb{E}(Y) [1 - (1 - \alpha)^n]}{n - \delta (n - 1) [1 - (1 - \alpha)^n]} - \varepsilon \equiv \hat{L}_{JL}(n, \alpha, \delta, Y^H, Y^L) \\ L &\leq \frac{[1 - (1 - \alpha)^n] Y^H}{n} - \varepsilon \equiv \tilde{L}_{JL}(n, \alpha, Y^H) \end{aligned}$$

Any feasible answer for the above problem must satisfy both constraints. Therefore we have to have $L \leq \min \{ \hat{L}_{JL}, \tilde{L}_{JL} \}$. There are two cases, it is either the case that $\hat{L}_{JL} \leq \tilde{L}_{JL}$ or the case that $\tilde{L}_{JL} \leq \hat{L}_{JL}$.

i) $\hat{L}_{JL} \leq \tilde{L}_{JL}$ iff

$$\delta \leq \frac{Y^H}{\mathbb{E}(Y) + \frac{n-1}{n} [1 - (1 - \alpha)^n] Y^H} \equiv \tilde{\delta}_{JL}(n, \alpha, Y^H, Y^L)$$

ii) $\tilde{L}_{JL} \leq \hat{L}_{JL}$ iff $\delta \geq \tilde{\delta}_{JL}$.

Thus for $\delta \leq \tilde{\delta}_{JL}$, JL lending is feasible for any $L \leq \hat{L}_{JL}$, and for $\delta \geq \tilde{\delta}_{JL}$, JL lending is feasible for any $L \leq \tilde{L}_{JL}$. \square

The first result of Proposition 1 shows that the maximum loan depends on the magnitude of δ . The second result of Proposition 1 suggests that the optimal repayment is decreasing in group size. Therefore a larger group can be charged less that in turn increases the borrowers welfare. Corollary 1 is a direct result from Proposition 1.

Corollary 1. *Individual lending is feasible iff $L \leq \alpha \delta \mathbb{E}(Y)$. For any $\alpha \neq 0$, the lender demands optimally the repayment $R = \frac{L+\varepsilon}{\alpha}$. The borrower's expected lifetime utility will be $V_{IL}^R = \frac{\mathbb{E}(Y) - (L+\varepsilon)}{1-\alpha\delta}$.*

We continue this section by trying to discover how the feasibility function and consequently feasibility of JL lending is affected by the group size and what is the optimal group size that results

in maximum feasibility. To be able to describe the properties of the feasibility function with respect to changes of n , in Lemma 1 and Lemma 2 we take a closer look at the changes of \hat{L} , \tilde{L} and $\tilde{\delta}$ with respect to changes of n when other parameters $(\alpha, \delta, Y^H, Y^L)$ are given.

Lemma 1. *Assume $\hat{L}_{JL}(n, \alpha, \delta, Y^H, Y^L)$ and $\tilde{L}_{JL}(n, \alpha, Y^H)$ are functions defined in Proposition 1.*

- 1) *There is a $\hat{\delta}_{JL}(n, \alpha)$ such that:*
 - i. *For any $0 < \delta < \hat{\delta}_{JL}(n, \alpha)$, $\hat{L}_{JL}(n, \alpha, \delta, Y^H, Y^L)$ is strictly decreasing in n .*
 - ii. *For any $\hat{\delta}_{JL}(n, \alpha) < \delta < \tilde{\delta}_{JL}(n, \alpha, Y^H, Y^L)$, $\hat{L}_{JL}(n, \alpha, \delta, Y^H, Y^L)$ strictly increasing in n .*
- 2) *$\tilde{L}_{JL}(n, \alpha, Y^H)$ is strictly decreasing in n .*

As we see in Lemma 1, changes of \hat{L} with respect to n is also affected by changes of $\hat{\delta}$ with respect to n . So we need to have understanding about how $\hat{\delta}(n, \alpha)$ changes with respect to changes of n .

Lemma 2. *Assume $\tilde{\delta}_{JL}(n, \alpha, Y^H, Y^L)$ and $\hat{\delta}_{JL}(n, \alpha)$ are functions defined in Proposition 1 and Lemma 1.*

- 1) *$\hat{\delta}_{JL}(n, \alpha)$ is strictly increasing in both n and α .*
- 2) *For any given n and α , $-\infty < \hat{\delta}_{JL}(n, \alpha) < 1$.*
- 3) *$\tilde{\delta}_{JL}(n, \alpha, Y^H, Y^L)$ is strictly decreasing in both n and α .*
- 4) *For any given n and α , $\frac{n}{2n-1} < \tilde{\delta}_{JL}(n, \alpha, Y^H, Y^L) < \frac{Y^H}{Y^L}$.*

A direct result of Lemma 2 is that interval $(\hat{\delta}_{JL}, \tilde{\delta}_{JL})$ becomes tighter by the increase of n or α , and it becomes wider by the decrease of n or α . Therefore to keep the interval $(\hat{\delta}_{JL}, \tilde{\delta}_{JL})$ non-empty, the larger the α is, the smaller the n must be chosen in order to offset the contraction of the interval resulted by large α . And in general, $n = 2$ always provides the widest interval for any given α . Below in Proposition 2, we prove that regardless of the magnitude of n , for small enough α , the interval $(\hat{\delta}_{JL}, \tilde{\delta}_{JL})$ is never empty. However if α becomes too small, then project may not be considered for financing.

Proposition 2. *Assume $\tilde{\delta}_{JL}(n, \alpha, Y^H, Y^L)$ and $\hat{\delta}_{JL}(n, \alpha)$ are functions defined in Proposition 1 and Lemma 1.*

- 1) *There exists $\bar{\alpha}$, such that for any $\alpha < \bar{\alpha}$, feasibility of JL is increasing in $n \in [2, N_{\alpha, \delta}]$, only if $\hat{\delta}_{JL}(n, \alpha) < \delta < \tilde{\delta}_{JL}(n, \alpha, Y^H, Y^L)$, where $N_{\alpha, \delta} = \min \left\{ \lfloor \hat{\delta}_{JL}^{-1}(\alpha, \delta) \rfloor, \lfloor \tilde{\delta}_{JL}^{-1}(\alpha, \delta, Y^H, Y^L) \rfloor \right\}$.*

2) For any $\alpha > \bar{\alpha}$, maximum feasibility happens at $n = 2$.

3) $\bar{\alpha}$ is approximately 0.5.

4) For very large n , feasibility of JL will be decreasing.

Proof. According to Proposition 1, JL is feasible iff $L \leq F_{JL}(n, \alpha, \delta, Y^H, Y^L)$, where

$$F_{JL}(n, \alpha, \delta, Y^H, Y^L) = \min \left\{ \hat{L}_{JL}, \tilde{L}_{JL} \right\} = \begin{cases} \hat{L}_{JL} & \text{if } \delta \leq \tilde{\delta}_{JL} \\ \tilde{L}_{JL} & \text{if } \delta \geq \tilde{\delta}_{JL} \end{cases}$$

According to Lemma 1, for given α and δ , if there are some n such that $\hat{\delta}_{JL} < \delta < \tilde{\delta}_{JL}$, then feasibility function $F_{JL}(n, \alpha, \delta, Y^H, Y^L)$ will be strictly increasing in n . Everywhere else $F_{JL}(n, \alpha, \delta, Y^H, Y^L)$ will be strictly decreasing in n .

1) For what n and α , feasibility function can be increasing? Let us first look at the extreme cases where α is very small or α is very large. We know from Lemma 2 that

$$a) \lim_{\alpha \rightarrow 0} \hat{\delta}_{JL} = -\infty \text{ and } \lim_{\alpha \rightarrow 0} \tilde{\delta}_{JL} = \frac{Y^H}{Y^L}.$$

$$b) \lim_{\alpha \rightarrow 1} \hat{\delta}_{JL} = \frac{Y^H}{Y^L} \text{ and } \lim_{\alpha \rightarrow 1} \tilde{\delta}_{JL} = \frac{n}{2n-1}.$$

Thus, for very small α , regardless of the magnitude of n , we have $\hat{\delta}_{JL} < \tilde{\delta}_{JL}$, and for very large α and for any $n > 1$, we have $\hat{\delta}_{JL} > \tilde{\delta}_{JL}$. We also know from Lemma 2 that $\hat{\delta}_{JL}$ and $\tilde{\delta}_{JL}$ are monotonic. Therefore, we should conclude that $\hat{\delta}_{JL}$ and $\tilde{\delta}_{JL}$ coincide only once at some critical $\bar{\alpha} \neq 0$. So for any given n , if $\alpha \in (0, \bar{\alpha})$, then $\hat{\delta}_{JL} < \tilde{\delta}_{JL}$. Consequently, for $\alpha \in (0, \bar{\alpha})$, feasibility function will be increasing in n and will be maximized at

$$N_{\alpha, \delta} = \max \left\{ n \mid \hat{\delta}_{JL} < \delta < \tilde{\delta}_{JL} \right\}$$

Our algorithm to find the maximum n when α and δ are given, would be as follows: start with $n = 2$ and increase n one by one until one of the $\hat{\delta}_{JL}$ or $\tilde{\delta}_{JL}$ equals δ so that n cannot be increased further. If $\hat{\delta}_{JL} = \delta$, then $N_{\alpha, \delta} = \lfloor \hat{\delta}_{JL}^{-1} \rfloor$ (note that we are only interested in n that is a natural number) and if $\tilde{\delta}_{JL} = \delta$, then $N_{\alpha, \delta} = \lfloor \tilde{\delta}_{JL}^{-1} \rfloor$ (see Figure 1 for intuition). Thus $N_{\alpha, \delta}$ can be rewritten as $N_{\alpha, \delta} = \min \left\{ \lfloor \hat{\delta}_{JL}^{-1} \rfloor, \lfloor \tilde{\delta}_{JL}^{-1} \rfloor \right\}$.

2) A direct result from the discussion in part 1 of this proof is that for $\alpha > \bar{\alpha}$, feasibility will be decreasing in n and so maximum feasibility is given at $n = 2$.

	$n = 2$	$n = 3$	$n = 4$	$n = 5$	\dots	$n = 10$	\dots	$n = 50$
$\alpha < 0.718$	✓							
$\alpha < 0.568$	✓	✓						
$\alpha < 0.477$	✓	✓	✓					
$\alpha < 0.415$	✓	✓	✓	✓				
$\alpha < 0.269$	✓	✓	✓	✓		✓		
$\alpha < 0.082$	✓	✓	✓	✓		✓		✓

Table 1: The range of feasible group sizes for some given α

3) What is the maximum α for which $\hat{\delta}_{JL} < \delta < \tilde{\delta}_{JL}$? For simplicity of calculations, assume $Y^L = 0$ and $0 < \alpha < 1$. $\hat{\delta}_{JL} < \tilde{\delta}_{JL}$ iff

$$\frac{(1 - \alpha)^n \ln(1 - \alpha)^n + [1 - (1 - \alpha)^n]}{[1 - (1 - \alpha)^n]^2} < \frac{n}{n\alpha + (n - 1)[1 - (1 - \alpha)^n]}$$

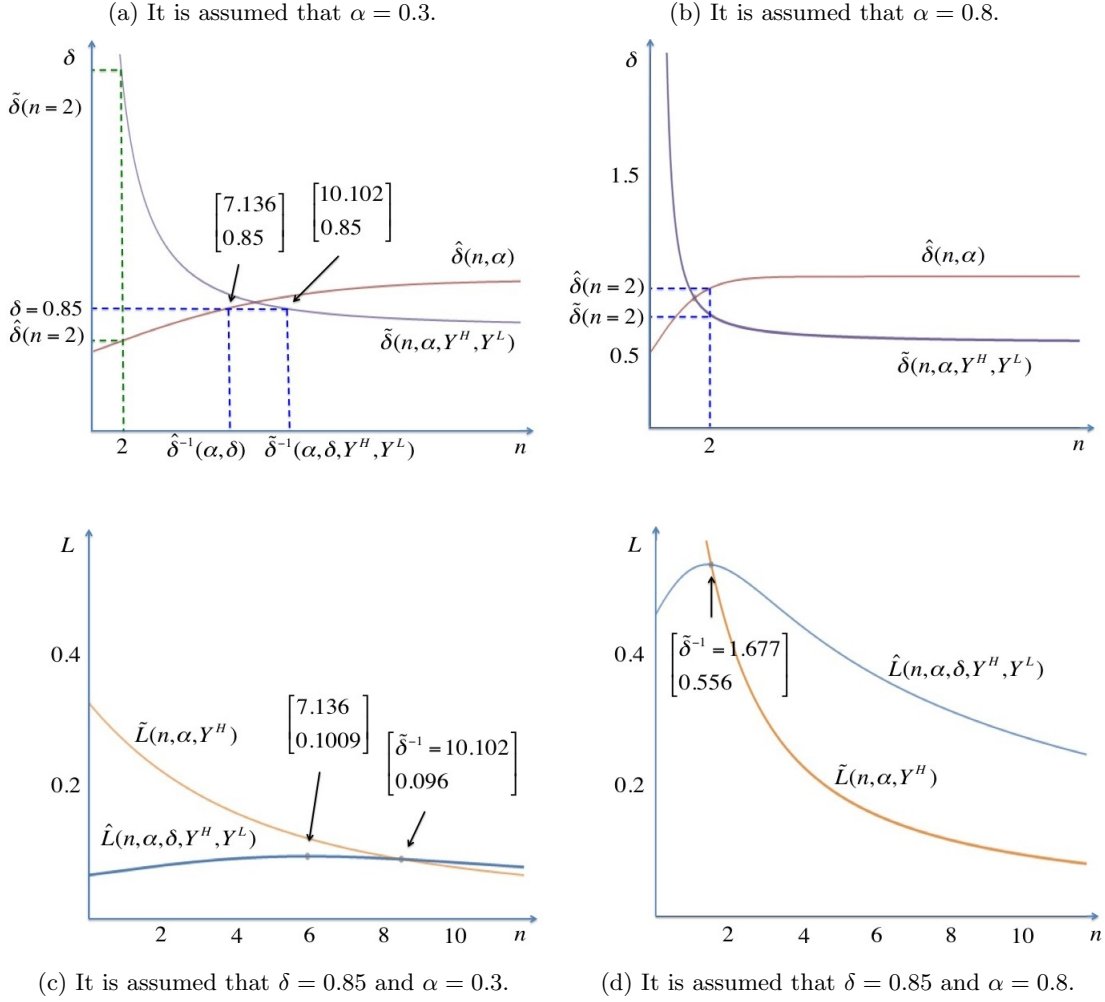
For $n = 2$, it can be easily verified that, the above inequality holds for any $\alpha < 0.718$. Put it differently, for $\alpha < 0.718$, the inequality holds at least for $n = 2$. We can see in Table 1 that the largest α for which the inequality holds for more than one n is $\alpha < 0.568$. So feasibility can be increasing in n only if $\alpha < 0.568$ and the given δ is such that $\hat{\delta}_{JL} < \delta < \tilde{\delta}_{JL}$. Note that when $Y^L \neq 0$, α becomes a slightly smaller. For example when Y^L is as large as $\frac{Y^H}{2}$, α is approximately 0.458.

4) If n grows very large, feasibility of JL will be decreasing in n : $\lim_{n \rightarrow \infty} \hat{\delta}_{JL} = 1$ and $\lim_{n \rightarrow \infty} \tilde{\delta}_{JL} = \frac{Y^H}{\mathbb{E}(Y) + Y^H} \leq 1$. So except for the case that $\alpha \rightarrow 0$ and $Y^L = 0$ simultaneously, for very large n , $(\hat{\delta}_{JL}, \tilde{\delta}_{JL})$ will be empty. \square

Proposition 2 is telling us that if the chance of success in the project is low (less than 50%), larger loans could be given to larger groups (see Figure 1, parts (a) and (c)). While for projects with higher chance of success (more than 50%), the maximum loan that can be given to a group of two members is higher than any other group (see Figure 1, parts (b) and (d)).

Proposition 2 also suggests that the group size cannot grow too large. Intuitively group size has two countervailing effects. On one side, larger group can provide stronger repayment insurance and is able to handle riskier projects and repay successfully. On the other side, a large group can be a threat towards feasibility of group lending. The threat comes from the point that each successful member is in charge of all defaulting peers and if everybody else has failed, he must repay the entire

Figure 1: Proposition 2.



loan of the group. And this becomes very difficult if the group is too large. Proposition 2 suggests an algorithm to find the optimal group size when parameters α and δ are given.

Figure 1 adds some intuition to Proposition 2. For simplicity it is assumed that $Y^H = 1$ and $Y^L = 0$. In parts (a) and (b), we depict $\tilde{\delta}$ and $\hat{\delta}$ with respect to n when chance of success in project is small, e.g. $\alpha = 0.3$, and when it is large, e.g. $\alpha = 0.8$, respectively. As shown in part (a), for $\alpha = 0.3$, there are some n for which the given $\delta = 0.85$ belongs to the interval $(\hat{\delta}_{JL}, \tilde{\delta}_{JL})$, and the largest of such n lies at $\lceil \hat{\delta}_{JL}^{-1} \rceil = \lceil 7.136 \rceil = 7$. Note that if the given δ is very close to 1, then the largest n lies at $\lceil \tilde{\delta}_{JL}^{-1} \rceil$. However in part (b), when α is large, for any $n > 1$, the interval $(\hat{\delta}_{JL}, \tilde{\delta}_{JL})$ is empty.

Parts (c) and (d) of Figure 1 depict \hat{L} and \tilde{L} with respect to n when chance of success in project

is small $\alpha = 0.3$, and when it is large $\alpha = 0.8$ respectively. As shown in part (c), when α is small, \hat{L} defines the boundary for maximum feasible loan, and it reaches its maximum at $n = 7$. Therefore $n = 7$ is the group size that maximizes the feasibility of JL. However in part (d), when α is large, \tilde{L} defines the boundary for maximum feasible loan which is decreasing in n . Therefore $n = 2$ is the optimum group size that maximizes the feasibility of JL.

Up to here, we discussed that feasibility of JL lending can be increasing in group size when chance of success in project is small. However we do not know yet if it can perform more efficiently than IL lending does. Are there circumstances under which JL lending outperforms IL lending? Proposition 3 proves formally that the lender can charge borrowers less under JL lending comparing to IL lending while staying still break-even. The reason for this may be: “no repayment” is something that happens less often under JL than IL. And in turn smaller amount of repayment leads to a higher level of welfare for the borrowers.

Proposition 3. *When both IL and JL contracts are feasible, then*

- 1) *Borrowers repayment amount is lower and his welfare is higher under JL than IL.*
- 2) *There exists $\underline{\alpha}$, such that for $\alpha < \underline{\alpha}$, JL is feasible for a larger range of loans than IL only if at least for some n , $\frac{\alpha n - [1 - (1 - \alpha)^n]}{\alpha(n-1)[1 - (1 - \alpha)^n]} < \delta < \frac{Y^H [1 - (1 - \alpha)^n]}{n\alpha\mathbb{E}(Y)}$.*
- 3) *For any $\alpha > \underline{\alpha}$, IL is feasible for a larger range of loan than JL.*
- 4) *$\underline{\alpha}$ is approximately 0.764.*
- 5) *For very large n , IL can always offer larger range of loans than JL.*

Proof. 1) When both IL and JL are feasible, then each borrower is supposed to repay $R = \frac{L+\varepsilon}{[1-(1-\alpha)^n]}$ under JL and $R = \frac{L+\varepsilon}{\alpha}$ under IL. Clearly he pays less under JL. Each borrower’s expected lifetime utility under JL is $V_{JL}^R = \frac{\mathbb{E}(Y) - (L+\varepsilon)}{1-\delta[1-(1-\alpha)^n]}$, and under IL is $V_{IL}^R = \frac{\mathbb{E}(Y) - (L+\varepsilon)}{1-\alpha\delta}$. Obviously, the expected lifetime utility of borrower is higher under JL.

2) For which one feasibility is easier to achieve, JL or IL? There are two cases:

i. If $\delta < \tilde{\delta}_{JL}$, then JL is feasible for all $L \leq \hat{L}_{JL} = \alpha\delta\mathbb{E}(Y) \times \frac{[1-(1-\alpha)^n]}{\alpha[n-\delta(n-1)[1-(1-\alpha)^n]]} - \varepsilon$, and IL is feasible for all $L \leq \alpha\delta\mathbb{E}(Y) - \varepsilon$. So JL can offer larger range of loans to borrowers than IL *iff*

$$\frac{[1 - (1 - \alpha)^n]}{\alpha [n - \delta (n - 1) [1 - (1 - \alpha)^n]]} > 1$$

that can be rewritten as

$$\delta > \frac{\alpha n - [1 - (1 - \alpha)^n]}{\alpha(n-1)[1 - (1 - \alpha)^n]}$$

Otherwise, IL can offer larger range of loans to borrowers than JL. Since we already have an upper bound for δ , it must be true that

$$\frac{\alpha n - [1 - (1 - \alpha)^n]}{\alpha(n-1)[1 - (1 - \alpha)^n]} < \delta < \tilde{\delta}_{JL}$$

ii. If $\delta > \tilde{\delta}_{JL}$, then JL is feasible for all $L \leq \tilde{L}_{JL} = \frac{Y^H[1 - (1 - \alpha)^n]}{n} - \varepsilon$, and again IL is feasible if $L \leq \alpha\delta\mathbb{E}(Y) - \varepsilon$. So JL can offer larger range of loan to each member than IL *iff*

$$\delta < \frac{Y^H [1 - (1 - \alpha)^n]}{n\alpha\mathbb{E}(Y)}$$

Since we already have a lower bound for δ , it must be true that

$$\tilde{\delta}_{JL} < \delta < \frac{Y^H[1 - (1 - \alpha)^n]}{n\alpha\mathbb{E}(Y)}$$

Comparing the results of parts (a) and (b), JL is feasible for larger amount of loans than IL *iff*

$$\frac{\alpha n - [1 - (1 - \alpha)^n]}{\alpha(n-1)[1 - (1 - \alpha)^n]} < \delta < \frac{Y^H[1 - (1 - \alpha)^n]}{n\alpha\mathbb{E}(Y)}$$

We call this condition the *relative feasibility condition* from now on. What are the circumstances for which the relative feasibility condition holds? A necessary condition to satisfy is

$$\frac{\alpha n - [1 - (1 - \alpha)^n]}{\alpha(n-1)[1 - (1 - \alpha)^n]} < \frac{Y^H[1 - (1 - \alpha)^n]}{n\alpha\mathbb{E}(Y)}$$

Both right-hand side and left-hand side of this inequality are strictly monotonic in $\alpha \in (0, 1)$. If α is very small, the inequality always holds:

$$\lim_{\alpha \rightarrow 0} \frac{\alpha n - [1 - (1 - \alpha)^n]}{\alpha(n-1)[1 - (1 - \alpha)^n]} = \frac{1}{2}$$

$$\lim_{\alpha \rightarrow 0} \frac{Y^H [1 - (1 - \alpha)^n]}{n\alpha \mathbb{E}(Y)} = \frac{Y^H}{Y^L}$$

If α is very large, the inequality never holds:

$$\lim_{\alpha \rightarrow 1} \frac{\alpha n - [1 - (1 - \alpha)^n]}{\alpha (n - 1) [1 - (1 - \alpha)^n]} = 1$$

$$\lim_{\alpha \rightarrow 1} \frac{Y^H [1 - (1 - \alpha)^n]}{n\alpha \mathbb{E}(Y)} = \frac{1}{n}$$

Therefore, we should conclude that there exists a critical $\underline{\alpha}$ such that for any $\alpha < \underline{\alpha}$, the relative feasibility condition holds.

3) A direct result of part 2 of this proof is that for any $\alpha > \underline{\alpha}$, then the relative feasibility function never holds and so IL performs better than JL in terms of larger feasible maximum loans.

4) What is the maximum α for which

$$\frac{\alpha n - [1 - (1 - \alpha)^n]}{\alpha (n - 1) [1 - (1 - \alpha)^n]} < \frac{Y^H [1 - (1 - \alpha)^n]}{n\alpha \mathbb{E}(Y)}$$

For simplicity of calculations, we assume $Y^L = 0$

$$\frac{\alpha n - [1 - (1 - \alpha)^n]}{\alpha (n - 1) [1 - (1 - \alpha)^n]} < \frac{[1 - (1 - \alpha)^n]}{n\alpha^2}$$

It is easy to verify that if $n = 2$, the inequality holds for $\alpha < 0.764$. Thus for $\alpha < 0.764$, JL does better than IL at least for groups of $n = 2$. As we can see in Table 2, for any other smaller α , there are more than one group size for which JL does better than IL. Therefore for $\alpha < 0.764$, JL outperforms IL if the given δ is such that

$$\frac{\alpha n - [1 - (1 - \alpha)^n]}{\alpha (n - 1) [1 - (1 - \alpha)^n]} < \delta < \frac{Y^H [1 - (1 - \alpha)^n]}{n\alpha \mathbb{E}(Y)}$$

Note that if $Y^L \neq 0$, then α must be a slightly smaller. As an instance if $Y^L = \frac{Y^H}{2}$, then we should have $\alpha < 0.697$.

	$n = 2$	$n = 3$	$n = 4$	$n = 5$	\dots	$n = 10$	\dots	$n = 50$
$\alpha < 0.764$	✓							
$\alpha < 0.634$	✓	✓						
$\alpha < 0.552$	✓	✓	✓					
$\alpha < 0.495$	✓	✓	✓	✓				
$\alpha < 0.349$	✓	✓	✓	✓		✓		
$\alpha < 0.150$	✓	✓	✓	✓		✓		✓

Table 2: The range of group sizes for some given α for which JL outperforms IL

5) For very large n , the relative feasibility condition never holds:

$$\lim_{n \rightarrow \infty} \frac{(n-1)[1-(1-\alpha)^n]^2}{\alpha n^2 - n[1-(1-\alpha)^n]} = \lim_{n \rightarrow \infty} \frac{[1-(1-\alpha)^n]^2}{\alpha n - [1-(1-\alpha)^n]} = 0$$

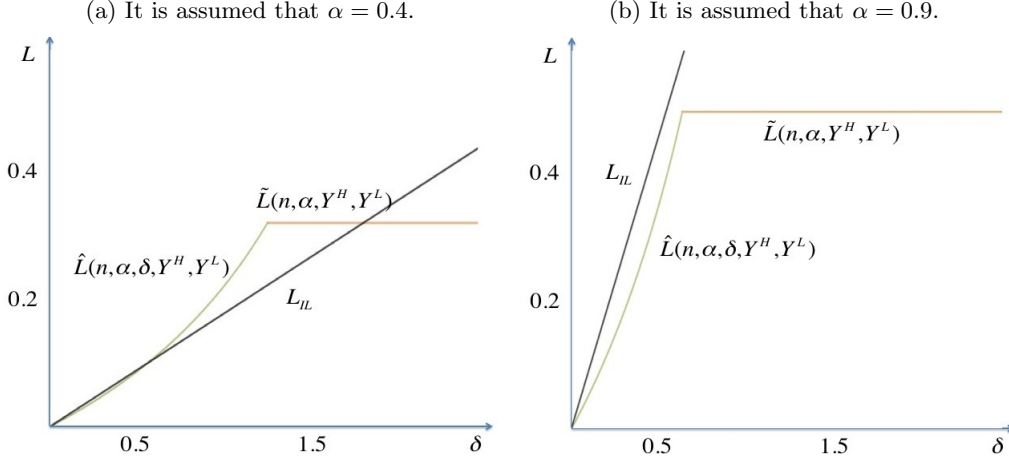
$$\lim_{n \rightarrow \infty} \frac{Y^H[1-(1-\alpha)^n]}{n\alpha\mathbb{E}(Y)} = 0$$

□

Proposition 3 shows that JL contract has a positive effect on borrower's welfare and repayment rate compared to IL contract. It suggests a necessary condition, the *relative feasibility condition*, under which JL contract can offer larger maximum loan than IL contract, and as discussed before in Proposition 2, the amount of loan can be increasing in the group size. Proposition 3 also discusses the circumstances that are necessary for the relative feasibility condition to be satisfied. It proves that if the chance of success in project is high (higher than 75%), the maximum loan that can be given to a borrower under IL contract is higher than JL contract. While for projects with lower chance of success (lower than 75%), larger loans could be given only under JL contracts. As it was expected, the group size cannot grow too large. JL contract with groups of more than 10 members can have better outcome than IL contract only when project has very little chances of success (35%), otherwise IL contract has higher outcome than JL with groups of too many members.

Figure 2 illustrates the situation discussed in Proposition 3. For simplicity it is assumed that $Y^H = 1$ and $Y^L = 0$. As shown in part (a), when chance of success in project is small enough so that the relative feasibility condition is satisfied (for example $\alpha = 0.4$), for any $0 < \delta < 1$, \hat{L} defines the boundary for maximum feasible loan under JL which is always equal to or larger than

Figure 2: Proposition 3.



maximum feasible loan under IL. However when chance of success in project is large (for example $\alpha = 0.9$), maximum feasible loan under IL is strictly higher than maximum feasible loan under JL.

5 Flexible Joint Liability (FJL) Contracts

The grim-trigger strategy played in JL lending maybe too harsh. Consider a situation that some borrowers default strategically, but it is still beneficial for the other members of the group to take care of the entire repayment and be entitled for a new loan next round. Moreover the strategically defaulting members may have good explanations and the other group members may want to punish them with less than grim-trigger. In this section, we examine joint liability under a strategy that is less severe than grim-trigger. In this type of lending players start in lending phase and they cooperate until someone defaults strategically, then they go to a punishment phase and exclude him for T periods of receiving loans. Later when punishment is served they let him to enter the game again. Timing is the same as before. The lender's problem in this case is similar to the JL case with a change in the incentive constraint, equation (3). Each successful borrower must have correct incentive not to default strategically and repay not only for himself but also for all the defaulting peers (even for the entire group, in worst case). Thus we must have

$$Y^H + \delta^{T+1}V_{FJL}^R < Y^H - nR + \delta V_{FJL}^R$$

that can be simplified to

$$nR < (1 - \delta^T) \delta V_{FJL}^R \quad (5)$$

Clearly if T is a large constant, then $1 - \delta^T \rightarrow 1$, and equation (5) will be equal to equation (3). Note that the above incentive constraint is more difficult to satisfy compared to the incentive constraint of JL lending, as it is less costly to default strategically under FJL.

Proposition 4. *There are $\tilde{\delta}_{FJL}(n, \alpha, \delta, T, Y^H, Y^L)$, $\tilde{L}_{FJL}(n, \alpha, Y^H)$ and $\hat{L}_{FJL}(n, \alpha, \delta, T, Y^H, Y^L)$ such that:*

a) *For any $\delta \geq \tilde{\delta}_{FJL}(n, \alpha, \delta, T, Y^H, Y^L)$, JL is feasible iff $L \leq \tilde{L}_{FJL}(n, \alpha, Y^H)$.*

b) *For any $\delta \leq \tilde{\delta}_{FJL}(n, \alpha, \delta, T, Y^H, Y^L)$, JL is feasible iff $L \leq \hat{L}_{FJL}(n, \alpha, \delta, T, Y^H, Y^L)$.*

Moreover, whenever FJL is feasible, for any $\alpha \neq 0$, the lender demands the optimal repayment $R = \frac{L+\epsilon}{[1-(1-\alpha)^n]}$ from each borrower; and the expected lifetime utility for each borrower will amount to $V_{FJL}^R = \frac{\mathbb{E}(Y)-(L+\epsilon)}{1-\delta[1-(1-\alpha)^n]}$.

Proof. In case of FJL, the optimal contract (L, R) is a solution to the problem below. Adopting the same method used in the proof of Proposition 1, we solve this problem:

$$\begin{aligned} \underset{L,R}{\text{maximize}} \quad & V_{FJL}^R = \frac{\mathbb{E}(Y) - R[1 - (1 - \alpha)^n]}{1 - \delta[1 - (1 - \alpha)^n]} \\ \text{s.t.} \quad & nR \leq (1 - \delta^T) \delta V_{FJL}^R \\ & nR \leq Y^H \\ & R \geq \frac{L + \epsilon}{1 - (1 - \alpha)^n} \end{aligned}$$

It is clear that V_{FJL}^R is decreasing in R , so the bank would like to set R as low as possible. The third constraint gives the minimum R required for breaking even, $R = \frac{L+\epsilon}{[1-(1-\alpha)^n]}$. As long as this R is below the upper limit expressed in the first and second constraints, lending is feasible, otherwise

it is not feasible. Replacing R in the above problem, we will have:

$$\begin{aligned}
\underset{L,R}{\text{maximize}} \quad V_{FJL}^R &= \frac{\mathbb{E}(Y) - (L + \varepsilon)}{1 - \delta [1 - (1 - \alpha)^n]} \\
L &\leq \frac{\delta (1 - \delta^T) \mathbb{E}(Y) [1 - (1 - \alpha)^n]}{n - [n\delta - \delta (1 - \delta^T)] [1 - (1 - \alpha)^n]} - \varepsilon \equiv \hat{L}_{FJL}(n, \alpha, \delta, T, Y^H, Y^L) \\
L &\leq \frac{Y^H [1 - (1 - \alpha)^n]}{n} - \varepsilon \equiv \tilde{L}_{FJL}(n, \alpha, Y^H)
\end{aligned}$$

A feasible L must satisfy $L \leq \text{Min} \{ \hat{L}_{FJL}, \tilde{L}_{FJL} \}$, in order to satisfy both constraints. There are two cases, it is either the case that $\hat{L}_{FJL} \leq \tilde{L}_{FJL}$ or the case that $\tilde{L}_{FJL} > \hat{L}_{FJL}$.

i) $\hat{L}_{FJL} \leq \tilde{L}_{FJL}$ iff

$$\delta \leq \frac{Y^H}{(1 - \delta^T) \mathbb{E}(Y) + \frac{n - (1 - \delta^T)}{n} [1 - (1 - \alpha)^n] Y^H} \equiv \tilde{\delta}_{FJL}(n, \alpha, \mu, Y^H, Y^L)$$

ii) $\tilde{L}_{FJL} \leq \hat{L}_{FJL}$ iff $\delta \geq \tilde{\delta}$.

Therefore if $0 \leq \delta \leq \tilde{\delta}_{FJL}$, FJL is feasible for any $L \leq \hat{L}_{FJL}$, and if $\tilde{\delta}_{FJL} \leq \delta < 1$, FJL is feasible for any $L \leq \tilde{L}_{FJL}$. \square

As the first part of Proposition 4 suggests, $\tilde{L}_{FJL} = \tilde{L}_{JL}$; and for large T , $\tilde{\delta}_{FJL} = \tilde{\delta}_{JL}$ and $\tilde{L}_{FJL} = \tilde{L}_{JL}$. Intuitively if the punishment phase is very long, we are again in the same situation as JL lending that borrowers used to play grim-trigger strategy to their peers. The second part of Proposition 4 shows that the optimal repayment R and borrowers' welfare V_{FJL}^R under FJL lending, are the same as the optimal repayment and borrowers' welfare calculated in Proposition 1 for JL lending.

We continue this section by investigating if and how the length of the punishment phase can affect the maximum feasible loan granted to each borrower, and whether these changes affect the optimal group size. To this end, first in Lemma 3, we look at the changes of \hat{L}_{FJL} and $\tilde{\delta}_{FJL}$ with respect to changes of n and α when other parameters $(\alpha, \delta, Y^H, Y^L)$ are given. Next, in Lemma 4, we compare \tilde{L}_{FJL} and $\tilde{\delta}_{FJL}$ with their counterparts, \tilde{L}_{JL} and $\tilde{\delta}_{JL}$.

Lemma 3. *Assume $\hat{L}_{FJL}(n, \alpha, \delta, T, Y^H, Y^L)$, $\tilde{\delta}_{FJL}(n, \alpha, \delta, T, Y^H, Y^L)$ are functions defined in*

Proposition 4.

1) There is a $\hat{\delta}_{FJL}(n, \alpha)$ such that:

i. For any $0 < \delta < \hat{\delta}_{FJL}(n, \alpha)$, $\hat{L}_{FJL}(n, \alpha, \delta, T, Y^H, Y^L)$ is strictly decreasing in n .

ii. For any $\hat{\delta}_{FJL}(n, \alpha) < \delta < \tilde{\delta}_{FJL}(n, \alpha, \delta, T, Y^H, Y^L)$, $\hat{L}_{FJL}(n, \alpha, \delta, T, Y^H, Y^L)$ is strictly increasing in n .

2) $\tilde{\delta}_{FJL}(n, \alpha, \delta, T, Y^H, Y^L)$ is strictly decreasing in both n and α .

3) For any given n and α , $\frac{n}{(2n-1) - (n-1)\delta^T} < \tilde{\delta}_{FJL} < \frac{Y^H}{(1-\delta^T)Y^L}$.

Lemma 3 shows that the behavior of \hat{L}_{FJL} and $\tilde{\delta}_{FJL}$ in response to changes of n and α are similar to what we saw in Lemmas 1 and 2 for \hat{L}_{JL} and $\tilde{\delta}_{JL}$. Note that neither of $\hat{\delta}$ and \tilde{L} depend on T , and they are the same for JL and FJL. We recall from Lemmas 1 and 2 that \tilde{L} is strictly decreasing in n , $\hat{\delta}$ is strictly increasing in both n and α , and for any given n and α , $\hat{\delta} < 1$.

Lemma 4. Assume $\hat{L}_{JL}(n, \alpha, \delta, Y^H, Y^L)$ and $\tilde{\delta}_{JL}(n, \alpha, Y^H, Y^L)$ functions defined in Proposition 1, and $\hat{L}_{FJL}(n, \alpha, \delta, T, Y^H, Y^L)$ and $\tilde{\delta}_{FJL}(n, \alpha, \delta, T, Y^H, Y^L)$ are functions defined in Proposition 4.

1) $\hat{L}_{FJL}(n, \alpha, \delta, T, Y^H, Y^L) \leq \hat{L}_{JL}(n, \alpha, \delta, Y^H, Y^L)$.

2) $\hat{L}_{FJL}(n, \alpha, \delta, T, Y^H, Y^L)$ is strictly decreasing in T .

3) $\tilde{\delta}_{FJL}(n, \alpha, \delta, T, Y^H, Y^L) \geq \tilde{\delta}_{JL}(n, \alpha, Y^H, Y^L)$.

4) $\tilde{\delta}_{FJL}(n, \alpha, \delta, T, Y^H, Y^L)$ is strictly decreasing in T .

Lemma 4 shows that the interval $(\hat{\delta}, \tilde{\delta}_{FJL})$ is wider than $(\hat{\delta}, \tilde{\delta}_{JL})$, that is $(\hat{\delta}, \tilde{\delta}_{FJL})$ can allow for larger group sizes than it was possible under JL. However, this lemma argues that $\tilde{\delta}_{FJL}$ is strictly decreasing in the length of the punishment period, T . Intuitively, when punishment phase is not that long and it is less costly for members to default strategically, we need a larger group to assure repayment.

Proposition 5. When both JL and FJL are feasible:

1) FJL is feasible for at most the same range of loans as JL.

2) Borrowers' repayment amount and their welfare are the same with JL and FJL.

Proof. First, we recall from Propositions 1 and 4, the maximum loan that can be offered to each borrower each time is the minimum between \hat{L} and \tilde{L} and it is dependent on the magnitude of $\tilde{\delta}$.

We also recall from Lemma 4 that $\tilde{\delta}_{JL} \leq \tilde{\delta}_{FJL}$.

1) In order to compare the feasible loans that can be offered under FJL and JL, we need to look at three different cases:

i. For any $\delta < \tilde{\delta}_{JL}$, JL is feasible for all $L \leq \hat{L}_{JL}$ and FJL is feasible for all $L \leq \hat{L}_{FJL}$. According to Lemma 4, $\hat{L}_{FJL} \leq \hat{L}_{JL}$. That is for any $\delta < \tilde{\delta}_{JL}$, JL can offer larger range of loans to each borrower than FJL.

ii. For any $\tilde{\delta}_{JL} \leq \delta \leq \tilde{\delta}_{FJL}$, FJL is feasible for all $L \leq \hat{L}_{FJL}$ and JL is feasible for all $L \leq \tilde{L}$. From the proof of Proposition 4, we know that for any $\delta \leq \tilde{\delta}_{FJL}$, $\hat{L}_{FJL} \leq \tilde{L}$. Therefore for any $\tilde{\delta}_{JL} \leq \delta \leq \tilde{\delta}_{FJL}$, JL can offer larger range of loans to each member than FJL.

iii. For any $\delta > \tilde{\delta}_{FJL}$, both JL and FJL are feasible for any $L \leq \tilde{L}$. That is for any $\delta > \tilde{\delta}_{FJL}$, both JL and FJL can offer similar range of loans to each borrower.

2) A comparison between the results of Propositions 1 and 4 can show that borrowers' repayment amount and their welfare are the same with JL and FJL. \square

Proposition 5 discusses that FJL has a disadvantage compared to JL in terms of the maximum loan that can be offered to borrowers. As shown, under FJL lending the range of feasible loans is smaller or equal to what could be offered under JL lending. However both types of lending contract necessitate the same repayment amount and provide the same lifetime benefit for the borrower. Therefore, from borrowers' point of view FJL and JL contracts are almost similar. This gives the opportunity to the lender to decide how much flexible she wants to be about strategic default.

6 Project Correlation

Although most of the literature on group lending in microfinance assumes independence between projects,³ it may be more realistic to assume that projects are correlated. Intuitively, being in charge of each other's repayment would increase the cooperation among group members, thus group members may actually contribute to each other's chance of success in project. Moreover, correlation can come from the macro environment or because the businesses are linked to each other. For example, when harvest is good in a region, all businesses related to agriculture are

³There are some exceptions that consider project correlations. As remarkable examples we could mention Laffont (2003) and Ahlin and Townsend (2007)

affected. In village economies, as money increases in the society, each little business becomes more attractive.

In this section, assuming that α is not a constant but an increasing function of n , we examine whether larger groups are always better and have higher outcomes when projects are positively correlated. We only consider the case of FJL contracts, as the JL contracts can be seen as special cases of FJL in which the punishment phase is very long, i.e. $T \rightarrow \infty$. We assume $\alpha(n)$ is a concave function that is also a realistic assumption, because a very large group should also deal with higher tensions and could have adverse effects. We also assume $Y^H = 1$ and $Y^L = 0$ for simplicity of calculations. Note that we don't lose generality with this later assumption, as we already assumed that outcome could only be high or low and nothing in between.

Since assuming project correlation doesn't affect our model, the results of Propositions 4 stays valid. In what follows, we examine the effect of project correlation on the maximum feasible loan under FJL. The following lemma provides us with some insight on where maximum feasible loan can be increasing in the group size.

Lemma 5. *Assume $\tilde{\delta}'_{FJL}(n, \alpha(n), \delta, T)$, $\hat{L}'_{FJL}(n, \alpha(n), \delta, T)$ and $\tilde{L}'_{FJL}(n, \alpha(n))$ are functions defined in Proposition 4 with the only difference that α is a function of n , and it is assumed that $Y^H = 1$ and Y^L .*

- 1) *If $\alpha'(n) < \frac{\alpha(n)}{n-(1-\delta^T)}$, then there exists $\hat{\delta}'_{FJL}(n, \alpha(n), \alpha'(n), \delta, T)$ such that*
 - i. *For any $0 < \delta < \hat{\delta}'_{FJL}(n, \alpha(n), \alpha'(n), \delta, T)$, $\hat{L}'_{FJL}(n, \alpha(n), \delta, T)$ is strictly decreasing in n .*
 - ii. *For any $\hat{\delta}'_{FJL}(n, \alpha(n), \alpha'(n), \delta, T) < \delta < \tilde{\delta}'_{FJL}(n, \alpha(n), \delta, T)$, $\hat{L}'_{FJL}(n, \alpha(n), \delta, T)$ is strictly increasing in n .*
- 2) *If $\alpha'(n) < \frac{1-(1-\alpha(n))^n + (1-\alpha(n))^n \ln(1-\alpha(n))^n}{n(1-\alpha(n))^{n-1}}$, then $\tilde{L}'_{FJL}(n, \alpha(n))$ is strictly decreasing in n .*

Lemma 5 shows that when projects are weakly correlated (i.e. correlation degree is lower than boundaries suggested in Lemma 5), the determinants of maximum feasible loan under FJL, i.e. $\hat{L}'_{FJL}(n, \alpha(n), \delta)$ and $\tilde{L}'_{FJL}(n, \alpha(n))$, still show behaviors similar to the case that projects are independent. Therefore, similar to the independent projects case, the maximum feasible loan could be increasing in group size for some midrange discount factor, more specifically for any $\delta \in \left(\hat{\delta}'_{FJL}, \tilde{\delta}'_{FJL}\right)$. Therefore, whether maximum loan is increasing in n , in case projects are weakly correlated, is dependent on $\left(\hat{\delta}'_{FJL}, \tilde{\delta}'_{FJL}\right)$ being non-empty. However, contrary to the case that

projects are independent, this is not the only time that maximum loan could increase in n .

From Lemma 5, it could also be inferred that if the projects are strongly correlated (i.e. correlation degree is higher than boundaries suggested in Lemma 5), then the behavior of \hat{L}'_{FJL} and \tilde{L}'_{FJL} will be reversed. In this case, for any $0 < \delta < \hat{\delta}'_{FJL}$, \hat{L}'_{FJL} will be strictly increasing in n ; and for any $\hat{\delta}'_{FJL} < \delta < \tilde{\delta}'_{FJL}$, \hat{L}'_{FJL} will be strictly decreasing in n ; moreover \tilde{L}'_{FJL} will be always strictly increasing in n . Put it concisely, when projects are strongly correlated, the determinants of maximum feasible loan under FJL and consequently the maximum feasible loan itself are always increasing in group size except for some mid range discount factor, $\delta \in (\hat{\delta}'_{FJL}, \tilde{\delta}'_{FJL})$. Proposition 6 proves that $(\hat{\delta}'_{FJL}, \tilde{\delta}'_{FJL})$ is non-empty when projects are weakly correlated and it is empty when they are strongly correlated.

Proposition 6. *When projects are correlated, feasibility of FJL is always increasing in n .*

Proof. 1) When projects are weakly correlated (i.e. correlation degree is lower than boundaries suggested in Lemma 5), feasibility of FJL is increasing in n as far as

$$\hat{\delta}'_{FJL}(n, \alpha(n), \alpha'(n), \delta, T) < \tilde{\delta}'_{FJL}(n, \alpha(n), \delta, T)$$

We know from Lemma 5 that $\alpha' < \frac{\alpha}{n-(1-\delta^T)}$, thus the above inequality can be written as

$$\alpha' > \frac{\alpha [1 - (1 - \alpha)^n] + \alpha (1 - \alpha)^n \ln(1 - \alpha)^n - \frac{\alpha [1 - (1 - \alpha)^n]^2}{\alpha (1 - \delta^T) + \frac{n-1+\delta^T}{n} [1 - (1 - \alpha)^n]}}{n [1 - (1 - \alpha)^n] + n^2 \alpha (1 - \alpha)^{n-1} - \frac{(n - 1 + \delta^T) [1 - (1 - \alpha)^n]^2}{\alpha (1 - \delta^T) + \frac{n-1+\delta^T}{n} [1 - (1 - \alpha)^n]}} \quad (6)$$

For large n , the right-hand side of inequality (6) is zero:

$$\begin{aligned} & \alpha [1 - (1 - \alpha)^n] + \alpha (1 - \alpha)^n \ln(1 - \alpha)^n - \frac{\alpha [1 - (1 - \alpha)^n]^2}{\alpha (1 - \delta^T) + \frac{n-1+\delta^T}{n} [1 - (1 - \alpha)^n]} \\ \lim_{n \rightarrow \infty} & \frac{n [1 - (1 - \alpha)^n] + n^2 \alpha (1 - \alpha)^{n-1} - \frac{(n-1 + \delta^T) [1 - (1 - \alpha)^n]^2}{\alpha (1 - \delta^T) + \frac{n-1+\delta^T}{n} [1 - (1 - \alpha)^n]}}{n + 0 - \frac{1}{2 - \delta^T}} \\ & = \lim_{n \rightarrow \infty} \frac{1 + 0 - \frac{1}{2 - \delta^T}}{n + 0 - \frac{1}{2 - \delta^T}} = 0 \end{aligned}$$

For small n , the right-hand side of inequality (6) is smaller than zero:

$$\begin{aligned} & \alpha [1 - (1 - \alpha)^n] + \alpha (1 - \alpha)^n \ln(1 - \alpha)^n - \frac{\alpha [1 - (1 - \alpha)^n]^2}{\alpha (1 - \delta^T) + \frac{n-1+\delta^T}{n} [1 - (1 - \alpha)^n]} \\ \lim_{n \rightarrow 1} & \frac{n [1 - (1 - \alpha)^n] + n^2 \alpha (1 - \alpha)^{n-1} - \frac{(n-1 + \delta^T) [1 - (1 - \alpha)^n]^2}{\alpha (1 - \delta^T) + \frac{n-1+\delta^T}{n} [1 - (1 - \alpha)^n]}}{n + 0 - \frac{1}{2 - \delta^T}} \\ & = \lim_{n \rightarrow 1} \frac{(1 - \alpha) \ln(1 - \alpha)}{2 - \delta^T} < 0 \end{aligned}$$

Thus the right-hand side of inequality (6) is always negative, while its left-hand side is always positive (we assumed that α is increasing in n). Therefore, inequality (6) is always valid.

2) From what we discussed in part 1 of this proposition can be understood that when projects are strongly correlated (i.e. correlation degree is higher than boundaries suggested in Lemma 5), feasibility of FJL is increasing in n iff

$$\hat{\delta}'_{FJL}(n, \alpha(n), \alpha'(n), \delta, T) > \tilde{\delta}'_{FJL}(n, \alpha(n), \delta, T)$$

As projects are strongly correlated, $\alpha' > \frac{\alpha}{n-(1-\delta^T)}$. Thus the above inequality can be also rewritten as inequality (6) that we already proved valid always. \square

Proposition 6 suggest that when projects are correlated, regardless of the degree of correlation, maximum loan that can be offered under FJL lending and consequently under JL lending is higher in larger groups.

7 Conclusion

Most of the existing theoretical papers in microfinance discuss JL lending in groups of only two members, although experimental and empirical studies talk about the importance of group size (Abbink et al., 2006 and Galak et al., 2011). Our results show that group size can be an influential factor in improving lending efficiency and the assumption $n = 2$ that is largely used in the literature may result in neglecting the effect of group size. In particular, we calculate the optimal group size endogenously, while deriving the optimal contract that maximizes the borrower's welfare subject to: repayment being affordable for each borrower; repayment being better than default for each borrower; and lender breaking even.

Our results show that although JL contracts are feasible under a smaller set of parameter values than IL contracts, the feasibility of JL contracts can increase in the group size for riskier projects, meaning that larger groups are more reliable to repay the loan when projects are risky, but the group size cannot grow too large. Our findings confirm the results of Conning (2005) and Ahlin (2015). These authors, although focused on different biases (free-riding and local borrower information respectively), also conclude that borrower's group cannot grow too large. Intuitively there are costs and benefits in being a member of a large group of borrowers. From one side, it enhances the chance of assured repayment for a defaulting member. On the other side, there is also a higher threat of repaying for other defaulting members of the group. While for risky projects the insurance provided by larger groups becomes increasingly more attractive, it is less necessary for low risk projects to be insured by many group members. In this paper, an algorithmic method is suggested to calculate the optimal group size given the amount of high and low returns of projects, the discount factor of borrowers for future loans, and the chance of success in projects.

Furthermore, addressing strategic default, we discuss that JL, when feasible, has a positive effect on borrower's welfare and repayment rate compared to IL. And we demonstrate that JL can also outperform IL in terms of maximum loan that can be offered to the borrower. This is in contrary to the existing literature that concludes that a lender may be better off by choosing IL over JL (Besley and Coate, 1995, Armendáriz de Aghion, 1999), unless borrowers are able to impose strong social sanctions on each other.

Besides comparing JL lending and IL lending contracts, our study engages in comparing FJL

lending with JL lending contracts and seeks to find out how much remission are possible to the defaulting borrowers. We discuss that from borrowers point of view FJL and JL contracts are the same, as their lifetime welfare and repayment amount stays the same under both contracts. From the lender point of view, however, these contracts are not the same and the maximum loan that can be offered to borrowers under FJL lending should be lower compared to JL lending. Moreover, in order to enhance the likelihood of repayment, larger groups should be formed, when punishment phase is not that long. Finally, we investigate the effect of project correlation on feasibility of joint liability contracts, and we argue that maximum loan that can be offered to joint liability groups could be increasing in the group size, and this result is robust to changes in the degree of correlation and borrowers' discount factor.

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8 Appendix

Proof of Lemma 1. 1)

$$\frac{\partial \hat{L}_{JL}}{\partial n} = \frac{\delta \mathbb{E}(Y) \left[- (1 - \alpha)^n \ln(1 - \alpha)^n + \delta [1 - (1 - \alpha)^n]^2 - [1 - (1 - \alpha)^n] \right]}{[n - \delta(n - 1) [1 - (1 - \alpha)^n]]^2}$$

$\frac{\partial \hat{L}_{JL}}{\partial n} < 0$ iff

$$- (1 - \alpha)^n \ln(1 - \alpha)^n + \delta [1 - (1 - \alpha)^n]^2 - [1 - (1 - \alpha)^n] < 0$$

that is $\frac{\partial \hat{L}_{JL}}{\partial n} < 0$ iff

$$\delta < \frac{[1 - (1 - \alpha)^n] + (1 - \alpha)^n \ln(1 - \alpha)^n}{[1 - (1 - \alpha)^n]^2} \equiv \hat{\delta}_{JL}(n, \alpha)$$

Thus for any $0 < \delta < \hat{\delta}_{JL}$, \hat{L}_{JL} is strictly decreasing in n , and for any $\hat{\delta}_{JL} < \delta < \tilde{\delta}_{JL}$, \hat{L}_{JL} is strictly increasing in n .

2)

$$\frac{\partial \tilde{L}_{JL}}{\partial n} = \frac{(1 - \alpha)^n [1 - \ln(1 - \alpha)^n] Y^H - Y^H}{n^2}$$

It is not difficult to show that for any $\alpha \neq 0$, $H(\alpha, n) = (1 - \alpha)^n [1 - \ln(1 - \alpha)^n] < 1$. Assume $\rho = (1 - \alpha)^n$, then $H(\rho) = \rho [1 - \ln \rho]$. Since $\frac{\partial H}{\partial \rho} = -\ln \rho$ is positive for any $0 < \rho < 1$, zero for $\rho = 1$, and negative for any $\rho > 1$, then $H(\rho)$ has a maximum at $\rho = 1$. Therefore $H(\rho) < H(\rho = 1)$ or $H(\rho) < 1$. ■

Proof of Lemma 2. 1)

$$\frac{\partial \hat{\delta}_{JL}}{\partial n} = (1 - \alpha)^n \ln(1 - \alpha)^n \times \frac{\ln(1 - \alpha)^n [1 + (1 - \alpha)^n] + 2[1 - (1 - \alpha)^n]}{[1 - (1 - \alpha)^n]^3}$$

$$\frac{\partial \hat{\delta}_{JL}}{\partial \alpha} = -n(1 - \alpha)^{n-1} \times \frac{\ln(1 - \alpha)^n [1 + (1 - \alpha)^n] + 2[1 - (1 - \alpha)^n]}{[1 - (1 - \alpha)^n]^3}$$

$\hat{\delta}_{JL}$ is strictly increasing in both n and α iff

$$\ln(1-\alpha)^n [1 + (1-\alpha)^n] + 2[1 - (1-\alpha)^n] < 0$$

that can be rewritten as

$$K(n, \alpha) \equiv \ln(1-\alpha)^n + \frac{2[1 - (1-\alpha)^n]}{[1 + (1-\alpha)^n]} < 0$$

It can be proved that for any $0 < \alpha < 1$, $K(n, \alpha) < 0$ as follows. Assume $\rho = (1-\alpha)^n$, then $K(\rho) = \ln \rho + \frac{2[1-\rho]}{[1+\rho]}$. Since for any $0 < \rho < 1$, $\frac{dK(\rho)}{d\rho} = \frac{[1-\rho]^2}{\rho[1+\rho]^2} > 0$, $K(\rho)$ is increasing in ρ . On the other hand $\lim_{\rho \rightarrow 1} K(\rho) = 0$. Therefore for any $0 < \rho < 1$, $K(\rho) < 0$, and so for any $0 < \alpha < 1$, $K(n, \alpha) < 0$.

2) $\hat{\delta}_{JL}$ is strictly increasing in α , so $\lim_{\alpha \rightarrow 0} \hat{\delta}_{JL} < \hat{\delta}_{JL} < \lim_{\alpha \rightarrow 1} \hat{\delta}_{JL}$, that is, $-\infty < \hat{\delta}_{JL} < 1$.

3) $\tilde{\delta}_{JL}$ is strictly decreasing in both n and α , simply because $[1 - (1-\alpha)^n]$ is strictly increasing in both n and α .

4) $\tilde{\delta}_{JL}$ is strictly decreasing in α , thus $\lim_{\alpha \rightarrow 1} \tilde{\delta}_{JL} < \tilde{\delta}_{JL} < \lim_{\alpha \rightarrow 0} \tilde{\delta}_{JL}$, that is $\frac{n}{2n-1} < \tilde{\delta}_{JL} < \frac{Y^H}{Y^L}$. ■

Proof of Lemma 3. 1)

$$\frac{\partial \hat{L}_{FJL}}{\partial n} = \frac{\delta(1-\delta^T) \mathbb{E}(Y) \left[- (1-\alpha)^n \ln(1-\alpha)^n + \delta [1 - (1-\alpha)^n]^2 - [1 - (1-\alpha)^n] \right]}{[n - \delta [n - (1-\delta^T)]] [1 - (1-\alpha)^n]^2}$$

$\frac{\partial \hat{L}_{FJL}}{\partial n} < 0$ iff

$$- (1-\alpha)^n \ln(1-\alpha)^n + \delta [1 - (1-\alpha)^n]^2 - [1 - (1-\alpha)^n] < 0$$

$\frac{\partial \tilde{L}_{FJL}}{\partial n} < 0$ iff

$$\delta < \frac{(1-\alpha)^n \ln(1-\alpha)^n + [1 - (1-\alpha)^n]}{[1 - (1-\alpha)^n]^2} \equiv \hat{\delta}(n, \alpha)$$

Therefore, \hat{L}_{FJL} is strictly decreasing in n for any $0 < \delta < \hat{\delta}(n, \alpha)$, and \tilde{L}_{FJL} is strictly increasing in n for any $\hat{\delta}(n, \alpha) < \delta < \tilde{\delta}_{FJL}$.

2) $\tilde{\delta}_{FJL}$ is strictly decreasing in both n and α , simply because its denominator is strictly increasing in both n and α .

3) Since $\tilde{\delta}_{FJL}$ is decreasing in α , we have $\lim_{\alpha \rightarrow 1} \tilde{\delta}_{FJL} < \tilde{\delta}_{FJL} < \lim_{\alpha \rightarrow 0} \tilde{\delta}_{FJL}$, that is

$$\frac{n}{(2n-1) - (n-1)\delta^T} < \tilde{\delta}_{FJL} < \frac{Y^H}{(1-\delta^T)Y^L} \quad \blacksquare$$

Proof of Lemma 4. 1) $\hat{L}_{FJL} \leq \hat{L}_{JL}$ iff

$$\frac{\delta(1-\delta^T)\mathbb{E}(Y)[1-(1-\alpha)^n]}{n-\delta[n-(1-\delta^T)][1-(1-\alpha)^n]} \leq \frac{\delta\mathbb{E}(Y)[1-(1-\alpha)^n]}{n-\delta(n-1)[1-(1-\alpha)^n]}$$

this inequality can be simplified to $(1-\delta^T) \leq 1$ that is always true.

2)

$$\frac{\partial \hat{L}_{FJL}}{\partial T} = \frac{n\delta^{T+1}(\ln \delta)\mathbb{E}(Y)[1-(1-\alpha)^n](\delta[1-(1-\alpha)^n]-1)}{(n-\delta[n-(1-\delta^T)][1-(1-\alpha)^n])^2}$$

$\frac{\partial \hat{L}_{FJL}}{\partial T}$ is always positive.

3) $\tilde{\delta}_{FJL} \geq \tilde{\delta}_{JL}$ iff

$$(1-\delta^T)\mathbb{E}(Y) + \frac{n-(1-\delta^T)}{n}[1-(1-\alpha)^n]Y^H \leq \mathbb{E}(Y) + \frac{n-1}{n}[1-(1-\alpha)^n]Y^H$$

this inequality can be simplified to

$$\frac{[1-(1-\alpha)^n]Y^H}{n} \leq \mathbb{E}(Y)$$

that is always true.

4)

$$\frac{\partial \tilde{\delta}_{FJL}}{\partial T} = \frac{Y^H \delta^T \ln \delta \left[\frac{1}{n}[1-(1-\alpha)^n]Y^H - \mathbb{E}(Y) \right]}{\left[(1-\delta^T)\mathbb{E}(Y) + \frac{n-(1-\delta^T)}{n}[1-(1-\alpha)^n]Y^H \right]^2}$$

$\frac{\partial \tilde{\delta}_{FJL}}{\partial T} < 0$ iff $\frac{1}{n}[1-(1-\alpha)^n]Y^H - \mathbb{E}(Y) < 0$ that is always true. \blacksquare

Proof of Lemma 5. 1)

$$\begin{aligned} \frac{\partial \hat{L}'_{FJL}}{\partial n} &= \frac{\delta \alpha' (1 - \delta^T) [1 - (1 - \alpha)^n] (n - \delta [n - (1 - \delta^T)]) [1 - (1 - \alpha)^n]}{(n - \delta [n - (1 - \delta^T)]) [1 - (1 - \alpha)^n]^2} \\ &\quad + \frac{n \delta \alpha (1 - \delta^T) [n \alpha' (1 - \alpha)^{n-1} - (1 - \alpha)^n \ln(1 - \alpha)]}{(n - \delta [n - (1 - \delta^T)]) [1 - (1 - \alpha)^n]^2} \\ &\quad - \frac{\delta \alpha (1 - \delta^T) [1 - (1 - \alpha)^n] (1 - \delta [1 - (1 - \alpha)^n])}{(n - \delta [n - (1 - \delta^T)]) [1 - (1 - \alpha)^n]^2} \end{aligned}$$

$\frac{\partial \hat{L}'_{FJL}}{\partial n} < 0$ iff

$$\begin{aligned} &\alpha' [1 - (1 - \alpha)^n] (n - \delta [n - (1 - \delta^T)]) [1 - (1 - \alpha)^n] \\ &\quad + n \alpha [n \alpha' (1 - \alpha)^{n-1} - (1 - \alpha)^n \ln(1 - \alpha)] \\ &\quad - \alpha [1 - (1 - \alpha)^n] (1 - \delta [1 - (1 - \alpha)^n]) \\ &< 0 \end{aligned}$$

Assuming that $n \alpha' - \alpha' (1 - \delta^T) - \alpha < 0$ or $\alpha' < \frac{\alpha}{n - (1 - \delta^T)}$, the above inequality can be simplified to

$$\delta < \frac{(n \alpha' - \alpha) [1 - (1 - \alpha)^n] + n \alpha [n \alpha' (1 - \alpha)^{n-1} - (1 - \alpha)^n \ln(1 - \alpha)]}{(n \alpha' - \alpha' (1 - \delta^T) - \alpha) [1 - (1 - \alpha)^n]^2} \equiv \hat{\delta}'_{FJL}(n, \alpha, \alpha', \delta, T)$$

Thus for any $0 < \delta < \hat{\delta}'_{FJL}$, \hat{L}'_{FJL} is strictly decreasing in n , and for any $\hat{\delta}'_{FJL} < \delta < \tilde{\delta}'_{FJL}$, \hat{L}'_{FJL} is strictly increasing in n .

2)

$$\frac{\partial \tilde{L}'_{FJL}}{\partial n} = \frac{n \alpha' (1 - \alpha)^{n-1} - (1 - \alpha)^n \ln(1 - \alpha)^n - [1 - (1 - \alpha)^n]}{n^2}$$

$\frac{\partial \tilde{L}'_{FJL}}{\partial n} < 0$ iff

$$\alpha' < \frac{1 - (1 - \alpha)^n + (1 - \alpha)^n \ln(1 - \alpha)^n}{n (1 - \alpha)^{n-1}}$$

Thus if α is such that the above inequality is satisfied, \tilde{L}'_{FJL} is strictly decreasing in n . ■