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Research Note

Strategic Bid-Shading and Sequential Auctioning with Learning from Past Prices

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This paper analyzes sequential auctioning of single units of an indivisible good to a fluctuating population composed of overlapping generations of unit-demand bidders. Two phenomena emergent in such a market are investigated: forward-looking bidding strategies, and closed-loop selling strategies that involve learning from past prices. The buyers shade their bids down, i.e., bid less than they would in a single isolated auction, whenever they expect the seller to sell another unit of the good in the near future. Unlike in exogenous sequences of auctions, the optimal bidding strategy thus depends on the seller's selling strategy. The converse dependence also occurs: the seller can learn about current demand from past realized prices, and sell only in periods with high-enough demand. Such learning depends on the extent of bid-shading because the seller needs to invert the bidding strategy to learn. In equilibrium, buyer bid-shading persists even when the seller does not sell in every period, but it is self-regulating in that it eventually vanishes when the existence of the market is threatened by low seller profits. In this sense, auction markets have a "self-preservation instinct." General properties of learning about current demand from past auction prices are also investigated and characterized.

Key words: optimal selling; sequential auctions; market equilibrium; bidding strategy; bid-shading
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1. Introduction

Suppose that a long-lived seller has a supply stream of an indivisible good, and she can auction each unit of the good to a population of bidders that fluctuates over time. For example, she may be an individual able to obtain a unique theatre ticket or a discounted digital camera every week, and resell them on eBay. Alternatively, she may be a state government that can specify a few highway-construction contracts every month, and "sell" them to the construction firm that submits the lowest bid. Suppose further that each bidder participates in the market more than one period, but only wants one unit of the good (the units are perfect substitutes or the bidders are capacity constrained), so the bidders who lose in today's auction remain active in future periods while the winner exits the market. The longer bidder life spans create an incentive for bidders to shade their current bids down in hopes of coming back later when competition has subsided: winning one unit immediately involves an opportunity cost of foregoing future auctions that may involve lower prices. The longer bidder life spans also make current prices informative about future demand: the seller can learn about demand from the final price of today's auction, and increase future profits by not procuring the good when the demand happens to be low.

The seller's selling decisions and the bidders' bid-shading are mutually related: the seller needs to take bid-shading into account because it reduces her profits relative to the profits available should each auction be held in isolation instead. Conversely, the bidders' strategy depends on the seller's strategy because the existence of future auctions depends on the future selling decisions of the seller. Intuitively, because bid-shading reduces seller profits, it reduces her incentive to sell, which should in turn reduce the incentives for bid-shading in the first place. In this article, I assess the equilibrium outcomes within a model of a monopolistic sequential auction market with overlapping generations of bidders (OLG) that captures the relationships between selling, learning, and bid-shading. Specifically, I ask: How can a seller learn from past prices to assess the current state of demand when the bidders are strategically bid-shading? What is the equilibrium bid-shading strategy when the seller is strategic and takes it into account? Can the seller somehow reduce the extent of bid-shading to increase her profits? Should policymakers or auctioneers try to somehow eliminate bid-shading in thin markets whose existence may be threatened by the lower bids? The main result is that bid-shading can be self-regulating, and gradually diminish as the overall gains from trade approach zero. The auction market thus has a "self-preservation instinct" in that the phenomenon of bid-shading does not reduce the ability of the market to exist when overall gains from trade are low. However, bid-shading does persist when gains from trade are high, and it can coexist with nontrivial selling patterns that arise when the seller avoids selling in low-demand periods. The self-preservation instinct is an important finding for policymakers and auctioneers, who may consider bid-shading to *always* reduce seller participation, and hence attempt to stop it through mechanism design (as in Juda and Parkes 2006).

Bid-shading is not only a normative theoretical construct, it was detected in the above-mentioned real-world auction markets (Jofre-Bonet and Pesendorfer 2003, Zeithammer 2006). Previous theoretical work on sequential auctions focused either on bid-shading in an exogenous sequence of auctions (see, for example, Milgrom and Weber 2000, Jeitschko 1999), or on strategic auctioning to short-lived buyers, who never want to shade their bids (see, for example, Vulcano et al. 2002). This paper provides the first model of a sequential auction with both endogenous strategic selling and forward-looking longer-lived buyers who can shade their bids. The model's contribution is the analysis of the best response of the seller to strategic bid-shading, and the exposition of a market equilibrium, in which bidders do not always shade. The most related model of bidding is Jeitschko (1999), who finds that relatively to exogenous and certain future supply, exogenous but uncertain future supply leads to a proportional bid increase. In contrast, high-valuation bidders shade more than low-valuation bidders here. The most related model of optimal sequential auctioning by Vulcano, van Ryzin, and Maglaras (2002) (VRM), who study a monopolist selling to unit-demand strategic buyers who each only lives for one period. While VRM's bidders do not shade their bids by assumption, strategic sequential auctioning has an effect on their bidding strategy because they are forward-looking: there is an incentive to overbid and make the seller sell more units in the current period than would be optimal for her.

Motivated by McAfee and Vincent (1997), this model does not consider reserve prices above cost, implicitly assuming they are not credible because the seller cannot commit to never reselling an unsold object. Nevertheless, the seller faces a problem analogous to the problem of a durable-good monopolist because bid-shading in early sequential auctions arises in expectation of a future option to buy for less. Therefore, in agreement with the Coase Conjecture (Coase 1972), the sequential-auction monopolist is unable to extract monopoly profits. As in Conlisk, Gerstner, and Sobel (1984) (CGS), the monopolist here faces recurrent entry of long-lived buyers. Unlike in CGS, the present OLG buyers do not accumulate indefinitely and eventually leave, a critical assumption that makes them change their bidding behavior as they age. However, the buyers do accumulate to some extent, and the phenomenon of "pulsing," i.e., selling every other period, is caused by analogous buyer-accumulation dynamics that cause CGS's cyclical pricing.

2. Basic Model

A monopolist seller lives for infinitely many periods in discrete time, and she can produce one unit of an indivisible good per period, at constant marginal cost $c > 0$ which is common knowledge. She decides in the beginning of every period whether to produce the good and immediately sell it by a second-price sealed-bid auction without a reserve. Bidders are risk neutral, live for two periods so they are first *new* and then *old*, and occur in overlapping generations in that a new generation of bidders enters the market every period. Each generation is of a valuation type $t \in \{L = Low, H = High\}$, $0 < L < H$, and $\Pr(H) = p$, where t is the utility of the product to a single bidder of that generation. There are two bidders in each generation,[1] and each of them has a unit demand, so the winner of each auction drops out of the market. Therefore, in each period, the seller faces two *new* bidders and one or two *old* bidders. Suppose that $H > c$ so that some trading occurs, and let $L < c$ to capture unprofitable bidders. Without loss of generality, set the scale of utility by $c = 1$ (the c label is used below whenever it makes exposition clearer). The seller and all buyers hold beliefs (p_0, p_1, p_2) about the number of *old High* bidders, where $p_i = \Pr(i \text{ } old \text{ } High \text{ bidders})$. Both the seller and all buyers hold the same beliefs and update them in the beginning of each period based on past prices,[2] starting with a (correct) belief $(1-p, 0, p)$ that there are two randomly drawn *old* bidders. Everyone discounts future utility exponentially at a factor of δ per period. In the auction, ties are resolved first in favor of *old* bidders, then randomly within a generation of bidders.[3] The model has only four free

[1] Two bidders per generation is the maximum nontrivial number given their perfect correlation. The assumption about the number of bidders per generation will be relaxed in §3 that discusses generalizations of the model.

[2] Price is a statistic always available to both the seller and the bidders, while other past bids may not be available.

[3] Resolving in favor of *old* bidders is a regularity assumption needed to reduce the number of ties inherent in the discrete model of demand used here. It will be relaxed in §3 that discusses generalizations of the model.

parameters: $H > 1 > L$, $0 < p < 1$, and $0 < \delta < 1$, and for every set of parameters, there is a unique Markov-Perfect Bayesian Nash equilibrium of the game between all the bidders and the seller. The bidding strategy as a best response to a selling strategy, and the selling strategy as a best response to a bidding strategy are discussed next, in turn.

2.1. Bidding Strategy as a Best Response to a Selling Strategy

All *old* bidders have a dominant strategy to bid their valuation because they are effectively in a single-period second-price auction. *New Low* bidders also bid their valuation because they face at least one *old* bidder bidding at least L, so bidding less than L would have no chance of winning. *New High* bidders, on the other hand, shade their bids down from their valuation iff they expect to make a positive surplus tomorrow conditional on losing today (this conditioning is critical):

Proposition 1. *Suppose that the seller strategy is to sell in period $t+1$ whenever $price_t \geq M_t$. Then, new High bidders shade their bids down in period t, and bid $(H-a) < H$ iff both of the following conditions hold*:

(1) *There is a chance there are no old High bidders in the current period*: $p_{0,t} > 0$.

(2) *The seller will sell again in $(t+1)$ after observing $price_t = H - a$: $M_t \leq H - a$, where $a = \delta(1-p)(H-L)/(1-\delta\omega p)$ and ω is the probability that new bidders also shade their bids down by a in period $(t+1)$. Thus, $(H-a) > L$. Otherwise, new High bidders bid their valuation H.*

The proof of Proposition 1 is in the appendix.

The *new High* bidders shade their bids when the surplus associated with losing is positive (i.e., not zero as in an isolated auction). The first condition in Proposition 1 ensures that there is a chance for one of the *new High* bidders to win the current auction when bidding less than H. The second condition ensures that the surviving loser will face a seller who will sell again. The bid-decrement a arises from an equilibrium condition that effectively forces the *new High* bidders to be indifferent between winning and losing their first auction. Proposition 1 is not novel in finding shading, it merely highlights the role of the seller in the game (through M_t), and shows how the phenomenon of bid-shading manifests itself under the present distributional assumptions. The infinite-horizon overlapping-generations model involves a feedback link not present in the standard finite-horizon single-generation setup used in the literature: when $\omega = 1$, the *new*-bidder strategy influences (softens) the future competition because the future *new* bidders also shade.

2.2. Learning from Prices: The Evolution of Beliefs About the Number of *Old High* Bidders

Everyone starts with the correct "uncertain" belief $(1-p, 0, p)$, and by construction returns to that belief whenever the seller does not sell in a period. When she does sell, learning from prices depends on whether or not the bidders are shading. The bid-shading possibility is discussed first. When *new High* bidders shade their bids, the bidding strategies of Proposition 1 lead to the prices shown in Table 1.

Given a price today and current beliefs (p_0, p_1, p_2), the Bayes rule updates beliefs about the *new High* bidders surviving until tomorrow (tomorrow's *old High* bidders) as follows:

Price $= H$: there must have been two *old High* bidders today, and both *new* bidders thus survive until tomorrow. No additional information is available about their type: $(p_0, p_1, p_2) \to (1-p, 0, p)$.

Price $= H - a$: the *new* bidders were *High*, and there was zero or one *old High* bidder. If zero, only one *new* (*High*) bidder survives, otherwise both *new* (*High*) bidders survive: $(p_0, p_1, p_2) \to (0, p_0/(p_0+p_1), p_1/(p_0+p_1))$.

Price $= L$: the *new* bidders were *Low*, and there was at most one *old High* bidder. Either way, there will be no *old High* bidders tomorrow: $(p_0, p_1, p_2) \to (1, 0, 0)$.

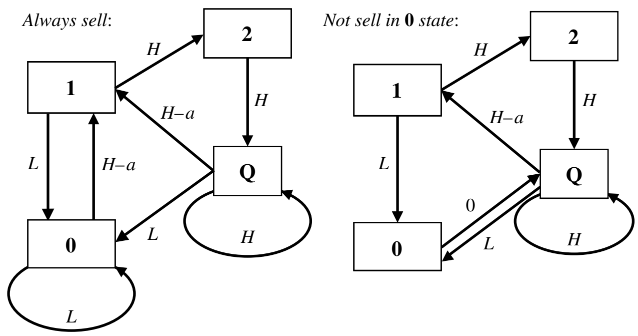
The key property of learning that allows for a closed-form solution of the model is the fact that when the *new* bidders are shading at least in the belief-state $\mathbf{Q} \equiv (1-p, 0, p)$, the beliefs can only be in four profit-relevant states $\{\mathbf{Q} \equiv (1-p, 0, p)$, $\mathbf{2} \equiv (0, 0, 1)$, $\mathbf{1} \equiv (0, 1, 0)$, $\mathbf{0} \equiv (1, 0, 0)\}$ along the equilibrium path: starting with any of the four states, the above rules of learning applied to any possible price L, $H-a$, or H lead again to one of the four states (see Figure 1). Note that *new* bidders do not shade in State **1** (Proposition 1).

Without bid-shading in **Q**, the seller's ability to learn is degraded because only two price levels can now occur. The learning process now visits infinitely many payoff-relevant states because along the "prices $= (H, H, H, \ldots)$" branch of the game, the probability of yet another price $= H$ is $p + (1-p)p_2$,

Table 1 Prices in the Auction Market with Bid-Shading

	New bidder type (probability)	
Number of *old High* bidders (*belief*)	*Low* $(1-p)$	*High* (p)
0 (p_0)	L	$H-a$
1 (p_1)	L	$H-a$
2 (p_2)	H	H

Figure 1 Belief-State Transitions in the Basic Model



Note. States are represented by square boxes, and transitions are represented with arrows and are labeled in italics with the respective prices.

where p_2 evolves nonrecurrently according to

$$(p_0, p_1, p_2) \to \left(\frac{(1-p)p_2}{pp_0+pp_1+p_2}, \frac{pp_0}{pp_0+pp_1+p_2}, \frac{pp_1+pp_2}{pp_0+pp_1+p_2}\right).$$

2.3. Selling Strategy as a Best Response to Bidding Strategy

The seller maximizes net present value of profits, taking into account her future learning. Her Markovian strategy amounts to selling as a function of the belief-vector (state): $(p_0, p_1, p_2) \to \{sell, not\ sell\}$. When *new* bidders are not bid-shading in state **Q**, a simple steady-state analysis of the model is not available because the beliefs visit infinitely many states as shown in the previous section. On the other hand, when the *new* bidders are shading in state **Q**, the game visits only the four belief-states $\{\mathbf{Q}, \mathbf{0}, \mathbf{1}, \mathbf{2}\}$, so the optimal steady-state profit function Π: $\{\mathbf{Q}, \mathbf{0}, \mathbf{1}, \mathbf{2}\} \to \boldsymbol{R}$ is characterized by a set of Bellman equations:

$$\Pi_0 = \max\{\delta\Pi_Q, p(H-a)+(1-p)L-c + \delta[p\Pi_1+(1-p)\Pi_0]\},$$
$$\Pi_1 = \max\{\delta\Pi_Q, pH+(1-p)L-c+\delta[p\Pi_2+(1-p)\Pi_0]\},$$
$$\Pi_2 = \max\{\delta\Pi_Q, H-c+\delta\Pi_Q\} = H-c+\delta\Pi_Q,$$
$$\Pi_Q = \max\{\delta\Pi_Q, pH+p(1-p)(H-a)+(1-p)^2L-c + \delta[p\Pi_Q+p(1-p)\Pi_1+(1-p)^2\Pi_0]\},$$

where $a = \delta(1-p)(H-L)$ because *new High* bidders shade their bids in **Q** and **0**, but not in **1**, so $\omega = 0$.

2.4. Equilibrium in the Game Between the Seller and the Bidders

It is intuitively clear and formally implied by the Bellman equations that the seller sells iff the past price is at least some level M. Therefore, Proposition 1 applies to the buyers' best response, and the overall equilibrium of the game between the seller and the buyers depends on the parameters of the game. The key determinant of the qualitative nature of the equilibrium is the *relative expected gain from trade* $\pi = (E(t)-L)/(c-L) = p(H-L)/(c-L)$. To characterize the equilibrium, it is useful to introduce notation for the short-run expected profits $\{R_0, R_1, R_Q\}$ of the seller in the $\{\mathbf{0}, \mathbf{1}, \mathbf{Q}\}$ states with bidders bid-shading in **0** and **Q**:

$$R_0 = p(H-a)+(1-p)L-c, \quad R_1 = pH+(1-p)L-c,$$

and

$$R_Q = pH + p(1-p)(H-a) + (1-p)^2L - c.$$

It is also useful to define four levels of π ("contours" in parameter space) that will mark seller indifference between selling and not selling in four different situations:

$$C_0 = 1/(1-2\delta(1-p)),$$
$$C_1 = 1+\delta^2 p(1-p)/(1+\delta^2(1-p)(2-p)),$$
$$C_{12} = 1-(\delta p(H-c))/(c-L), \quad \text{and}$$
$$C_Q = 1+\delta p(1-p)(1+\delta p)/(2-p+\delta(1-p)((2+\delta)p-1).$$

Given $\{\pi, R_k, C_k\}$, the equilibrium is characterized as follows:

Proposition 2 (Equilibrium Characterization). *The Markov-Perfect Bayesian Nash equilibrium of the auction market depends on the model parameters as follows*:

- *When the relative gains from trade are so large that* $\pi \geq C_0$, *the seller sells in every period and the bidders shade their bids down in* **Q** *and* **0**. *The seller makes*

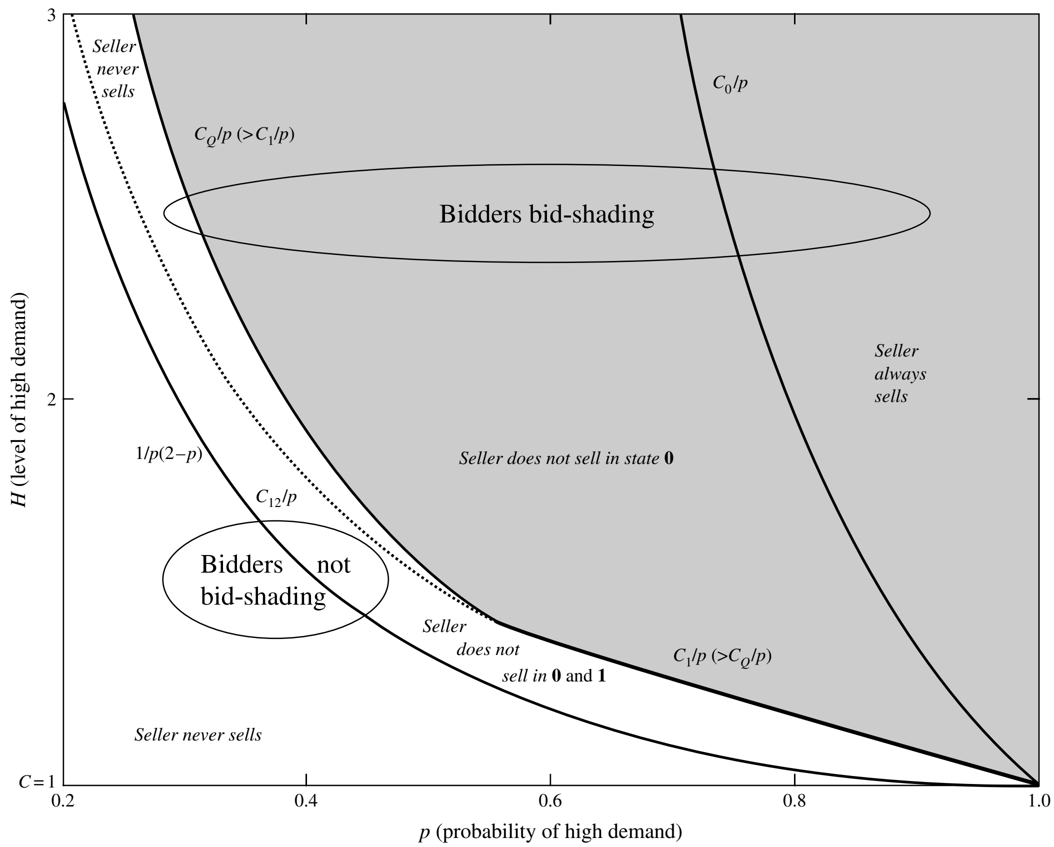
$$\Pi_Q^{all} = \frac{R_Q - \delta p(1-p)^2(H-L)}{1-\delta}.$$

- *When the relative gains from trade are medium such that* $\max[C_1, C_Q] \leq \pi < C_0$, *the seller does not sell in state* **0** (*after price L*), *sells in* $\{\mathbf{2}, \mathbf{1}, \mathbf{Q}\}$, *and the bidders shade their bids in* **Q**. *The seller makes*

$$\Pi_Q^{not0} = \frac{R_Q + \delta p(1-p)[R_1 + \delta p(H-c)]}{(1-\delta)[1+\delta(1+\delta p)(1-p)]}.$$

- *When the relative gains from trade are so small that* $1/(2-p) \leq \pi < \min(C_Q, C_{12})$ *or* $C_Q < \pi < C_1$, *the seller does not sell after prices L or* $(H-a)$, *and bidders therefore do not shade their bids down. The seller either uses the pulsing strategy of selling every other period, or also sells in other informational states associated with the new learning environment without bid-shading.*
- *When the relative gains from trade are even smaller such that* $\pi < 1/(2-p)$, *or when the threat of bid-shading prevents selling in* **Q** *because* $C_Q > \pi > C_{12}$, *the seller never sells.*

Figure 2 Equilibrium Characterization



Notes. Illustration of Proposition 2 by fixing remaining parameters to $L=0$, $\delta=0.9$, thus implying $\pi=pH$, hence the scaling by p of the C_k boundaries. The shaded area shows where bidders shade their bids in equilibrium, with the general bidder strategies shown in ovals. The seller strategies are shown in italics. The dashed line is the $c_{12} > C_1$ contour.

The proof of Proposition 2 is in the appendix.

It can be shown that $C_0 > \max[C_1, C_{12}, C_Q]$, $C_1 < C_Q \Leftrightarrow C_1 < C_{12}$, $\min[C_{12}, C_Q] > 1/(2-p)$, and hence the areas of the parameter space delineated in the proposition are exhaustive and mutually exclusive. Figure 2 illustrates Proposition 2, fixing $\delta = 0.9$ and $L = 0$ to allow two-dimensional plotting of the contours. Changing these parameters would move the curves around the space, but it would preserve their relative positions. Intuition for Proposition 2 is discussed next.

2.4.1. High Gains from Trade. As long as the seller always sells (which happens as long as selling in state **0** is better than waiting a period, i.e., when $\pi \geq C_0$), the bidders obviously shade their bids in **Q** and **0**. The profit function Π_Q^{all} says that the net present value of profits obtained from always selling is somewhat less than collecting R_Q in every period, by a factor related to the magnitude of bid-shading a.

2.4.2. Medium Gains from Trade. From Proposition 1, the bidders continue shading in **Q** as long as the seller sells in state **1**, even when the seller withholds supply in state **0**. The numerator of profit function Π_Q^{not0} is R_Q plus the expected value of continuing through the right flow diagram in Figure 1, and the denominator adjusts the frequency of this single "**Q** to **Q** lap" payoff. There are three jointly necessary and sufficient conditions for this pattern of selling to be optimal: First, profitability must be low enough such that selling in **0** is no longer preferred to waiting for the demand side to "refill" with bidders ($\pi < C_0$). Second, profitability must be high enough such that selling in **1** is preferred to waiting one period ($\pi > C_1$). Third, profitability must be also high enough for selling in **Q** to generate positive profits in the first place ($\pi > C_Q$) when state **1** is more lucrative than state **Q**.

2.4.3. Low Gains from Trade. When profitability is such that $C_1 > \pi > C_Q$, a phenomenon central to the thesis of this paper occurs: the seller sells in **Q** but not in **1**. The bidders understand the seller's strategy, and hence refrain from bid-shading ($M > H - a$ in Proposition 1). Note that the seller withholds supply in state **1** because she prefers the profit in **Q** delayed by one period to the profit in **1** now, not because she wants to "punish" bid-shading behavior. Therefore, bid-shading ceases not because the seller can actually detect it perfectly (by observing a price strictly between L and H), but because selling with only one *old High* bidder is simply not profit maximizing. Section 3.4 will illustrate this intuition further on a model with a continuous distribution of valuations, where the seller cannot detect bid-shading. It is not immediately clear what happens in a world without bid-shading because the seller's ability to learn from prices decreases, and the number of payoff-relevant states becomes infinite as explained in §2.2. It is clear that as the profitability of the auction market decreases further, the seller eventually exits completely. The least enthusiastic seller who still participates in the auction market uses a pulsing strategy, always waiting one period for the demand side to refill with two old bidders, with no bidders shading. Pulsing is profitable as long as $\pi \geq 1/(2-p)$.

2.4.4. Unprofitable Market. The market can be unprofitable for two reasons: First, it can be that $\pi < 1/(2-p)$, and even the maximum (four) number of bidders bidding their valuations does not result in an expected revenue exceeding c, and so even the pulsing strategy is unprofitable. Second, the market can be unprofitable when $\Pi_Q^{not0} < 0$ ($\Leftrightarrow \pi < C_Q$), but also $\pi > C_{12}$, guaranteeing that the seller would sell in **1** and **2** only. Then, the seller is unable to eliminate bid-shading, and pulsing profits become unattainable.

3. Alternative Model Specifications

Let N bidders enter the auction market every period and live for two periods; let each bidder have unit demand and a unit valuation drawn from some distribution F. The basic model assumes that $N = 2$, F is Bernoulli(p), and the valuations of bidders are perfectly correlated within each generation. This section discusses how the conclusions change when the support of F exceeds c and all bidders are thus profitable, when bidders have simpler beliefs than the seller, when they are not perfectly correlated within each generation, when F is a continuous distribution, and when there are multiple competing sellers.

3.1. No Unprofitable Bidders

One way to interpret the $L < c$ assumption is that the seller's ability to produce the good—the number of units per unit of time—can exceed the new demand that arrives to the market, hence making the seller's participation decision nontrivial. Then, bid-shading diminishes as the gains from trade decrease because the seller finds it less and less profitable to sell in every period—the chance of a loss from a price $L < c$ looms larger and larger, so the seller prefers to wait for the demand-side competition to increase through accumulation of generations. When $\min[support(F)] = L > c$, the seller cannot lose money by selling, so her implicit "threat" of not selling again tomorrow is not credible. In particular, the Π^{all} and Π^{not0} formulas of Proposition 2 still hold, and it is straightforward to show that $L > c$ implies $\Pi_Q^{not0} > 0$ and $\Pi_1^{not0} > \delta\Pi_Q^{not0}$, so the seller will never be able to eliminate bid-shading. However, she may not necessarily sell in every period because $\Pi_0^{all} > \delta\Pi_Q^{all}$ is not guaranteed: the incentive to increase demand-side competition through waiting in low-demand periods still remains, but this reduction in selling is not sufficient to reduce the extent of bid-shading. Therefore, the possibility of an ex post loss is a necessary condition for most of the results in this paper.

3.2. Simpler Beliefs About Remaining *Old* Bidders

The assumption that bidders can learn as much as the seller from past prices can be replaced with the assumption that new bidders are unaware of the seller's state or the past prices, and always believe that the old bidders are of type *Low* with the "prior" probability p. The resulting model, solved and discussed in detail in the online supplement (provided in the e-companion),[4] is simpler than the basic model because *new High* bidders either shade in every state or do not shade at all. Proposition 1 still applies, and the bid-shading decrement a obviously involves $\omega = 1$. The flow diagram in Figure 1 still describes the transition of the seller's beliefs, with state **1** altered to produce price $H - a$ instead of H, thus making the tie-breaking assumption explained in Footnote 3 unnecessary. Finally, the overall equilibrium is qualitatively the same as in Proposition 2, with the simplification that there is no $C_Q > \pi > C_{12}$ "wedge" due to the fact that state **1** is now always less lucrative than **Q**.

3.3. Mutually Independent Bidders

When the within-generation perfect-correlation assumption is relaxed and the two bidders within each generation are assumed independent of each other, the model remains tractable and analogous to the basic model. The key reason is that the seller can still only be in a finite number of states, and she still effectively uses a cutoff selling strategy. The buyers, on the other hand, have more scope for obtaining surplus. As the profitability of the market shrinks, the seller will eventually make bid-shading unprofitable for the buyers, either by using a pulsing strategy or by only selling after the highest possible price is observed. This version of the model is developed in the online supplement. It is an important case because it demonstrates that the *new High* bidders shade their bids because there is a chance of surplus in the next period, not because they know each other's valuations exactly.

[4] An electronic companion to this paper is available as part of the online version that can be found at http://mansci.journal.informs.org/.

3.4. Continuous Distribution of Valuations

Bernoulli's discreetness leads to a distribution of prices with finite support, making the seller's learning task simple, and even allowing the seller to detect bid-shading when it occurs (by observing a price between L and H). It is thus important to explain how equilibrium bid-shading is attenuated as expected gains from trade become smaller even when F is continuous, and the seller is thus unable to detect and "punish" bid-shading. It is also important to generalize the properties of seller learning from the prices shown in Figure 1.

Assume that each generation consists of N bidders with valuations drawn i.i.d. from any continuous distribution on the $[0, 1]$ interval. This specification of demand is similar to Milgrom and Weber (2000), with two additional complications: Milgrom and Weber set $\delta = 1$, and they do not consider entry in the second period—two features necessary for any infinite-horizon model. On the supply side, assume that the seller can use a first-price sealed-bid auction to sell one unit of the good in each period, an auction format which involves increasing pure bidding strategies of both the *old* and the *new* bidders.[5]

The participants' beliefs about *old* bidders in period t are a distribution function W_t of the maximum valuation among the *old* bidders: $W_t(x) = \Pr(\max(v_{old} \text{ in } t) \leq x)$. Given the pure increasing bidding strategies and the first-price auction format, price p_t is the upper bound on everyone's bids at time t, implying $\bar{v}_{t+1} = b_1^{-1}(p_t \mid W_t)$—the maximum possible valuation of *new* bidders in t (*old* bidders in $t+1$) given p_t. Given $\bar{v}_{t+1}$, W_{t+1} depends on the (unobserved) age of the winner in t who bid p_t: $W_{t+1}(x) = T(W_t \mid \bar{v}_{t+1})(x) = \alpha_t H_1(x \mid x < \bar{v}_{t+1}) + (1-\alpha_t)H_2(x \mid \bar{v}_{t+1})$, where $\alpha_t = \Pr(\textit{old} \text{ won}_t \mid p_t, W_t)$, H_1 is the truncated distribution of the highest valuation within a generation of bidders, and H_2 is the conditional distribution of the second-highest valuation within a generation given the first-highest valuation: $H_1(x \mid x < \bar{v}) = (F(x)/F(\bar{v}))^N$, $H_2(x \mid \bar{v}) = (F(x)/F(\bar{v}))^{N-1}$. Because H_1 and H_2 are common knowledge, the set of all possible beliefs W_t can be parametrized by points $(\alpha, \bar{v}) \in [0, 1]^2$, with the transition T depending on the exact shape of the bidding strategies. The transformation T plays the role of the basic model's learning process illustrated in Figure 1, and shares many qualitative properties with it: When the seller does not sell in period t, the belief transitions to a belief-state $\mathbf{Q}$ analogous to the $\mathbf{Q}$ state in the basic model, namely, the belief $(1, 1)$ that the highest of the remaining *old* bidders in $t+1$ is the maximum of N valuations sampled independently from F. When the seller does sell in period t, everyone updates their beliefs using T with α_t determined by Bayes Rule. Very high prices transition the seller to the $\mathbf{Q}$ state for the same reason that the price of H in the basic model transitioned the seller to the $\mathbf{Q}$ state: because new bidders never bid more than $\bar{b}_1(W) = b_1(1 \mid W) < b_2(1 \mid W)$, $p_t > \bar{b}_1(W)$ indicates an *old* bidder winning ($\alpha_t = 1$) and all *new* bidders surviving until tomorrow. Analogously, with prices $(H - a)$ and L in the basic model, a price $p_t < \bar{b}_1(W)$ places a binding constraint on the surviving bidders' valuations and invites the possibility that a new bidder is the winner. α_t is then state dependent: for example, $\alpha_t = 0$ when $p_t > b_2(\bar{v}_t \mid W_t)$, but $\alpha_t > 0$ when $p_t \leq b_2(\bar{v}_t \mid W_t)$. The α_t update when $p_t \leq b_2(\bar{v}_t \mid W_t)$ is discussed in more detail in the online supplement (the evolution of α_t is cumbersome to express).

The supply side of a continuous model is most parsimoniously captured by a seller, whose profit-maximizing behavior leads to a continuous "selling-probability" function $\lambda(w) = \Pr(sell_t \mid W_t = w)$ nondecreasing in $\bar{v}$, where w is parametrized as $w = (\alpha, \bar{v})$. For example, one can assume that the production cost c is a temporary random shock drawn in the beginning of each period, and a private information of the seller. Given λ that characterizes the seller's tomorrow's selling, the best response of the bidders to the seller can be characterized by the following analogue of Proposition 1 for the continuous setting:

Proposition 3. *For every pair of selling-probability functions λ and μ such that λ involves less selling than μ in every state: $\lambda(w) < \mu(w)\ \forall w \in [0, 1]^2$, the new bidders bid more as their best response to λ than to μ: $b_1^\lambda(v) > b_1^\mu(v)\ \forall v \in [0, 1]$.*

The proof of Proposition 3 is in the online supplement.

The intuition is straightforward: overall less selling tomorrow means a lower chance of a positive surplus tomorrow, hence decreasing the opportunity cost of winning today, hence making winning today more attractive. Unlike in the discrete world of Proposition 1, where bid-shading is either "on" or "off," the reduction of bid-shading is gradual in the continuous world. In the limit as the market profitability approaches zero, bid-shading is still eliminated completely because the seller pulses, setting $\lambda \equiv 0$.

Finding the best response of the seller to $b_1^\lambda(v)$ characterized in Proposition 3 and its proof is difficult analytically in the continuous setting, but straightforward conceptually. In particular, the Bellman analysis still applies: The seller maximizes the net present value of profits Π, starting by assumption in state $\mathbf{Q}$.

[5] The continuous model needs to have a symmetric equilibrium pure bidding strategy to preserve informativeness of past prices. Such a pure strategy exists in first-price, but not in second-price auctions: in the latter, the price-setting bidder survives until the next period, and hence always has an incentive to deceive the seller into selling too often in the next period. Mixed strategies result.

Therefore, the profitability of any state W is captured by Bellman equations analogous to those of the basic model:

$$\Pi(W) = \max_{(sell,\, not\, sell)} \{E[p + \delta\Pi(T(W \mid p))] - c,\ \delta\Pi(\mathbf{Q})\}.$$

The exact properties of these functions under specific distributional assumptions can be numerically approximated by simulation because the space of beliefs can be parametrized to the unit square, but such an analysis is beyond the scope of this paper (see also the online supplement).

3.5. Generalizations of the Supply-Side Assumptions

The qualitative predictions of the basic model are robust to seller-side competition because bidding described in Proposition 1 remains the best response to the aggregate supply arising from the selling strategies of competing sellers. The competition among multiple sellers makes it harder but not impossible to withhold current supply to increase future profits. Because the market can only support one seller profitably near the zero-pulsing-profit contour of Proposition 2, the sellers have an incentive to coordinate when gains from trade are low, effectively sharing the market. One way to model such coordination is a model in continuous time with random recurrent arrival of sellers to the marketplace and observable entry. When coordination is thus managed, bid-shading still ceases as gains from trade decrease because the monopoly profit analysis applies to the total market profit (near the zero-profit contour).

Additional sellers result in additional units sold away from the zero-profit contour because of the tragedy of the commons: each seller collects the entire contemporaneous profit, but only part of the continuation profit. Hence, there is an increased incentive to sell now relatively to the monopoly situation. The two-seller model is developed in detail in the online supplement.

Throughout this paper, the seller decides in each period whether or not to sell in the auction market, and a sale is always via an auction without a reserve price. An extension of the proposed model to auctions with reserve prices would introduce the issues of storage, multi-unit auctions, and ratcheting (Caillaud and Mezzetti 2004). Such an extension is unlikely to change the qualitative conclusions of the basic model: Because an anticipated future reserve price reduces the expected future surplus in case of a loss today, reserve prices should further attenuate bid-shading, working together with the threat of not selling at all. However, reserve prices are unlikely to have a large quantitative impact on the seller's profits as argued by McAfee and Vincent (1997), who show that the effectiveness of reserve prices is greatly diminished in a sequential context without commitment not to resell unsold units.

4. Discussion

In sequential auction markets, long-lived buyers interact with a long-lived seller within a Markov-perfect Bayesian Nash equilibrium that has several nontrivial properties. When the seller has a marginal cost of participating, it is in her interest to learn about the current level of demand, and sell only when demand is high enough. When losing bidders remain active in the future, recent past prices are informative about demand, enabling such learning. When losing bidders remain active in the future and each of them only demands one unit, they shade their bids down to account for anticipated future options to buy another unit of the good for a lower price. However, bid-shading does not occur *always* because the seller does not always sell and the bidders need to take the seller's strategy into account. In equilibrium, the resulting bidding strategy must be a best response to the selling strategy, and vice versa. The main insight of this paper is that strategic sellers can regulate the extent of bid-shading when the overall gains from trade in the market are relatively low. In particular, the sequential auction market has a "self-preservation instinct" in that the phenomenon of bid-shading does not reduce its ability to exist. This is an important finding for policymakers and auctioneers: somehow stopping bid-shading from happening would not result in new auction markets—it would only increase the volume of sales in some existing markets, namely, the more profitable ones. More generally, this paper demonstrates that any model of bidding within an auction market must take the selling strategy into account and vice versa.

There are two related seller strategies that effectively reduce the buyer's incentives to shade bids, and hence regulate the extent of bid-shading: pulsing (selling every other period) and only selling when recent prices were high enough. When these strategies maximize seller profits, they reduce the extent of bid-shading because they credibly reduce the expected future bidder surplus. Pulsing can be profit maximizing because not selling in a period both reduces the extent of bid-shading and allows the demand side of the market to refill with more new buyers. Pulsing eliminates bid-shading completely because it spaces sales in the auction market enough from each other to let each buyer experience only one auction during their limited participation time window. Then, buyers have nothing to gain from bid-shading, and bid-shading therefore "switches itself off." The online supplement shows that pulsing is more likely to occur when the seller can commit in advance to the timing of sales, but as long as profits are low enough, pulsing occurs even when the seller makes a selling decision in every period. The closed-loop selling strategy with learning is not useful when gains from trade are very high or very low because the seller always sells or pulses, respectively. However, intermediate gains from trade make learning useful. Interestingly, bid-shading can coexist with a discriminating seller that learns from prices and avoids selling in low-demand periods. Therefore, the ability of the closed-loop selling strategy to discourage bid-shading is limited to markets with low gains from trade. Both of these selling strategies highlight a general intuition about optimal selling in sequential auction markets, namely, the benefit of spacing sales apart from each other in time, especially when the general profitability of the market is low for the seller.

This paper also analyzes a phenomenon inherent to sequential auctions—learning from past prices—and shows that learning needs to be analyzed within a buyer-seller equilibrium model because the seller's ability to learn depends on the strategy of the bidders, which itself depends on the strategy of the seller, which in turn depends on the learning. A past realized price is an upper bound on the past bidders' bids, and hence implies an upper bound on their valuations. A general property of learning from an upper bound is that the lower the bound, the more informative the learning. Therefore, the seller can assess a low state of demand more accurately than a high state of demand. The second observation about the seller's learning process is that learning about new bidders is a function of the current beliefs about the old bidders. Therefore, the beliefs evolve over time, and today's belief is not a zero-order Markov process that only depends on yesterday's price. In particular, lower prices today imply not only lower residual demand tomorrow, but also more accurate learning going forward from tomorrow. Finally, the seller's learning process is sometimes particularly enabled by the phenomenon of bidder bid-shading as bid-shading bidders bid differently throughout their lifetimes, starting low and ending higher, and the desirable high-valuation bidders are thus more likely to both survive and be detected from relatively high prices.

This work contributes to the management science literature by exploring a market institution of large and growing importance—the sequential auction. The model illustrates why and how to take the other side of the market into account when formulating one's own strategy, and what are the properties of the resulting equilibrium. The proposed model abstracts away from several complexities of real-world marketplaces, so it can only provide qualitative properties of optimal buying and selling, not concrete quantitative prescriptions for managerial action. The strongest assumption employed by the present model is that the seller is a monopolist. While this assumption fits specialized procurement auctions like the highway-construction market well, more work is needed to characterize concrete seller strategies in hypercompetitive markets like eBay. This paper shows that the "self-preservation instinct" result persists even with two competing sellers. More work is needed to capture the dynamic competition among more sellers.

5. Electronic Companion

An electronic companion to this paper is available as part of the online version that can be found at http://mansci.journal.informs.org/.

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Appendix. Proofs of Propositions 1 and 2 (Proof of Proposition 3 is in the Online Supplement)

Proof of Proposition 1. Consider a *new High* bidder, and suppose that all other bidders are playing the proposed strategies. Note that the expected payoff from bidding $H-a$ is equal to a whether or not the bidder wins the first or the second round: when the probability that future *new* bidders also shade their bids is ω, a satisfies $a=\delta[(1-p)(H-L)+\omega pa]$. Consider deviations from bidding $H-a$ depending on whether there are *old High* bidders in the current period. When there are *old High* bidders, no deviation to bid $b'\leq H$ changes the expected surplus of zero (both *new High* bidders survive and compete in the second period). When there are no *old High* bidders, (1) deviating to $b'<H-a$ can only reduce the payoff because the bidder now loses for sure and gets either a or nothing in the future, depending on whether the resulting price b' is enough for the seller to sell again. (2) Deviating to $H-a<b'\leq H$ leaves the bidder indifferent because he loses and makes a for sure. Uniqueness follows from the uniqueness of a. □

Proof of Proposition 2. Bid-shading persists in **Q** and **0** iff the seller sells in the **1** state (Proposition 1). Suppose first that the seller sells in **1** ($\Pi_1>\delta\Pi_Q$) and so there is bid-shading. Then, the Bellman equations for all but Π_0 are determined. This leaves two cases: the seller either does or does not sell in the **0** state.

Case 1. *Bid-shading, seller always sells* (*the "all" superscript on* Π *is omitted for clarity for Case 1)*:

$$\Pi_0=p(H-a)+(1-p)L-c+\delta[p\Pi_1+(1-p)\Pi_0],$$

$$\Pi_1=pH+(1-p)L-c+\delta[p(H-c+\delta\Pi_Q)+(1-p)\Pi_0],$$

$$\Pi_Q = pH + p(1-p)(H-a) + (1-p)^2 L - c + \delta[p\Pi_Q + p(1-p)\Pi_1 + (1-p)^2\Pi_0],$$

where $a = \delta(1-p)(H-L)$ and $\Pi_0 > \delta\Pi_Q$, $\Pi_1 > \delta\Pi_Q$, and $\Pi_Q > 0$.

The solution is

$$\Pi_Q = \frac{[1-\delta(1-p)]R_Q + \delta(1-p)[R_1 - p(1-p)a]}{1-\delta} = \frac{R_Q - \delta p(1-p)^2(H-L)}{1-\delta}.$$

The binding constraint is $\Pi_0 > \delta\Pi_Q \Leftrightarrow p(H-L)[1-2\delta\cdot(1-p)] > c-L$, which implies both $\Pi_1 > \delta\Pi_Q$ and $\Pi_Q > 0$. With $L=0$ and $c=1$, $\Pi_0 > \delta\Pi_Q \Leftrightarrow H > 1/(p[1-2\delta(1-p)])$ and the denominator must be positive.

Case 2. *Bid-shading, seller withholds supply in the* **0** *state* ($\Pi_0 = \delta\Pi_Q$):

$$\Pi_1^{not0} = pH + (1-p)L - c + \delta[p(H-c+\delta\Pi_Q^{not0}) + (1-p)\delta\Pi_Q^{not0}],$$
$$\Pi_Q^{not0} = pH + p(1-p)(H-a) + (1-p)^2L - c + \delta[p\Pi_Q^{not0} + p(1-p)\Pi_1^{not0} + (1-p)^2\Pi_0],$$

$\Pi_1^{not0} > \delta\Pi_Q^{not0}$, $\Pi_Q^{not0} > 0$, and $\delta\Pi_Q^{not0} > p(H-a) + (1-p)L - c + \delta[p\Pi_1^{not0} + (1-p)\delta\Pi_Q^{not0}]$.

The solution is

$$\Pi_Q^{not0} = \frac{R_Q + \delta p(1-p)[R_1 + \delta p(H-c)]}{(1-\delta)[1+\delta(1+\delta p)(1-p)]}$$

and the last constraint is just the converse of the constraint $\Pi_0 > \delta\Pi_Q$ in Case 1.

$$\Pi_1^{not0} > \delta\Pi_Q^{not0} \;\Leftrightarrow\; \frac{p(H-L)}{c-L} > \frac{1+\delta^2 p(1-p)}{1+\delta^2(1-p)(2-p)} = C_1.$$

Selling in **Q** occurs when

$$\Pi_Q^{not0} > 0 \;\Leftrightarrow\; \frac{p(H-L)}{c-L} > \frac{1+\delta p(1-p)(1+\delta p)}{2-p+\delta(1-p)((2+\delta)p-1)} = C_Q,$$

where the denominators of C_t must be positive. With $L=0$ and $c=1$, $\Pi_1^{not0} > \delta\Pi_Q^{not0} \Leftrightarrow pH > C_1$, $\Pi_Q^{not0} > 0 \Leftrightarrow pH > C_Q$.

Case 3. *No bid-shading, seller withholds supply in the* **0** *and* **1** *states*: When $\Pi_Q^{not0} > 0$ and $\Pi_1^{not0} < \delta\Pi_Q^{not0}$, the seller sells in **Q**, and the bidders do not shade their bids (Proposition 1). No closed-form Bellman analysis is possible because the beliefs can be in infinitely many states (§4.2). The lower bound of the Case 3 configuration is discussed below in Case 4(b).

Case 4(a). *Bid-shading preventing sale despite gains from trade positive*: When $\Pi_Q^{not0} < 0$, the seller exits the market whenever she would sell only in states **1** and **2** before exiting the market because the bidders would continue bid-shading in the initial state **Q**. Such a "one last sale" would be profitable when

$$pH + (1-p)L - c + \delta p(H-c) > 0 \;\Leftrightarrow\; \frac{p(H-L)}{c-L} > 1 - \frac{\delta p(H-c)}{c-L},$$

which holds with $\Pi_Q^{not0} < 0$ for low p.

Case 4(b). *Low gains from trade prevent sale*: When $[\Pi_Q^{not0} > 0$ and $\Pi_1^{not0} < \delta\Pi_Q^{not0}]$ or $[\Pi_Q^{not0} < 0$ and $p(H-L)/(c-L) < 1-(\delta p(H-c))/(c-L)]$, the seller exits the market when pulsing becomes unprofitable when even the pulsing strategy of selling every other period is unprofitable, namely, when

$$(1-p)^2L + [1-(1-p)^2]H - c < 0 \;\Leftrightarrow\; \frac{p(H-L)}{c-L} < \frac{1}{2-p}. \quad \square$$

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