THE JOURNAL OF FINANCE • VOL. XXXV, NO. 1 • MARCH 1980

Retractable and Extendible Bonds: The Canadian Experience

A. L. ANANTHANARAYANAN and EDUARDO S. SCHWARTZ*

I. Introduction

SINCE THE PUBLICATION OF the seminal papers by Black and Scholes [3] and Merton [16] on the pricing of options and corporate liabilities, their basic framework has been extended and applied to a variety of problems in finance.[1] More recently, the same framework has been used for the valuation of interest dependent claims, and in particular for the pricing of default free bonds.[2] These securities (generally government bonds of various types) are valued by treating them as "contingent" upon the course of one or more interest rates, along with suitable assumptions about the term structure of interest rates.

Brennan and Schwartz [4] assume that the value of a default free bond is a function solely of the instantaneous interest rate and time to maturity, and show that various types of bonds—savings bonds, retractable and extendible bonds, and callable bonds—all follow the same partial differential equation as discount bonds, the distinguishing feature being the associated boundary conditions.

Taking into account the considerable theoretical work that has been done, there is relatively little published empirical research testing these models. Most of the empirical work in the area of contingent claims analysis has been on the stock options market,[3] with the exception of Ingersoll [12] on the pricing of dual fund shares and Brennan and Schwartz [6] on the valuation of Canadian Federal Government coupon bonds.

In this paper contingent claims analysis is applied to the valuation of retractable and extendible bonds and the resultant model is then applied to price Government of Canada bonds.

An extendible bond is a medium term debt obligation that gives the holder the option of extending the term of the instrument, on or before a fixed date at a predetermined coupon rate. For example, the 5½ percent Government of Canada extendible bond was issued on October 1, 1959. It was exchangeable on or before June 1, 1962 into 5½ percent bonds maturing October 1, 1975. Thus the three-

* Univeristy of British Columbia. This research was partially supported by a Dean Witter Foundation grant through the Institute of Business and Economic Research, University of California, Berkeley; and a Leslie G. J. Wong Summer Research Grant through the University of British Columbia. The paper was written while E. Schwartz was a visiting assistant professor at the University of California, Berkeley. The authors gratefully acknowledge the helpful comments of Michael J. Brennan and Richard Brealey. The authors retain responsibility for remaining errors.

[1] See Smith (20) for an excellent survey of the work in this area up to 1976.

[2] See, for example, Cox, Ingersoll and Ross (8), Richard (19), Vasicek (21), and Brennan and Schwartz (6).

[3] See, for example, Black and Scholes (2), Brennan and Schwartz (5), and Galai (11).

year initial bond was extendible into a 16 year bond, at the holder's option. A retractable, on the other hand, gives the holder the option of electing an earlier maturity. From the viewpoint of both practical investment and valuation-theory, the two instruments are similar.

An extendible bond can be viewed as the shorter-term bond plus a call option to buy the longer term bond at the face value of the bond up to the extension date; and a retractible bond can be viewed as a long-term bond plus a put option to sell the bond at face value on the retraction date.

Extendible and retractable bonds were first issued in Canada in 1959 by the Federal Government, and while there were additional issues in the mid-sixties, these instruments have been used more widely in the high interst rate period that has prevailed since 1969/70. As of March 31, 1977, the total amount of federal government extendibles and retractables outstanding amounted to $6.25 billion dollars. In addition to the federal government, provincial governments and private corporations in Canada have also issued these types of bonds,[4] with amounts outstanding of $2.5 billion and $2.22 billion, respectively, as of March 31, 1977. To avoid dealing with the problem of default risk, this paper will concentrate on federal government bonds.

II. The Model

The basic feature of the pricing model of default free bonds is the stochastic nature of the instantaneously risk-free rate of interest. In this study it is assumed that this interest rate follows a diffusion process that can be described by the stochastic differential equation

$$dr = \beta(r)\, dt + \gamma(r)\, dz \tag{1}$$

where $\beta(r)$ and $(\gamma(r))^2$ represent the instantaneous drift and variance respectively of the process and dz is a Gauss-Wiener process with $E[dz] = 0$, and $E[dz^2] = dt$. Additional restrictions will be imposed on $\beta(r)$ and $\gamma(r)$ in the following sections to give empirical content to the model; for the present there is nothing to be gained by restricting the generality of process (1).

Assuming that the value of any default-free bond is only a function of r and time to maturity, τ; i.e., $B(r, \tau)$,[5] then using Ito's Lemma (MacKean [14]), the instantaneous price change on the bond is given by

$$\frac{dB}{B} = \mu_B\, dt + \sigma_B\, dz \tag{2}$$

where

$$\mu_B = (\beta B_1 - B_2 + \tfrac{1}{2}\gamma^2 B_{11})/B$$

$$\sigma_B = \gamma B_1/B$$

[4] The referee has pointed out that the Belgian government has also issued retractable bonds.

[5] See Vasicek (21), Merton (18), and Brennan and Schwartz (4) for a more detailed discussion of this point and the hedging arguments that follow, and Brennan and Schwartz (6) for a situation where two stochastic interest rates are considered.

and subscripts denote partial derivatives of the bond price with respect to its first (r) and second (τ) arguments. The arguments of the functions have been omitted for simplicity.

Given the above assumptions, the instantaneous returns on any two default free bonds are perfectly correlated (i.e., they have the same stochastic process dz), It is then possible to use the familiar hedging arguments from option pricing theory[6] to form a portfolio of two bonds (with different maturities and/or different types of bonds) in proportions such that the instantaneous return on this hedge portfolio be non-stochastic. To avoid arbitrage profits, the instantaneous return on the hedge portfolio must equal the instantaneously risk-free rate of interest. These arguments lead to the expression

$$\frac{(\mu_B + C/B) - r}{\sigma_B} = \phi(r) \tag{3}$$

where C is the continuous coupon rate paid to the holder of the bond.[7] $(\mu_B + C/B)$ represents the total instantaneous expected return on the bond, including price appreciation (μ_B) and coupon yield (C/B).

Expression (3) implies that the total instantaneous return on the bond in excess of the risk-free rate (which may reasonably be called a "risk premium") per unit of "risk" (measured by σ_B), is independent of the time to maturity of the bond and applies to all default free bonds. Then $\phi(r)$ is a function that depends on the preferences of the market participants and is a kind of "price of instantaneous standard deviation risk." Further assumptions with respect to the functional form of $\phi(r)$ will be required subsequently for the empirical application of the model. Note that $\phi(r)$ has been written as a function solely of r and not also of t, i.e., it is assumed to be stationary through time. This comes about from (1), where $\beta(r)$ and $\gamma(r)$ have been assumed stationary.

Substituting μ_B and σ_B from (2) into (3) yields the partial differential equation for the bond price

$$\tfrac{1}{2}\gamma^2 B_{11} + (\beta - \gamma\phi)B_1 - rB - B_2 + C = 0 \tag{4}$$

In equilibrium, all default free bonds follow the same valuation equation. What distinguishes among them, are the boundary conditions that each has to satisfy.

The terminal value of the bond at maturity determines the first boundary condition

$$B(r, 0) = F \tag{5}$$

where F is the face value of the bond. This boundary condition is relevant for all default free bonds, given that the face value is guaranteed at maturity.

In the case of default free retractable/extendible bonds, the extension/retraction option imposes an additional boundary condition:

$$B(r, \tau_e^+) = \max[B(r, \tau_e^-), F] \tag{6}$$

[6] Assuming that the relevant assumptions of the option pricing model hold. For a discussion of these assumptions, see Black and Scholes (3) or Merton (16).

[7] The assumption of continuous coupon payments is not necessary for the development of the model; it is used because it simplifies the boundary conditions that follow. Because bonds are usually traded at market price plus accrued coupons, this assumption is quite reasonable.

where τ_e^+ represents the instant in time just prior to the decision point and τ_e^- represents the instant in time just after the decision point. Note that in this case τ represents the time to maturity of the longer-term bond and τ_e the extension/retraction (time to maturity) date. Expression (6) implies that at the decision point the bond holder will optimally choose the longer-term bond only if at that time its value is greater than the face value; if this is not the case, he will choose to retract the bond (or not to extend it). This formulation assumes that the retraction/extension option has to be exercised at a single point in time.[8]

In addition to the above boundary conditions, whenever the partial differential equation is solved by numerical methods,[9] the interest rate boundaries ($r = 0$ and $r = \infty$) are also required. The latter conditions are obtained as "natural" boundaries of equation (4) once β and γ have been specialized.

To solve equation (4), it is necessary to know, in addition to the boundary conditions and the specialized form of β and γ, the form of $\phi(r)$, the latter depending upon factors such as investors' attitudes to risk. In section 4, a sample of straight-coupon bonds is used to determine a particular form of $\phi(r)$.

So far the model has been developed on the assumption of no taxes, either on income (coupons and interest) or capital gains. It is possible to incorporate taxes into the valuation equation, along the lines of Ingersoll [12], making the following assumptions:

a) Income taxes are payable on a continuous basis and at a fixed rate. This implies that there is some "average" tax rate over all investors.
b) Capital gains taxes are also paid continuously and at a fixed rate. This is a very restrictive assumption, because in reality capital gains taxes are paid only when gains are actually realized by a sale, but it is required to ensure a unique equilibrium bond value using the continuous hedging approach.

Applying the same hedging arguments as before, it is possible to show that in the presence of taxes the value of a default-free bond follows the partial differential equation

$$\tfrac{1}{2}\gamma^2(1-T)B_{11} + [(1-T)\beta - \gamma\phi]B_1 + (1-R)(C - rB) - (1-T)B_2 = 0 \quad (7)$$

where R and T are the tax rates on income and capital gain, respectively. Equation (7) is subject to the same boundary conditions (5), always, and (6), depending on the type of bond considered.

The tax assumptions are very restrictive. It is an empirical question, however, whether it is better to ignore taxes altogether or to incorporate them into the valuation equation with the current assumptions. This question is addressed later.

III. The Interest Rate Process

To apply the model developed in the previous section, it is necessary to specify the parameters $\beta(r)$ and $\gamma(r)$ of the interest rate process. Lacking a well-developed

[8] For the case where the option could be exercised over a period of time, rather than at a point in time, see Ananthanarayanan (1). In general it is optimal to exercise the option at the last possible date.

[9] This will be the case in the applications that follow because the equation obtained has no analytical solution.

theory of growth under uncertainty,[10] the only restriction which may be imposed is that interest rates should never become negative, to avoid dominance by money. The process assumed in this study is expressible as

$$dr = m(\mu - r)\,dt + \sigma\sqrt{r}\,dz \tag{8}$$

which implies

$$\beta(r) = m(\mu - r), \qquad m, \mu > 0$$

$$\gamma(r) = \sigma\sqrt{r}, \qquad \sigma > 0$$

This process has the desirable mean reverting property, when $r > \mu$ ($< \mu$), the drift is negative (positive), so that the deterministic movement of interest rate is always toward the central tendency, μ. The parameter m controls the speed of adjustment towards μ. The functional form of $\gamma(r)$ assures that, for certain values of the parameters of the process, $\gamma(r) \to 0$ as $r \to 0$, and, given that the drift is positive at this point, interest rates can never become negative.[11]

The parameters of expression (8) were determined using the yield to maturity on the 91-day (Canadian) treasury bill as a proxy for the instantaneous interest rate. 990 weekly observations, starting January 7, 1959 to December 21, 1977 were used in the estimation.

Since the stochastic process (8) has a known transition probability density function, the parameters of the process were estimated by maximizing the joint likelihood function of the data.[12] The resulting values for the parameters were:

$$m = 0.25\cdot 10^{-2}, \qquad \mu = 0.13\cdot 10^{-2}, \qquad \sigma^2 = 0.69\cdot 10^{-6}$$

These parameter values satisfy the condition required to make $r = 0$ a natural boundary and imply a central tendency value of 6.7 percent per annum. All further analyses on bond valuation made use of these parameter values.

IV. The Utility Dependent Function

To be able to solve equation (4), it is necessary to make an assumption about the functional form of $\phi(r)$. Under one possible version of the Pure Expectations hypothesis,[13] the instantaneous return on bonds of all maturities is the same, and equal to the instantaneously risk-free rate, r. From (3) this would imply that $\phi(r) = 0$. This would not be the case, however, if some kind of "liquidity premium" were present in the market.

Except for making ϕ a constant,[14] the simplest possible functional form for $\phi(r)$,

[10] However, see Merton (17) and Cox, Ingersoll and Ross (8).

[11] See Feller (10) for an extensive study of this process and a proof that when $2m\mu \geq \sigma^2$, $r = 0$ is a natural boundary.

[12] For a detailed discussion of the estimation procedure and a justification for using the treasury bill yield as a proxy for the instantaneous interest rate, see Anathanarayanan (1).

[13] See Cox, Ingersoll and Ross (8) for different interpretations of the Pure Expectations hypothesis and their consistency with no arbitrage profits in continuous time.

[14] Both Vasicek (21) and Brennan and Schwartz (4) assume ϕ = constant.

and the one here assumed is:

$$\gamma(r)\cdot\phi(r) = -k_1 - k_2 r \tag{9}$$

The main reason for assuming a linear relationship between r and $\gamma\phi$ (instead of ϕ alone), is the computational tractability of the solution to equation (4) under this assumption. Given k_1 and k_2, partial differential equation (4) subject to the appropriate boundary conditions, has a closed form solution for a pure discount bond (i.e., $C = 0$). This result is given in the Appendix.

The fact that a pure discount bond has an analytical solution under this assumption permits a simple method of estimating the parameters k_1 and k_2 from market data. The price of a bond paying a continuous coupon, $P(r, \tau, C)$ may be represented by

$$P(r, \tau, C) = \int_{\tau}^{0} C\, B(r, t)\, dt + B(r, \tau) \tag{10}$$

where $B(\cdot, \cdot)$ represents the price of a pure discount bond,[15] and is given by equation (A-1) in the Appendix.

A sample of weekly market prices on 18 straight coupon bonds was used to estimate k_1 and k_2. There were in total 6,662 observations. Corresponding to any choice of k_1 and k_2 (and given the parameters of the interest process, the current interest rate, time to maturity, and coupon rate), the model price of any straight coupon can be computed using equation (10)[16].

The model considered was

$$\ln(P_i') = \ln(P_i) + \epsilon_i \tag{11}$$

where P_i' and P_i are market and model prices respectively, and $\epsilon_i \sim N(0, \delta^2)$; $\text{cov}(\epsilon_i, \epsilon_j) = 0$ for $i \neq j$. Given that P_i is a nonlinear function of the parameters k_1 and k_2, the estimation of these parameters is the standard problem of coefficient estimation in a non-linear regression framework. The maximum likelihood estimates, after testing for heteroscedasticity, autocorrelation, and normality of the residuals, was found to be[17]

$$k_1 = 0.309\cdot 10^{-5}$$

$$k_2 = -0.154\cdot 10^{-2}$$

These values of k_1 and k_2 imply a particular shape of the term structure which depends on the current instantaneous interest rate. Figures 1 and 2 show the term structure curves for different values of the instantaneous interest rate. The yield to maturity on a pure discount bond is computed from

$$R(r, \tau) = -\frac{1}{\tau}\ln[B(r, \tau)] \tag{12}$$

The limiting value of $R(r, \tau)$ as $\tau \to \infty$ (R INF in the figures) is obtained from equation (A-1) in the Appendix—taking appropriate limits—and equation (12).

[15] For simplicity, in what follows the face value of all bonds has been arbitrarily set equal to 1.

[16] This was done by numerical integration.

[17] For a detailed discussion of the procedure and the appropriate references, see Ananthanarayanan (1).

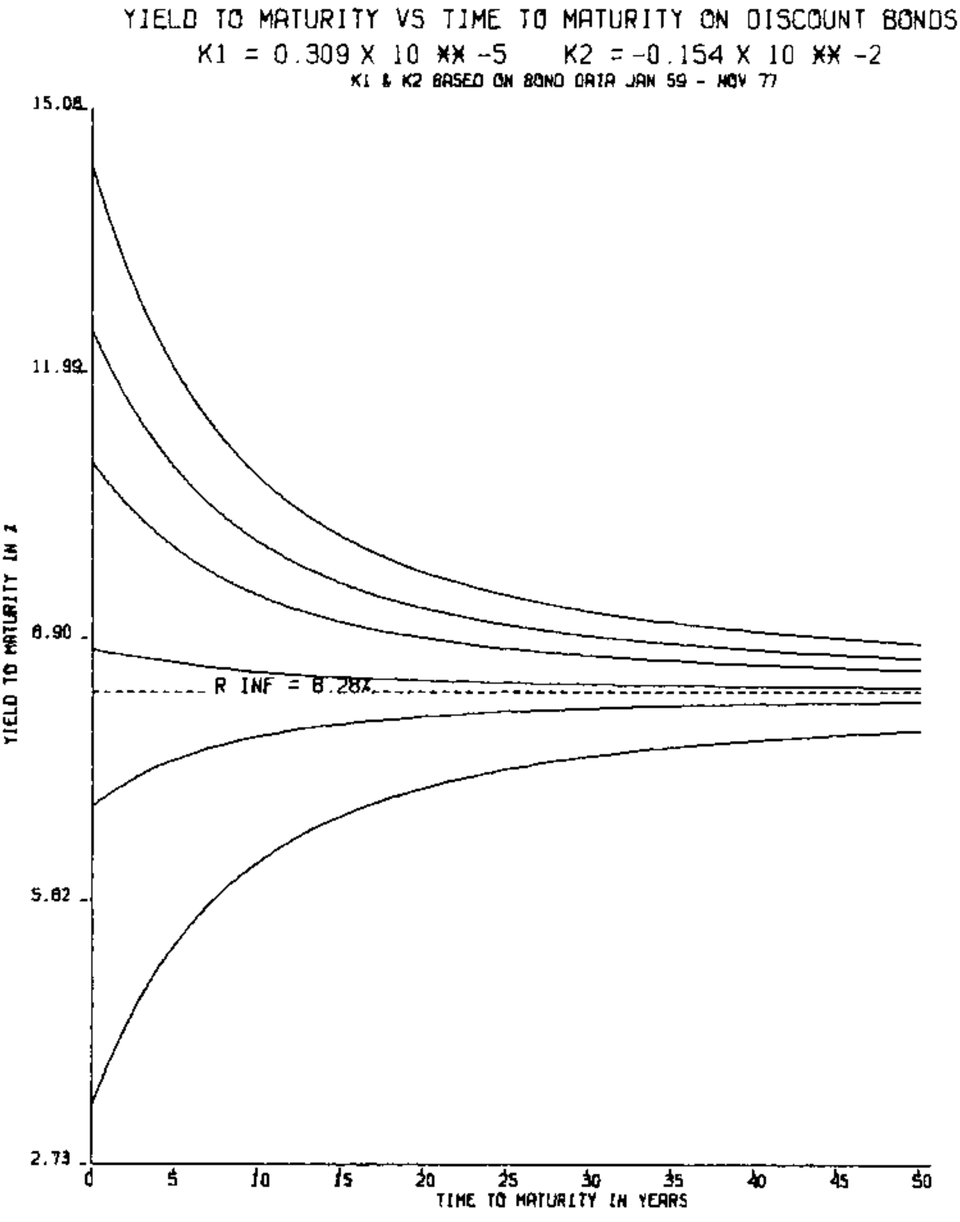

Figure 1. Yield to Maturity vs Time to Maturity on Discount Bonds
K1 = 0.309×10^{-5} K2 = -0.154×10^{-2}
K1 & K2 Based on Bond Data Jan 59–Nov 77

As Ingersoll [13] has pointed out, the term structure corresponding to interest rate process (8) and the functional form of $\phi(r)$ given by (9), could have a humped shape, as shown in Figure 2. When comparing Figures 1 and 2, care must be taken to note the large difference between the two in the scale along the Y-axis.

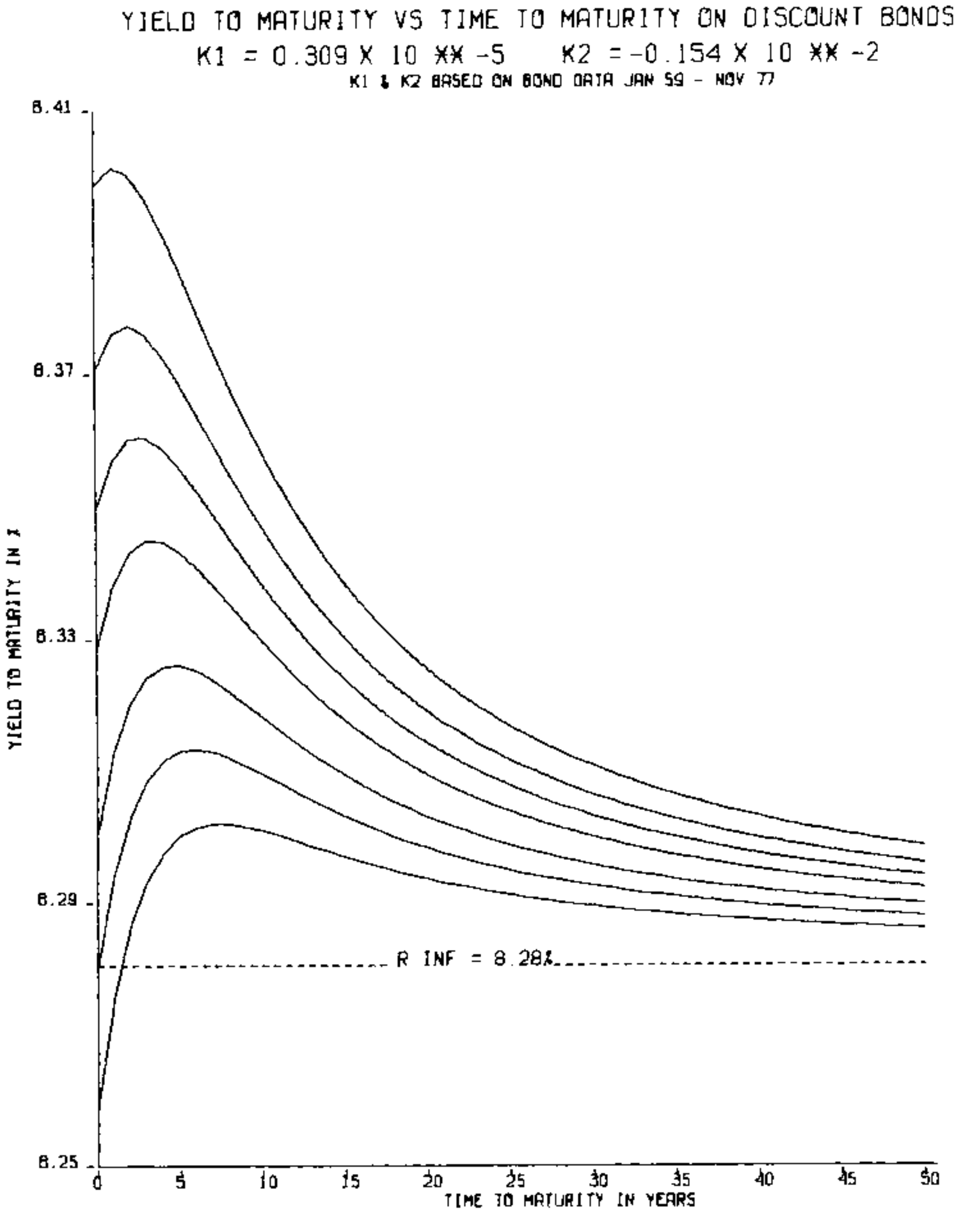

Figure 2. Yield to Maturity vs Time to Maturity on Discount Bonds
$K1 = 0.309 \times 10^{-5}$ $K2 = -0.154 \times 10^{-2}$
K1 & K2 Based on Bond Data Jan 59-Nov 77

V. Valuation of Extendible/Retractable Bonds

5.1 *Data*

All federal retractable/extendible bonds issued by the government of Canada were included in the study. Weekly price data were collected for each bond, starting within a week of the date of issue and extending to the extension or retraction date.

5.2 *Bond Valuation Models*

Three variants of the basic bond valuation model described in section 2 were used in the empirical tests.

a) Bond valuation under the pure expectations hypothesis. As mentioned in section 4, the assumption that the instantaneous return on bonds of all maturities is equal, implies that the utility dependent function $\phi(r) = 0$.

b) Bond valuation under the liquidity premium assumptions described in section 4.

c) Bond valuation with taxes. In this variant partial differential equation (7) was solved maintaining the assumptions about liquidity premium of variant b).[18] Four particular cases were considered. The first two cases include taxes on co ipons and interest at a rate of 50 percent and 25 percent, respectively, but not on capital gains. Cases three and four consider an income tax rate of 25 percent and capital gains tax rates of 10 percent and 20 percent, respectively.

In all cases, the corresponding differential equation and boundary conditions were solved for the 20 bonds in the sample using numerical procedures (given that no closed form solution exists). The solution technique used was the standard implicit differencing approach,[19] in which the partial derivatives of the equation are approximated by differences.

5.3 *Comparisons*

To compare model prices with market prices the mean square error (MSE) was computed

$$\text{MSE} = \frac{1}{T}\sum_{i=1}^{T}(G_i' - G_i)^2 \tag{13}$$

where G_i' and G_i are, respectively, the market and model prices, and T is the number of observations considered (the analysis was done on a per bond basis and on the overall data). The MSE (or its square root: RMSE) is broadly indicative of the lack of fit between the model and the market prices. Further, a simple regression of market prices on model prices permits the decomposition of the MSE into three components[20]

$$G_i' = a + bG_i + \epsilon_i \tag{14}$$

then

$$\text{MSE} = (\bar{G}' - \bar{G})^2 + \left[(1-b)^2\left\{\frac{1}{T}\sum_{i=1}^{T} G_i^2 - \bar{G}^2\right\} - 2(1-b)^2\bar{G}^2\right] + \frac{1}{T}\sum_{i=1}^{T}\epsilon_i^2 \tag{15}$$

[18] For simplicity, taxes were not taken into account in estimating the parameters of the model.

[19] See McCracken and Dorn (15) and Brennan and Schwartz (4) for details.

[20] See Ingersol (12) for a similar approach.

where $\bar{G}'$ and $\bar{G}$ stand for the means of the market and model prices.

The three component parts may be identified as:

i) The part due to bias—attributable to a difference between the mean levels.
ii) The part due to the slope being different from one ($b \neq 1$).
iii) The part due to residual error.

It should be clear that all these comparisons of model and market prices represent a joint test of three hypotheses: a) the generating process for the instantaneous interest rate; b) the model of the term structure of interest rates; and c) the bond-option valuation model.

In sub-section 5.7 some tests designed to measure the value of the extendible/retractable option are presented.

5.4 *Results*

The results of regression (14) and the error decomposition (15) for the total sample data are presented in Table I. Column one gives the results for the pure expectations model, column two for the liquidity premium model, and columns three to six the four cases when taxes were considered. Partial results for individual bonds for the same models are presented in Table II. This Table gives summary statistics of the mean error between model and market prices $\left(\frac{1}{T}\sum_{i=1}^{T}(G_i' - G_i)\right)$ for individual bonds.

The predominant element of the MSE in the pure expectations model is the bias. This is also indicated by noting that the mean error is consistently negative for all bonds (see Table II). This variant of the model overprices the bonds, which implies that the market's expected yield on the bonds is higher than that assumed in the model. A possible explanation of this is the existence of some liquidity or term premium.

As expected, the liquidity premium model clearly outperforms the pure expec-

Table I

Comparison of Model and Market Prices (All Models) (Summary Based on All Bonds in the Sample)

Model	Pure Exp.	Liq. Prem.	Rev. Tax (50%)	Rev. Tax (25%)	C.G. Tax (10%)	C.G. Tax (20%)	Mov. Avg.	"Naive"
R^2	0.391	0.491	0.306	0.311	0.332	0.357	0.254	0.371
RMSE	10.253	3.944	3.781	4.611	4.513	4.412	4.965	4.346
Mean Error	−7.570	0.778	−0.905	−1.621	−1.258	−0.751	−2.075	−0.841
Estimated Slope	0.301	0.546	0.678	0.479	0.478	0.477	0.469	0.444
(S.E. of Slope)	0.006	0.009	0.017	0.012	0.011	0.011	0.015	0.010
Est. Intercept	68.183	46.170	31.876	51.725	52.057	52.360	52.520	55.716
(S.E. of Intr)	0.718	0.978	1.825	1.285	1.216	1.144	1.546	1.042
Fraction of Error Due to Bias	0.545	0.038	0.057	0.123	0.077	0.029	0.174	0.037
$b \neq 1$	0.352	0.383	0.085	0.304	0.343	0.387	0.250	0.462
Res. Variance	0.102	0.577	0.857	0.572	0.579	0.583	0.574	0.500
Misspec Error	94.386	6.573	2.035	9.096	8.576	8.114	10.483	9.445
Resid. Error	10.748	8.986	12.264	12.170	11.797	11.360	14.168	9.448

Table II

Predicted and Actual Bond Prices for Alternative Models. Summary Statistics of Mean Error for Individual Bonds

Model	Minimum	Maximum	Mean	Std. Dev.	Percentiles 20th	50th	80th
Pure Exp.	−18.62	0.36	−6.24	4.81	−7.15	−5.66	−4.13
Liq. Prem.	−3.22	2.93	1.16	1.75	−0.27	1.66	2.45
Rev. Tax (50%)	−4.24	2.99	−0.28	2.37	−2.67	0.28	2.21
Rev. Tax (25%)	−7.04	2.56	−0.86	2.93	−3.14	−1.47	1.91
C.G. Tax (10%)	−6.76	2.76	−0.53	2.83	−2.76	−1.25	1.94
C.G. Tax (25%)	−6.28	3.03	−0.08	2.72	−2.01	−0.77	2.44
Mov. Avg.	−6.74	5.15	−1.43	3.41	−5.16	−1.82	0.66
"Naive"	−8.01	0.89	−1.60	2.93	−3.68	−0.09	0.44

tations model. In this case the mean error is slightly positive (.78%), but the estimated slope ($b = .55$) is still far from unity.

None of the tax models seems clearly to outperform the liquidity premium model.

5.5 *The "Moving Average" Model*

A sensitivity analysis performed on all the parameters of the model (m, μ, σ, α, k_1, k_2) indicated that the interest rate process parameter μ (central tendency) has the most significant impact on model bond prices. Increasing (decreasing) μ would lead to an across-the-board decrease (increase) in bond values. It was felt, therefore, that the assumption of time homogeneity of the parameters, and particularly of μ, might be an important misspecification. An approximate method of relaxing this assumption is to use different μ's for different time periods. Those bonds for which enough information was available (fifteen bonds) were valued by means of the liquidity premium model, using as μ the average value of the short-term interest rate in the two years immediately prior to the date of issue. This value of μ was maintained constant for the life of that bond.[21] As shown in column seven of Table I, and row seven of Table II, it does not appear that this approach significantly improves the fit between market and model prices.

5.6 *The "Naive" Model*

As a bench mark against which to compare the performance of the bond valuation models developed in this paper, an ad hoc procedure—referred to as the "naive" model—was used. It is based on an approach suggested by Dipchand and Hanrahan [9] and consists essentially in assigning as the value of the extendible/retractable bond the larger of the two possible straight bond values (the short and the long bonds) determined by the option to extend or retract. To

[21] A more accurate procedure would have been to change μ each year for all bonds, but this would have made more difficult the numerical solution to differential equation (4) by making the parameters a function of time.

obtain the values of the straight bonds, their yield to maturity was computed from a regression equation for the yield curve.[22]

The regression model used to determine the yield, y, was

$$y = a_1 + a_2 r + a_3 \tau + a_4 \sqrt{\tau} + a_5 \tau^2 + a_6 \tau^3 + a_7 \log \tau. \tag{16}$$

Assuming continuous coupon payments at a rate C, and a face value of the bond of \$100, the relationship between the yield and the price of the bond is given by,

$$B_r = \frac{C}{y}(1 - e^{-y\tau}) + 100e^{-y\tau}. \tag{17}$$

The sample of 18 straight coupon bonds was used to determine the coefficients of regression (16). Given the price of the bond, B_r, at each point in time (17) was solved to obtain the corresponding yield (y), which was then used in regression (16). The coefficients obtained for the total sample and the corresponding t statistic (in parenthesis) are

$a_1 =$	0.0509	(16.81)
$a_2 =$	0.7049	(138.12)
$a_3 =$	−0.0067	(−12.39)
$a_4 =$	0.0184	(11.78)
$a_5 =$	0.0005	(14.20)
$a_6 =$	−0.0019	(−15.34)
$a_7 =$	−0.0271	(−9.80).

These regression coefficients were then used with equation (16) to compute, at every point in time, two yields to maturity for each extendible bond in the sample,[23] corresponding to each of the alternative maturities. Using equation (17), the values of the long and short bond were obtained. The "naive" model price of the extendible, at every point in time, is then set at the higher of these two bond values.

The results of regressing the market prices on these model prices are reported in the last column of Table I, and the last row of Table II. A cursory examination of the results indicates that the naive model performs almost as well as the substantially more sophisticated and complex models developed in this study (compare it, for example, with the liquidity model). It can be argued, however, that the partial equilibrium models developed in this paper—unlike the naive model—are amenable to considerable improvement.[24]

5.7 *The Extendible/Retractable Option*

There are two ways to compute the value of the extendible/retractable option. One is to assume a short term bond plus a call option to buy the long term bond

[22] The ad hoc regression equation relating yield to maturity on straight bonds to current interest rate and various functions of time to maturity used in this study is very similar to the one developed by Bell Canada's Bond Research Division.

[23] The retractable was excluded from this test because it had several retraction dates.

[24] Perhaps the most promising improvement of the basic model is the extension of the state space to two stochastic interrelated interest rate processes. See, for example, Brennan and Schwartz (7) where this approach has been used in the valuation of Savings Bonds.

at the extension/retraction date, and the other is to assume a long term bond plus a put option to sell the long term bond at the extension/retraction date. These two ways of looking at the problem represent the familiar put-call parity relationship.

During the sample period considered, interest rates were generally increasing and for the sample of bonds available the short term bond maturity was usually substantially shorter than the maturity of the long term bond. Because of these factors the call option value for some of the bonds was very close to zero ("deep out of the money"). For this reason it was easier to evaluate and make comparisons using the put option value, and this was done in what follows.

To test the ability of the model to evaluate the option element of the bond price one would like to compare the predicted difference between prices of an extendible/retractable and a straight (but otherwise identical) bond with the actual difference between these two bonds. Unfortunately, comparable straight bonds do not exist for the sample considered, so the next best alternative was to compare the estimated option value with the difference between the actual price of the extendible/retractable and the model price of an otherwise identical straight bond.

The results of regressing "actual" option values (as defined above) on predicted option values for individual bonds[25] are presented in Table III. Predicted option values were computed using the liquidity premium model and model prices of the straight bonds were computed using equation (10) and equation (A-1) in the Appendix. Even though for some bonds, the model is able to explain a relatively large proportion of the variation in the option values, in three cases (two of them statistically significant) the regression coefficients are negative. These results indicate that the single interest rate model provides, at best, a weak explanation of the variations in the option value of extendible bonds.

VI. Tests of Market Efficiency

The hedging arguments developed in section 2 to derive the partial differential equation governing the value of a default free bond, required the formation of a portfolio of two bonds in proportions such that the instantaneous return on the portfolio be nonstochastic. From (2) it is easy to show that these proportions are

$$x_B = -x_G \frac{G_1}{G} \frac{B}{B_1} \tag{18}$$

where x_B and x_G are, respectively, the dollar investments in bond B (say a straight coupon bond) and G (say a retractable/extendible bond), and B_1 and G_1 are partial derivatives with respect to the first argument (the interest rate).

It is then possible to form a zero net investment portfolio by investing an amount x_G in bond G, x_B in bond B, and $-(x_G + x_B)$ in the riskless asset. If the market is efficient with respect to the information contained in the model, the return on this riskless zero net investment portfolio should not be significantly different from zero.[26]

[25] Again the retractable was excluded from this test because it had several retraction dates.

[26] For an extensive discussion of these tests of market efficiency as they apply to the stock option markets, see Black and Scholes (2), Galai (11), and Brennan and Schwartz (5).

Table III

Predicted and "Actual" Option Values for the Liquidity Premium Model. Summary Statistics of Regression Results for Individual Bonds. (Alpha and Beta are the Coefficients from Regression of "Actual" Values on Predicted Values)

					Percentiles		
	Minimum	Maximum	Mean	Std. Dev.	20th	50th	80th
Alpha	−4.57	6.33	1.03	3.26	−1.91	0.98	4.69
Beta	−2.59	6.53	1.35	1.80	0.72	1.22	2.08
R^2	0.00	0.99	0.41	0.28	0.15	0.34	0.64

Table IV

Return on Zero Investment Portfolio.
Summary of Statistics for Individual Bonds for the Liquidity Premium Model.

					Percentile		
	Minimum	Maximum	Mean	Std. Dev.	20th	50th	80th
A. Based on Constant Long Position in Bond							
Mean Return	−0.148	0.0695	0.0148	0.0219	−0.019	0.0054	0.0231
t-Statistic	−0.04	0.23	0.06	0.08	−0.01	0.01	0.08
B. Based on Varying Position in Bond							
Mean Return	−0.0948	0.0086	−0.0215	0.0272	−0.0429	−0.0167	−0.0039
t-Statistic	−0.26	0.02	−0.07	0.09	−0.19	−0.04	−0.01

The sample of straight coupon bonds and retractable/extendible bonds was used to perform a number of tests on market efficiency[27], only two of which are reported here, since the results on all of them were very similar.

In the first test it was assumed that at the beginning of each week a long position in G was taken by buying one bond at the market price (i.e., $x_G = G$). Then x_B is computed from (18) by using the market price of bond B, and the partial derivatives of the bond prices with respect to r obtained from the numerical solution to the respective bond valuation equations. A short position in an amount x_B is taken in a straight coupon bond, and the balance is made up by an investment in the riskless asset. At the end of the period, the portfolio is assumed to be liquidated at the then-existing market prices, and the return on the portfolio over the one period is computed. A new portfolio is then formed and the procedure is repeated until the end of the data on each bond. Panel A of Table IV presents summary statistics of mean returns (and t-statistic) on these hedges for individual bonds for the liquidity premium model. The clear indication is that the returns to the zero-investment hedge portfolio are not significantly different from zero. The same results can be observed when at each point in time a portfolio of these hedges is formed with all bonds outstanding at that time. Table V presents these results for all models where the appropriate correction for heteroscedasticity, caused by the different numbers of hedge portfolios in each period, was performed (the dollar return in each period was weighted by $1/\sqrt{N}$, where N represents the number of bonds outstanding in each period).

[27] Note that the tests of market efficiency have used data from the same test period used to estimate the process parameters.

Table V

Return on Zero Net Investment Portfolio (Based on a Constant Long Position in the Generic Bond) By Aggregating Over All Bonds (Results for all Models)

Model	Mean Return ($)	Standard Deviation of Return	t-Statistic
Pure Exp.	−0.0197	0.955	−0.021
Liq. Prem.	0.0228	0.476	0.048
Rev. Tax (50%)	0.0352	0.486	0.072
Rev. Tax (25%)	0.0270	0.451	0.060
CG. Tax (10%)	0.0258	0.457	0.056
CG. Tax (20%)	0.0242	0.466	0.052
Moving Average	0.0113	0.484	0.023

The purpose of the second test was to see if the model could be used to identify over and underpriced bonds. This test is similar to the previous one, only that at each period a long (short) position in G is taken if its model price is lower (higher) than the market price at that point. If the return on the hedge portfolio based on this strategy were to result in a statistically significant increase in the mean return over the strategy of a constant long position in G, it would imply that the model can be used to identify over-priced/under-priced bonds. Summary statistics of this test for individual bonds for the liquidity premium model are presented in panel B of Table IV. Here again, the mean return appears to be insignificantly different from zero. It does not seem possible to profit from the differences between market and model prices.

VII. Summary and Conclusions

The research herein described represents a first attempt to apply contingent claims analysis to the valuation of retractable and extendible bonds. Three hypotheses have been tested jointly: the mathematical structure of the model, the methodology used to measure the parameters of the model, and the efficiency of the market in pricing bonds. The tentative results of this study indicate that, even in the absence of transaction costs and taxes, the differences between model and market prices cannot be exploited to make arbitrage profits.

With respect to the mathematical structure of the model, considerable improvement can be obtained by extending the state space from one interest rate to two, thus avoiding the objectionable implication that instantaneous returns on bonds of all maturities are perfectly correlated. The particular stochastic specification of interest rates to be used in the model is also an area which requires further research. Finally, additional work should be done on the elements of the model which depend upon investors' attitudes toward risk.

The above comments apply not solely to the valuation of extendible and retractable bonds, but also to the valuation of all types of bonds. It is the belief of the authors of this paper that the application of contingent claims analysis to bond valuation is a fruitful area of future research.

Appendix

The solution to equation (4) subject to boundary condition (5), given the interest rate dynamics (8) and the "liquidity premium" assumption (9), for a pure discount bond (with $C = 0$ and $F = 1$), is, as pointed out by Ingersoll [13]

$$B(r, \tau) = [H(\tau)]^{\frac{2m'\mu'}{\sigma^2}} \exp[\eta m'\mu'\tau + \eta r\{1 - H(\tau)e^{-A\tau}\}] \qquad \text{(A-1)}$$

where

$$m' = (m - k_2)$$
$$\mu' = (m\mu + k_1)/m'$$
$$\eta = [m' - (m'^2 + 2\sigma^2)^{1/2}]/\sigma^2$$
$$A = (m'^2 + 2\sigma^2)^{1/2}$$
$$H(\tau) = [1 + (m' - A)(1 - \exp(-A\tau))/2A]^{-1}$$

From $(A - 1)$ it can be shown that

$$\frac{B_1}{B} = \eta[1 - H(\tau)\exp(-A\tau))] = g(\tau) \qquad \text{(A-2)}$$

the ratio B_1/B is independent of r and strictly a function of time to maturity. Then (9) implies that the "risk premium" is given by

$$\mu_B - r = -(k_1 + k_2 r)g(\tau) \qquad \text{(A-3)}$$

References

1. A. L. Ananthanarayanan. "A Stochastic Specification of Short Term Interest Rate Process and the Pricing of Extendible and Retractable Bonds." Unpublished PhD dissertation, University of British Columbia, (1978).
2. F. Black and M. Scholes. "The Valuation of Options Contracts and a Test of Market Efficiency." *Journal of Finance*; Volume 27, Number 2 (May 1972).
3. ———. "The Pricing of Options and Corporate Liabilities." *Journal of Political Economy*. Volume 81, Number 3 (May–June 1973).
4. M. J. Brennan and E. S. Schwartz. "Savings Bonds, Retractable Bonds and Callable Bonds." *Journal of Financial Economics*, Volume 5, Number 1 (August 1977).
5. ———. "The Valuation of American Put Options." *Journal of Finance*, Volume 32, Number 2 (May 1977).
6. ———. "A Continuous Time Approach to the Pricing of Bonds." *Journal of Banking and Finance*, Volume 3, Number 2 (July 1979).
7. ———. "Savings Bonds in Europe and North America." Unpublished manuscript, (1978).
8. J. C. Cox, J. E. Ingersoll, and S. A. Ross. "A Theory of the Term Structure of Interest Rates." Research Paper No. 468, Stanford University (1978).
9. C. R. Dipchand and J. R. Hanrahan. "The Value of the Extendible Option on a Bond." Presented at the FMA meeting, Seattle, (1977).
10. W. Feller. "Two Singular Diffusion Problems." *Annals of Mathematics*, Volume 54, Number 1 (July 1951).
11. D. Galai. "Test of Market Efficiency of the Chicago Board Options Exchange." *Journal of Business*, Volume 50, Number 2 (April 1977).

12. J. Ingersoll. "A Theoretical and Empirical Investigation of the Dual Purpose Funds: An Application of Contingent Claims Analysis." *Journal of Financial Economics*, Volume 3, Number 1/2 (Jan/March 1976).
13. ———. "Interest Rate Dynamics, the Term Structure, and the Valuation of Contingent Claims." Unpublished manuscript, University of Chicago, (1976).
14. H. D. McKean Jr.. *Stochastic Integrals* (New York: Academic Press, 1968).
15. D. D. McCracken and W. M. Dorn. *Numerical Methods and Fortran Programming* (New York: John Wiley & Sons, Inc., 1964).
16. R. C. Merton. "Theory of Rational Option Pricing." *Bell Journal of Economics and Management Science*, Volume 4, Number 1 (Spring 1973).
17. ———. "An Asymptotic Theory of Growth under Uncertainty." *Review of Economic Studies*, Volume 42, Number 3 (July 1975).
18. ———. "On the Pricing of Contingent Claims and the Modigliani-Miller Theorem." *Journal of Financial Economics*, Volume 5, Number 2 (November 1977).
19. S. F. Richard. "Analytical Model of the Term Structure of Interest Rates." Working Paper No. 19-76-77, Carnegie Mellon University, (1976).
20. C. W. Smith. "Option Pricing: A Review." *Journal of Financial Economics*, Volume 3, Number 1/2 (Jan/March 1976).
21. O. Vasicek. "An Equilibrium Characterization of the Term Structure." *Journal of Financial Economics*, Volume 5, Number 2 (November 1977).