MANAGEMENT SCIENCE
Vol. 13, No. 9, May, 1967
Printed in U.S.A.

STOCHASTIC PROGRAMMING WITH ASPIRATION OR FRACTILE CRITERIA*†

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The general linear programming problem is considered in which the coefficients of the objective function to be maximized are assumed to be random variables with a known multinormal distribution. Three deterministic reformulations involve, respectively, maximizing the expected value, the α-fractile (α fixed, $0 < \alpha < \frac{1}{2}$), and the probability of exceeding a predetermined level κ of payoff. In this paper the author's previous work on "bi-criterion programs" is specialized to give an algorithm for routinely and efficiently solving the second and third reformulations. A by-product of the calculations in each case is the tradeoff-curve between the criterion being maximized and expected value. The intimate relationships between all three reformulations are illuminated, with the cumulative effect of considerably lessening the burden on the decision-maker to preselect with finality a particular value of α or κ.

1. Introduction

Consider the ill-defined stochastic program

(1) $$\text{Maximize}_{x \varepsilon X}\ p^t x,$$

where x is an n-vector, p is a random n-vector with a known distribution, and X is a known closed convex subset of R^n. A choice of x must be made before observing p (if, indeed, it ever is observed). Three deterministic reformulations of (1) are:

(2) $$\text{Maximize}_{x \varepsilon X}\ E(p^t x)$$

where E denotes expected value;

(3) $$\text{Maximize}_{x \varepsilon X}\ F_\alpha(p^t x),$$

where $0 < \alpha < 1$ is a predetermined constant and $F_\alpha(p^t x)$ is the α-fractile of $p^t x$; and

(4) $$\text{Maximize}_{x \varepsilon X}\ P_\kappa(p^t x),$$

where $P_\kappa(p^t x)$ is the probability that $p^t x$ equals or exceeds a predetermined "aspiration" level κ of payoff. The expected value reformulation (2) has the computational advantage that it yields a linear maximand, whereas the fractile and "aspiration" reformulations (which can be more appealing in certain situations) yield nonlinear maximands.

* Received August 1966 and revised December 1966.

† Supported partly by the Office of Naval Research (Task No. 047-041) and by the Western Management Science Institute under a grant from the Ford Foundation.

This paper illuminates the intimate relationships between (2), (3) and (4), and specializes the author's previous work on "bi-criterion programs" [7] in order to obtain an algorithm for routinely solving (3) and (4) by parametric quadratic programming. We take X to be polyhedral as well as convex throughout this paper for the sole reason that efficient parametric quadratic programming routines are presently available only for the case of linear constraints. The algorithm presented here, although turning out to have much in common with certain previous approaches to (3) or (4), offers a unified, non-graphical, and computationally efficient approach to (3) and (4). It has the additional advantage of yielding as a by-product of the calculations the tradeoff curve between the criterion function being maximized and $E(p^tx)$.

For simplicity we assume that p is multinormal with mean vector μ and covariance matrix Σ, so that p^tx is $N(\mu^tx, x^t\Sigma x)$; that (2) has a finite optimal value v; and that $x^t\Sigma x > 0$ on X.

We conclude this Introduction with some discussion of the fractile and aspiration criteria, including a survey of known computational approaches.

The Fractile Criterion

Since the .5-fractile (i.e., the median) of a normally distributed random variable coincides with its mean, (3) with $\alpha = .5$ is identical with (2). Maximizing the α-fractile with $\alpha < .5$ should appeal to some conservative decision-makers because it tends to control the lower tail of the distribution of payoffs.[1]

It is easy to show that

$$F_\alpha(p^tx) = \mu^tx + \Phi^{-1}(\alpha)(x^t\Sigma x)^{1/2},$$

where Φ is the Standardized Normal Distribution Function. When $0 < \alpha < .5$, as we assume henceforth, $\Phi^{-1}(\alpha) < 0$ and (3) is a concave program, since $(x^t\Sigma x)^{1/2}$ is convex [8, p. 195]. Note, however, that the criterion function is not differentiable when $x^t\Sigma x$ vanishes, thus limiting the applicability of gradient-type optimization procedures. Eisenberg [6], Sinha [11], and others have stepped into the breach with theoretical results that are designed to facilitate a computational solution.

Computational procedures for (3) have been offered by Kataoka [8], who proposed and partially justified two iterative procedures that can be viewed as discretized variants of the algorithm derived here, and by Sinha [11], who developed an elaborate specialized duality theory that leads to a computational solution involving linear and quadratic programming. Formulation (3) can also be solved by an obvious geometrical construction that requires the graph of the (E, σ)-tradeoff curve associated with (1), i.e., the image of all feasible x with the property that a higher value of μ^tx can be attained only at the expense of a higher value of $(x^t\Sigma x)^{1/2}$, and a lower value of $x^t\Sigma x^{1/2}$ only at the expense of a lower value of μ^tx. The (E, σ)-tradeoff curve is most conveniently obtained by a

[1] Cf. Baumol [3], who seems to be getting at this idea in the context of the portfolio selection problem, which is a special but important case of (1).

square-root transformation from the (E, σ^2)-tradeoff curve. The latter should be computed, as pointed out by Markowitz [9], by parametric quadratic programming. This use of parametric quadratic programming is at the heart of the present approach to (3) and (4), but we do not require a graphical construction and only compute a relevant subset of the (E, σ^2)-tradeoff curve.

The Aspiration Criterion

Clearly

$$P_\kappa(p^t x) = 1 - \Phi((\kappa - \mu^t x)/(x^t \Sigma x)^{1/2}) \equiv \Phi((\mu^t x - \kappa)/(x^t \Sigma x)^{1/2}).$$

Since Φ is strictly increasing, (4) has the same optimal solution set as

$$\text{(4A)} \qquad \text{Maximize}_{x \varepsilon X} \ (\mu^t x - \kappa)/(x^t \Sigma x)^{1/2}.$$

When the aspiration level κ is taken as v (the optimal value of (2)), it is easily seen that (4A) has an optimal value 0 and that this value is achieved for those feasible x for which $\mu^t x = v$. Hence (4A) with $\kappa = v$ has the same optimal solution set as (2), and the maximum probability in (4) is $\frac{1}{2}$ (since $\Phi(0) = \frac{1}{2}$). As before, we are interested in the conservative decision-maker, who would most likely take $\kappa < v$. Such a choice is necessary and sufficient for the maximum probability in (4) to exceed 0.5, and we assume it henceforth.

The aspiration criterion program (4A) can be solved by a simple geometrical construction noticed by Roy [10], who presented it for a special case, providing that the (E, σ)-tradeoff curve is available. The method can be modified to work almost as easily with the (E, σ^2)-tradeoff curve, which as we pointed out above is easier to compute.

See Charnes and Cooper [4] for additional discussion of the aspiration criterion. They also show how to reduce (4A) to a program that is linear except for one quadratic constraint. If this quadratic constraint we dealt with by the standard trick of taking it up into the objective function with an undetermined multiplier and applying quadratic programming, the result could be an algorithm that closely resembles the one given here.

Remark. It is easy to show that the fractile and aspiration criteria are reciprocal in the following sense: (a) for each fixed α, every optimal solution of (3) is also optimal in (4) with κ equal to the optimal value of (3); and (b) for each fixed κ, every optimal solution of (4) is also optimal in (3) with α equal to 1 minus the optimal value of (4). This reciprocity holds, in fact, for arbitrary distributions of p.

2. The Algorithm

In this section we view (3) and (4) as if they were bi-criterion programs and give an algorithm for each by specializing the following result from [7]. Let it be desired to solve

$$\text{(5)} \qquad \text{Maximize}_{x \varepsilon \chi} \ u(p_1(f_1(x)), p_2(f_2(x))),$$

where χ is a non-empty compact convex set in R^n, f_1, f_2, $p_1(f_1)$, $p_2(f_2)$ are concave on χ, p_1 and p_2 are strictly increasing on the image of χ under f_1 and f_2, respectively, u is non-decreasing and quasiconcave (see, e.g., [1]) on the convex hull of the image of χ under $(p_1(f_1), p_2(f_2))$, and all functions are continuous. Think of $u(\cdot, \cdot)$ as a utility function defining a preference ordering over pairs of values of the two criterion functions f_1 and f_2, on which the scale transformations p_1 and p_2 have been performed. Assume that a parametric programming algorithm is available for solving

$$\text{(6)} \qquad \text{Maximize}_{x \varepsilon \chi}\ \gamma f_1(x) + (1 - \gamma) f_2(x)$$

for each value of the parameter γ in the unit interval, and that the resulting optimal solution function $x^*(\gamma)$ would be continuous on [0, 1]. Then the function

$$U(\gamma) = u(p_1(f_1(x^*(\gamma))), p_2(f_2(x^*(\gamma))))$$

is continuous and unimodal on [0, 1], and if γ^* maximizes $U(\gamma)$ on [0, 1] then $x^*(\gamma^*)$ is optimal in (5).

Solving (3)

Consider now (3). Put

$$\begin{aligned} \chi &= X \\ f_1(x) &= \mu^t x \\ f_2(x) &= -x^t \Sigma x \\ p_1(f_1) &= f_1 \\ p_2(f_2) &= -(-f_2)^{1/2} \\ u(p_1(f_1), p_2(f_2)) &= p_1(f_1) - \Phi^{-1}(\alpha) p_2(f_2). \end{aligned}$$

It is easy to verify that all of the assumptions required of (5), which has been made to coincide with (3), are satisfied save possibly one: the boundedness of χ. However, the only need for boundedness is to ensure that certain suprema are achieved. The attainment of all suprema here follows from the non-negativity of $(x^t \Sigma x)^{1/2}$, our assumption that (2) has a finite optimal value v, and the fact [2, Th. 1.7] that a concave quadratic polynomial bounded above on a convex polyhedral set achieves its constrained supremum.

The parametric program (6) becomes

$$\text{(7)} \qquad \text{Maximize}_{x \varepsilon X}\ \gamma \mu^t x - (1 - \gamma) x^t \Sigma x,$$

a parametric quadratic program. Several algorithms are available for it (e.g., [9], [12], [5]), and they all yield an optimal solution function $x^*(\gamma)$ that is continuous on [0, 1]. In fact, a simple reparameterization of $x^*(\gamma)$ is piecewise linear on [0, 1] (see, e.g., [7] or [12]), thus enabling a very convenient representation of $x^*(\gamma)$ in terms of its values at the points of slope change. Hence the method quoted above applies, and it is easy to see how to solve (3) efficiently with the aid of any parametric programming code for (7). One may solve[2] (7) with $\gamma = 1$ and decrease γ until the unimodal function $F_\alpha(p^t x^*(\gamma))$ achieves its maximum on [0, 1]. When the maximizing γ is reached,[3] say γ_α, the parametric programming is stopped because the optimal solution $x^*(\gamma_\alpha)$ of (3) has been found. For more complete details see [7].

Solving (4)

The assumption $\kappa < v$ guarantees that the optimal value of (4A) is > 0. Therefore one can restrict attention in (4A) to feasible x such that $\mu^t x - \kappa \geqq 0$. In this region the maximand is quasi-concave. Put[4]

$$\begin{aligned} \chi &= \{x \varepsilon X : \mu^t x - \kappa \geqq 0\} \\ f_1(x) &= \mu^t x \\ f_2(x) &= -x^t \Sigma x \\ p_1(f_1) &= f_1 - \kappa \\ p_2(f_2) &= -(-f_2)^{1/2} \\ u(p_1(f_1), p_2(f_2)) &= p_1(f_1)/(-p_2(f_2)). \end{aligned}$$

It is easy to verify that all assumptions required of (5), which has been made to coincide with (4A), are satisfied save the boundedness of χ. As before, this seeming difficulty is eliminated by the fact that all pertinent constrained suprema are achieved. Furthermore, (6) is again the parametric quadratic program (7). Thus to solve (4A) one may solve (7) for $\gamma = 1$ and decrease γ until the unimodal function $(\mu^t x^*(\gamma) - \kappa)/(x^*(\gamma)^t \Sigma x^*(\gamma))^{1/2}$ reaches its maximum on [0, 1]. When the maximizing γ is reached,[5] say γ_κ, the optimal solution $x^*(\gamma_\kappa)$ has been found. Again, consult [7] for details.

3. Interpreting the Intermediate Quantities

The algorithm for solving (3) or (4) given in the previous section involves the computation of $x^*(\gamma)$ (an optimal solution of the parametric quadratic program (7)) from right to left on the interval $[\gamma_\alpha, 1]$ in one case and on $[\gamma_\kappa, 1]$ in the other. Although $x^*(\gamma_\alpha)$ and $x^*(\gamma_\kappa)$ are optimal solutions to (3) and (4), the intermediate $x^*(\gamma)$ are also of interest. Our main result is that the image in R^2 of $[\gamma_\alpha, 1]$ under $(E(p^t x^*(\gamma)), F_\alpha(p^t x^*(\gamma)))$ is the complete (E, F_α)-tradeoff curve, and that the image in R^2 of $[\gamma_\kappa, 1]$ under $(E(p^t x^*(\gamma)), P_\kappa(p^t x^*(\gamma)))$ is the complete (E, P_κ)-tradeoff curve.[6]

[2] For a reason that will become apparent in the next section, $\gamma = 1$ is a natural starting point—although any other value in the unit interval could be used.

[3] If the maximizing value is not unique, let γ_α be the largest.

[4] Technically, (7) must now include the constraint $\mu^t x - \kappa \geqq 0$. Since we shall take γ to be decreasing from 1, this constraint will never be binding before termination and can therefore be dropped.

[5] If the maximizing value is not unique, let γ_κ be the largest.

[6] Naturally $x^*(\gamma)$ also yields a portion of the (E, σ^2)-tradeoff curve. This fact is at the center of Markowitz's approach to portfolio selection [9], but is of little more than passing interest here.

It will be notationally convenient in the sequel to refer to the following reparameterized version of (7):

$$\text{(7A)} \qquad \text{Maximize}_{x \varepsilon X}\, \mu^t x - \beta x^t \Sigma x\,,$$

where the parameter β traverses $[0, \infty)$. For fixed $\beta \,\varepsilon\, [0, \infty)$, (7A) is equivalent to (7) with $\gamma = (1 + \beta)^{-1}$, and therefore has an optimal solution $x^*(1/(1 + \beta))$ which with some abuse of notation we henceforth call simply $x^*(\beta)$. Let $\beta_\alpha = (1 - \gamma_\alpha)/\gamma_\alpha$ and $\beta_\kappa = (1 - \gamma_\kappa)/\gamma_\kappa$. Then (3) or (4) are solved by computing $x^*(\beta)$ from left to right on $[0, \beta_\alpha]$ or $[0, \beta_\kappa]$, respectively.

Lemma. For $\beta \geqq 0$, an optimal solution $x^*(\beta)$ of (7A) satisfies the following:

A. $x^*(\beta)$ is optimal in (3) with $\alpha = \Phi(-2\beta(x^*(\beta)^t \Sigma x^*(\beta))^{1/2})$

B. $x^*(\beta)$ is optimal in (4) with $\kappa = \mu^t x^*(\beta) - 2\beta x^*(\beta)^t \Sigma x^*(\beta)$

C. $2\beta(x^*(\beta)^t \Sigma x^*(\beta))^{1/2}$, considered as a function of β, is non-decreasing on $[0, \beta_\alpha]$, assumes the value 0 only at $\beta = 0$, and assumes the value $-\Phi^{-1}(\alpha)$ only at $\beta = \beta_\alpha$.

Proof. Part A can be proven by setting up the appropriate identification between the Kuhn-Tucker conditions for (7A) and (3) (cf. Theorem 3 of [8]). Part B follows from Part A and Remark (a) at the end of Section 1.

The proof of Part C requires two applications of a method often used for obtaining monotonicity results in parametric programming. The first establishes the inequality

$$x^*(\beta')^t \Sigma x^*(\beta') \leqq x^*(\beta^0)^t \Sigma x^*(\beta^0),$$

where $0 \leqq \beta^0 < \beta' \leqq \beta_\alpha$. By part A, $x^*(\beta^0)$ maximizes

$$\mu^t x - 2\beta^0(x^*(\beta^0)^t \Sigma x^*(\beta^0))^{1/2}(x^t \Sigma x)^{1/2}$$

over all feasible x. A similar assertion holds for $x^*(\beta')$. A second application yields

$$(\beta'(x^*(\beta')^t \Sigma x^*(\beta'))^{1/2} - \beta^0(x^*(\beta^0)^t \Sigma x^*(\beta^0))^{1/2})((x^*(\beta')^t \Sigma x^*(\beta'))^{1/2} - (x^*(\beta^0)^t \Sigma x^*(\beta^0))^{1/2}) \leqq 0.$$

The desired monotonicity of $2\beta(x^*(\beta)^t \Sigma x^*(\beta))^{1/2}$ follows from this inequality with the help of the first. That the value 0 is assumed only at $\beta = 0$ follows from the assumption that $x^t \Sigma x > 0$ for all feasible x; that the value $-\Phi^{-1}(\alpha)$ is assumed only at β_α follows from the nature of the algorithm for solving (3). The proof is complete.

This lemma gives an additional useful interpretation to the intermediate $x^*(\beta)$. In the course of solving (3) with $\alpha = \alpha_0$ $(0 < \alpha_0 < \frac{1}{2})$ by the method of Section 2, one automatically solves (3) with each value of α between α_0 and $\frac{1}{2}$ and (4) with each value of κ between $F_{\alpha_0}(p^t x^*(\beta_{\alpha_0}))$ (which must be $< v$) and v. Similarly, in the course of solving (4) with $\kappa = \kappa_0$ $(\kappa_0 < v)$ by the method of section 2, one automatically solves (4) with each value of κ between κ_0 and v and (3) with each value of α between $1 - P_{\kappa_0}(p^t x^*(\beta_{\kappa_0}))$ (which must be $< \frac{1}{2}$) and $\frac{1}{2}$.

Theorem 1. As β traverses $[0, \beta_\alpha]$, $(\mu^t x^*(\beta), F_\alpha(p^t x^*(\beta)))$ traces in R^2 the complete (E, F_α)-tradeoff curve.

Proof. Since $x^*(0)$ is optimal in (2), $x^*(\beta_\alpha)$ is optimal in (3), and $(\mu^t x^*(\beta), F_\alpha(p^t x^*(\beta)))$ on $[0, \beta_\alpha]$ determines a curve by the continuity of $x^*(\beta)$, it is sufficient to show that $x^*(\beta)$ is (E, F_α)-efficient on $(0, \beta_\alpha)$. To show this it is clearly sufficient to show that for $0 < \beta < \beta_\alpha$ there exists a scalar λ, $0 < \lambda < 1$, such that $x^*(\beta)$ maximizes $(1 - \lambda)\mu^t x + \lambda F_\alpha(p^t x)$ over all feasible x. This maximand simplifies to $\mu^t x + \lambda\Phi^{-1}(\alpha)(x^t\Sigma x)^{1/2}$. For $\lambda = 2\beta(x^*(\beta)^t\Sigma x^*(\beta))^{1/2}/(-\Phi^{-1}(\alpha))$, part A of the lemma implies that $x^*(\beta)$ indeed maximizes it. Part C of the lemma implies that this choice of λ satisfies $0 < \lambda < 1$. Hence $x^*(\beta)$ is (E, F_α)-efficient for $0 < \beta < \beta_\alpha$, and the proof is complete.

Theorem 2. As β traverses $[0, \beta_\kappa]$, $(\mu^t x^*(\beta), P_\kappa(p^t x^*(\beta)))$ traces in R^2 the complete (E, P_κ)-tradeoff curve.

Proof. Since $x^*(0)$ is optimal in (2), $x^*(\beta_\kappa)$ is optimal in (4), and $(\mu^t x^*(\beta), P_\kappa(p^t x^*(\beta)))$ on $[0, \beta_\kappa]$ determines a curve by the continuity of $x^*(\beta)$, it is enough to show that $x^*(\beta)$ is (E, P_κ)-efficient on $(0, \beta_\kappa)$. To show this we shall resort to the definition of (E, P_κ)-efficiency and demonstrate: for $0 < \beta < \beta_\kappa$, one has

$$P_\kappa(p^t x) \leqq P_\kappa(p^t x^*(\beta)) \tag{8}$$

for all feasible x such that

$$\mu^t x \geqq \mu^t x^*(\beta), \tag{9}$$

with equality in (8) implying equality in (9). Clearly one may use $(\mu^t x - \kappa)/(x^t\Sigma x)^{1/2}$ in place of $P_\kappa(p^t x)$ in (8), which then becomes

$$(\mu^t x - \kappa)/(x^t\Sigma x)^{1/2} \leqq (\mu^t x^*(\beta) - \kappa)/(x^*(\beta)^t\Sigma x^*(\beta))^{1/2}. \tag{8A}$$

Let β satisfy $0 < \beta < \beta_\kappa$. By part *A* of the lemma,

$$\mu^t x - 2\beta(x^*(\beta)^t\Sigma x^*(\beta))^{1/2}(x^t\Sigma x)^{1/2} \leqq \mu^t x^*(\beta) - 2\beta x^*(\beta)^t\Sigma x^*(\beta) \tag{10}$$

for all feasible x. It is therefore sufficient to show (8A) for all pairs $(\mu^t x, (x^t\Sigma x)^{1/2})$ satisfying (9) and (10), with equality in (8A) implying equality in (9). It is now convenient to change notation. Let $y_1 = \mu^t x - \mu^t x^*(\beta)$, $y_2 = (x^t\Sigma x)^{1/2}$. Then the proposition to be shown can be written as follows after rearranging (8A):

$$\begin{aligned} &y_1(x^*(\beta)^t\Sigma x^*(\beta))^{1/2} - (\mu^t x^*(\beta) - \kappa)y_2 \\ &\qquad \leqq -(\mu^t x^*(\beta) - \kappa)(x^*(\beta)^t\Sigma x^*(\beta))^{1/2} \end{aligned} \tag{8B}$$

for all $(y_1, y_2) \geqq 0$ satisfying

$$y_1 - 2\beta(x^*(\beta)^t\Sigma x^*(\beta))^{1/2}y_2 \leqq -2\beta x^*(\beta)^t\Sigma x^*(\beta), \tag{10A}$$

with equality in (8B) implying $y_1 = 0$. This proposition is readily demonstrated from the inequality $2\beta x^*(\beta)^t\Sigma x^*(\beta) < \mu^t x^*(\beta) - \kappa$, which follows from the assumption that $\kappa < v$, the continuity of $x^*(\beta)$, and part B of the lemma. This completes the proof.

Theorems 1 and 2 reveal why the initial value of β was chosen to be 0. Any non-negative initial value could be chosen and the algorithm would still solve (3) (resp. (4)), perhaps even with less calculation; but if the initial value exceeds β_α(resp. β_κ), then the intermediate $x^*(\beta)$ will no longer be (E, F_α) (resp. (E, P_κ))-efficient.

The results of Theorems 1 and 2 and Parts A and B of the lemma considerably lessen the burden on the decision-maker to preselect with finality a particular value of α or κ. The availability of a piecewise linear reparameterization of $x^*(\beta)$ facilitates the practical use of these results.

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