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SOLVING BICRITERION MATHEMATICAL PROGRAMS†

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It often happens in applications of mathematical programming that there are two incommensurate objective functions to be extremized, rather than just one. One thus encounters bicriterion programs of the form of equation (1),

$$\text{maximize}_{x \in X} h[f_1(x), f_2(x)],$$

where h is an increasing utility function, preferably quasiconcave, defined on outcomes of the concave objective functions f_1 and f_2, and x is a decision n-vector constrained to the convex set X. It is shown how such programs can be numerically solved if a parametric programming algorithm is available for the parametric subproblem

$$\text{maximize}_{x \in X} \alpha f_1(x) + (1-\alpha) f_2(x). \qquad (0 \leqq \alpha \leqq 1)$$

A natural byproduct of the calculations is a relevant portion of the 'trade-off curve' between f_1 and f_2. Outlines of several algorithms for solving equation (1) under various special assumptions and a numerical example are presented to illustrate the application of the theory developed herein. A useful extension is presented that permits nonlinear scale changes to be made on the f_i.

IN THIS paper we study bicriterion mathematical programs of the form

$$\text{maximize}_{x \in X} h[f_1(x), f_2(x)], \tag{1}$$

where f_1 and f_2 are real-valued concave criterion (payoff) functions of the n-vector x of decision variables that are constrained to lie in a convex subset X of E^n, and h is a real-valued increasing (i.e., monotone nondecreasing in each argument) ordinal utility indicator function defined on the pairs of achievable values for f_1 and f_2. We present a simple method for solving (1) based on any known parametric programming algorithm for the parametric subproblem

$$\text{maximize}_{x \in X} \alpha f_1(x) + (1-\alpha) f_2(x), \tag{P_α}$$

where the parameter α varies over the unit interval. When f_1 and f_2 are

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linear and X is a convex polyhedron, for example, (1) is reduced essentially to a standard parametric linear program even though h is nonlinear. Thus parametric linear programming routines can be modified to solve this important class of nonlinear (even nonconcave) programs. A natural by-product of the calculations is a relevant portion of the tradeoff-curve between f_1 and f_2.

When h is also known to be quasiconcave (i.e., there is a diminishing marginal rate of substitution between f_1 and f_2), a property shared by most utility functions arising in practice, it is shown how to substantially reduce the amount of computational work necessary to solve (1). Some examples of nonlinear programs that are or can be viewed as bicriterion programs with quasiconcave h are:

A. $\text{maximize}_{x \in X} \min\{f_1(x), f_2(x)\}$.

B. $\text{maximize}_{x \in X} f_1(x)/f_2(x)$,
where $f_1(x)<0$ and $f_2(x)>0$ on X.

C. $\text{maximize}_{x \in X} v_1[f_1(x)]^{\beta_1}+v_2[f_2(x)]^{\beta_2}$,
where v_1, v_2, β_1, $\beta_2>0$ and $f_1(x)$, $f_2(x)>0$ on X.

D. $\text{maximize}_{x \in X} - v_1 \exp[-f_1(x)] - v_2 \exp[-f_2(x)]$,
where v_1, $v_2>0$, $f_i(x)=-\sum_j x_j q_{ij}$, $(i=1, 2)$ and $q_{ij}>0$, $x_j \geqq 0$; this program is equivalent to

$$\text{minimize}_{x \in X} v_1 \prod_j (q_{1j})^{x_j}+v_2 \prod_j (q_{2j})^{x_j}.$$

E. $\text{maximize}_{x \in X} [f_1(x)]^{\beta}[f_2(x)]$,
where $\beta>0$ and $f_1(x)$, $f_2(x)>0$ on X.

Example A is a special case of the commonly occurring Chebyshev problem. Example B is a generalized 'fractional programming' problem that has been extensively investigated for the special case in which f_1 and f_2 are linear and X is a convex polyhedron.† The other combinations of sign restrictions can also be handled. Example C embodies an additive utility function commonly used in the theory of consumer's choice.[7] Example D, in the equivalent form, arises in redundancy allocation and target-assignment contexts.‡ A numerical example based on Example E is given below.

In the next section, the necessary theory is developed. It is then used to construct outlines of several algorithms for solving (1) under various special assumptions on f_1, f_2, and X. They are based on parametric linear programming,[4] on WOLFE's method of parametric quadratic programming,[14]

† Several methods for solving the linear fractional programming problem are available, most of them based on linear programming techniques. For a brief guide to the literature, *see* reference 9, p. 197. For a discussion of more general fractional programs, *see* reference 8.

‡ By an appropriate change of variables,[11] Example D can also be cast as a linearly separable concave program.

and on any method of parametric concave programming that yields a continuous optimal solution function for (P_α) (e.g., reference 5). A numerical example is given. Finally, it is shown that the advantages of a quasiconcave h hold under a weaker condition than quasiconcavity.

DEVELOPMENT

IN ADDITION to the assumptions stated in the first paragraph, it will be convenient to avoid questions of the attainment of suprema by assuming throughout this paper that the feasible region X is compact (closed and bounded) and nonempty as well as convex, that the f_i are continuous as well as concave† on X, and that h is continuous as well as increasing in f_1 and f_2 on the attainable payoff set $f[X]$. We denote by f the vector-valued function (f_1, f_2), and by $f[X]$ the image in E^2 of X under f. A point $x^0 \in X$ is said to be *efficient* if and only if there does not exist another point $x' \in X$ such that $f_i(x') \geqq f_i(x^0)$, $i=1, 2$, with strict inequality holding for at least one i; in other words, if and only if $f(x^0)$ is in the admissible set. The set of optimal solutions of (P_α) for a fixed value of α is denoted by $X^*(\alpha)$, and any n-valued function $x^*(\alpha)$ on [0, 1] that satisfies $x^*(\alpha) \in X^*(\alpha)$ for each α is called an *optimal solution function* of (P_α).

The first two lemmas provide the primary motivation for a computational approach to solving (1) in terms of the parametric program (P_α).

LEMMA 1. *At least one point at which $h[f(x)]$ achieves its maximum over X is efficient.*

Proof. By the compactness of X and the continuity of f and h, (1) has at least one optimal solution x^0. Similarly, there exists a point x' that maximizes $f_1(x)+f_2(x)$ over X subject to the additional constraints $f_i(x) \geqq f_i(x^0)$, $i=1, 2$. Now x' is easily seen to be efficient, for the contrary contradicts the choice of x'. Finally we observe that x' must also solve (1); for x' is feasible in (1), and $f_i(x') \geqq f_i(x^0)$, $i=1, 2$, implies, by the fact that h is increasing, that $h[f(x')] = h[f(x^0)]$.

LEMMA 2. *If x^0 is efficient, then there exists a scalar α^0 in the unit interval such that x^0 is an optimal solution of (P_{α^0}).*

Proof. The proof uses the concavity of f and convexity of X in an essential way, and is an application of a basic separation property of convex sets (*see*, e.g., reference 10, p. 217).

Lemmas 1 and 2 imply

THEOREM 1. *An optimal solution of (1) is found among the optimal*

† A function $f(x)$ on a convex set X is said to be concave if $x^1, x^2 \in X$, $x^1 \neq x^2$, imply $f[tx^1+(1-t)x^2] \geqq tf(x^1)+(1-t)f(x^2)$ for all $0<t<1$. An important property of concave functions is that a nonnegative linear combination of such functions is always concave. This property is *not* shared by quasiconcave functions (cf. footnote p. 46).

solutions of (P_α) *for some* α *in the unit interval. More precisely, if* α^* *is optimal in*

$$\text{maximize}_{\alpha \in [0,1]} H(\alpha), \tag{2}$$

where we define $H(\alpha)$ *on the unit interval by*

$$H(\alpha) = \text{maximum}_{x \in X^*(\alpha)} h[f(x)], \tag{3}$$

then (1) *is solved by any point* $x \in X^*(\alpha^*)$ *satisfying* $h[f(x)] = H(\alpha^*)$.

That $H(\alpha)$ is well-defined follows from the nonemptiness and compactness of X and the continuity of f and h, which imply that $h[f(x)]$ is continuous on the nonempty and compact set $X^*(\alpha)$. $H(\alpha)$ achieves its maximum by Lemmas 1 and 2 or by the fact that it can be shown to be an upper semicontinuous function on the compact set [0, 1].

The computational usefulness of (2) depends primarily on how readily $H(\alpha)$ can be computed on the unit interval. If it can be computed easily, then (2) is likely to be a quite efficient means of solving (1), for finding the maximum of $H(\alpha)$ is but a one-dimensional maximization problem. Before taking up the question of how to compute $H(\alpha)$, we point out an easy partial converse of Lemma 2 that partly justifies the assertion made earlier concerning the availability of a portion of the tradeoff-curve between f_1 and f_2 as a by-product of the calculations for solving (1): Every point of $X^*(\alpha)$ is efficient when α satisfies $0<\alpha<1$, and some point of $X^*(\alpha)$ is efficient when $\alpha=0$ or 1.

Computing $H(\alpha)$

Let α be fixed in the unit interval. It might be feared that computing $H(\alpha)$, when (P_α) does not have a unique optimal solution, requires not only finding all optimal solutions of (P_α) in order to get $X^*(\alpha)$, but also solving a maximization problem of the same form as (1) itself; for it was noted that $X^*(\alpha)$ is a nonempty and compact subset of X, and from the convexity of X and the concavity of $\alpha f_1(x)+(1-\alpha)f_2(x)$ it follows that $X^*(\alpha)$ is also convex. Fortunately, however, it turns out that computing $H(\alpha)$ is not nearly so difficult as this observation would seem to indicate. The results of the following theorem show that $H(\alpha)$ can usually be computed on [0, 1] with little, if any, extra work beyond finding by parametric programming *any* optimal solution function $x^*(\alpha)$ of (P_α) on [0, 1].

THEOREM 2. *Let* $x^*(\alpha)$ *be any optimal solution function for* (P_α) *on* [0, 1] *that is continuous everywhere except possibly for a finite number of simple discontinuities interior to the unit interval. For each point* α' *of discontinuity, define* $\underline{x}^*(\alpha')$ *and* $\bar{x}^*(\alpha')$ *as the left-hand and right-hand limits of* $x^*(\alpha)$ *at* α'*, respectively. Then*

(i). $H(\alpha)=h\{f[x^*(\alpha)]\}$ *at every point of continuity in* [0, 1].

(ii). *If α' is a point of discontinuity then*

$$H(\alpha') = \text{maximum}_{t \in [0,1]} h\{f[t\underline{x}^*(\alpha') + (1-t)\,\bar{x}^*(\alpha')]\} \qquad (3.1)$$

and

$$H(\alpha') = \text{maximum}_{t \in [0,1]} h\{tf[\underline{x}^*(\alpha')] + (1-t)f[\bar{x}^*(\alpha')]\}. \qquad (3.2)$$

This theorem applies to all parametric programming algorithms known to the author in the sense that when they are applicable to (P_α), they all produce an optimal solution function $x^*(\alpha)$ that is continuous everywhere on the unit interval except possibly for a finite number of simple discontinuities.† These discontinuities can be taken to be interior to the unit interval without loss of generality because of Lemma 5 below, which implies that $x^*(\alpha)$ can be replaced by its right-hand limit at 0 and left-hand limit at 1 if a discontinuity exists at either of these points. Note that to compute $H(\alpha)$ for a point of discontinuity one has a choice of solving either of the two *one-dimensional* maximization problems (3.1) and (3.2).

The burden of the remainder of this subsection is to establish the results of Theorem 2.

It is convenient to write (3) in the alternate form

$$H(\alpha) = \text{maximum}_{y \in f[X^*(\alpha)]} h(y), \qquad (4)$$

where $f[X^*(\alpha)]$ is the image of $X^*(\alpha)$ under f and y is a generic element of E^2.

LEMMA 3. *For each fixed value of α satisfying $0<\alpha<1$, $f[X^*(\alpha)]$ is either a singleton or a compact line segment of nonzero length in E^2 with normal $(\alpha, 1-\alpha)$. In the latter case, if $f(x^1)$ and $f(x^2)$ are the endpoints of the line segment, then $f[tx^1+(1-t)x^2] = tf(x^1)+(1-t)f(x^2)$ for all t satisfying $0 \leqq t \leqq 1$.*

Proof. Let $0<\alpha<1$ be fixed. By definition, $X^*(\alpha)$ is the optimal solution set of (P_α). Hence

$$\alpha f_1(x) + (1-\alpha)f_2(x) = v(\alpha)$$

for all $x \in X^*(\alpha)$, where $v(\alpha)$ is the optimal value of (P_α). Hence $f[X^*(\alpha)]$ is a subset of the line $\{y=(y_1, y_2) \in E^2\colon \alpha y_1 + (1-\alpha)y_2 = v(\alpha)\}$. Suppose that $f(x^1) \neq f(x^2)$, where $x^1, x^2 \in X^*(\alpha)$. Let t be any real number in the unit interval. Then $[tx^1+(1-t)x^2] \in X^*(\alpha)$ by convexity, and $f_i[tx^1+(1-t)x^2] \geqq tf_i(x^1)+(1-t)f_i(x^2)$, $i=1, 2$, by concavity. Hence

$$v(\alpha) = \alpha f_1[tx^1+(1-t)x^2] + (1-\alpha)f_2[tx^1+(1-t)x^2]$$
$$\geqq \alpha[tf_1(x^1)+(1-t)f_1(x^2)]$$

† In fact one suspects that the exploitation of possible continuity in the optimal solution of (P_α) as α varies is *necessary* for a successful parametric programming algorithm.

$$+(1-\alpha)[tf_2(x^1)+(1-t)f_2(x^2)]$$
$$=t[\alpha f_1(x^1)+(1-\alpha)f_2(x^1)]$$
$$+(1-t)[\alpha f_1(x^2)+(1-\alpha)f_2(x^2)]$$
$$=tv(\alpha)+(1-t)v(\alpha)=v(\alpha),$$

from which it follows (recall that $0<\alpha<1$) that

$$f[tx^1+(1-t)x^2]=tf(x^1)+(1-t)f(x^2).$$

It remains only to show that $f[X^*(\alpha)]$ is compact. This follows immediately from the continuity of f and the compactness of $X^*(\alpha)$.

This lemma implies that to compute $H(\alpha)$ for fixed α satisfying $0<\alpha<1$ it is sufficient to know at most two points in $X^*(\alpha)$: any *one* point if $f[X^*(\alpha)]$ is a singleton, and any *two* points x^1 and x^2 that each map into a different endpoint otherwise. In the first case, from (4) we see that $H(\alpha)=h[f(x)]$ for any $x \in X^*(\alpha)$; and in the second case, we have

$$H(\alpha)=\text{maximum}_{t \in [0,1]}\, h[tf(x^1)+(1-t)f(x^2)] \qquad (5)$$

or, interestingly enough, the alternative

$$H(\alpha)=\text{maximum}_{t \in [0,1]}\, h\{f[tx^1+(1-t)x^2]\}. \qquad (6)$$

LEMMA 4. *If $x^*(\alpha)$ is any optimal solution function for (P_α) on* [0, 1], *then $f_1[x^*(\alpha)]$ {resp. $f_2[x^*(\alpha)]$} is monotonically nondecreasing (resp. nonincreasing) on* [0, 1].

The routine proof is omitted.

LEMMA 5. *$X^*(\alpha)$ is an upper semicontinuous mapping*† *on* [0, 1].

Proof. Apply Theorem 4 of sec. 1.8 of DEBREU,[3] p. 19, to (P_α).

LEMMA 6. *Let $x^*(\alpha)$ be an optimal solution function for (P_α) that is continuous on the unit interval except possibly for a finite number of simple discontinuities. At each α_0 satisfying* $0<\alpha_0<1$:

A. If $x^*(\alpha)$ is continuous at α_0, then $f[X^*(\alpha_0)]$ is a singleton;

B. If $x^*(\alpha)$ has a simple discontinuity at α_0, then $\underline{x}^*(\alpha_0)\equiv \lim_{\alpha\to\alpha_0-} x^*(\alpha)$ and $\bar{x}^*(\alpha_0)\equiv\lim_{\alpha\to\alpha_0+} x^*(\alpha)$ are both in $X^*(\alpha_0)$, and $f[X^*(\alpha_0)]$ is a compact line segment (possibly of zero length) with end points $f[\underline{x}^*(\alpha_0)]$ and $f[\bar{x}^*(\alpha_0)]$.

At the endpoints of the unit interval:

C. $\lim_{\alpha\to 0+} x^*(\alpha)$ is in $X^*(0)$ and is efficient;

D. $\lim_{\alpha\to 1-} x^*(\alpha)$ is in $X^*(1)$ and is efficient.

† The definition of upper semicontinuity for set-valued functions that we use is that of Debreu.[3] As applied to $X^*(\alpha)$, upper semicontinuity at $\alpha_0 \in [0, 1]$ means $<\alpha^i>\to\alpha_0$, where $\alpha^i \in [0, 1]$, and $<x^*(\alpha^i)>\to x^0$, where $x^*(\alpha^i) \in X^*(\alpha^i)$, implies: $x^0 \in X^*(\alpha_0)$.

Proof. Let $x^*(\alpha)$ be continuous at α_0 satisfying $0 < \alpha_0 < 1$. Suppose, contrary to A, that there exists $x^0 \in X^*(\alpha_0)$ such that $f(x^0) \neq f[x^*(\alpha_0)]$. Since $f(x^0)$ and $f[x^*(\alpha_0)]$ must both lie on a line through $f(x^0)$ with normal $(\alpha_0, 1-\alpha_0)$, either $f_1(x^0) < f_1[x^*(\alpha_0)]$ or $f_1[x^*(\alpha_0)] < f_1(x^0)$. In the first case, by the continuity of $f_1[x^*(\alpha)]$ at α_0 there exists a number $\hat{\alpha}$ satisfying $\hat{\alpha} < \alpha_0$ such that $f_1[x^*(\hat{\alpha})] > f_1(x^0)$. But this contradicts the monotonicity of f_1 proved in Lemma 4. A similar contradiction can be obtained in the second case. This proves part A.

Let $x^*(\alpha)$ have a simple discontinuity at a point α_0 satisfying $0 < \alpha_0 < 1$. By Lemma 3, $f[X^*(\alpha_0)]$ is a compact line segment. Denote $\lim_{\alpha \to \alpha_0^-} x^*(\alpha)$ [resp. $\lim_{\alpha \to \alpha_0^+} x^*(\alpha)$] by $\underline{x}^*(\alpha_0)$ [resp. $\bar{x}^*(\alpha_0)$]. From Lemma 5, $\underline{x}^*(\alpha_0)$ and $\bar{x}^*(\alpha_0)$ are in $X^*(\alpha_0)$. It remains to show that $f[\underline{x}^*(\alpha_0)]$ and $f[\bar{x}^*(\alpha_0)]$ are the endpoints of $f[X^*(\alpha_0)]$. Suppose the contrary. Then there exists $x^0 \in X^*(\alpha_0)$ such that $f_1(x^0) < f_1[\underline{x}^*(\alpha_0)]$ and $f_2(x^0) > f_2[\underline{x}^*(\alpha_0)]$, or $f_1[\bar{x}^*(\alpha_0)] < f_1(x^0)$ and $f_2[\bar{x}^*(\alpha_0)] > f_2(x^0)$. We shall consider the first case and construct the contradiction that there exists a value of α such that

$$\alpha f_1(x^0) + (1-\alpha) f_2(x^0) > \alpha\, f_1[x^*(\alpha)] + (1-\alpha) f_2[x^*(\alpha)].$$

A similar construction leads to a contradiction for the second case.

For all $\alpha \in (0, 1)$, we have

$$\begin{aligned}
&\alpha\{f_1(x^0) - f_1[x^*(\alpha)]\} + (1-\alpha)\{f_2(x^0) - f_2[x^*(\alpha)]\} \\
&\quad = (\alpha - \alpha_0 + \alpha_0)\{f_1(x^0) - f_1[x^*(\alpha)]\} + (1 - \alpha + \alpha_0 - \alpha_0)\{f_2(x^0) - f_2[x^*(\alpha)]\} \\
&\quad = (\alpha - \alpha_0)\{f_1(x^0) - f_1[x^*(\alpha)] + f_2[x^*(\alpha)] - f_2(x^0)\} \\
&\qquad + \{\alpha_0[f_1(x^0) - f_1[x^*(\alpha)]] + (1-\alpha_0)[f_2(x^0) - f_2[x^*(\alpha)]]\} \\
&\quad \geqq (\alpha - \alpha_0)\{f_1(x^0) - f_1[x^*(\alpha)] + f_2[x^*(\alpha)] - f_2(x^0)\},
\end{aligned}$$

where the last inequality follows from the fact that the quantity in large curly brackets is nonnegative [recall that x^0 solves $(P\alpha_0)$]. By the left continuity of $f[x^*(\alpha)]$ at α_0 and the fact that $\{f_1(x^0) - f_1[\underline{x}^*(\alpha_0)]\}$ and $\{f_2[\underline{x}^*(\alpha_0)] - f_2(x^0)\}$ are both negative, the desired inequality is established for all α less than but sufficiently near α_0. This completes the proof of part B.

Finally we prove part C. A similar argument proves part D. By Lemma 5, $\bar{x}^*(0) \equiv \lim_{\alpha \to 0^+} x^*(\alpha)$ is in $X^*(0)$. Suppose that $\bar{x}^*(0)$ is not efficient. Then there exists a point $x^0 \in X$ such that $f_1(x^0) > f_1[\bar{x}^*(0)]$ and $f_2(x^0) = f_2[\bar{x}^*(0)]$ {since $\bar{x}^*(0)$ solves (P_0), $f_2(x^0) > f_2[\bar{x}^*(0)]$ is impossible}. Thus $x^0 \in X^*(0)$. Let $\hat{\alpha} > 0$ be such that $f_1(x^0) > f_1[x^*(\hat{\alpha})]$. This contradicts the monotonicity of f_1 established in Lemma 4.

Theorem 2, except for part (i) for $\alpha = 0$ or 1, is established by parts A and B of Lemma 6 in conjunction with Lemma 3. For $\alpha = 0$ or 1, part (i) is established by parts C and D of Lemma 6 in conjunction with the easy result that $H(\alpha) = h[f(x)]$ for any efficient $x \in X^*(\alpha)$ when $\alpha = 0$ or 1.

The Case in Which h is Quasiconcave

In this subsection we introduce the additional hypothesis that h is quasiconcave† on the convex hull‡ F of the admissible payoff set. Quasiconcavity is a weaker property than concavity, and is almost universally assumed as a property of utility indicator functions in consumer demand theory of traditional economic analysis. Five examples of quasiconcave h were given in the Introduction.

An immediate consequence of this additional hypothesis, in the presence of our previous assumptions, is that $h[f(x)]$ is now quasiconcave on X (*see* e.g., BERGE,[2] p. 207). Although (1) now becomes susceptible to various direct (nonparametric) approaches to quasiconcave programming, the approach represented by Theorem 1 can be very efficient when an efficient parametric programming algorithm is available for (P_α)—especially in view of Theorem 3 below. Theorem 3 establishes the unimodality of $H(\alpha)$, thereby enabling attention to be restricted to a subset of the unit interval when (2) is being executed.

LEMMA 7. *Let* x^i *solve* $(P\alpha^i)$, $i=0, 1, 2$, *where* $0\leqq\alpha^1<\alpha^0<\alpha^2\leqq 1$. *Then there exists a number* t, $0\leqq t\leqq 1$, *such that*

$$f_i(x^0)\geqq tf_i(x^1)+(1-t)f_i(x^2). \qquad (i=1, 2)$$

Proof. Denote $f(x^i)$ by f^i, $i=0, 1, 2$. We assume that f^0 does not coincide with either f^1 or f^2, for otherwise the conclusion of the lemma would be trivially true. Suppose that the conclusion is false. Then there does not exist a number $t\geqq 0$ that satisfies the following system of inequalities:

$$t(f_1^1-f_1^2)\leqq(f_1^0-f_1^2), \qquad (7)$$

$$t(f_2^1-f_2^2)\leqq(f_2^0-f_2^2), \qquad (8)$$

$$t\leqq 1. \qquad (9)$$

By a standard theorem on nonnegative solutions to linear inequalities there exist nonnegative real numbers s_1, s_2, and s_3 such that

$$(f_1^1-f_1^2)s_1+(f_2^1-f_2^2)s_2+s_3\geqq 0, \qquad (10)$$

and

$$(f_1^0-f_1^2)s_1+(f_2^0-f_2^2)s_2+s_3<0. \qquad (11)$$

Multiplying (11) by -1 and adding the result to (10), one obtains

$$(f_1^1-f_1^0)s_1+(f_2^1-f_2^0)s_2>0. \qquad (12)$$

Using the fact that $s_3\geqq 0$, from (11) one obtains

† $h(y)$ is quasiconcave on the convex set F if and only if $\{y \in F: h(y)\geqq k\}$ is a convex set for all real k. An equivalent definition is that $h[ty^1+(1-t)y^2]\geqq \text{Min}\ \{h(y^1), h(y^2)\}$ for all y^1, y^2 in F and $0<t<1$. For further discussion, *see* reference 1.

‡ The convex hull of a subset of Euclidean space is the smallest convex set containing that set.

$$(f_1^2-f_1^0)s_1+(f_2^2-f_2^0)s_2>0. \tag{13}$$

Now s_1 and s_2 cannot both vanish. Dividing (12) and (13) by (s_1+s_2), recalling that s_1, $s_2 \geqq 0$, and defining ξ as $s_1/(s_1+s_2)$, one obtains

$$(f_1^1-f_1^0)\xi+(f_2^1-f_2^0)(1-\xi)>0, \tag{14}$$

$$(f_1^2-f_1^0)\xi+(f_2^2-f_2^0)(1-\xi)>0, \tag{15}$$

and $0 \leqq \xi \leqq 1$.

Define $v_j(\alpha) \equiv \alpha f_1^j+(1-\alpha)f_2^j$, $j=0, 1, 2$. By the definitions of x^i, $i=0, 1, 2$, $v_j(\alpha^j) \geqq v_k(\alpha^j)$ for $j=0, 1, 2$ and $k \neq j$. Thus

$$v_1(\alpha^1)-v_0(\alpha^1) \geqq 0, \tag{16}$$

$$v_1(\alpha^0)-v_0(\alpha^0) \leqq 0, \tag{17}$$

$$v_2(\alpha^2)-v_0(\alpha^2) \geqq 0, \tag{18}$$

$$v_2(\alpha^0)-v_0(\alpha^0) \leqq 0. \tag{19}$$

Now (14) and (15) may be written as

$$v_1(\xi)-v_0(\xi)>0. \tag{20}$$

$$v_2(\xi)-v_0(\xi)>0. \tag{21}$$

By the linearity of $v_1(\alpha)-v_0(\alpha)$ in α, (16), (17), and (20) imply that $\xi<\alpha^0$ (recall that $\alpha^1<\alpha^2$). Similarly, (18), (19), and (21) imply that $\xi>\alpha^0$. This contradiction implies that the conclusion of the lemma must be true.

THEOREM 3. *Assume that h is quasiconcave on F. If $x^*(\alpha)$ is* any *optimal solution function of (P_α) on* [0, 1], *then $h\{f[x^*(\alpha)]\}$ is unimodal on* [0, 1].

Proof. Let $0 \leqq \alpha^1<\alpha^0<\alpha^2 \leqq 1$, and let $x^i \in X^*(\alpha^i)$, $i=0, 1, 2$. By Lemma 7, there exists a number t, $0 \leqq t \leqq 1$, such that

$$f_i(x^0) \geqq tf_i(x^1)+(1-t)f_i(x^2),\ i=1, 2.$$

Thus

$$h[f(x^0)] \geqq h[tf(x^1)+(1-t)f(x^2)]$$
$$\geqq \min\{h[f(x^1)], h[f(x^2)]\},$$

where the first inequality holds because h is increasing and the second because it is quasiconcave. This shows that $h\{f[x^*(\alpha)]\}$ is unimodal on [0, 1].

It is of some interest to note that the proofs of Theorem 3 and Lemma 7 do not make use of the concavity of the f_i or the convexity of X.

EXEMPLARY ALGORITHMS

IN THIS section we apply the theorems of the last to show how known parametric programming algorithms can be used to solve (1) in the manner suggested by Theorem 1. For illustrative purposes we choose parametric linear programming, Wolfe's method of parametric quadratic programming, and the author's method of parametric concave programming. The algorithms presented below are given in outline form, with no attempt made to give details of the most efficient organization of the computations.

Parametric Linear Programming

In this subsection we assume that f_1 and f_2 are linear and that X is determined by linear inequality constraints, so that parametric linear programming[4] can be used to produce an optimal solution function $x^*(\alpha)$ for (P_α) on [0, 1]. It is well known that $x^*(\alpha)$ will be piecewise constant, and that without loss of generality it can be assumed to be of the form

$$x^*(\alpha)=x^i \qquad \text{for} \qquad \alpha^i \leqq \alpha < \alpha^{i+1}, \qquad (i=0, \cdots, N),$$

where $0<\alpha^1<\cdots<\alpha^N<1$ (N finite and possibly 0) are the points of discontinuity and we have put $\alpha^0=0$ and $\alpha^{N+1}=1$. Also, $x^*(1)=x^N$. Thus by Theorem 2 we have $H(\alpha)=h[f(x^i)]$ for $\alpha^i<\alpha<\alpha^{i+1}$, $i=0, \cdots, N$, $H(0)=h[f(x^0)]$, and $H(1)=h[f(x^N)]$. If $N=0$, then obviously x^0 is optimal in (1). If $N\geqq 1$, then we have $\underline{x}(\alpha^i)=x^{i-1}$ and $\bar{x}^*(\alpha^i)=x^i$ for $i=1, \cdots, N$; consequently, (3.1) and (3.2) become

$$H(\alpha^i)=\text{maximum}_{t\,\epsilon\,[0,1]}\, h\{f[tx^{i-1}+(1-t)x^i]\}, \tag{22}$$

$$H(\alpha^i)=\text{maximum}_{t\,\epsilon\,[0,1]}\, h[tf(x^{i-1})+(1-t)f(x^i)], \tag{23}$$

for $i=1, \cdots, N$. Since $x^*(\alpha)$ is piecewise constant we see that when $N\geqq 1$, $H(\alpha)$ achieves its maximum at a point of discontinuity α^{i*}; therefore the point $t^*x^{i*-1}+(1-t^*)x^{i*}$ is optimal in (1), where t^* satisfies $H(\alpha^{i*})=h\{f[t^*x^{i*-1}+(1-t^*)x^{i*}]\}$ or, alternatively, $H(\alpha^{i*})=h[t^*f(x^{i*-1})+(1-t^*)f(x^{i*})]$ [cf. (5) and (6)]. We thus obtain the following algorithm.

Algorithm 1

Step 1. Solve (P_α) by parametric linear programming to obtain α^i and x^i, $i=0, \cdots, N$, computing the quantities $H(\alpha^i)$, $i=1, \cdots, N$ by (22) or (23) as the calculations progress. If $N=0$, *stop*; x^0 is optimal in (1). If $N\geqq 1$, then go to *step 2*.

Step 2. Let $H(\alpha^{i*})$ be the largest of the quantities computed at *step 1*. Then $t^*x^{i*-1}+(1-t^*)x^{i*}$ is optimal in (1), where t^* is defined as in the text so as to achieve $H(\alpha^{i*})$. *Stop.*

If h is quasiconcave, then because of the consequent monotonicity of $H(\alpha)$ it is rarely necessary to solve (P_α) on the entire unit interval, or to compute all of the $H(\alpha^i)$. In Algorithm 2, which exploits the quasiconcavity of h in the obvious way, it is assumed for simplicity of exposition that the parameter α increases, starting from the value 0. A similar algorithm can easily be constructed to cover the more general case in which α has an arbitrary starting value and can decrease as well as increase [the closer the starting value is to the one that maximizes $H(\alpha)$, the less work is required to solve (1) by this approach]. This same remark applies to Algorithm 4.

Algorithm 2

Step 1. Solve (P_0) to obtain x^0. Put $I = 0$ and $\underline{I}=1$.

Step 2. Solve (P_α) by parametric linear programming as α increases above α^I until either $\alpha=1$ or α^{I+1} is encountered. In the first case, go to *step 4*; in the second, determine x^{I+1} and go to *step 3*.

Step 3. Compare $h[f(x^I)]$ with $h[f(x^{I+1})]$:

a. If $h[f(x^I)]<h[f(x^{I+1})]$, increase I by 1, put $\underline{I}=I$, and return to *step 2*;

b. If $h[f(x^I)]=h[f(x^{I+1})]$, increase I by 1 and return to *step 2*;

c. If $h[f(x^I)]>h[f(x^{I+1})]$, increase I by 1 and go to *step 4*.

Step 4. If $I=0$, *stop*; x^0 is optimal in (1). If $I\geqq 1$, then $H(\alpha^i)>h[f(x^I)]$ for at most one i, $\underline{I}\leqq i\leqq I$. If strict inequality is achieved for no such i, then x^I is optimal in (1); if strict inequality is achieved for i_*, then $t^*x^{i_*-1}+(1-t^*)x^{i_*}$ is optimal in (1), where t^* achieves the maximum in (22) or (23). *Stop.*

REMARK. In both of these algorithms, a one-dimensional maximization problem [(22) or (23)] must be solved each time an $H(\alpha^i)$ is required. Frequently these one-dimensional problems are trivial; in linear fractional programming, for example (Example B in the Introduction with f_1 and f_2 linear), $H(\alpha^i)$ is just the larger of $f_1(x^i)/f_2(x^i)$ and $f_1(x^{i-1})/f_2(x^{i-1})$. But even when they are not, various methods are available,[13] such as Fibonnacci search when h is quasiconcave.

Parametric Quadratic Programming

In this subsection we assume that $f_1(x)$ is linear, that $f_2(x)$ is a negative semidefinite quadratic form, and that X is determined by linear inequality constraints. Then (P_α) can be solved on [0, 1] by Wolfe's method of parametric quadratic programming (his so-called 'long form'),[14] among others, for an optimal solution function $x^*(\alpha)$ that is continuous on [0, 1]. By Theorem 2, $H(\alpha)=h\{f[x^*(\alpha)]\}$ on [0, 1], and therefore the point x^* in the image of [0, 1] under $x^*(\alpha)$ which maximizes $h[f(x)]$ is also optimal in (1). Now from Wolfe's results it follows easily that this image set is of the form $\bigcup_{i=0}^{N} \overline{x^i, x^{i+1}}$, where $\overline{x^i, x^{i+1}}$ is a line segment in E^n with endpoints x^i and x^{i+1} and N is a positive integer. The points $x^i (i=0, 1, \cdots, N+1)$ are determined serially, in order of increasing superscript, from the modified

Simplex procedure employed by Wolfe† [$x^i \equiv x^*(\alpha^i)$ for certain α^i satisfying $0=\alpha^0<\alpha^1<\cdots<\alpha^N<\alpha^{N+1}=1$]; a termination signal accompanies the determination of x^{N+1}. Putting these observations together, we obtain

Algorithm 3

Step 1. Solve (P_α) on [0, 1] to obtain x^i, $i=0, 1, \cdots, N+1$, computing the quantities

$$\eta^i \equiv \text{maximum}_{0\leqq\lambda\leqq 1}\, h\{f[\lambda\, x^{i-1}+(1-\lambda)x^i]\} \tag{24}$$

as the calculations proceed.

Step 2. If η^{i*} is the largest of the η^i (ties are immaterial) then $\lambda^* x^{i*-1}+(1-\lambda^*)x^{i*}$ is optimal in (1), where λ^* achieves the maximum in (24). *Stop.*

If h is quasiconcave, then $h\{f[x^*(\alpha)]\}$ is unimodal, and an improved version of Algorithm 3 can be constructed that bears much the same relation to it as Algorithm 2 does to Algorithm 1:

Algorithm 4

Step 1. Solve (P_0) and obtain x^0. Put $\underline{I}=1$ and $I=0$.

Step 2. If $I=N+1$, go to *step 4*; otherwise, determine x^{I+1} and go to *Step 3.*

Step 3. Same as *Step 3* of Algorithm 2.

Step 4. Compute the quantities η^i, defined by (24), $\underline{I}\leqq i\leqq I$. Terminate as in *Step 2* of Algorithm 3.

The remark following Algorithm 2 is appropriate here also with regard to computing the η^i, especially when h is quasiconcave$\cdots$for then $h\{f[\lambda x^{i-1}+(1-\lambda)x^i]\}$ is unimodal in λ on [0, 1].

More General Parametric Concave Programming

When X is determined by concave inequality ($\geqq$) constraints and certain additional hypotheses are satisfied, the author's algorithm[5] can be used to solve (P_α) on [0, 1]. The $x^*(\alpha)$ so produced is continuous on [0, 1]. By Theorems 1 and 2, $x^*(\alpha^*)$ solves (1), where α^* maximizes $h\{f[x^*(\alpha)]\}$ on [0, 1]. When h is quasiconcave, the unimodality of $h\{f[x^*(\alpha)]\}$ simplifies the search for α^*.

NUMERICAL EXAMPLE

IN THIS section we give a numerical illustration of what amounts to Algorithm 2, based on Example E of the Introduction with linear f's and constraints. It will be of interest to note that the (unique) optimal solution is

† Actually, Wolfe's algorithm is addressed to a reparameterized version of (P_α), but this causes no essential difficulty.

not an extreme point of the convex polyhedral feasible region, and therefore could not be found by any algorithm that considers only vertices (cf. reference 12). We shall make free use of the mechanics of parametric linear programming, the main computational tool. The problem we shall solve is:

maximize$[32-40x_2+23x_3-7x_4]^{2/3}[32-10x_1-4x_2+7x_3-7x_4]$, subject to the usual nonnegativity constraints and the following linear equality constraints given in detached coefficient form:

x_1	x_2	x_3	x_4	x_5	x_6	=	rhs.
2	1	−0.37	0.37	0	4		9.08
1	3	−0.91	−1.91	0	5		8.44
1	−1	−0.97	−0.07	1	−1		1.88
1	1	−0.81	0.19	0	1		4.04

Following Example E, we put

$$f_1(x)=32-40x_2+23x_3-7x_4,$$

$$f_2(x)=32-10x_1-4x_2+7x_3-7x_4,$$

$$h(f_1, f_2)=[f_1]^{2/3}[f_2].$$

The parametric program (P_α) is, in this case, a *parametric linear program* $(0\leqq\alpha\leqq1)$ with the above linear constraints:

$$\text{maximize}(-10x_1-4x_2+7x_3-7x_4)+\alpha(10x_1-36x_2+16x_3).$$

Solving (P_0), we get (using one popular form of 'simplex tableau') Tableau 1.

TABLEAU 1

x_1	x_2	x_3	x_4	x_5	x_6	=	rhs.
1	0	−23/50	27/50	0	0		132/50
0	1	−65/100	35/100	0	0		60/100
0	0	1/25	1/25	1	0		16/25
0	0	15/50	−35/50	0	1		40/50
0	0	1/5	1/5	0	0		

The final basic solution x^0 is also optimal for α in an interval $[0, \alpha^1]$, where one easily computes $\alpha^1=0.0278$. Hence

$$H(\alpha)=h[f(x^0)]=(8)^{2/3}(16/5)=12.8 \quad \text{on} \quad [0, 0.0278),$$

for $\quad f_1(x^0)=32-40(60/100)+23(0)-7(0)=8,$

$$f_2(x^0)=32-10(132/50)-4(60/100)+7(0)-7(0)=16/5.$$

At $\alpha=0.0278$, the bottom row becomes

0	0	5/18	0	0	0

and to solve (P_α) for $\alpha>0.0278$ one pivots on the (2, 4) element of the tableau above to obtain Tableau 2.

TABLEAU 2

x_1	x_2	x_3	x_4	x_5	x_6	=	rhs.
1	−54/35	19/35	0	0	0		60/35
0	100/35	−65/35	1	0	0		60/35
0	−4/35	4/35	0	1	0		20/35
0	2	−1	0	0	1		2
0	0	5/18	0	0	0		

We find that this basic solution x^1 is optimal for $0.0278\leqq\alpha\leqq 0.053=\alpha^2$. Hence $H(\alpha)=h[f(x^1)]=(20)^{2/3}(20/7)=21.1$ on $(0.0278, 0.053)$.

Since $H(\alpha)$ has not yet decreased, we continue. At α^2, the bottom row dictates a pivot on the (1, 3) element, which results in Tableau 3.

TABLEAU 3

x_1	x_2	x_3	x_4	x_5	x_6	=	rhs.
35/19	−54/19	1	0	0	0		60/19
65/19	−46/19	0	1	0	0		144/19
−4/19	4/19	0	0	1	0		4/19
35/19	−16/19	0	0	0	1		98/19
0	0.52	0	0	0	0		

We find that the basic solution x^2 is optimal for $0.053\leqq\alpha\leqq 0.0736=\alpha^3$. Hence

$$H(\alpha)=(51.6)^{2/3}(1.05)=14.55 \quad \text{on} \quad (0.053, 0.0736).$$

From what we have determined about $H(\alpha)$ thus far, in light of its unimodality, we deduce that it achieves its maximum at either $\alpha^1=0.0278$ or $\alpha^2=0.053$. Arbitrarily examining α^2 first, by calculus we easily find from (22) that $t^*=0.75$ and $H(\alpha^2)=22$. Thus $\alpha^*=\alpha^2=0.053$, and the solution to our example is given by

$$x^* = 0.75x^1 + 0.25x^2 = (1.29, 0, 0.79, 3.18, 0.48, 2.79),$$

where x^1 and x^2 are obtained from the last two tableaus. Note that x^* is *not* an extreme point of the feasible region.

AN EXTENSION

FOR SOME applications (e.g., in stochastic programming[6]), it is necessary to have the following generalization of Theorem 3, since no amount of ingenuity will suffice to allow the program of interest to be written in the form (1) with the requisite assumptions satisfied. It does not require h to be quasiconcave, although h must still be nondecreasing.

THEOREM 3A. *Assume that* $h[f_1(x), f_2(x)]$ *can be written* $u[p_1(f_1(x)), p_2(f_2(x))]$, *where* u *is nondecreasing, quasiconcave, and continuous on the convex hull of the image of* X *under* $[p_1(f_1), p_2(f_2)]$, *and* p_1 *and* p_2 *are strictly increasing and continuous functions on the image of* X *under* (f_1, f_2) *such that* $p_1(f_1)$ *and* $p_2(f_2)$ *are concave on* X. *If* $x^*(\alpha)$ *is* any *optimal solution function of* (P_α) *on* $[0, 1]$, *then* $h\{f[x^*(\alpha)]\}$ *is unimodal on* $[0, 1]$.

Proof. Let α^1, α^0, and α^2 satisfy $0 \leqq \alpha^1 < \alpha^0 < \alpha^2 \leqq 1$, and let $x^i \in X^*(\alpha^i)$, $i = 0, 1, 2$. If $f(x^0) = f(x^1)$ or $f(x^0) = f(x^2)$, then obviously $h[f(x^0)] \geqq \min \{h[f(x^1)], h[f(x^2)]\}$. We shall show that this conclusion holds when $f(x^0) \neq f(x^1)$ and $f(x^0) \neq f(x^2)$, thereby showing that $h\{f[x^*(\alpha)]\}$ is unimodal on $[0, 1]$.

To proceed we must observe that under our assumptions on p_1 and p_2, the parametric program

$$(Q_\lambda) \text{maximize}_{x \in X} \lambda\, p_1[f_1(x)] + (1 - \lambda) p_2[f_2(x)]$$

has the same properties as (P_α) does, if λ is viewed as taking the place of α and $p_i(f_i)$ is viewed as taking the place of f_i, $i = 1, 2$. Hence Lemmas 2 through 7 also hold for (Q_λ) as well as (P_α), with the obvious changes in notation.

If $0 < \alpha^1 < \alpha^0 < 1$, then x^1 and x^0 must be efficient with respect to f_1 and f_2, and therefore also with respect to $p_1(f_1)$ and $p_2(f_2)$, in view of the strictly increasing nature of p_1 and p_2. By Lemma 2 applied to (Q_λ), x^1 and x^0 solve that program for some λ^1 and λ^0 in the unit interval. Applying Lemma 5 to (P_α) in view of $\alpha^1 < \alpha^0$, the fact that $f(x^0) \neq f(x^1)$, the strictly increasing nature of p_1 and p_2, and Lemma 5 to (Q_λ), in that order, it follows that $\lambda^1 < \lambda^0$. If $\alpha^1 = 0$, then since p_2 is increasing x^1 solves (Q_λ) with $\lambda = 0$. Thus for $0 \leqq \alpha^1 < \alpha^0 < 1$ we have proved the existence of λ^1 and λ^0 satisfying $0 \leqq \lambda^1 < \lambda^0 \leqq 1$ such that x^i solves (Q_λ) for λ^i, $i = 0, 1$. Similarly we can prove that for $0 < \alpha^0 < \alpha^2 \leqq 1$ there exists λ^2 satisfying $\lambda^0 < \lambda^2 \leqq 1$ such that x^2 solves (Q_λ) for λ^2.

The remainder of the proof follows exactly that of Theorem 3, with (Q_λ) taking the place of (P_α).

Most of the results of this paper can be generalized to the case of more than two f_i. The computational advantages of the present approach seem to diminish sufficiently rapidly with increasing dimension, however, so as not to warrant an explicit treatment of the more general case here.

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