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ELEMENTS OF LARGE SCALE MATHEMATICAL PROGRAMMING

Part II: Synthesis of Algorithms and Bibliography*†‡§

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The problem manipulations and solution strategies of Part I of this paper are now further illustrated by combining them in various ways to yield several known algorithms. The main object is not an exposition of these algorithms, although this is certainly important; rather, we wish to focus on the principal *patterns* in which manipulations and strategies can be assembled. These patterns constitute the real common denominators in the literature on large-scale programming. See Table 2 in Part I.

4. Synthesizing Algorithms from Manipulations and Strategies

It is beyond the scope of this effort to exemplify all of the important patterns of manipulations and strategies. We shall limit our discussion to five key ones:

1. PROJECTION, OUTER LINEARIZATION/RELAXATION
2. PROJECTION/PIECEWISE
3. INNER LINEARIZATION/RESTRICTION
4. PROJECTION/FEASIBLE DIRECTIONS
5. DUALIZATION/FEASIBLE DIRECTIONS

The first pattern is illustrated in §4.1 by Benders' Partitioning Procedure for what might be called semilinear programs; the second is illustrated in §4.2 by Rosen's Primal Partition Programming algorithm for linear programs with block-diagonal structure; the third in §4.3 by Dantzig-Wolfe Decomposition; the fourth in §4.4 by a procedure the author recently developed for nonlinear programs with multidivisional structure; and the fifth in §4.5 by the "local" approach discussed by Takahashi for concave programs with "complicating" constraints. Another key pattern, OUTER LINEARIZATION/RELAXATION, was already illustrated in §3.3 with reference to Kelley's cutting-plane method. In addition, it is indicated in §4.2 how Rosen's algorithm can be used to illustrate the pattern DUALIZATION/PIECEWISE, and in §4.3 how Dantzig-Wolfe Decomposition can be used to illustrate DUALIZATION, OUTER LINEARIZATION/RELAXATION.

The discussion of the various algorithms is as uncluttered by detail as we have been

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‡ The reader is assumed to be familiar with Part I of this paper, which is immediately preceding in this issue.

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able to make it. There is little or no mention of how to find an initial feasible solution,[1] the details of computational organization, or questions of theoretical convergence. The reader is invited to ponder such questions in the light of the concepts and results advanced in the previous two sections, and then to consult the original papers.

4.1 [*Benders* 62]

One might refer to

$$\text{(4.1)} \qquad \text{Maximize}_{x \geqq 0; y \in Y} \quad c^t x + f(y) \quad \text{s.t.} \quad Ax + F(y) \leqq b$$

as a *semi-linear* program because it is a linear program in x when y is held fixed temporarily. The algorithm of [Benders 62] for this problem can be recovered by applying the pattern PROJECTION, OUTER LINEARIZATION/RELAXATION. Specifically, project (4.1) onto the space of the y variables, outer-linearize the resulting supremal value function in the maximand, and apply the Relaxation strategy to the new constraints arising as a consequence of Outer Linearization. Assume for simplicity that (4.1) is feasible and has finite optimal value.

Projection onto the space of the y variables yields

$$\text{(4.2)} \qquad \text{Maximize}_{y \in Y} \quad [f(y) + \text{Sup}_{x \geqq 0} \ \{c^t x \quad \text{s.t.} \quad Ax \leqq b - F(y)\}].$$

Note that the supremal value function appearing in the maximand corresponds to the linear program

$$\text{(4.3)} \qquad \text{Maximize}_{x \geqq 0} \quad c^t x \quad \text{s.t.} \quad Ax \leqq b - F(y).$$

This program is parameterized nonlinearly in the right-hand side by y, and our assumption implies that it has a finite optimum for at least one value of y. By the Dual Theorem, therefore, the dual linear program

$$\text{(4.4)} \qquad \text{Minimize}_{u \geqq 0} \quad u^t(b - F(y)) \quad \text{s.t.} \quad u^t A \geqq c^t$$

must be feasible (for all y). Let $\langle u^1, \cdots, u^p \rangle$ be the extreme points and $\langle u^{p+1}, \cdots, u^{p+q} \rangle$ representatives of the extreme rays of the feasible region of (4.4) (cf. Theorem 3). Again using the Dual Theorem, we see that (4.3) is feasible if and only if (4.4) has finite optimal value, that is, if and only if y satisfies the constraints

$$\text{(4.5)} \qquad (u^j)^t(b - F(y)) \geq 0, \qquad j = p + 1, \cdots, p + q.$$

Since we take the supremal value function in (4.2) to be $-\infty$ for y such that (4.3) is infeasible—see §2.1—we may append the constraints (4.5) to (4.2). Thus Projection applied to (4.1) yields (4.2) subject to the additional constraints (4.5).

Next we outer-linearize the supremal value function appearing in (4.2). It is easy to see, referring to (4.4), that its value is precisely

$$\text{(4.6)} \qquad \text{Minimum}_{1 \leqq j \leqq p} \quad \{(u^j)^t(b - F(y))\}$$

for all y feasible in (4.2) with (4.5) appended. (Strictly speaking, it is accurate to call this Outer "Linearization" only if F is linear.) With this manipulation, (4.2) becomes

$$\text{(4.7)} \quad \text{Maximize}_{y \in Y} \quad [f(y) + \text{Minimum}_{1 \leq j \leq p} \quad \{(u^j)^t(b - F(y))\}] \quad \text{s.t.} \quad (4.5)$$

[1] If one exists, it can usually be found by applying the algorithm itself to a suitably modified version of the given problem.

or, with the help of an elementary manipulation based on the fact that a minimum is really a greatest lower bound,

$$\text{(4.8)}\quad \begin{aligned} \text{Maximize}_{y\in Y;y_0}\ f(y) + y_0 \quad \text{s.t.}\quad y_0 \le (u^j)^t(b - F(y)),\ \ j = 1, \cdots, p,\\ (u^j)^t(b - F(y)) \ge 0,\ \ j = p+1, \cdots, p+q. \end{aligned}$$

This is the master problem to be solved.

Relaxation is a natural strategy for (4.8); it avoids having to determine in advance all of the vectors $u^j, j = 1, \cdots, p + q$. To test the feasibility of a trial solution $(\hat{y}_0, \hat{y})$, where $\hat{y} \in Y$, one solves the linear subproblem (4.4) with y equal to $\hat{y}$. If the infimal value is greater than or equal to $\hat{y}_0$, then $(\hat{y}_0, \hat{y})$ is feasible and therefore optimal in (4.8); $\hat{y}$, along with $\hat{x}$ equal to the optimal dual variables of (4.4), is an optimal solution of the given problem (4.1). If, on the other hand, the infimal value is less than $\hat{y}_0$, then a violated constraint of (4.8) is produced (some u^j with $1 \le j \le p$ is found if the infimal value is finite, while $p + 1 \le j \le p + q$ if it is $-\infty$). Of course, f, F, and Y must satisfy the obvious convexity assumptions if dropping amply satisfied constraints is to be justified. These assumptions will probably have to hold anyway if the relaxed problems based on (4.8) are to be concave programs (remember $u^j \geqq 0$). There is, however, at least one other interesting case: if Y is a discrete set, say the integer points of some convex polytope, while f and F are linear, then (4.8) is a pure (except for y_0) integer linear program (see [Balinski and Wolfe 63], [Buzby, Stone and Taylor 65]).

The present development seems preferable to the original one since: (a) it justifies dropping amply satisfied constraints from successive relaxed versions of (4.8); (b) it retains $f(y)$ in its natural position in the criterion function of (4.8) (Benders' version of (4.8), which is also equivalent to (4.7), has y_0 alone as the criterion function and an added term $f(y)$ in the right-hand side of each of the first p constraints); and (c) its comparative simplicity suggests a generalization, with the help of nonlinear duality theory, permitting nonlinearities in x [Geoffrion 70].

4.2 [*Rosen* 64]

The algorithm of [Rosen 64] for the linear program

$$\text{(4.9)}\quad \text{Maximize}_{x,y}\ \ b_0{}^t y + \textstyle\sum_{i=1}^{l} b_i{}^t x_i \quad \text{s.t.}\quad x_i{}^t A_i + y^t D_i \leqq c_i{}^t,\quad i = 1, \cdots, l$$

illustrates the pattern PROJECTION/PIECEWISE. Assume for simplicity that (4.9) is feasible and has finite optimal value.

Projection onto the y variables yields the master problem

$$\text{(4.10)}\quad \text{Maximize}_y\ \ [b_0{}^t y + \textstyle\sum_{i=1}^{l} \text{Sup}_{x_i}\, \{b_i{}^t x_i \quad \text{s.t.}\quad x_i{}^t A_i \leqq c_i{}^t - y^t D_i\}],$$

where we have separated the supremum in the maximand (this separation is perhaps the main justification for using Projection).

The Piecewise strategy is appropriate for (4.10) because each supremal value in the maximand is piecewise-linear as a function of y. This follows from the elementary theory of linear programming, as we now explain. Let $\hat{y}$ be feasible in (4.10) in the sense that the maximand is not $-\infty$. Then each of the l linear programs appearing in the maximand must have a finite optimal value, and by the Dual Theorem this optimal value must be equal to that of the dual linear program

$$\text{(4.11)}\quad \text{Minimize}_{u_i \geqq 0}\ \ (c_i{}^t - \hat{y}^t D_i)u_i \quad \text{s.t.}\quad A_i u_i = b_i\,.$$

Let the vector $\hat{u}_i$ be an optimal solution of this program, and let the corresponding basis matrix be B_i. Since changes in y cannot affect the feasibility of $\hat{u}_i$, the optimal value of (4.11)—which is equal to the value of ith supremal value function of (4.10) at y—must be

$$(c_i^t - y^t D_i)\hat{u}_i \tag{4.12}$$

so long as the "reduced costs" remain of the correct sign, that is, so long as y satisfies the condition

$$(c_i^t - y^t D_i)^B B_i^{-1}(A_i)_{.j} - (c_i^t - y^t D_i)_j \leq 0, \quad \text{all nonbasic } j, \tag{4.13}$$

where the superscript B masks all but the basic components of $(c_i^t - y^t D_i)$. Thus the master problem (4.10), confined to the linear "piece" containing $\hat{y}$, becomes the linear program

$$\text{Maximize}_y \quad b_0^t y + \textstyle\sum_{i=1}^{l} (c_i^t - y^t D_i)\hat{u}_i \quad \text{s.t.} \quad (4.13), \qquad i = 1, \cdots, l. \tag{4.14}$$

This shows that Step 2 of the Piecewise strategy can be accomplished by linear programming. Rosen actually solves the dual of (4.14). His Theorems 1 and 2 concern Step 3 (cf. the discussion following (3.4) in §3.1).

It is interesting to note that if we had started with the dual of (4.9)—a block-diagonal linear program with coupling constraints—we would obtain precisely the same procedure as the one just described by dualizing with respect to the coupling constraints only [Geoffrion 69] and then invoking the Piecewise strategy. In this way [Rosen 64] could also be used to illustrate the pattern DUALIZATION/PIECEWISE.

4.3 *Dantzig-Wolfe Decomposition*

Dantzig-Wolfe Decomposition is archetypical of the pattern INNER LINEARIZATION/RESTRICTION. Mechanized pricing plays a prominent role. We shall illustrate this pattern first with the algorithm of [Dantzig and Wolfe 60] for a purely linear program, then with the algorithm of [Dantzig 63a, Ch. 24] for a nonlinear program, and finally with a variation of the latter in which not all nonlinear functions need be inner-linearized.

It is interesting to note that Dantzig-Wolfe Decomposition can also be viewed as an instance of the pattern DUALIZATION, OUTER LINEARIZATION/RELAXATION. In the context of (4.15), for example, one would dualize with respect to the constraints $\bar{A}x \leqq \bar{b}$, outer-linearize the resulting minimand in the obvious way, and then apply Relaxation.

[*Dantzig and Wolfe* 60]. The well-known Dantzig-Wolfe decomposition approach for linear programs will be explained in terms of the linear program

$$\text{Maximize}_{x \geqq 0} \quad c^t x \quad \text{s.t.} \quad Ax \leqq b, \qquad \bar{A}x \leqq \bar{b}, \tag{4.15}$$

where we have arbitrarily divided the constraints into two groups. With the definition

$$X \triangleq \{x \geqq 0 : Ax \leqq b\}, \tag{4.16}$$

we may write (4.15) as

$$\text{Maximize}_{x \in X} \quad c^t x \quad \text{s.t.} \quad \bar{A}x \leqq \bar{b}. \tag{4.17}$$

Since X is a convex polytope, we know (Theorem 3) that it admits an exact inner linearization using only a finite number of points. Invoking this representation for X, we obtain a new master linear program with a vast number of variables to which Restriction can be applied in the form of the Simplex Method. It turns out that the pricing operation (cf. §3.2) can be accomplished by solving a linear subproblem whose feasible region is X. The details are as follows.

Assume that X is not empty and also, for ease of exposition only, that X is bounded. Then X can be represented in terms of its extreme points $\langle x^1, \cdots, x^p\rangle$, and (4.17) can be written as the equivalent master linear program

$$(4.18) \quad \text{Maximize}_{\alpha \geqq 0} \quad c^t\left(\sum_{j=1}^p \alpha_j x^j\right) \quad \text{s.t.} \quad \sum_{j=1}^p \alpha_j = 1, \qquad \bar{A}\left(\sum_{j=1}^p \alpha_j x^j\right) \leqq \bar{b}.$$

The Simplex Method for this problem corresponds to Restriction with respect to the constraints $\alpha \geqq 0$.[2] To describe how the pricing operation can be mechanized, we shall use the familiar terminology of linear programming rather than the general terminology of Restriction. The optimality conditions at the general iteration are $u \geqq 0$ and

$$(4.19) \qquad u_0 + u^t\bar{A}x^j - c^t x^j \geq 0, \qquad j = 1, \cdots, p,$$

where u_0 and the vector u are the current Simplex multipliers. Condition (4.19) is equivalent to

$$[u_0 + \text{Minimum}_{1 \leq j \leq p} \quad \{(u^t\bar{A} - c^t)x^j\}] \geq 0$$

or, since $\langle x^1, \cdots, x^p\rangle$ span X, to

$$(4.20) \qquad [u_0 + \text{Min}_{x \in X} \quad (u^t\bar{A} - c^t)x] \geq 0.$$

The linear program in this expression is a valid replacement for the finite minimum in the previous expression because the minimum of a linear function over X occurs at an extreme point. Thus we see how to test optimality when the Simplex Method is applied to (4.18). If either $u \geqq 0$ or (4.20) fails to hold, a profitable nonbasic variable satisfying the usual criterion for the entering variable is obtained automatically: if the greatest violation occurs in $u \geqq 0$, introduce the corresponding slack variable; if in (4.20), introduce the variable α_{j_0}, where x^{j_0} is an optimal basic feasible solution of the linear program in (4.20) (the extremal function coefficient of α_{j_0} is $c^t x^{j_0}$, and the technological coefficient column is unity followed by $\bar{A}x^{j_0}$).

Thus there is no difficulty in carrying out the Simplex Method applied to (4.18). Each iteration requires solving the linear subproblem in (4.20).[3] This approach may possess an advantage over the direct application of the Simplex Method to (4.15) when the subproblem has some special structure. For example, if (4.15) is a transportation problem with additional constraints, then the subproblem becomes a pure transportation problem if $\bar{A}$ is taken to comprise the additional constraints. Another example is the case in which A is block-diagonal, for then the subproblem separates into k independent smaller linear programs. In general, one should select a grouping of the constraints (in terms of A and $\bar{A}$) that isolates a special structure, and then exploit this structure in dealing with (4.20). See [Broise, Huard and Sentenac 68], [Orchard-Hays 68, §10.4] for additional discussion based on computational experience.

[*Dantzig* 63a, *Chapter* 24]. Now consider a nonlinear version of (4.17), namely

$$(4.21) \qquad \text{Maximize}_{x \in X} \quad f(x) \quad \text{s.t.} \quad g_i(x) \leq b_i, \qquad i = 1, \cdots, m,$$

[2] Actually, the inequality constraints involving $\bar{A}$ are also normally considered as candidates for restriction to equality. The latter constraints can be excluded, if desired, from the candidates for restriction by giving $u \geqq 0$ priority over (4.19) in determining the entering basic variable. Such a modification is necessary, as we shall see later in this subsection, when nonlinear functions are inner-linearized.

[3] The subproblem need be solved from scratch only at the first iteration; thereafter, restarting or parametric techniques can be used to recover an optimum as u changes from iteration to iteration.

where X is a convex set, f is concave on X, and g_i is convex on X. Dantzig and Wolfe's approach [Dantzig 63a, Chapter 24] for this problem can be viewed as follows. Let f and each g_i be approximated by Inner Linearization over an arbitrarily fine base $\langle x^1, x^2, \cdots \rangle$ in X, so that (4.21) is approximated as closely as desired (in principle, at least) by the linear master problem

$$(4.22)\quad \text{Maximize}_{\alpha \geqq 0} \ \textstyle\sum_j \alpha_j f(x^j) \quad \text{s.t.} \quad \sum_j \alpha_j = 1, \quad \sum_j \alpha_j g_i(x^j) \leq b_i, \quad i = 1, \cdots, m.$$

We say "in principle" because we do not wish to actually evaluate f and each g_i at every point in the base, or even specify the base explicitly. Hence it is natural to solve (4.22) by Restriction with the constraints $\alpha \geqq 0$ as the candidates for restriction to equality (when α_j is restricted to 0, the values $f(x^j)$ and $g_i(x^j)$ are not needed). A very natural way to do this is to employ the Simplex Method with a priority convention to ensure that the restricted problems are truly optimized: slack variables corresponding to the g_i constraints must be given priority over structural variables in determining which variable is to enter a basis. Any feasible solution of (4.21) can be used to find an initial basic feasible solution, and at the general iteration the optimality criterion or pricing problem is (cf. (4.19)) $u_i \geq 0$ $(1 \leq i \leq m)$ and

$$(4.23)\quad u_0 + \textstyle\sum_{i=1}^m u_i g_i(x^j) - f(x^j) \geq 0, \quad \text{all } j,$$

where $u_0, u_1, \cdots, u_m$ are the current Simplex multipliers. By the priority convention, we may assume that $u_i \geq 0$ $(1 \leq i \leq m)$. Note that (4.23) is intimately related (cf. (4.20)) to the convex subproblem

$$(4.24)\quad \text{Minimize}_{x \in X} \ \textstyle\sum_{i=1}^m u_i g_i(x) - f(x).$$

If u_0 plus the optimal value of this problem is nonnegative, then (4.23) holds and an optimal solution of (4.21) is at hand ($x^* = \sum_j \hat{\alpha}_j x^j$, where $\hat{\alpha}$ is the current and optimal solution of (4.22)); otherwise, an optimal or near-optimal solution $\hat{x}$ of (4.24) can be profitably added to the current explicit base by introducing the corresponding α_j into the basis in the usual way after evaluating $f(\hat{x})$ and $g_i(\hat{x})$. In practice, termination would take place as soon as the value of the current approximation to an optimal solution of (4.21)—the quantity $f(\sum_j \hat{\alpha}_j x_j)$—approaches closely enough the following easily demonstrated upper bound for the true optimal value:

$$(4.25)\quad \textstyle\sum_{i=1}^m u_i b_i - \text{Min}_{x \in X}[\sum_{i=1}^m u_i g_i(x) - f(x)].$$

This approach is particularly attractive when the structure is such that (4.24) is relatively tractable by comparison with (4.21); for example, when X is an open set and f and g_i are differentiable, or when (4.24) is separable into several independent subproblems.

A Variant. It is interesting to observe that Inner Linearization need not be applied to *all* nonlinear functions of (4.21).[4] An advantage can sometimes be gained by inner-linearizing only a subset of the nonlinear functions, say $g_1, \cdots, g_{m_1}$ $(m_1 < m)$. Then instead of (4.22) we have the concave master problem

$$(4.26)\quad \begin{aligned} &\text{Maximize}_{\alpha \geqq 0} f(\textstyle\sum_j \alpha_j x^j) \quad \text{s.t.} \quad \sum_j \alpha_j = 1, \\ &\textstyle\sum_j \alpha_j g_i(x^j) \leq b_i, \quad i = 1, \cdots, m_1, \\ &g_i(\textstyle\sum_j \alpha_j x^j) \leq b_i, \quad i = m_1 + 1, \cdots, m. \end{aligned}$$

[4] In [Whinston 66], for example, the objective function of a block-diagonal quadratic program with coupling constraints is not inner-linearized.

Again we wish to apply Restriction with only the nonnegativity constraints $\alpha \geqq 0$ as candidates for restriction to equality. The Simplex Method can no longer be adapted to this purpose, however, since (4.26) is not a linear program. Implementation requires a concave programming algorithm for solving the restricted versions of (4.26) and also a means of mechanizing the pricing operation. We need not discuss the first requirement. The second involves being able to determine the prices μ_j^s for all j in S, where S is the current set of indices for which α_j is restricted to value 0. This can be done as follows [Holloway 69]. Let α^s be the optimal solution to (4.26) with the additional restrictions $\alpha_j = 0$ for $j \in S$, and let $u_0^s, u_1^s, \cdots, u_m^s$ be the associated optimal multipliers (which must exist if a constraint qualification is satisfied). Then, assuming all functions are continuously differentiable, the price μ_j^s associated with $\alpha_j = 0$ is given for all $j \in S$ by

$$\mu_j^s = u_0^s - \nabla f(x^s)x^j + \sum_{i=1}^{m_1} u_i^s g_i(x^j) + \sum_{i=m_1+1}^{m} u_i^s \nabla g_i(x^s)x^j, \tag{4.27}$$

where

$$x^s \triangleq \sum_{j \notin S} \alpha_j^s x^j. \tag{4.28}$$

It follows that the pricing problem can be solved by optimizing the convex ($u_i^s \geq 0$) subproblem

$$\text{Minimize}_{x \in X} \quad -\nabla f(x^s)x + \sum_{i=1}^{m_1} u_i^s g_i(x) + \sum_{i=m_1+1}^{m} u_i^s \nabla g_i(x^s)x. \tag{4.29}$$

Compare with (4.24). If f were inner-linearized too, the first term of the maximand of (4.29) would be $-f(x)$.

Which of all given constraints should be incorporated into X, and which of the remainder and whether f itself should be inner-linearized, depends mainly on the availability of efficient algorithms for the resulting versions of (4.29) and (4.26) with $\alpha_j = 0$ for $j \in S$.

4.4 [*Geoffrion* 68b; §4]

A quite general problem with multidivisional structure is

$$\text{Maximize}_x \quad \sum_{i=1}^{k} f_i(x_i) \quad \text{s.t.} \quad H_i(x_i) \geqq 0, \quad i = 1, \cdots, k \qquad \sum_{i=1}^{k} G_i(x_i) \geqq b, \tag{4.30}$$

where f_i, h_{ij} and g_{ij} are all concave differentiable functions of the vector x_i. The subscript i can be thought of as indexing the individual divisions, which are linked together only by coupling constraints. The approach of [Geoffrion 68b; §4] is an application of the pattern PROJECTION/FEASIBLE DIRECTIONS. The optimization of (4.30) is carried out largely at the divisional level subject to central coordination.

First (4.30) is projected onto the space of its coupling constraints. This requires introducing the vectors $y_1, \cdots, y_k$:

$$\text{Maximize}_{x,y} \quad \sum_{i=1}^{k} f_i(x_i) \quad \text{s.t.} \quad H_i(x_i) \geqq 0, i = 1, \cdots, k; G_i(x_i) \geqq y_i, i = 1, \cdots, k; \sum_{i=1}^{k} y_i \geqq b. \tag{4.31}$$

In effect, this changes the given problem from one with coupling constraints to one with coupling variables, since (4.31) separates into k separate problems if y is held fixed temporarily. One may interpret y_i as a vector of resources and tasks assigned to the ith division. Projection of this problem onto y yields the master problem

$$\text{Maximize}_y \quad \sum_{i=1}^{k} v_i(y_i) \quad \text{s.t.} \quad \sum_{i=1}^{k} y_i \geqq b, \tag{4.32}$$

where v_i is defined as the supremal value of the parameterized divisional problem

$$\text{Maximize}_{x_i} \quad f_i(x_i) \quad \text{s.t.} \quad (H_i(x_i) \geqq 0, \qquad G_i(x_i) \geqq y_i. \tag{4.33}$$

Now we wish to apply the Feasible Directions strategy to (4.32). The idea of this strategy, it will be recalled, is to generate an improving sequence of feasible points, with each new point determined from the previous one by selecting an improving feasible direction and then maximizing along a line emanating in this direction. The latter maximization is only one-dimensional, and can easily be essentially decentralized to the divisional level. The chief difficulty with this strategy concerns how to find a good improving feasible direction, for the maximand $\sum_{i=1}^{k} v_i(y_i)$ is not everywhere differentiable and is available only implicitly in terms of the divisional problems (4.33). It can nevertheless be shown [ibid., §4.2], using the theory of subgradients for concave functions and the optimality conditions associated with (4.33), that the following explicit linear program yields an improving feasible direction z^0 for (4.32) at a feasible point y^0; moreover, z^0 is *best* among all feasible directions in that it maximizes the initial rate of improvement of $\sum_{i=1}^{k} v_i(y_i)$:

$$
\begin{aligned}
&\text{Maximize}_{w,z} \quad \textstyle\sum_{i=1}^{k} \nabla f_i^0 w_i \quad \text{s.t.} \\
&\qquad \nabla g_{ij}^0 w_i - z_{ij} \geq 0, \qquad i = 1, \cdots, k \\
&\qquad\qquad\qquad\qquad\qquad\qquad j \text{ such that } g_{ij}^0 = y_{ij}^0 \\
&\qquad \nabla h_{ij}^0 w_i \geq 0, \qquad i = 1, \cdots, k \\
(4.34)\quad &\qquad\qquad\qquad\qquad\qquad\qquad j \text{ such that } h_{ij}^0 = 0 \\
&\qquad \textstyle\sum_{i=1}^{k} z_{ij} \geq 0, \qquad j \text{ such that } \sum_{i=1}^{k} y_{ij}^0 = b_j \\
&\qquad -1 \leq z_{ij} \leq 1, \qquad \text{all } i \text{ and } j.
\end{aligned}
$$

Here ∇g_{ij}^0 refers to a row vector that is the gradient of g_{ij} evaluated at an optimal solution of (4.33) with $y_i = y_i^0$, and the other superscripted quantities have similar definitions. The vector w_i has the same dimension as x_i. This subproblem enables the Feasible Directions strategy for (4.32) to be carried out.

4.5 [*Takahashi* 64]

Consider

$$(4.35) \qquad \text{Maximize}_x \quad f(x) \quad \text{s.t.} \quad H(x) = 0, \qquad G(x) = 0,$$

where f is concave and all constraints are linear. Suppose that the G constraints are *complicating* in the sense that the problem would be much easier if they were not present. For instance, the complicating constraints may be the coupling constraints of a structure similar to the one in the previous subsection, or they may spoil what would otherwise be a special structure for which efficient solution methods would be available. The pattern of the "local" approach of [Takahashi 64] for this problem is DUALIZATION/FEASIBLE DIRECTIONS.

The dual of (4.35) with respect to the complicating constraints only yields (see, e.g., [Rockafellar 68] or (Geoffrion 69]) the following problem in the space of the dual variables λ (a vector whose dimension matches G):

$$(4.36) \qquad \text{Minimize}_\lambda \quad v(\lambda),$$

where $v(\lambda)$ is defined as the supremal value of the parameterized problem

$$(4.37) \qquad \text{Maximize}_x \quad f(x) + \lambda^t G(x) \quad \text{s.t.} \quad H(x) = 0.$$

Note that (4.37) is of the same form as (4.35) except the complicating constraints are now part of the criterion function.

To apply the Feasible Directions strategy to (4.36), we must be able to identify an improving feasible direction. *Any* direction is feasible, of course, since λ is unconstrained. When f is strictly concave, it can be shown that v is differentiable. Its gradient at a point λ^0 is simply $G(x^0)$, where x^0 is the optimal solution of (4.37) with $\lambda = \lambda^0$. Hence the Feasible Directions strategy can be carried out for (4.36) using the negative of the gradient of v as the improving feasible direction. Actually, Takahashi proposes a short-step method rather than requiring a one-dimensional minimization to be performed in order to determine step size. The procedure may be summarized as follows.

1. Choose a starting point λ^0.
2. Solve (4.37) with $\lambda = \lambda^0$ for its optimal solution x^0. If $G(x^0) = 0$, then x^0 is optimal in (4.35); otherwise, go on to Step 3.
3. Let $\lambda' = \lambda^0 - \zeta G(x^0)$, where ζ is a small positive constant, and return to Step 2 with λ' in place of λ^0.

5. Conclusion

We have attempted to develop a framework of unifying concepts that comprehends much of the literature on large-scale mathematical programmning. If we have been successful, the nonspecialist should have an overview of the field that facilitates further study, and the advanced reader should feel that he has a deeper understanding of previously familiar algorithms and that he perceives new commonalities among approaches that heretofore seemed to be related only vaguely if at all.

In addition, we hope that the framework will suggest a variety of worthwhile topics for investigation. The problem manipulations and solution strategies discussed here all invite further study, and others should be added to the fold so that additional algorithms can be encompassed. The algorithms falling within the purview of each particular manipulation/strategy pattern (cf. Table 2) should be studied carefully in relation to one another, with the aim of learning how "best" to use the tactical options of the pattern and organize the computations for various classes of problems.

The relationships between ostensibly different patterns also warrant further study. We mentioned in §3.3 that Restriction (Relaxation) is essentially equivalent to Dualization followed by Relaxation (Restriction), and other equivalences were briefly noted in §4.2 and §4.3. Many others exist; for example, it has often been observed that Dantzig-Wolfe and Benders Decomposition are dual to one another in an appropriate sense. The results of [Zoutendijk 60; Secs. 9.4, 10.3, 11.4] are in this spirit, even if they do not specifically involve algorithms for large-scale programming. Knowledge of such relations reduces the number of essentially different patterns to be considered, and enables meaningful comparisons among the remainder.

Investigations along these lines should help civilize the jungle of extant algorithms and pave the way for truly significant computational studies.

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