

# Mitigating the Double-Blind Effect in Opaque Selling: Inventory Information, Customer Preferences, and Options Pricing

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## Abstract

Opaque selling is a marketing strategy under which firms sell their end-of-the-season inventory of different products as a single probabilistic good and customers find out what those products are only *after* they have made the purchase. Opaque selling usually entails a “double-blind” process: customers do not know the inventory level of each product, and the seller does not know about each buyer’s product preference. This double-blind process can result in customer dissatisfaction when customers do not receive the products they were hoping for, and the seller is unable to allocate the right product to the right customer. To reduce customer dissatisfaction arising from the double-blind process, we examine whether the seller should: (1) reveal its inventory information to customers, (2) solicit each customer’s product preference, or (3) charge customers a discretionary options fee in return for not allocating a certain product to them. To examine these issues, we analyze a single period model in which a firm sells leftover inventory of two distinctive products as a single probabilistic goods to rational customers. We show that revealing inventory information or soliciting customer preferences results in higher customer surplus, but lower revenue for the firm. However, the firm can strictly increase its revenue by charging customers a discretionary options fee which ensures that a certain product will not be allocated. When the options fee is product dependent, such practice will benefit the firm the most when the inventory levels of different products are relatively balanced and their valuations are strongly skewed. When the options fee is product independent, however, the firm gains the most when the inventory levels of different products are relatively balanced but their valuations are not too strongly skewed.

Keywords: Opaque Selling, Double Blind, Inventory Information, Customer Preferences, Options Pricing

## 1 Introduction

Firms often use markdown pricing, also known as fire sale, to dispose of their leftover inventory (e.g., toys and apparels) or unsold capacity (e.g., airline seats and hotel rooms). For example,

according to Fisher (2006), 26% of fashion goods are sold at markdown prices. However, this selling strategy can cannibalize regular sales because it provides an incentive for customers to wait for discounts instead of buying at the regular price earlier in the selling season. This form of strategic waiting can hurt sellers' revenue significantly. According to McWilliams (2004), 26% of Best Buy's customers wait for markdowns. To discourage strategic waiting, researchers have proposed different ideas. The first idea is to use product scarcity as a mechanism to urge customers to buy early. Liu and van Ryzin (2008) show that, by reducing the number of units available for sale at the beginning of the selling season, customers are motivated to buy early. In the same vein, Yin et al. (2009) show that fewer customers will wait strategically when a store displays fewer items to create an impression of product scarcity. The second idea is to sell callable reservations where a customer can reserve an item at the clearance price during the season and will receive the reserved item only when the reserved item remains unsold at the regular price at the end of the season (Gallego et al., 2008 and Elmaghraby et al., 2009).

The third idea is known as opaque selling: this is when a store sells its leftover inventory of different products as a single probabilistic good under which the seller will reveal the product identity only *after* the customers have purchased the probabilistic good. The probabilistic goods appeal to bargain hunters who do not have a strong product preference (as opposed to regular customers who might have a strong product preference)<sup>1</sup>. Hence, selling probabilistic goods at a discount can reduce cannibalization of regular products at regular price. Jiang (2007) shows that opaque selling can enable a seller to increase revenue by discriminating different customer segments.

In recent years, firms are increasingly adopting the opaque selling strategy to dispose of leftover inventories. In the travel industry, some airlines are selling unsold tickets at low prices as "blind tickets" without telling the passengers what the actual destination is in advance. For example, customers who purchase blind tickets from Eurowings airline in Europe will only find out what the actual destination is after making the purchase. Virgin Australia will only inform passengers about the exact destination a few days before departure. Hence, opaque selling enables airlines to sell their distressed seats as blind tickets at a discount to leisure travelers without cannibalizing the sales of regular tickets to business travelers. In the United States, Pack Up + Go (packupgo.com) offers a "mystery 3-day getaway package" (flight + hotel) for a low price: customers can select a theme but they do not know the destination until a few days

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<sup>1</sup>For example, intermediaries (e.g., Priceline and Hotwire) sell unsold airline seats or hotel rooms through opaque selling that tend to appeal to leisure travelers, while regular airline tickets or hotel reservation appeals to business travelers.

before departure (Li and Tang, 2017).

Other industries are also adopting the opaque selling strategy. In the toy industry, Lego sells mini-figures of different movies characters (e.g., Batman and Ninjago) through Amazon and Toys “R” Us in a “blind bag” and customers do not know which mini-figures are in each bag. Disney sells its Funko Heroes mini-figures in blind bags as well. In the apparel industry, Fay et al. (2015) report that many online retailers (e.g., lane4swim.com, swimoutlet.com, and store.americanapparel.net) sell certain leftover inventories of apparel and shoes with known sizes in a blind bag; however, the colors, patterns, and styles are unknown to the customers prior to purchase.

While opaque selling enables a seller to sell their leftover inventories as a single probabilistic good without cannibalizing regular sales during the selling season, there is a double-blind effect inherent in this practice that can create customer dis-satisfaction. First, in the absence of inventory information, customers may not receive the products they were hoping for. Second, without information about customer preferences, the seller may allocate the wrong products to customers. Should the double-blind effect be mitigated, and if so, how? In this study, we examine three research questions:

1. **Inventory Information.** Should the firm disclose its inventory information so that customers can make informed purchasing decisions about probabilistic goods?
2. **Customer Preferences.** Should the firm solicit customer product preference information and promise customers that it will do its best to allocate the inventory according to their preferences?
3. **Options Fee.** Should the firm charge customers a discretionary fee allowing them to eliminate certain product options of their own choosing? This mechanism enables the seller to identify customer preferences and at the same time obtain extra revenue.

The answers to these questions are not at all obvious. From the perspective of customers, knowing the inventory levels enables them to make informed purchasing decisions<sup>2</sup>. However, inventory information increases the heterogeneity in customer valuation: a customer’s valuation of the probabilistic good increases (decreases) if the inventory of his or her preferred product is higher (lower). Consequently, the firm may price the probabilistic good differently. As for

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<sup>2</sup>To use product scarcity as a mechanism to deter waiting, some on-line retailers such as Amazon and Bonton disclose the inventory level of a product when there are only a few units left (Park et al. 2017). However, we are not aware of this practice in opaque selling yet.

the firm, it is not clear whether it can benefit from the increase in heterogeneity in customer valuation.

Solicitation of customer preferences and allocation based on their preferences could increase customer welfare. For example, Pack Up + Go solicits customer preferences by asking customers to complete a survey before they purchase their mystery vacation packages. In this case, customers will become more hopeful about receiving their preferred product and hence they will value the probabilistic good more. However, we also need to understand how the firm prices differently with the preference information and examine whether the firm can benefit from improved allocation.

Finally, the firm can charge a discretionary options fee that will allow customers to eliminate a certain product option of their own choosing. Obviously, only customers with a relatively strong product preference will pay this options fee, which would allow the seller to find out which products the customer prefer. For example, Eurowings offers an options fee of five euros that allows a blind ticket buyer to eliminate a certain destination (Post 2010, and Post and Spann, 2012). However, the impact of this discretionary options fee on the selling price of the probabilistic good is not clear. It is also not clear when the seller can benefit the most from offering this options fee, if at all.

To examine the above three research questions, we develop a single-period model in which a firm sells its inventory of two distinctive products as a single probabilistic good to rational customers. We show that revealing inventory information can increase customer welfare, but the firm's revenue suffers. Soliciting preference information free of charge has a similar effect: customers are better off, but the seller is worse off. However, we find that the firm can always improve its revenue by charging a discretionary options fee that allows customers to eliminate a certain product option.

Our analytical results provide two managerial insights. First, there is no-short term economic gain for the firm from revealing its inventory information to and/or soliciting product preferences from customers. The real incentive for the firm to do so is that it would increase customer trust and satisfaction, which would improve customer loyalty and generate positive customer reviews and hence benefit the firm in a long term. Second, the firm can obtain a higher revenue by charging a discretionary options fee which would allow customers to exclude certain product options. When the options fee is product dependent, we find that the firm gains the most when the inventories of different products are relatively balanced and their valuations are strongly skewed. When the options fee is product independent, however, the firm gains the

most when the inventories of different products are relatively balanced but their valuations are not too strongly skewed.

This paper is organized as follows. We review the related literature in Section 2. We provide model preliminaries and an overview of results in Section 3. We analyze the equilibrium outcomes for the case when the seller reveals inventory information and/or solicits customer preference and compare these outcomes analytically in Sections 4 and 5. We then analyze the case when the firm charges a discretionary options fee in Section 6. We conclude in Section 7 along with a discussion about the limitations of our model and potential future research.

## 2 Literature Review

This paper is related to two streams of research: revenue management and opaque selling. Most of the revenue management literature focuses on the use of different pricing strategies to manage the sales of its fixed and limited capacity over time (Talluri and van Ryzin, 2004). This literature usually assumes that demand is unaffected by the inventory level even though there is anecdotal evidence (Fisher 2006; McWilliams 2004) and empirical evidence (e.g., Park et al. 2017) suggesting the opposite. Liu and van Ryzin (2008) propose a capacity rationing strategy by creating product scarcity during the regular season. In the same vein, Yin et al. (2009) show analytically that, by displaying fewer units in a store, the seller can send a signal about product scarcity to reduce strategic waiting.

Besides product scarcity, researchers have examined various pricing strategies for improving the seller's revenue. When all customers are strategic and demand is known to the seller, Gallego et al. (2008) show that it is optimal for the firm to sell its inventory at a single-price (i.e., no price markdown). However, when only a fraction of customers is strategic and/or when demand is uncertain, they show that a two-price markdown policy is optimal. Also, Gallego et al. (2008) and Elmaghraby et al. (2009) show that the seller can increase its expected revenue further by issuing callable reservations.

In addition to the above selling strategies, researchers have examined the implications of Priceline's and Hotwire's opaque selling strategies for selling airline tickets and hotel rooms. Jiang (2007) is among the first to present a single-period model and shows that opaque selling can enable the seller to increase its revenue by segmenting the market: charge a discounted price in the opaque market and a regular price in the transparent market (i.e., traditional sales). Fay and Xie (2008) examine two selling mechanisms of two component products: traditional selling and opaque selling. Under the traditional selling strategy, the firm sets the price for

each component. However, under opaque selling, the firm sets three prices: one for each component and one for the probabilistic good associated with the two component products. By using a Hotelling model with a “fit-cost-loss coefficient”, they show that the introduction of an additional probabilistic good can enable the firm to obtain a higher revenue through price discrimination.

Because opaque selling disguises the product identity, it reduces product differentiation and intensifies price competition when multiple competing firms sell probabilistic goods. By incorporating the issue of competition in a deterministic model, Fay (2008) shows that opaque selling can reduce industry profits unless there is significant firm loyalty. When customers have different firm preferences, Shapiro and Shi (2008) show that opaque selling can improve industry profits because the firms can set different prices according to customer preferences. By considering a two-period model in which a firm sells through a transparent channel in the first period and it has the option to sell through an opaque channel in the second period, Jerath et al. (2009, 2010) establish the conditions under which it is optimal for the firm to switch to selling in the opaque channel in the second period.

The aforementioned literature on opaque selling examines the strategic implications of selling probabilistic goods in different market conditions: monopolistic or oligopolistic, single-period or two-period, deterministic or stochastic demand, etc. To complement this body of research, some of the recent literature deal with different operational issues arising from opaque selling. First, Anderson and Xie (2012) use **a combination of logistic regression and dynamic programming to illustrate how a firm can sell its probabilistic goods through dynamic pricing.** Second, Anderson and Xie (2014) present a stylized model in which a firm can sell its product via three channels: a transparent channel, an opaque channel, and an opaque bidding channel where customers specify the price they are willing to pay (i.e., name your own price). By examining how each selling mechanism segments the market, they characterize **the optimal sales channel for the firm.** Similarly, Chen et al. (2014) study the impact of different selling mechanisms of an opaque reseller on competing service providers. Fay et al. (2015) relax the assumption made in the opaque selling literature that the product mix of the probabilistic goods is given by showing that the seller can improve its revenue further from selling the optimal probabilistic goods that is generated from the optimal product mix.

Our paper complements the existing research on opaque selling by examining three mechanisms that are intended to reduce the negative effects arising from opaque selling and the associated double-blind process. Rather than focusing on the issues of dynamic pricing, dynamic

selling channels, and product assortments examined by Anderson and Xie (2012), Anderson and Xie (2014) and Fay et al. (2015), we focus on the implications of inventory information, customer preferences, and options fee. Our model setup is similar to that in Fay and Xie (2008): a single-period, deterministic demand, Hotelling model with the fit-cost-loss coefficient. However, we generalize their model by allowing different maximum valuations and different inventory levels for different products. Such generalization is critical for the central research questions we ask. It allows us to examine how the double blind process affects the way customers form expectations about the product they will receive and the way the seller allocates products to customers.

### 3 Model Preliminaries and Overview of Results

Consider the situation when a firm sells its leftover inventories  $n_A$  and  $n_B$  of products A and B, respectively, as a single probabilistic good at a single price  $p$  (a decision variable). Without loss of generality, we scale the inventory of the opaque product  $n_A + n_B = 1$ , and assume that  $n_A > 0.5 > n_B$ . Hence, the probability that the opaque product is actually product A is  $\phi = \frac{n_A}{n_A + n_B} = n_A > 0.5$ . Notice that  $\phi$  plays two roles: it represents the probability as well as the inventory level of product A. Therefore,  $1 - \phi$  represents the probability that the opaque product is product B as well as the inventory level of product B.

We assume that customers are distributed uniformly over  $[0, 1]$  as in the Hotelling model, and we scale the market size<sup>3</sup> to 1. Over the Hotelling line  $[0, 1]$ , products A and B are located at the end points, 0 and 1, respectively. The maximum values of A and B are  $V_A$  and  $V_B$ , respectively. To model each customer's preference, we assume that there exists a fit-cost-loss coefficient  $t \in (0, 1]$  so that, for a customer located at  $x \in [0, 1]$ , the utility of receiving product  $j$  is given by  $v_j(x)$  (Figure 1) where:

$$v_j(x) = \begin{cases} V_A - tx & \text{if } j = A; \\ V_B - t(1 - x) & \text{if } j = B; \end{cases} \quad (1)$$

Throughout this paper, we assume that  $V_A, V_B, t$  satisfy three basic assumptions:

1.  $V_A \geq t$  and  $V_B \geq t$  so that a customer's valuation  $v_j(x)$  of product  $j = A, B$  is positive for any  $x \in [0, 1]$ .
2.  $V_A > V_B - t$  so that the customer located at  $x = 0$  would prefer A over B.

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<sup>3</sup>The market size is the same as the total inventory. The cases when they are not the same will be discussed in Section 7.

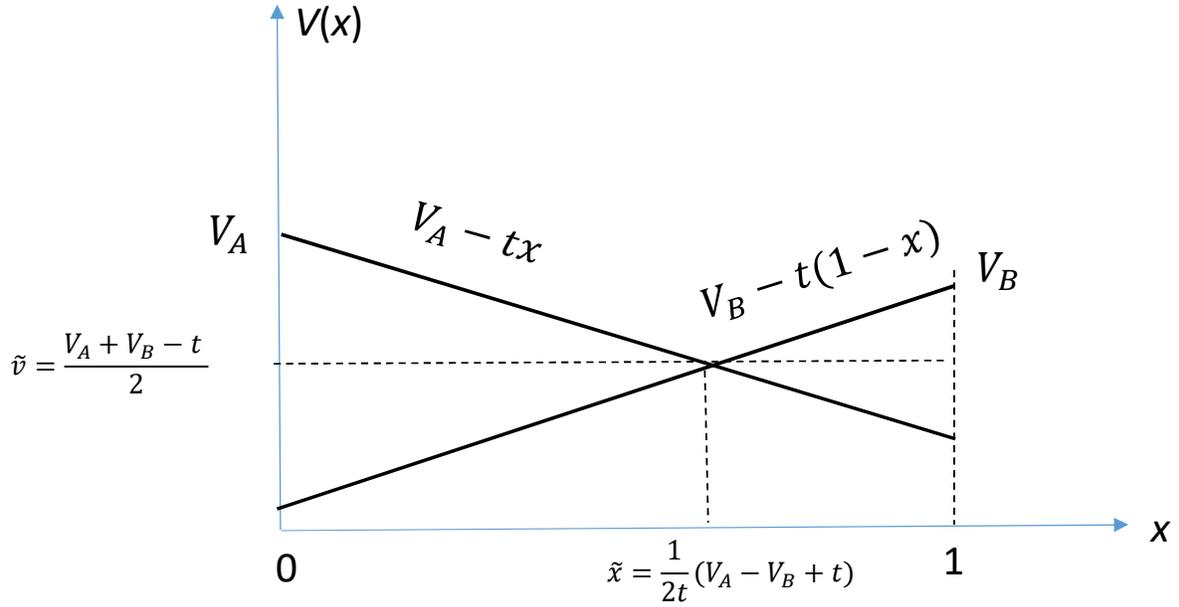


Figure 1: Valuation

3.  $V_B > V_A - t$  so that the customer located at  $x = 1$  would prefer B over A.

Let us define  $\tilde{x} \equiv \frac{V_A - V_B + t}{2t}$  as the location of a customer whose valuations of both products are the same with value  $\tilde{v} \equiv v_A(\tilde{x}) = \frac{V_A + V_B - t}{2}$ . Hence, we can observe from Figure 1 that customers located at  $x \in [0, \tilde{x}]$  prefer A, while customers located at  $x \in (\tilde{x}, 1]$  prefer B. In other words, there are  $\tilde{x}$  customers who prefer A and  $(1 - \tilde{x})$  customers who prefer B.

In the opaque selling literature (e.g., Fay and Xie 2008), it is assumed that the firm knows the distribution of customer valuations. However, there is an inherent double-blind process in traditional opaque selling. The first blind process is due to the fact that the firm does not know each customer's product preference unless the firm solicits this information from customers. The second blind process is due to the fact that customers do not have any inventory information about  $\phi$  and  $1 - \phi$  unless the firm discloses this information. These observations motivate us to examine whether the firm should reduce or eliminate this double-blind process by disclosing inventory information and/or soliciting customer preferences to make the opaque selling process more transparent.

### 3.1 Inventory Information and Customer Preferences

To examine the implications of making the opaque selling process more transparent, we will examine four different settings based on whether the seller discloses inventory information or

not, and whether the seller solicits customer product preference or not. These four settings can be described as follows.

1. **The firm does not reveal inventory information or solicit customer product preferences (Setting NN).** When customers do not have inventory information, they would simply assume that the products have similar inventory levels and that the probability of receiving product A is equal to  $\hat{\phi} = 0.5$ .<sup>4</sup> At the same time, without knowing customers' product preferences, the firm will allocate the products to customers randomly so that the actual probabilities that a customer will receive product A and product B are  $\phi$  and  $1 - \phi$  respectively.
2. **The firm reveals inventory information but does not solicit customer product preferences (Setting RN).** In this setting, customers know the actual inventory levels but the seller does not know their product preferences. As such, customers know that the seller will allocate product A to them randomly and the actual probabilities that a customer will receive product A and product B are  $\phi$  and  $1 - \phi$  respectively.
3. **The firm does not reveal inventory information but it does solicit customer product preferences (Setting NR).** Without inventory information, customers would simply assume that the products have the similar inventory levels as in Setting NN. Unlike in Setting NN, however, the seller now knows customer product preferences and will try to allocate to customers their preferred product. Among those customers who purchase the opaque product, let  $l_A$  ( $l_A \leq \tilde{x}$ ) and  $l_B$  ( $l_B \leq 1 - \tilde{x}$ ) be the number of customers who prefer A and B, respectively (Figure 1). Because the  $l_A$  customers would assume that the inventory levels of the two products are similar, they would estimate the probability of receiving product A to be equal to  $\hat{\phi}_A = \frac{\min\{0.5, l_A\}}{l_A}$ . Similarly, the  $l_B$  customers who purchase the opaque product would estimate the probability of receiving product A to be equal to  $\hat{\phi}_B = 1 - \frac{\min\{0.5, l_B\}}{l_B}$ . The firm can use the inventory information and customer preferences to make its product allocation. Hence, the corresponding actual probabilities of receiving product A are equal to  $\phi_A = \frac{\min\{\phi, l_A\}}{l_A}$  and  $\phi_B = 1 - \frac{\min\{1-\phi, l_B\}}{l_B}$ .
4. **The firm reveals inventory information and solicits customer product preferences (Setting RR).** This setting is similar to Setting NR except that customers can

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<sup>4</sup>Throughout this paper, we will use  $\hat{z}$  to denote the customer's estimated value of a quantity  $z$ . For example,  $\hat{\phi} = 0.5$  is the customer's estimated value of  $\phi$ . We need to distinguish between the estimated probability  $\hat{\phi}$  and the actual probability  $\phi$  of receiving A because the customer's purchasing decision is based on the estimated probability  $\hat{\phi}$ , while the customer surplus is based on the actual probability  $\phi$ .

use the actual inventory information to compute the probability of receiving product A correctly; that is,  $\hat{\phi}_A = \phi_A$  and  $\hat{\phi}_B = \phi_B$ .

For setting  $i$ ,  $i = NN, RN, NR, RR$ , we use the corresponding information structure and the customer's estimated probability of receiving product A (i.e.,  $\hat{\phi}$  and  $\phi$  in Settings NN and RN, and  $\hat{\phi}_A$  and  $\hat{\phi}_B$  for those customers who prefer A and B in Settings NR and RR) to determine the optimal selling price  $p^i$ , optimal revenue  $\pi^i$ , and customer surplus  $CS^i$ . Then we compare these outcomes across different settings in Section 4. Our analytical comparisons yield the following results: (a) disclosing inventory information and/or soliciting customer preference can result in lower revenue for the seller; (b) disclosing inventory information and/or soliciting customer preference will always result in higher customer surplus; (c) which type of information benefits the customers more would depend on the inventory composition.

### 3.2 Options Fee

Instead of making the opaque selling process more transparent (by revealing inventory information and/or soliciting customer preferences), we consider an alternative mechanism in which the firm charges a discretionary options fee that can enable a customer to eliminate a certain product option.<sup>5</sup> We consider two different settings:

1. **Product-dependent options fee.** In addition to the selling price of the opaque product  $p$ , the firm charges an additional options fee  $k_A$  or  $k_B$ , dependent of the product. Hence, the firm needs to decide on the price  $p$  and two product-specific options fees. This setting is equivalent to the case when the firm sells product A at  $p + k_A$ , product B at  $p + k_B$ , and the opaque product at  $p$ .<sup>6</sup>
2. **Product-independent options fee.** In addition to the selling price of the opaque product  $p$ , the firm charges an options fee  $k$  that is product-independent. This setting captures the mechanism by which Eurowings sells its blind tickets (Post 2012).

For these two settings, we determine the optimal selling price, the optimal options fee(s), and the firm's optimal revenue. Also, by considering Setting NN as the benchmark, we show analytically that, when selling a single opaque product, the firm can strictly increase its revenue

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<sup>5</sup>We consider this form of options fee because it has been used in practice (e.g., Eurowings). Also, a firm is likely to face backlashes from customers if it charges customers for accessing its inventory information and/or declaring their product preferences.

<sup>6</sup>While this setting is akin to the model considered by Fay and Xie (2008), we generalize their model by allowing for different product valuations ( $V_A, V_B$ ) and different inventory levels ( $\phi, 1 - \phi$ ).

by charging a discretionary options fee in both settings. Hence, our result for the product-independent options fee provides a plausible explanation for why Eurowings is able to obtain a higher revenue when selling its blind tickets (Post, 2012).

## 4 Mechanisms for Reducing the Double Blind Effect: Inventory Information and Customer Preference

We now examine the four settings defined in Section 3.1 that entail whether the seller discloses inventory information or not, and whether the seller solicits customer preference or not.

### 4.1 NN: The firm does not disclose inventory information or solicit customer product preferences

Setting NN is our benchmark in which the firm sells the opaque product without disclosing inventory information or soliciting customer product preferences. As discussed in Section 3.1, all customers would estimate the probability of receiving product A to be equal to  $\hat{\phi} = 0.5$  in Setting NN. Combining this observation with (1), the expected valuation of the opaque product for any customer located at  $x \in [0, 1]$  is equal to  $V(x)$ , where:

$$V(x) = \hat{\phi}v_A(x) + (1 - \hat{\phi})v_B(x) = \frac{V_A + V_B - t}{2} = \tilde{v}.$$

Observe from Figure 2 that the valuation  $V(x) = \tilde{v}$  for all customers  $x \in [0, 1]$ . Hence, all

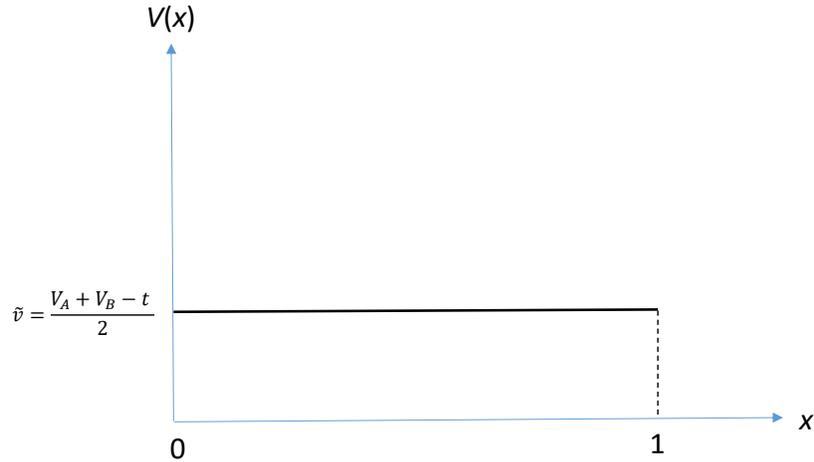


Figure 2: Without any information

customers will buy the opaque product if  $p \leq \tilde{v}$ . This observation implies that:

**Proposition 1** (*Setting NN*) *When the firm does not disclose inventory information nor solicit customer product preference, it is optimal for the firm to set its price for the opaque product at  $p^{NN} = \tilde{v} = \frac{V_A + V_B - t}{2}$ . In this case, all customers will purchase the product and the firm's optimal revenue will be  $\pi^{NN} = p^{NN} \cdot 1 = \frac{V_A + V_B - t}{2} = \tilde{v}$ .*

When all customers purchase the opaque product and the firm has no knowledge about customer preferences, the firm will allocate product A to customers randomly according to the actual probability  $\phi \geq 0.5$ . In this case, by using the valuation given in (1) and the selling price  $p^{NN}$  as stated in Proposition 1, the customer surplus associated with Setting NN, denoted by  $CS^{NN}$ , is:

$$CS^{NN} = \int_0^1 [(V_A - tx - p^{NN})\phi + (V_B - t(1-x) - p^{NN})(1-\phi)]dx = \frac{(V_A - V_B)(2\phi - 1)}{2}. \quad (2)$$

Although the selling price  $p^{NN}$  and the seller's revenue  $\pi^{NN}$  are strictly positive (Proposition 1), equation (2) shows that the customer surplus can be negative when  $V_B > V_A$ . When  $V_B > V_A$ , most customers prefer product B (Figure 1), but they are more likely to receive product A than B because  $\phi \geq 0.5$ . These observations motivate us to consider the following questions: Should the firm reveal inventory information so that customers can make informed purchasing decisions about the probabilistic goods? Should the firm solicit customer product preferences to improve its product allocation? How would these initiatives affect the firm's revenue? We will examine these questions in the remainder of this section.

## 4.2 RN: The firm discloses inventory information without soliciting customer product preference

As discussed in Section 3.1, customers know the exact inventory level in Setting RN and they know that the actual probability of receiving product A is  $\phi$ . From (1), the expected valuation of any customer located at  $x \in [0, 1]$  is equal to  $V(x)$ , where:

$$V(x) = \phi v_A(x) + (1 - \phi)v_B(x) = V_A - (1 - \phi)(V_A - V_B) - (1 - \phi)t - (2\phi - 1)tx. \quad (3)$$

Because  $\phi \geq 0.5$ ,  $V(x)$  is decreasing in  $x$  (Figure 3). Relative to Figure 2 for the benchmark case (Setting NN), Figure 3 suggests that disclosing inventory information can create heterogeneity in customer valuation. Heterogeneous customer valuations can create an opportunity for the firm to obtain a higher revenue by setting a higher price to discriminate certain customers. We will examine this opportunity next.

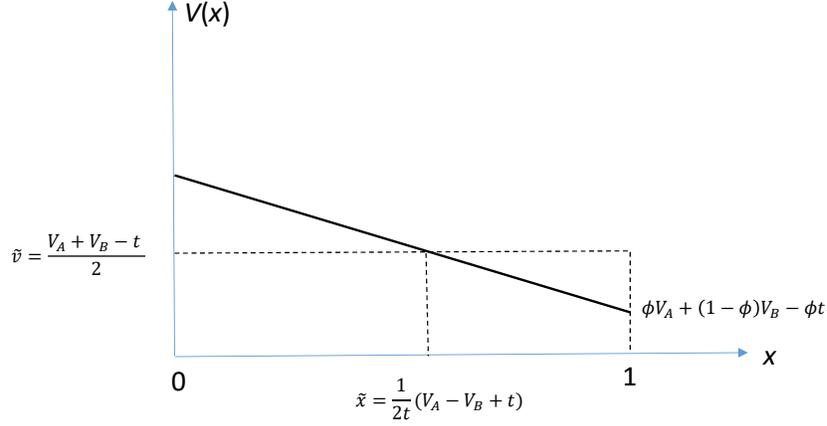


Figure 3: With inventory information

Observe from (3) and Figure 3 that  $V(x)$  is bounded below by  $\phi V_A + (1 - \phi)V_B - \phi t$ . This means that there is no reason for the firm to charge a price below this bound, or that the optimal price must satisfy:

$$p \geq \phi V_A + (1 - \phi)V_B - \phi t. \quad (4)$$

For any given price  $p$  that satisfies (4), we can use (3) to show that the demand for the opaque product is equal to

$$\frac{[V_A - (1 - \phi)(V_A - V_B) - (1 - \phi)t - p]}{t(2\phi - 1)}.$$

Hence, the firm's problem can be formulated as:

$$\pi^{RN} = \max_p p \cdot \frac{[V_A - (1 - \phi)(V_A - V_B) - (1 - \phi)t - p]}{t(2\phi - 1)}, \text{ s.t. (4).}$$

The optimal price and profit are summarized in the following proposition:

**Proposition 2** (*Setting RN*) *When the firm discloses inventory information without soliciting customer product preferences, its optimal price  $p^{RN}$  and optimal revenue  $\pi^{RN}$  satisfy:*

$$p^{RN} = \begin{cases} V_A - (1 - \phi)(V_A - V_B) - \phi t & \text{if } V_A - (1 - \phi)(V_A - V_B) \geq (3\phi - 1)t; \\ \frac{1}{2}[V_A - (1 - \phi)(V_A - V_B) - (1 - \phi)t] & \text{if } V_A - (1 - \phi)(V_A - V_B) \in [(1 - \phi)t, (3\phi - 1)t], \end{cases}$$

$$\pi^{RN} = \begin{cases} V_A - (1 - \phi)(V_A - V_B) - \phi t & \text{if } V_A - (1 - \phi)(V_A - V_B) \geq (3\phi - 1)t; \\ \frac{[V_A - (1 - \phi)(V_A - V_B) - (1 - \phi)t]^2}{4t(2\phi - 1)} & \text{if } V_A - (1 - \phi)(V_A - V_B) \in [(1 - \phi)t, (3\phi - 1)t]. \end{cases}$$

Also, the optimal price and profit in Setting RN are lower than those in the benchmark Setting NN; that is,  $p^{RN} < p^{NN}$  and  $\pi^{RN} < \pi^{NN}$ .

Observe from Proposition 2 that, in case 1 (i.e., when  $V_A - (1 - \phi)(V_A - V_B) \geq (3\phi - 1)t$ ), the optimal price  $p^{RN}$  is set at the lower bound of  $V(x)$  as shown in Figure 3. In this case, all customers will purchase the opaque product at  $p^{RN}$ . Because  $p^{RN} < p^{NN}$  and because all customers will purchase the product, we can conclude that: (a) the firm earns less by disclosing inventory information than it does in the benchmark Setting NN; and (b) customers obtain a higher surplus when the firm discloses inventory information than they do in the benchmark case. Specifically,  $CS^{RN} = \int_0^1 [(V_A - tx - p^{RN})\phi + (V_B - t(1 - x) - p^{RN})(1 - \phi)]dx$ . By substituting  $p^{RN} = V_A - (1 - \phi)(V_A - V_B) - \phi t$ , we get  $CS^{RN} = \frac{t(2\phi - 1)}{2}$ .

Next, let us consider case 2 in which the optimal price  $p^{RN}$  is higher than the lower bound of  $V(x)$  hence some customers will not purchase the product. This observation and the fact that  $p^{RN} < p^{NN}$  enable us to conclude that the firm also earns less by disclosing inventory information than it does in the benchmark case. Finally, let us compute the customer surplus. Specifically, customers with  $V(x) \geq p^{RN}$  will purchase the product. It follows from (3) and  $p^{RN}$  as stated in case 2 that customers located between  $[0, \bar{x}]$  will purchase the product, where  $\bar{x} = \frac{V_A - (1 - \phi)(V_A - V_B) - (1 - \phi)t}{2t(2\phi - 1)}$ . In this case, the corresponding customer surplus in case 2 is given by:

$$\begin{aligned} CS^{RN} &= \int_0^{\bar{x}} [(V_A - tx - p^{RN})\phi + (V_B - t(1 - x) - p^{RN})(1 - \phi)]dx \\ &= \frac{[(t + V_A - V_B)\phi + (V_B - t)]^2}{8t(2\phi - 1)}. \end{aligned} \quad (5)$$

By comparing (5) and (2), it can be shown that  $CS^{RN} > CS^{NN}$ . In summary, we get:

**Corollary 1** (*Setting RN*) *When the firm reveals inventory information, the customer surplus  $CS^{RN}$  satisfies:*

$$CS^{RN} = \begin{cases} \frac{t(2\phi - 1)}{2} & \text{if } \phi \leq \min(1, \frac{t + V_B}{3t - V_A + V_B}); \\ \frac{[(t + V_A - V_B)\phi + (V_B - t)]^2}{8t(2\phi - 1)} & \text{if } \phi \geq \min(1, \frac{t + V_B}{3t - V_A + V_B}). \end{cases}$$

*The firm's optimal revenue is lower (i.e.,  $\pi^{RN} < \pi^{NN}$ ) and the customer surplus is higher when the firm discloses inventory information without soliciting customer product preferences in Setting RN than they are in the benchmark Setting NN.*

Revealing inventory information can create heterogeneity in customer valuation. However, as shown in Figure 3, the lower bound of customer valuation  $\phi V_A + (1 - \phi)V_B - \phi t$  in Setting RN is lower than the constant valuation  $\tilde{v} = \frac{V_A + V_B - t}{2}$  in Setting NN. In this case, if the firm wants to cover the whole market, it has to set a lower price in Setting RN than in Setting NN

and earns less in Setting RN. Otherwise, the market would only be partially covered, which is suboptimal. This explains why the firm is worse off when it reveals inventory information. In other words, heterogeneous customer valuations harm the firm.

It is interesting to observe from Corollary 1 that while  $CS^{NN}$  can be negative when  $V_B > V_A$ ,  $CS^{RN}$  is always positive regardless of the values of  $V_A$  and  $V_B$ . When the customers possess inventory information, their valuation of the opaque product is computed based on accurate information and hence the customer surplus cannot be lower than zero, which is the customer surplus when they make no purchase.

### 4.3 NR: The firm does not disclose inventory information but it does solicit customer product preferences

Even without inventory information, customers would know that the firm will allocate the product based on customer product preferences in Setting NR as discussed in Section 3.1. Specifically recall from Section 3 (Figure 2) that customers located at  $x \in [0, \tilde{x}]$  prefer A and customers located at  $x \in [\tilde{x}, 1]$  prefer B, where  $\tilde{x} = \frac{(V_A - V_B + t)}{2t}$ . Also, among those who have purchased the opaque product, let  $l_A$  and  $l_B$  be the number of customers who prefer A and B respectively, where  $l_A \leq \tilde{x}$  and  $l_B \leq (1 - \tilde{x})$ . In this case, the  $l_A$  customers located at  $x \in [0, \tilde{x}]$  would estimate the probability of receiving product A to be equal to  $\hat{\phi}_A = \frac{\min\{0.5, l_A\}}{l_A}$ . Similarly, the  $l_B$  customers located at  $x \in [\tilde{x}, 1]$  would estimate the probability of receiving product A to be equal to  $\hat{\phi}_B = 1 - \frac{\min\{0.5, l_B\}}{l_B}$ . However, the firm has both the inventory information and customer preferences. Hence, the corresponding actual probabilities of receiving product A are equal to  $\phi_A$  and  $\phi_B$ .

To begin, let us use (1) and the customers' estimated probabilities of receiving A (i.e.,  $\hat{\phi}_A$  for customers located below  $\tilde{x}$  and  $\hat{\phi}_B$  for customers located above  $\tilde{x}$  as defined above) to determine the expected valuation  $V(x)$  for any customer  $x \in [0, 1]$ , where  $V(x) = \hat{\phi}_A v_A(x) + (1 - \hat{\phi}_A) v_B(x)$  for  $x \in [0, \tilde{x}]$  and  $V(x) = \hat{\phi}_B v_A(x) + (1 - \hat{\phi}_B) v_B(x)$  for  $x \in [\tilde{x}, 1]$ . More formally, we have:

$$V(x) = \begin{cases} V_A - (1 - \hat{\phi}_A)(V_A - V_B) - (1 - \hat{\phi}_A)t - (2\hat{\phi}_A - 1)tx & \text{if } x \leq \tilde{x} \\ V_A - (1 - \hat{\phi}_B)(V_A - V_B) - (1 - \hat{\phi}_B)t - (2\hat{\phi}_B - 1)tx & \text{if } x > \tilde{x}. \end{cases} \quad (6)$$

Figure 4 depicts the expected valuation  $V(x)$  as given in (6). By comparing Figure 4 and Figure 2, we can conclude that, when customers know that the firm will try to allocate the product according to their preferences, they become more hopeful about receiving their preferred product. Consequently, the expected valuation  $V(x)$  is higher in Setting NR than in Setting NN. Hence, we can use the same argument as before to show that the firm will always set its price

$p \geq \tilde{v} \equiv \frac{V_A + V_B - t}{2}$ . For a given price  $p$ , the customers located between  $[0, l_A]$  and  $[1 - l_B, 1]$  will buy the opaque product and the rest will not. If  $p = \tilde{v}$ , the market is fully covered (i.e., all customers will purchase the product) and if  $p > \tilde{v}$ , the market **it** partially covered. By comparing the firm's revenue associated with these two cases, we obtain:

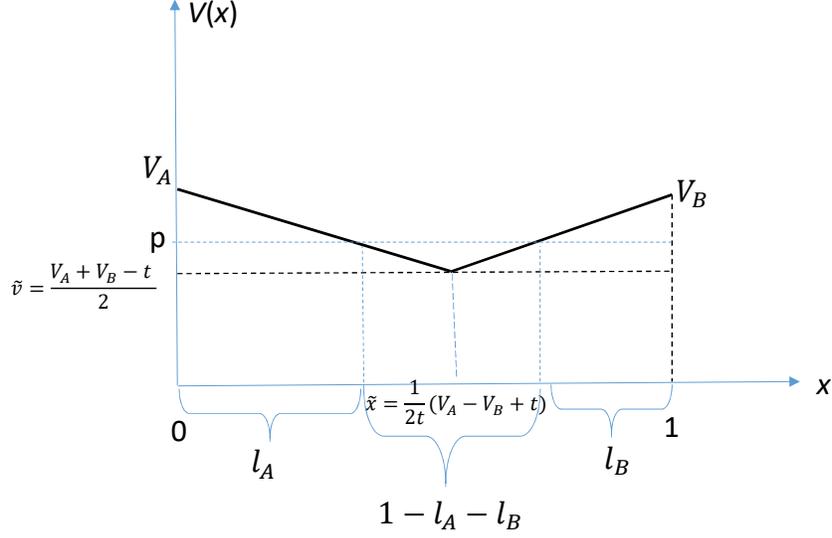


Figure 4: With preference information

**Proposition 3** (*Setting NR*) *When the firm solicits customer product preference without disclosing inventory information, it is optimal for the firm to set its price for the opaque product at  $p^{NR} = \tilde{v} = \frac{V_A + V_B - t}{2} = p^{NN}$ . In this case, all customers will purchase the product and the firm's optimal revenue will be  $\pi^{NR} = p^{NR} \cdot 1 = \frac{V_A + V_B - t}{2} = \tilde{v} = \pi^{NN}$ .*

From Propositions 1 and 3, we notice that the optimal price  $p^{NR} = p^{NN} = \tilde{v}$  and the firm's optimal revenue  $\pi^{NR} = p^{NN}$ . Hence, the firm cannot increase its revenue simply by allocating the product according to customer preference information solicited from customers. The reason is as follows. First, as shown in Figure 4, the lower bound of customer's valuation is equal to  $\tilde{v}$  in Setting NR. Therefore, if the firm wants to cover the whole market, it has to set  $p^{NR} = p^{NN} = \tilde{v}$  and earns the same revenue as in Setting NN. If the firm sets a higher price so that  $p^{NR} > \tilde{v}$ , the gain with the higher price is offset by the loss in customer demand. Consequently, setting  $p^{NR} = p^{NN} = \tilde{v}$  is optimal and the firm earns the same revenue as in Setting NN.

By computing the customer surplus **CS<sup>NR</sup>** and comparing it against **CS<sup>NN</sup>** as given in (2), we obtain:

**Corollary 2** (*Setting NR*) *When the firm solicits customer product preference without disclosing inventory information in Setting NR, the corresponding customer surplus  $CS^{NR}$  satisfies:*

$$CS^{NR} = \begin{cases} \frac{t\phi - (1-\phi)(V_A - V_B)}{2} & \text{if } V_A \geq V_B + (2\phi - 1)t, \\ \frac{\phi(V_A - V_B) + (1-\phi)t}{2} & \text{if } V_A \leq V_B + (2\phi - 1)t. \end{cases} \quad (7)$$

Also,  $CS^{NR} \geq CS^{NN}$  so that, relative to Setting NN, the firm can increase customer surplus by allocating the product according to customer preference information solicited from customers.

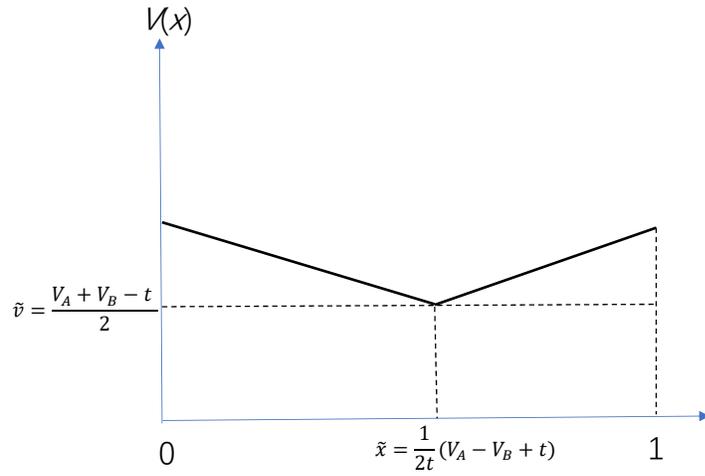
Proposition 3 states that, in Setting NR, it is optimal to sell the probabilistic good at the same price as in Setting NN so that  $p^{NR} = p^{NN} = \tilde{v}$ . While customers are paying the same price in both settings, Corollary 2 states that they will obtain a higher surplus in Setting NR because the firm allocates the products according to customer preferences as opposed to randomly in Setting NN.

#### 4.4 RR: The firm discloses inventory information and solicits customer product preference

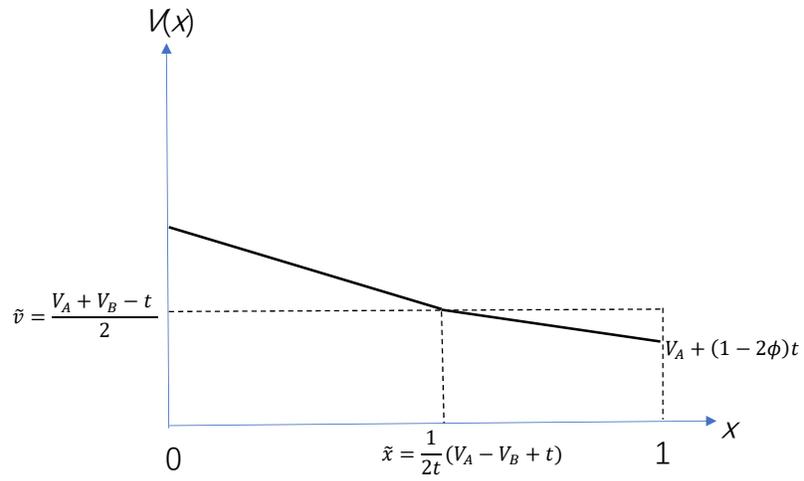
This setting is akin to Setting NR except that customers can now use the inventory information to estimate the allocation probabilities  $\phi_A$  and  $\phi_B$  accurately. Specifically, customers located at  $x \in [0, \tilde{x}]$  who prefer A would estimate the probability of receiving A correctly so that  $\hat{\phi}_A = \phi_A$ . Also, customers located at  $x \in [\tilde{x}, 1]$  who prefer B would estimate the probability of receiving A correctly so that  $\hat{\phi}_B = \phi_B$ . Combining this observation with (1), we can derive the expected valuation of any customer located at  $x \in [0, 1]$  as:

$$V(x) = \begin{cases} \phi_A v_A(x) + (1 - \phi_A) v_B(x) & \text{if } x \leq \tilde{x}; \\ \phi_B v_A(x) + (1 - \phi_B) v_B(x) & \text{if } x > \tilde{x}. \end{cases} \quad (8)$$

Observe from (1) and (8) that the slope of  $V(x)$  as a function of  $x$  is equal to  $(1 - 2\phi_A)t$  for  $x \in [0, \tilde{x}]$  and  $(1 - 2\phi_B)t$  for  $x \in [\tilde{x}, 1]$ . Also, because  $\phi \geq 0.5$ ,  $\phi_A \geq 0.5$ . However,  $\phi_B$  can be smaller or greater than 0.5, depending on which is greater,  $1 - \phi$  or  $l_B$ . Therefore, we have two cases to consider. In Figure 5 (I), we depict  $V(x)$  for the case when  $\phi_B \leq 0.5$ . Similarly, we depict  $V(x)$  in Figure 5 (II) for the case when  $\phi_B \geq 0.5$ . When  $\phi_B \leq 0.5$ , Figure 5 (I) resembles Figure 4 and it is optimal for the firm to set  $p^{RR} = \tilde{v}$ . Now let's consider the case when  $\phi_B \geq 0.5$ . Observe that Figure 5 (II) resembles Figure 3 except that the slope of  $V(x)$  is not constant in Setting RR (so that the lower bound of  $V(x)$  is higher in Setting RR). However, we can use the same argument to show that the firm will always set its price  $p \geq V_A + (1 - 2\phi)t$  and we can formulate the firm's problem in a similar fashion as in Setting RN. By comparing the firm's revenue associated with these two cases, we obtain:



(I)



(II)

Figure 5: With both inventory and preference information

**Proposition 4** (Setting RR) *When the firm discloses inventory information and solicits customer product preference, the optimal price  $p^{RR}$  and the firm's optimal revenue  $\pi^{RR}$  satisfy:*

1. *If  $\phi \in [\frac{1}{2}, \frac{3t+V_A-V_B}{4t}]$ , then  $p^{RR} = \frac{V_A+V_B-t}{2} = \tilde{v}$  and  $\pi^{RR} = p^{NR} \cdot 1 = \frac{V_A+V_B-t}{2} = \tilde{v} = \pi^{NN}$ .*
2. *If  $\phi \in [\frac{3t+V_A-V_B}{4t}, 1]$ , then  $p^{RR} < \frac{V_A+V_B-t}{2} = \tilde{v}$  and  $\pi^{RR} < \pi^{NN}$ .*

The result stated in Proposition 4 is based on the following intuition. Consider the first statement when the inventory levels are relatively balanced so that  $\phi$  is slightly above 0.5. In this case, inventory information is not that informative because customers would have guessed that  $\phi = 0.5$  in Setting NR. Hence, this case is akin to Setting NR and Proposition 4 is analogous to Proposition 3. Next, when the inventory levels are highly imbalanced (i.e.,  $\phi$  is closed to 1), inventory information will lower the expected valuations for those customers with a strong preference for B (i.e., those customers located near 1) because they know that the chances of receiving B are slim. As shown in Figure 5 (II), customer valuation falls below  $\frac{V_A+V_B-t}{2} = \tilde{v}$  for those customers located near 1. Because the lower bound of the valuation is below  $\tilde{v}$ , it is optimal for the firm to set its optimal selling price at  $\pi^{RR} < \tilde{v} = \pi^{NN}$  for the same reason as that given in Setting RN. Hence, we can conclude that the firm earns the same revenue as in Setting NN when the inventory levels are relatively balanced; however, the firm earns a slightly lower revenue when the inventory levels are highly imbalanced.

By using the same approach as before, we can compute the customer surplus  $CS^{RR}$ . Comparing it against  $CS^{NN}$  as given in (2) gives:

**Corollary 3** (Setting RR) *When the firm discloses inventory information and solicits customer product preference in Setting RR, the corresponding customer surplus  $CS^{RR}$  satisfies:*

$$CS^{RR} = \begin{cases} CS^{NR} & \text{if } \frac{1}{2} \leq \phi \leq \frac{3t+V_A-V_B}{4t}, \\ \frac{(3\phi-2)t-(1-\phi)(V_A-V_B)}{8t} & \text{if } S_1 \text{ holds,} \\ \frac{(t+V_A-V_B)^2}{8t} & \text{if } \frac{3t+V_A-V_B}{4t} \leq \phi \leq \frac{3t-V_B}{2t} \text{ and } V_A + V_B \leq 3t, \\ \frac{V_A^2-4(1-\phi)(V_A-V_B)t-4(1-\phi)(2-\phi)t^2}{8t} & \text{if } S_2 \text{ holds.} \end{cases} \quad (9)$$

The two conditions ( $S_1$  and  $S_2$ ) are  $S_1 = \{\phi \geq \frac{3t+V_A-V_B}{4t} \text{ and } V_A \geq 2t\} \cup \{\frac{3t+V_A-V_B}{4t} \leq \phi \leq \frac{V_A}{2t}, V_A + V_B \geq 3t \text{ and } V_A \leq 2t\}$  and  $S_2 = \{\frac{V_A}{2t} \leq \phi \leq 1, V_A + V_B \geq 3t \text{ and } V_A \leq 2t\} \cup \{\phi \geq \frac{3t-V_B}{2t} \text{ and } V_A + V_B \leq 3t\}$ . Also,  $CS^{RR} \geq CS^{NN}$  so that, relative to Setting NN, the firm can increase customer surplus by allocating the product according to customer preference information solicited from customers.

In Setting RR, when the firm charges a selling price that is lower than that in Setting NN (as stated in Proposition 4), customers can enjoy a higher surplus.

In this section, we have shown that the firm cannot be better off by revealing inventory information and/or soliciting customer preferences; i.e.,  $\pi^{RN} < \pi^{NN}$  (Proposition 2),  $\pi^{NR} = \pi^{NN}$  (Proposition 3),  $\pi^{RR} = \pi^{NN}$  when  $\phi$  is low and  $\pi^{RR} < \pi^{NN}$  when  $\phi$  is high (Proposition 4). In fact, the firm is likely to be strictly worse off by revealing inventory information to customers. This is because, when customers possess inventory information, their valuation  $V(x)$  can drop below  $\frac{V_A + V_B - t}{2} = \tilde{v}$  especially for those customers located near 1 who strongly prefer B. When that happens, the firm usually ends up dropping the selling price below  $\tilde{v} = \pi^{NN}$  in order to cover the whole market. Hence, the firm will end up earning less than in the benchmark case. However, soliciting customer preferences does not change revenue levels for the firm. Therefore, through our analysis, it becomes clear that the firm does not accrue any no short-term economic gain from revealing its inventory information to and/or soliciting product preferences from customers.

While the firm is either unaffected or worse off when it discloses inventory information or solicits customer preferences, Corollaries 1, 2, and 3 assert that customers can obtain a higher surplus. Therefore, the firm can improve customer satisfaction and loyalty through inventory information disclosure and/or customer preference solicitation, which benefits the firm in the long term.

## 5 Comparisons of Customer Surplus

While disclosing inventory information and/or soliciting customer preferences can improve customer surplus, it is not clear which type of information is more beneficial? Also, it is not clear if more information (as in Setting RR) would generate a higher customer surplus. Figure 6 depicts the customer surplus associated with Settings RN, NR and RR as stated in Corollaries 1, 2, and 3. Through analytical comparison, we obtain:

### Proposition 5

1.  $CS^{RN} < CS^{NR}$  when  $0.5 \leq \phi < \frac{2t}{3t - (V_A - V_B)}$ ; otherwise  $CS^{RN} \geq CS^{NR}$ .
2.  $CS^{RR} \geq CS^{NR}$ .
3.  $CS^{RR} > CS^{RN}$  when  $0.5 \leq \phi < \frac{2t}{3t - (V_A - V_B)}$ ; otherwise  $CS^{RR} \leq CS^{RN}$ .

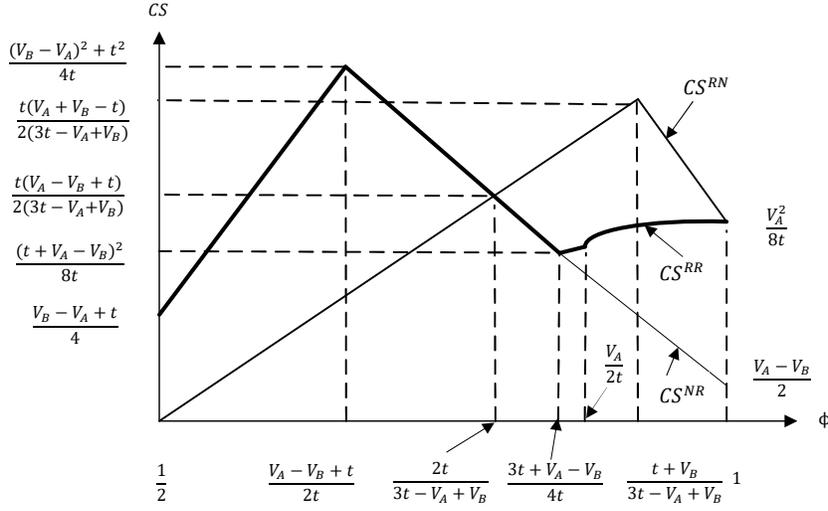


Figure 6: Comparison of Customer Surplus

The consumer surplus depends on three factors: price, market coverage, and allocation of products. Under both Settings RN and NR, the consumer surplus first increases and then decreases as  $\phi$  increases from 0.5 to 1. For Setting RN, this is because the price first decreases and then increases and the market coverage decreases when  $\phi$  increases. Furthermore, when  $\phi$  is only slightly greater than 0.5, there is no information asymmetry, and hence the consumer surplus is small. For Setting NR, the price is independent of  $\phi$  and the market is always fully covered. But when  $\phi$  increases, the benefit of more efficient allocation of inventory first increases and then decreases. In the extreme case when  $\phi = 1$ , all customers will receive product A anyway and there is no room for improving the allocation (Figure 6). This is what happens in Part (1) of Proposition 5. The idea behind Part (3) is similar. Finally, Part (2) is simply because the price under Setting RR is always no greater than that under Setting NR.

In summary, for customer surplus, which type of information is more beneficial depends on the composition of inventory.

## 6 Options Fee

So far, we have shown that the firm cannot generate a higher revenue by disclosing inventory information or soliciting customer preferences to make the opaque selling process more transparent. Instead of disclosing inventory information and/or soliciting customer preferences, we now consider an alternative in which the firm charges a discretionary options fee that allows

a customer to eliminate a certain product option of his or her own choosing. Specifically, we consider two settings as discussed in Section 3: (a) The options fee is product dependent (Case PD); and (b) the options fee is product independent (Case PI). Throughout this section, we allow  $\phi \in [0, 1]$  (instead of  $\phi \geq 0.5$ ).

## 6.1 PD: A product-dependent options fee

In addition to the selling price of the opaque product  $p$ , consider the situation where the firm charges an additional product-specific options fee  $k_A$  or  $k_B$  that enables a customer to eliminate a product option. In this case, if a customer pays  $p + k_A$ , then he or she can avoid receiving product B (and receive product A for certain). Similarly, a customer will receive B if he or she pays  $p + k_B$ .<sup>7</sup> Let  $x_A$  be the total number of customers who pay  $p + k_A$  to obtain A and define  $x_B$  similarly. By superimposing customer valuation as given in (1) along with the corresponding options price  $(p_A, p_B)$ , it is easy to observe from Figure 7 that  $p + k_A = V_A - tx_A$  and  $p + k_B = V_B - tx_B$ .

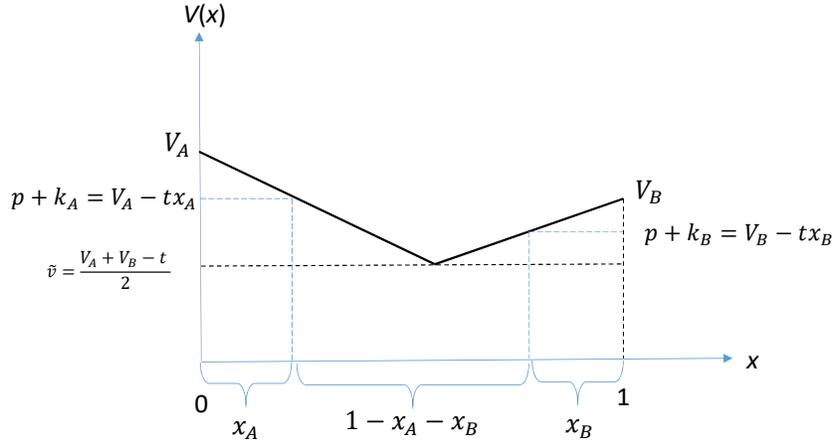


Figure 7: Product-Dependent Options Pricing

By noting that the customer valuation is bounded below by  $\tilde{v}$  (Figure 7), the firm should always set  $p \geq \tilde{v}$ . In this case, the firm's problem can be formulated as:

$$\max_{p, k_A, k_B} (p + k_A)x_A + (p + k_B)x_B + p(1 - x_A - x_B)$$

subject to  $p \geq \tilde{v}$ ,  $p + k_A = V_A - tx_A$ ,  $p + k_B = V_B - tx_B$ ,  $x_A \leq \phi$  and  $x_B \leq 1 - \phi$ . For any

<sup>7</sup>This setting is similar to the case when the firm sells product A at  $p_A = p + k_A$ , product B at  $p_B = p + k_B$ , and the opaque product at  $p$  as examined by Fay and Xie (2008). However, our setting is more general because we allow for different product valuations  $(V_A, V_B)$  and different inventory levels  $(\phi, 1 - \phi)$ .

given options fees  $(k_A, k_B)$ , it is optimal for the firm to set  $p^* = \tilde{v}$ . By substituting  $p^* = \tilde{v}$ , the firm's problem can be reduced to:

$$\tilde{v} + \max_{k_A, k_B} \{k_A x_A + k_B x_B\}, \text{ s.t. } x_A \leq \phi, x_B \leq 1 - \phi, \quad (10)$$

where  $\tilde{v} + k_A = V_A - t x_A$ ,  $\tilde{v} + k_B = V_B - t x_B$ , and  $\tilde{v} = \frac{V_A + V_B - t}{2}$ . As  $\pi^{NN} = \tilde{v}$  (Proposition 1), we can define  $\pi^{PD} = \max_{k_A, k_B} \{k_A x_A + k_B x_B\}, \text{ s.t. } x_A \leq \phi, x_B \leq 1 - \phi$ . As such  $\pi^{PD}$  represents the additional revenue to be obtained by the firm when it charges a product-dependent options fee. The optimal options fees and additional profit are presented in the following proposition.

**Proposition 6** *When the firm sells product A at  $p + k_A$ , product B at  $p + k_B$  and the opaque product at  $p$ , it is optimal to set  $p^* = \tilde{v}$ . Also, the optimal options fee  $(k_A^*, k_B^*)$  and the firm's optimal additional revenue  $\pi^{PD}$  satisfy:*

$$(k_A^*, k_B^*) = \begin{cases} \left( \frac{V_A - V_B + t}{2} - t\phi, \frac{V_B - V_A + t}{4} \right) & \text{if } \phi \leq \frac{V_A - V_B + t}{4t}; \\ \left( \frac{V_A - V_B + t}{4}, \frac{V_B - V_A + t}{4} \right) & \text{if } \phi \in \left( \frac{V_A - V_B + t}{4t}, \frac{V_A - V_B + 3t}{4t} \right]; \\ \left( \frac{V_A - V_B + t}{4}, \frac{V_B - V_A + t}{2} - t(1 - \phi) \right) & \text{if } \phi > \frac{V_A - V_B + 3t}{4t}. \end{cases}$$

$$\pi^{PD} = \begin{cases} \frac{(V_A - V_B)^2 + t^2}{8t} - t\left(\phi - \frac{V_A - V_B + t}{4t}\right)^2 & \text{if } \phi \leq \frac{V_A - V_B + t}{4t}; \\ \frac{(V_A - V_B)^2 + t^2}{8t} & \text{if } \phi \in \left( \frac{V_A - V_B + t}{4t}, \frac{V_A - V_B + 3t}{4t} \right]; \\ \frac{(V_A - V_B)^2 + t^2}{8t} - t\left(1 - \phi - \frac{V_B - V_A + t}{4t}\right)^2 & \text{if } \phi \geq \frac{V_A - V_B + 3t}{4t}. \end{cases}$$

Also, in all three cases,  $(k_A^*, k_B^*) > (0, 0)$ , and  $\pi^{PD} > 0$ .

Proposition 6 shows that options fees allow the firm to price discriminate customers with a strong product preference (Fay and Xie 2008). Those customers who strongly prefer one product over the other will pay an options fee to remove their less desired product, while those customers who only moderately prefer one product over the other will take a chance and purchase the opaque product without paying an options fee. Such a price discrimination tool is most effective in raising the firm's profit when the inventories of A and B are balanced, as Proposition 6 demonstrates, and the maximum additional profit the firm can achieve is  $\frac{(V_A - V_B)^2 + t^2}{8t}$ . The more skewed the valuations of the two products, the higher the additional profit.

## 6.2 PI: A product-independent options fee

We now consider a variant in which the discretionary options fee is product-independent; that is, the additional options fee  $k$  is the same regardless of which product the customer wants to eliminate. This particular form of strategy is currently adopted by Eurowings and the fee is five euros for removing each destination. In our context, a customer can obtain product A for

sure by paying  $(p + k)$ , product B for sure by paying  $(p + k)$ , and either product by paying  $p$  (Figure 8). By using the exact same approach as before, it is easy to show that  $p^* = \tilde{v}$ . The

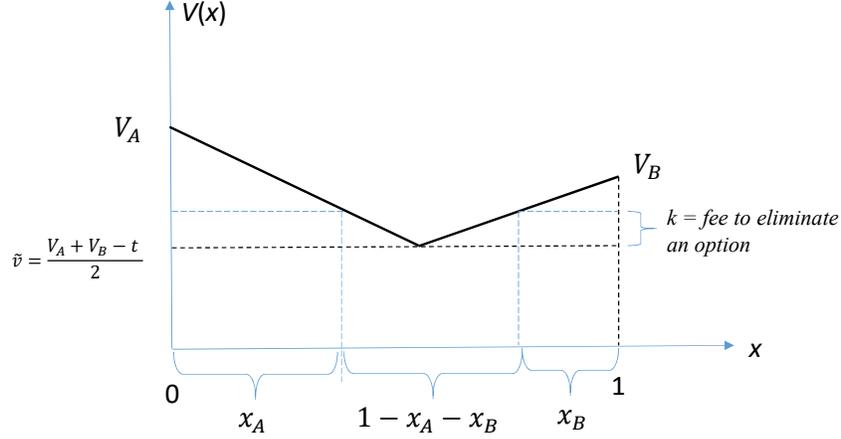


Figure 8: Product-Independent Options Pricing

remaining issue is to determine the optimal options fee  $k^*$  and the additional revenue, denoted by  $\pi^{PI}$ . To begin, without loss of generality, let us assume that  $V_A \geq V_B$ . In this case, it is easy to observe from Figure 8 that we need to consider two cases: (1)  $p^* + k = \tilde{v} + k \leq V_B$ , and (2)  $V_B < p^* + k = \tilde{v} + k \leq V_A$ . In the first case when  $\tilde{v} + k \leq V_B$ , some customers would pay  $p^* = \tilde{v}$  to buy the opaque product, some would pay  $p^* + k$  to eliminate B and hence receive A (i.e.,  $x_A > 0$ ), and some would pay  $p^* + k$  to eliminate A and receive B (i.e.,  $x_B > 0$ ). However, in the second case when  $V_B < p^* + k \leq V_A$ , from Figure 8 we can see that some customers would pay  $p^* = \tilde{v}$  to buy the opaque product, some would pay  $p^* + k$  to eliminate B and receive A (i.e.,  $x_A > 0$ ), but no one would pay  $p^* + k$  to eliminate A (i.e.,  $x_B = 0$ ). By considering these cases we have:

**Proposition 7** *Suppose  $V_A \geq V_B$  without loss of generality. Customers can eliminate one product option by paying an options fee  $k$ . Then the optimal price for the opaque product  $p^* = \tilde{v}$ . The optimal product-independent options fee  $k^*$  and the optimal additional revenue  $\pi^{PI}$  satisfy:*

1. When  $V_A - V_B \in [0, \frac{t}{3}]$ ,

$$(\pi^{PI}, k^*) = \begin{cases} \left( -\frac{(t\phi - \frac{V_A - V_B + t}{4})^2}{t} + \frac{(V_A - V_B + t)^2}{16t}, \frac{V_A - V_B + t}{2} - t\phi \right) & \text{if } \phi \leq \frac{V_A - V_B}{t} \\ \left( -\frac{(2V_A - 2V_B + t - 4t\phi)^2}{8t} + \frac{t}{8}, \frac{V_A - V_B + t}{2} - t\phi \right) & \text{if } \frac{V_A - V_B}{t} \leq \phi \leq \frac{V_A - V_B}{2t} + \frac{1}{4} \\ \left( \frac{t}{8}, \frac{t}{4} \right) & \text{if } \frac{V_A - V_B}{2t} + \frac{1}{4} \leq \phi \leq \frac{V_A - V_B}{2t} + \frac{3}{4} \\ \left( -\frac{(4t\phi - 2V_A + 2V_B - 3t)^2}{8t} + \frac{t}{8}, t\phi - \frac{V_A - V_B + t}{2} \right) & \text{if } \phi \geq \frac{V_A - V_B}{2t} + \frac{3}{4} \end{cases}$$

2. When  $V_A - V_B \in [(\frac{t}{3}, (\sqrt{2} - 1)t]$ ,

$$(\pi^{PI}, k^*) = \begin{cases} \left( -\frac{(t\phi - \frac{V_A - V_B + t}{4})^2}{t} + \frac{(V_A - V_B + t)^2}{16t}, \frac{V_A - V_B + t}{2} - t\phi \right) & \text{if } \phi \leq \frac{V_A - V_B}{4t} + \frac{1}{4} \\ \left( \frac{(V_A - V_B + t)^2}{16t}, \frac{V_A - V_B + t}{4} \right) & \text{if } \frac{V_A - V_B}{4t} + \frac{1}{4} \leq \phi \leq \frac{V_A - V_B}{2t} + \frac{1}{4} \\ \left( \frac{t}{8}, \frac{t}{4} \right) & \text{if } \frac{V_A - V_B}{2t} + \frac{1}{4} \leq \phi \leq \frac{V_A - V_B}{2t} + \frac{3}{4} \\ \left( -\frac{(4t\phi - 2V_A + 2V_B - 3t)^2}{8t} + \frac{t}{8}, t\phi - \frac{V_A - V_B + t}{2} \right) & \text{if } \frac{V_A - V_B}{2t} + \frac{3}{4} \leq \phi \leq \alpha \\ \left( \frac{(V_A - V_B + t)^2}{16t}, \frac{V_A - V_B + t}{4} \right) & \text{if } \phi \geq \alpha \end{cases}$$

$$\text{where } \alpha = \frac{4(V_A - V_B) + 6t + \sqrt{-2(V_A - V_B)^2 - 4t(V_A - V_B) + 2t^2}}{8t}.$$

3. When  $V_A - V_B \in [(\sqrt{2} - 1)t, t]$ ,

$$(\pi^{PI}, k^*) = \begin{cases} \left( -\frac{(t\phi - \frac{V_A - V_B + t}{4})^2}{t} + \frac{(V_A - V_B + t)^2}{16t}, \frac{V_A - V_B + t}{2} - t\phi \right) & \text{if } \phi \leq \frac{V_A - V_B}{4t} + \frac{1}{4} \\ \left( \frac{(V_A - V_B + t)^2}{16t}, \frac{V_A - V_B + t}{4} \right) & \text{if } \phi \geq \frac{V_A - V_B}{4t} + \frac{1}{4} \end{cases}$$

In all cases, the optimal options fee  $k^* > 0$  and the optimal additional revenue  $\pi^{PI} > 0$ .

When does the firm benefit the most from charging an options fee in this case? According to the above proposition, when  $V_A - V_B \in [(\sqrt{2} - 1)t, t]$ , the firm's additional profit is the highest when the inventory of product A,  $\phi$ , is high enough and the highest additional profit is  $\frac{(V_A - V_B + t)^2}{16t}$ . When  $V_A - V_B \in [0, (\sqrt{2} - 1)t]$ , the firm benefits the most when  $\frac{V_A - V_B}{4t} + \frac{1}{4} \leq \phi \leq \frac{V_A - V_B}{2t} + \frac{1}{4}$ ; that is, when the inventories of products A and B are balanced, and the maximum additional profit is  $t/8$ , which is greater than  $\frac{(V_A - V_B + t)^2}{16t}$ . When  $V_A - V_B \in [(\sqrt{2} - 1)t, t]$ , the valuations are strongly skewed and  $k^* + p^* > V_B$ . In this case, no customers pay the options fee to eliminate product A. The additional profit in this case is not as high as that in the case where there are both customers who pay the options fee to eliminate product A and customers who do the same to eliminate product B. In summary, the ideal environment for the firm to adopt product-independent options pricing is when the valuations of the two products are not too skewed and the inventories are balanced.

We can also compare the maximum additional profit in this case (i.e.,  $t/8$ ) with that in the case when the firm charges a different options fee for eliminating a different product (i.e.,  $\frac{(V_A - V_B)^2 + t^2}{8t}$ ). The flexibility in charging a different options fee for eliminating a different product allows the firm to gain an additional profit of  $\frac{(V_A - V_B)^2}{8t}$ .

Charging an options fee for customers to remove a product choice in the opaque product can increase the firm's revenue, whether the fee is product dependent or not. This is because it allows price discrimination based on the strength of preference. Although Fay and Xie (2008) have shown that price discrimination based on the strength of preference is an effective tool for

firms to increase revenue, we look at the issue from a different angle. First, there are firms that sell exclusively opaque goods. We are interested in whether inventory information, preference information, or options pricing can benefit them. Fay and Xie (2008), however, use traditional selling as a benchmark (i.e., setting a price for each component good) and show that firms can increase their revenue by adding an opaque product in their product portfolio. Second, although the benefit of options pricing in our paper and that of selling an opaque product along with component good are both driven by price discrimination, our model setup allows us to go one step further by examining when such price discrimination is most beneficial to firms.

## 7 Conclusion

In this paper, we have articulated the double-blind process (i.e., customers do not have inventory information and the firm does not know their product preferences) arising from opaque selling and discussed its negative effects resulting from customer dissatisfaction (due to wrong expectations) and customer disappointment (due to product misallocation). To reduce these negative effects, we first examine whether the firm should reveal inventory information and whether it should solicit customer preferences. We have found that the firm is either unaffected or worse off (in terms of revenue) if it discloses inventory information or solicits customer preferences. However, we have shown that customers can obtain a higher surplus when the firm discloses inventory information and/or solicits customer preferences. Therefore, while the firm cannot improve revenue in the short term, it can improve customer satisfaction and loyalty and generate positive customer reviews through inventory information disclosure and/or customer preference solicitation, which can improve its long-term success. Also, as an alternative to disclosing inventory information and/or soliciting customer preferences, we consider another selling strategy in which the firm charges a discretionary options fee that allows a customer to eliminate a certain product option of their own choosing. By examining the case when this options fee is product dependent (and product independent), we have shown that charging an options fee can enable a firm to price discriminate those customers with a strong product preference. Also, we have found this selling strategy to be most effective when the valuations of the two products are similar and when their inventory levels are relatively balanced.

In our analysis, we have assumed that the market size is the same as the total inventory. If the total inventory is smaller than the market size, the firm sells while stock lasts. In this case, some customers will be rejected, but if the inventory allocation is done at once in the end, then those who make a purchase are still located uniformly over  $[0, 1]$ . Therefore, our earlier analysis

applies after rescaling the market size. If the total inventory is greater than the market size, the firm should release exactly enough units to cover the entire market, and our earlier analysis also applies. An interesting question is how many units of product A and product B the firm should release. If the firm is using options pricing, then Propositions 6 and 7 provide direct answers.

Our model represents an initial attempt to examine how inventory information and customer product preference information affect customers' purchasing decisions, the firm's product allocation policy, and the firm's pricing strategy. As such, our model has several limitations. First, we have adopted the Hotelling model to capture the heterogeneity among customers. However, the Hotelling model is based on the assumption that a customer's preference changes uniformly across locations. Therefore, a natural future research direction would be to examine other types of demand models. Second, we have focused our analysis on the case when the firm sells the leftover inventory of two distinctive products as a single probabilistic good. As a future research topic, it would be interesting to examine the multi-product case in which the firm can sell multiple probabilistic goods associated with different attributes (e.g., colors and styles). Third, our model is based on the assumption that customers are rational and risk-neutral. Hence, a natural extension is to consider the analysis when they are risk-averse or when they have bounded rationality (Huang and Yu, 2014). Finally, opaque selling has been viewed as a way to price discriminate without cannibalizing the sales of regular products at regular price. However, the practice has been viewed as deceptive. Perhaps opaque selling can be branded as fun and exciting experience for customers. For example, instead of selling mini-figures as known products, Lego and Disney are selling them as probabilistic goods in blind bags to create excitement. Also, McDonald's has been selling their happy meals with probabilistic toys (e.g., Hello Kitty and Snoopy) with great success over the years. Hence, it would be of interest to identify conditions under which opaque selling can generate more demand than traditional selling. Overall, opaque selling creates many research avenues for researchers to explore.

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## Appendix: Proofs.

### 7.1 Proof of Proposition 1

The proof is immediate and is therefore omitted.

### 7.2 Proof of Proposition 2

The expression for the optimal price can be shown by comparing the interior solution and the lower bound. The expression for the optimal profit follows by substituting the optimal price into the objective function. The conclusion  $p^{RN} < p^{NN}$  is immediate. The profit is lower because the price is lower and the sale is not higher.

### 7.3 Proof of Corollary 1

We have already derived the expression for  $CS^{RN}$ . It remains to show that  $CS^{RN} > CS^{NN}$  when  $\phi \geq \min(1, \frac{t+V_B}{3t-V_A+V_B})$ . From (5) and (2) we have:

$$\begin{aligned} CS^{RN} - CS^{NN} &= \frac{[(t + V_A - V_B)\phi + (V_B - t)]^2}{8t(2\phi - 1)} - \frac{(V_A - V_B)(2\phi - 1)}{2} \\ &= \frac{[(t + V_A - V_B)\phi + (V_B - t)]^2 - 4t(V_A - V_B)(2\phi - 1)^2}{8t(2\phi - 1)}. \end{aligned}$$

When  $V_A < V_B$ ,  $CS^{RN} \geq CS^{NN}$  always holds because  $CS^{NN} \leq 0$  and  $CS^{RN} \geq 0$ .

When  $V_A \geq V_B$ ,  $CS^{RN} \geq CS^{NN}$  if and only if

$$(t + V_A - V_B)\phi + (V_B - t) - 2(2\phi - 1)\sqrt{(V_A - V_B)t} \geq 0.$$

Define

$$g(\phi) = (t + V_A - V_B - 4\sqrt{(V_A - V_B)t})\phi + V_B - t + 2\sqrt{(V_A - V_B)t}.$$

We have

$$g(0) = V_B - t + 2\sqrt{(V_A - V_B)t} \geq 0,$$

and

$$g(1) = V_A - 2\sqrt{(V_A - V_B)t},$$

which is also positive because

$$V_A^2 - 4(V_A - V_B)t = (V_A - 2t)^2 + 4t(V_B - t) \geq 0.$$

So  $g(\phi)$  is positive for all  $\phi \in [0, 1]$  and hence  $CS^{RN} \geq CS^{NN}$ .

## 7.4 Proof of Proposition 3

For any selling price  $p \geq \tilde{v}$ , it is evident from Figure 4 that customers located within  $[0, l_A]$  and  $[1 - l_B, 1]$  will purchase the product but customers located within  $[l_A, 1 - l_B]$  will not, where  $l_A \leq \tilde{x} \equiv \frac{t+V_A-V_B}{2t}$  and  $l_B < (1 - \tilde{x}) = \frac{t-V_A+V_B}{2t}$ . We now consider two cases: (1) when  $V_A \geq V_B$ , and (2) when  $V_A < V_B$ . We will focus on case (1). The proof for case (2) is omitted to avoid repetition. When  $V_A \geq V_B$ , we need to consider two scenarios: (a)  $l_A \leq 0.5$  so that  $\hat{\phi}_A = 1$ ; and (b)  $0.5 < l_A \leq \tilde{x}$  so that  $\hat{\phi}_A < 1$ .

Scenario (a). Consider the case when  $V_A \geq V_B$  and  $l_A \leq 0.5$ . In this case,  $\hat{\phi}_A = \frac{\min\{0.5, l_A\}}{l_A} = 1$ . By substituting  $\hat{\phi}_A = 1$  into (6), we get  $p = V(l_A) = V_A - tl_A$ . From Figure 4, we also know that  $p = V_B - tl_B$ , which implies that  $l_B = l_A - \frac{V_A-V_B}{t}$ . In this case, the total demand  $l_A + l_B = [2l_A - \frac{V_A-V_B}{t}]$  and the firm's revenue can be written as a function of  $l_A$  as  $\pi(l_A) = (V_A - tl_A)(2l_A - \frac{V_A-V_B}{t})$ . Because  $V_A \geq V_B$ ,  $l_A \leq 0.5$ , and  $V_A \geq t$  (assumption 1), it is easy to check that the firm's revenue function is increasing in  $l_A$ . Hence, we can conclude that the optimal  $l_A$  must be at least 0.5. This observation implies that it is sufficient for us to consider Scenario (b).

Scenario (b). Consider the case when  $V_A \geq V_B$  and  $0.5 \leq l_A \leq \tilde{x}$ . In this case,  $\hat{\phi}_A = \frac{\min\{0.5, l_A\}}{l_A} = \frac{1}{2l_A}$ . By substituting  $\hat{\phi}_A = \frac{1}{2l_A}$  into (6), we get  $p = (V_B - 2t + tl_A + \frac{V_A-V_B+t}{2l_A})$ . Also, from Figure 4, we also know that  $p = V_B - tl_B$ , which implies that  $l_B = 2 - l_A - \frac{V_A-V_B+t}{2l_A t}$ . In this case, the total demand  $l_A + l_B = (2 - \frac{V_A-V_B+t}{2l_A t})$ , and the firm's revenue  $\pi$  can be expressed a function of  $l_A$ , where  $\pi = p \cdot (l_A + l_B) = (V_B - 2t + tl_A + \frac{V_A-V_B+t}{2l_A})(2 - \frac{V_A-V_B+t}{2l_A t})$ . Because  $V_A \geq V_B$  and  $V_A \geq t$  (assumption 1), it is easy to check that the firm's revenue function is increasing in  $l_A$ . Hence, we can conclude that the optimal  $l_A = \tilde{x}$ . Hence, we can also conclude that, when  $V_A \geq V_B$ , it is optimal for the firm to set  $p^{NR} = \tilde{v}$  to cover the whole market. Using the same approach we can show that when  $V_A < V_B$  it is also optimal for the firm to set  $p^{NR} = \tilde{v}$  to cover the whole market. The details are omitted. We have completed our proof.

## 7.5 Proof of Corollary 2

According to Proposition 3, the market is fully covered. The number of customers who prefer A is  $l_A = \tilde{x}$  and the number of customers who prefer B is  $l_B = 1 - \tilde{x}$ , where  $\tilde{x} = \frac{V_A-V_B+t}{2t}$ . Also, knowing customer preferences, the firm will allocate the product accordingly so that the actual probability of receiving product A for customers located within  $[0, \tilde{x}]$  is equal to  $\phi_A = \frac{\min\{\phi, l_A\}}{l_A}$  and the "actual" probability of receiving product A for customers located within  $[\tilde{x}, 1]$  is equal to  $\phi_B = 1 - \frac{\min\{1-\phi, l_B\}}{l_B}$ . Combining these observations with the actual valuation  $v_j(x)$  given

in (1), the customer surplus satisfies:

$$CS^{NR} = \int_0^{\tilde{x}} [(V_A - tx - \tilde{v})\phi_A + (V_B - t(1-x) - \tilde{v})(1 - \phi_A)]dx \\ + \int_{\tilde{x}}^1 [(V_A - tx - \tilde{v})\phi_B + (V_B - t(1-x) - \tilde{v})(1 - \phi_B)]dx$$

Consider the following cases:

Case 1:  $V_A \geq V_B + (2\phi - 1)t$ . We have  $\phi_A = \frac{2\phi t}{t + V_A - V_B}$ ,  $\phi_B = 0$ , and  $CS^{NR} = \frac{t\phi - (1-\phi)(V_A - V_B)}{2}$ .

Case 2:  $V_B \leq V_A \leq V_B + (2\phi - 1)t$ . We have  $\phi_A = 1$ ,  $\phi_B = \frac{V_B - V_A + (2\phi - 1)t}{t - V_A + V_B}$ , and  $CS^{NR} = \frac{\phi(V_A - V_B) + (1-\phi)t}{2}$ .

Case 3:  $V_A \leq V_B$ . We have  $\phi_A = 1$ ,  $\phi_B = \frac{V_B - V_A + (2\phi - 1)t}{t - V_A + V_B}$ , and  $CS^{NR} = \frac{\phi(V_A - V_B) + (1-\phi)t}{2}$ .

This proves the first statement.

It remains for us to compare the customer surplus between  $CS^{NR}$  given above and  $CS^{NN}$  given in (2). For Case 1,  $CS^{NR} - CS^{NN} = \frac{\phi[t - (V_A - V_B)]}{2} \geq 0$ . For Cases 2 and 3,  $CS^{NR} - CS^{NN} = \frac{(1-\phi)[V_A - V_B + t]}{2} \geq 0$ .

## 7.6 Proof of Proposition 4

The firm chooses  $l_A$ ,  $l_B$ , and price to maximize its profit subject to  $l_A \leq \frac{t + V_A - V_B}{2t}$  and  $l_B \leq \frac{t - V_A + V_B}{2t}$ .

When  $\phi \leq \frac{t + V_A - V_B}{2t}$ ,  $1 - \phi \geq \frac{t - V_A + V_B}{2t}$  holds and  $\phi_B = 0$ ; when  $\frac{t + V_A - V_B}{2t} \leq \phi$ ,  $\phi_A = 1$  holds. Depending on the sign of  $\phi_B - 1/2$ , the expected valuation  $V(x)$  is either (I) or (II) in Figure 5. We now consider the following cases.

Case 1:  $\phi \leq \frac{t + V_A - V_B}{2t}$ , which is equivalent to  $V_A \geq V_B + (2\phi - 1)t$ . The profit function has a different expression depending on the value of  $l_A$ .

Case 1.1: When  $l_A \leq \phi$ ,  $\phi_A = 1$ ,  $\phi_B = 0$ . The customers located at  $l_A$  and  $1 - l_B$  are indifferent between buying and not buying. Therefore,  $p = V_A - tl_A$  and  $l_A = \frac{V_A - V_B}{t} + l_B$ . The profit function is

$$\pi^{RR} = p(l_A + l_B) = (V_A - tl_A)[2l_A - \frac{V_A - V_B}{t}]$$

The firm maximizes the above profit subject to the constraint that  $l_A \leq \phi$ . It is easy to show that  $\pi^{RR}$  is increasing in  $l_A$  for  $l_A \leq \phi \leq \frac{t + V_A - V_B}{2t}$  and hence the optimal  $l_A = \phi$

Case 1.2: When  $\phi \leq l_A < \frac{t + V_A - V_B}{2t}$ ,  $\phi_B = 0$ ,  $\phi_A = \phi/l_A$ , and  $p = V_B - tl_B$ . Together with  $V_A - (1 - \phi_A)(V_A - V_B) - (1 - \phi_A)t - (2\phi_A - 1)tl_A - p = 0$ , we can express  $l_B$  as a function of  $l_A$ :

$$l_B = 1 + 2\phi - l_A - \frac{V_A - V_B + t}{l_A t} \phi.$$

The profit function as a function of  $l_A$  is then written as:

$$\pi^{RR} = [V_B - (1 + 2\phi)t + tl_A + \frac{V_A - V_B + t}{l_A}\phi](1 + 2\phi - \frac{V_A - V_B + t}{l_A t}\phi)$$

Consider the first and second derivatives of the profit function:

$$\frac{d\pi^{RR}}{dl_A} = (t - \frac{V_A - V_B + t}{l_A^2}\phi)(1 + 2\phi - \frac{V_A - V_B + t}{l_A t}\phi) + [V_B - (1 + 2\phi)t + tl_A + \frac{V_A - V_B + t}{l_A}\phi]\frac{V_A - V_B + t}{tl_A^2}\phi,$$

and

$$\frac{d^2\pi^{RR}}{dl_A^2} = \frac{2\phi(V_A - V_B + t)}{l_A^4}[l_A(2 + 4\phi - \frac{V_B}{t}) - \frac{3\phi}{t}(V_A - V_B + t)].$$

If  $2 + 4\phi - \frac{V_B}{t} \leq 0$ , then the second derivative is negative. Otherwise,

$$\begin{aligned} \frac{d^2\pi^{NR}}{dl_A^2} &= \frac{2\phi(V_A - V_B + t)}{l_A^4}[l_A(2 + 4\phi - \frac{V_B}{t}) - \frac{3\phi}{t}(V_A - V_B + t)] \\ &\leq \frac{2\phi(V_A - V_B + t)}{l_A^4}[\frac{V_A - V_B + t}{2t}(2 + 4\phi - \frac{V_B}{t}) - \frac{3\phi}{t}(V_A - V_B + t)] \\ &= \frac{\phi(V_A - V_B + t)^2}{tl_A^4}(2 - 2\phi - \frac{V_B}{t}), \end{aligned}$$

which is also negative because  $V_B \geq t$  and  $\phi \geq 1/2$ . So the profit function is concave for  $l_A \in (\phi, \frac{t+V_A-V_B}{2t}]$ . Consider the first order derivative evaluated at  $l_A = \frac{t+V_A-V_B}{2t}$ :

$$\begin{aligned} \frac{d\pi^{NR}}{dl_A} &= \frac{t}{t + V_A - V_B}[(2\phi + 1)V_A + (2\phi - 1)V_B + t - 6t\phi] \\ &\geq \frac{2\phi t}{t + V_A - V_B}[(2(V_B - t) + (2\phi - 1)t)] \geq 0 \end{aligned}$$

where the first inequality holds because  $V_A \geq V_B + (2\phi - 1)t$  and the second inequality holds because  $V_B \geq t$  and  $\phi \geq 1/2$ . Hence the optimal solution is  $l_A = \frac{t+V_A-V_B}{2t}$  and  $l_B = 1 - l_A$ . In other words, it is optimal to fully cover the market,  $p^{RR} = p^{NN}$ , and hence  $\pi^{RR} = \pi^{NN}$ .

Case 2:  $\frac{t+V_A-V_B}{2t} \leq \phi \leq \frac{3t+V_A-V_B}{4t}$ . In this case,  $1 - \phi \leq \frac{t-V_A+V_B}{2t} \leq 2(1 - \phi)$  holds and  $\phi_A = 1$ .

Case 2.1: When  $l_B \leq 1 - \phi$ ,  $\phi_B = 0$ . Together with  $\phi_A = 1$ , there is no opaqueness in the product and  $l_A = \frac{V_A-V_B}{t} + l_B$ . The profit function is

$$\pi^{RR} = (V_B - tl_B)[\frac{V_A - V_B}{t} + 2l_B]$$

with the constraint that  $l_B \leq 1 - \phi$ . The unconstrained optimizer is  $l_B = \frac{3V_B-V_A}{4t} \geq \frac{t-(V_A-V_B)}{2t} \geq 1 - \phi$  because  $\min\{V_A, V_B\} \geq t$ . So the optimal  $l_B = 1 - \phi$ .

Case 2.2: When  $1 - \phi \leq l_B \leq \frac{t - V_A + V_B}{2t}$ ,  $\phi_B = 1 - \frac{1 - \phi}{l_B}$  and  $\phi_B \leq 1/2$  because  $l_B \leq 2(1 - \phi)$ . We have  $p = V_A - tl_A$ . Together with  $V_A - (1 - \phi_B)(V_A - V_B) - (1 - \phi_B)t - (2\phi_B - 1)t(1 - l_B) - p = 0$ , we can express  $l_B$  as a function of  $l_A$ :

$$l_B = \frac{(1 - \phi_B)(V_A - V_B) + \phi_B t - tl_A}{(2\phi_B - 1)t}.$$

Together with the fact that  $\phi_B = 1 - \frac{1 - \phi}{l_B}$ , the function  $l_A$  is further simplified as:

$$l_A = 3 - 2\phi - l_B - \frac{(1 - \phi)(t - V_A + V_B)}{tl_B}.$$

The profit function is written as:

$$\pi^{RR} = [V_A - (3 - 2\phi)t + tl_B + \frac{(1 - \phi)(t - V_A + V_B)}{l_B}][3 - 2\phi - \frac{(1 - \phi)(t - V_A + V_B)}{tl_B}].$$

Consider the derivatives of the profit function:

$$\begin{aligned} \frac{d\pi^{RR}}{dl_B} &= [t - \frac{(1 - \phi)(t - V_A + V_B)}{l_B^2}][3 - 2\phi - \frac{(1 - \phi)(t - V_A + V_B)}{tl_B}] \\ &\quad + [V_A - (3 - 2\phi)t + tl_B + \frac{(1 - \phi)(t - V_A + V_B)}{l_B}]\frac{(1 - \phi)(t - V_A + V_B)}{tl_B^2}, \end{aligned}$$

and

$$\frac{d^2\pi^{RR}}{dl_B^2} = \frac{2(1 - \phi)(t - V_A + V_B)}{l_B^4} [l_B(6 - 4\phi - \frac{V_A}{t}) - \frac{3}{t}(1 - \phi)(t - V_A + V_B)].$$

If  $6 - 4\phi - V_A/t \leq 0$ , then the second derivative is negative. Otherwise,

$$\begin{aligned} \frac{d^2\pi^{RR}}{dl_B^2} &= \frac{2(1 - \phi)(t - V_A + V_B)}{l_B^4} [l_B(6 - 4\phi - \frac{V_A}{t}) - \frac{3}{t}(1 - \phi)(t - V_A + V_B)] \\ &\leq \frac{2(1 - \phi)(t - V_A + V_B)}{l_B^4} [\frac{t - V_A + V_B}{2t}(6 - 4\phi - \frac{V_A}{t}) - \frac{3}{t}(1 - \phi)(t - V_A + V_B)] \\ &= \frac{(1 - \phi)(t - V_A + V_B)^2}{2tl_B^4} (2\phi - \frac{V_A}{t}). \end{aligned}$$

The condition  $6 - 4\phi - V_A/t \leq 0$  means that  $2\phi - V_A/t \leq -6(1 - \phi) \leq 0$ , so the profit function is concave.

Consider the first-order derivative evaluated at  $l_B = \frac{t - V_A + V_B}{2t}$ :

$$\begin{aligned} \frac{d\pi^{RR}}{dl_B} &= \frac{t}{t - V_A + V_B} [(1 - 2\phi)V_A + (3 - 2\phi)V_B + (6\phi - 5)t] \\ &\geq \frac{t}{t - V_A + V_B} \{(1 - 2\phi)[V_B + (2\phi - 1)t] + (3 - 2\phi)V_B + (6\phi - 5)t\} \\ &= \frac{2(1 - \phi)t}{t - V_A + V_B} [2(V_B - t) + (2\phi - 1)t] \geq 0, \end{aligned}$$

where the first inequality holds because  $\phi \geq 1/2$  and  $\phi \geq \frac{t+V_A-V_B}{2t}$  and the second inequality holds because  $V_B \geq t$  and  $\phi \geq 1/2$ . Therefore the optimal solution is  $l_B = \frac{t-V_A+V_B}{2t}$  and  $l_A = 1 - l_B$ . In other words, it is optimal to fully cover the market,  $p^{RR} = p^{NN}$ , and hence  $\pi^{RR} = \pi^{NN}$ .

Case 3:  $\phi \geq \frac{3t+V_A-V_B}{4t}$ . In this case,  $1 - \phi \leq 2(1 - \phi) \leq \frac{t-V_A+V_B}{2t}$  holds and  $\phi_A = 1$ .

Case 3.1:  $l_B \leq 1 - \phi$ . This case is the same as Case 2.1. The profit is increasing and hence the highest profit is achieved at  $l_B = 1 - \phi$ .

Case 3.2: When  $1 - \phi \leq l_B \leq 2(1 - \phi)$ ,  $\phi_B = 1 - \frac{1-\phi}{l_B}$  and  $\phi_B \leq 1/2$  because  $l_B \leq 2(1 - \phi)$ . This case is the same as Case 2.2 except that  $l_B \leq 2(1 - \phi)$  instead of  $l_B \leq \frac{t-V_A+V_B}{2t}$ . From Case 2.2, we know that  $\frac{d\pi^{RR}}{dl_B} \geq 0$  at  $l_B = \frac{t-V_A+V_B}{2t}$ . Because  $l_B \leq 2(1 - \phi) \leq \frac{t-V_A+V_B}{2t}$ , we know that the optimal profit in this case is obtained at  $l_B = 2(1 - \phi)$ .

Case 3.3: When  $2(1 - \phi) \leq l_B \leq \frac{t-V_A+V_B}{2t}$ ,  $\phi_A = 1$ ,  $\phi_B = 1 - \frac{1-\phi}{l_B}$  and  $\phi_B \geq 1/2$  because  $l_B \geq 2(1 - \phi)$ . In this case,  $l_A = \frac{t+V_A-V_B}{2t}$  and  $p^{RR} = \phi_B V_A + (1 - \phi_B) V_B - (1 - \phi_B)t - (2\phi_B - 1)t(l_A + l_B) = \frac{V_A+V_B-t}{2} + 2(1 - \phi)t - tl_B$ . The firm maximizes the profit:

$$\pi^{RR} = \left[ \frac{V_A + V_B - t}{2} + 2(1 - \phi)t - tl_B \right] \left( l_B + \frac{t + V_A - V_B}{2t} \right)$$

subject to constraint  $2(1 - \phi) \leq l_B \leq \frac{t-V_A+V_B}{2t}$ . The unconstrained optimizer is  $\frac{V_B-(2\phi-1)t}{2t}$ . When  $V_A + V_B \geq 3t$  and  $\frac{3t+V_A-V_B}{4t} \leq \phi \leq \min(1, \frac{V_A}{2t})$ , the optimal  $l_B = \frac{t-V_A+V_B}{2t}$  and it is optimal to fully cover the market. When  $V_A + V_B \geq 3t$  and  $\min(1, \frac{V_A}{2t}) \leq \phi \leq 1$ , the optimal  $l_B = \frac{V_B-(2\phi-1)t}{2t}$ . When  $V_A + V_B \leq 3t$  and  $\frac{3t+V_A-V_B}{4t} \leq \phi \leq \frac{3t-V_B}{2t}$ , the optimal  $l_B = 2(1 - \phi)$ . When  $V_A + V_B \leq 3t$  and  $\frac{3t-V_B}{2t} \leq \phi \leq 1$ , the optimal  $l_B = \frac{V_B-(2\phi-1)t}{2t}$ . In Case 3,  $p^{RR}$  is always smaller than  $p^{NN}$  and  $\pi^{RR} < \pi^{NN}$ .

In terms of coverage, partial coverage is optimal if and only if  $V_A+V_B \geq 3t$  and  $\min(1, \frac{V_A}{2t}) \leq \phi \leq 1$  or when  $V_A + V_B \leq 3t$  and  $\phi \geq \frac{3t+V_A-V_B}{4t}$ . When  $V_A + V_B \geq 3t$  and  $\min(1, \frac{V_A}{2t}) \leq \phi \leq 1$  or  $V_A + V_B \leq 3t$  and  $\frac{3t-V_B}{2t} \leq \phi \leq 1$ ,  $\pi^{RR} = \frac{[V_A+2t(1-\phi)]^2}{4t}$  and when  $V_A + V_B \leq 3t$  and  $\frac{3t+V_A-V_B}{4t} \leq \phi \leq \frac{3t-V_B}{2t}$ ,  $\pi^{RR} = \frac{(V_A+V_B-t)[V_A-V_B+(5-4\phi)t]}{4t}$ .

In summary, when  $\phi \in [1/2, \frac{3t+V_A-V_B}{4t}]$ ,  $\pi^{RR} = \pi^{NN}$ . Otherwise,  $\pi^{RR} < \pi^{NN}$ . Partial coverage is optimal when  $V_A + V_B \geq 3t$  and  $\min(1, \frac{V_A}{2t}) \leq \phi \leq 1$  or when  $V_A + V_B \leq 3t$  and  $\phi \geq \frac{3t+V_A-V_B}{4t}$ .

## 7.7 Proof of Corollary 3

Case 1.  $\frac{1}{2} \leq \phi \leq \frac{3t+V_A-V_B}{4t}$ . From Proposition 4, we know that  $p^{RR} = p^{NR}$  and the market is fully covered in both Settings RR and NR. Further, the probability of getting A or B is the same under the two settings. Therefore,  $CS^{RR} = CS^{NR}$ .

Case 2. Under Condition  $S_1$ ,  $CS^{RR}$  is

$$CS^{RR} = \int_0^{\tilde{x}} [(V_A - tx - p^{RR})\phi_A + (V_B - t(1-x) - p^{RR})(1 - \phi_A)]dx \\ + \int_{\tilde{x}}^1 [(V_A - tx - p^{RR})\phi_B + (V_B - t(1-x) - p^{RR})(1 - \phi_B)]dx$$

with  $\phi_A = 1$ ,  $\phi_B = \frac{V_B - V_A + (2\phi - 1)t}{t - V_A + V_B}$  and  $p^{RR} = V_A + (1 - 2\phi)t$ . After simplification, we have  $CS^{RR} = \frac{(3\phi - 2)t - (1 - \phi)(V_A - V_B)}{2}$ .

Case 3.  $\frac{3t + V_A - V_B}{4t} \leq \phi \leq \frac{3t - V_B}{2t}$  and  $V_A + V_B \leq 3t$ . The customer surplus in this case is:

$$CS^{RR} = \int_0^{\tilde{x}} (V_A - tx - p^{RR})dx + \int_{\tilde{x}}^{\tilde{x} + 2(1 - \phi)} [(V_A - tx - p^{RR})\phi_B \\ + (V_B - t(1-x) - p^{RR})(1 - \phi_B)]dx$$

with  $p^{RR} = \frac{V_A + V_B - t}{2}$  and  $\phi_B = \frac{1}{2}$ . Or,  $CS^{RR} = \frac{(t + V_A - V_B)^2}{8t}$ .

Case 4. Under Condition  $S_2$ , the customer surplus in this case is:

$$CS^{RR} = \int_0^{\tilde{x}} (V_A - tx - p^{RR})dx + \int_{\tilde{x}}^{\tilde{x} + \frac{V_B - (2\phi - 1)t}{2t}} [(V_A - tx - p^{RR})\phi_B \\ + (V_B - t(1-x) - p^{RR})(1 - \phi_B)]dx$$

with  $p^{RR} = \frac{V_A}{2} + (1 - \phi)t$  and  $\phi_B = \frac{V_B - t}{V_B - (2\phi - 1)t}$ . After simplification, we have

$$CS^{RR} = \frac{V_A^2 - 4(1 - \phi)(V_A - V_B)t - 4(1 - \phi)(2 - \phi)t^2}{8t}.$$

Because  $CS^{NR} \geq CS^{NN}$  (Corollary 2), to show  $CS^{RR} \geq CS^{NN}$ , it suffices to show  $CS^{RR} \geq CS^{NR}$ . Consider the following three scenarios.

Scenario (a). When  $\frac{1}{2} \leq \phi \leq \frac{3t + V_A - V_B}{4t}$  or Condition  $S_1$  is satisfied, the market is fully covered under both Settings RR and NR. We have  $CS^{RR} \geq CS^{NR}$  because  $p^{RR} \leq p^{NR} = \tilde{v}$ .

Scenario (b). Consider the case when  $\frac{3t + V_A - V_B}{4t} \leq \phi \leq \frac{3t - V_B}{2t}$  and  $V_A + V_B \leq 3t$ . In this case, we have  $\phi \geq \frac{V_A - V_B + t}{2t}$ . Then  $CS^{RR} - CS^{NR} = \frac{(t - V_A + V_B)[4t\phi - (3t + V_A - V_B)]}{8t} \geq 0$  because  $\phi \geq \frac{3t + V_A - V_B}{4t}$  and  $t - V_A + V_B \geq 0$ .

Scenario (c). Consider the case when Condition  $S_2$  is satisfied. We also have  $\phi \geq \frac{V_A - V_B + t}{2t}$  in this case. Then  $CS^{RR} - CS^{NR} = \frac{f_1(\phi)}{8t}$  with  $f_1(\phi) = -4t^2\phi^2 + 16t^2\phi + (V_A^2 - 4tV_A + 4V_Bt - 12t^2)$ . It is easy to check that  $f_1(\phi)$  is decreasing when  $\phi \geq 1/2$  and  $f_1(1) = (V_A - 2t)^2 + 4t(V_B - t) \geq 0$ . Therefore  $CS^{RR} \geq CS^{NR}$ .

## 7.8 Proof of Proposition 5

Part 1. We have  $\frac{V_A - V_B + t}{2t} - \frac{t + V_B}{3t - (V_A - V_B)} = -\frac{f_2(V_A)}{2t(3t - V_A + V_B)}$  with the function  $f_2(V_A) = V_A^2 - 2(t + V_B)V_A + (4tV_B - t^2 + V_B^2)$  and the function  $f_2(V_A) \geq 0$  because  $V_A \in [V_B - t, V_B + t]$ . Therefore  $\frac{V_A - V_B + t}{2t} \leq \min(1, \frac{t + V_B}{3t - (V_A - V_B)})$  holds because  $\frac{V_A - V_B + t}{2t} \leq 1$  and  $\frac{V_A - V_B + t}{2t} \leq \frac{t + V_B}{3t - (V_A - V_B)}$ . We now compare  $CS^{RN}$  and  $CS^{NR}$  in the following cases:

Case 1:  $\phi \leq \frac{V_A - V_B + t}{2t}$ . We have  $CS^{RN} - CS^{NR} = \frac{-(1-\phi)(t - V_A + V_B)}{2} \leq 0$ .

Case 2:  $\frac{V_A - V_B + t}{2t} \leq \phi \leq \min(1, \frac{t + V_B}{3t - (V_A - V_B)})$ . We have  $CS^{RN} - CS^{NR} = \frac{(3t - V_A + V_B)\phi - 2t}{2}$ . It is easy to check that  $\frac{V_A - V_B + t}{2t} \leq \frac{2t}{3t - (V_A - V_B)} \leq \min(1, \frac{t + V_B}{3t - (V_A - V_B)})$  holds. Therefore  $CS^{RN} \leq CS^{NR}$  when  $\phi \leq \frac{2t}{3t - (V_A - V_B)}$  and  $CS^{RN} \geq CS^{NR}$  when  $\phi \geq \frac{2t}{3t - (V_A - V_B)}$ .

Case 3:  $\phi \geq \min(1, \frac{t + V_B}{3t - (V_A - V_B)})$ . We have  $CS^{RN} - CS^{NR} = \frac{1}{8t(2\phi - 1)}f_3(V_A)$  with

$$f_3(V_A) = \phi^2 V_A^2 - 2\phi[(V_B + 3t)\phi - (V_B + t)]V_A + (1 - \phi)^2 V_B^2 + 2t(3\phi^2 - 1)V_B - (1 - \phi)(9\phi - 5)t^2.$$

It is easy to check that  $\Delta = 16t\phi^2(2\phi - 1)(t - V_B) \leq 0$  because  $\phi \geq \min(1, \frac{t + V_B}{3t - (V_A - V_B)}) \geq \frac{1}{2}$  and  $V_B \geq t$ . Therefore  $f_3(V_A) \geq 0$  and  $CS^{RN} \geq CS^{NR}$  holds.

Part 2. The conclusion that  $CS^{RR} \geq CS^{NR}$  has already been proved when proving Corollary 3.

Part 3. Consider the following scenarios.

Scenario (a):  $\frac{1}{2} \leq \phi \leq \frac{3t + V_A - V_B}{4t}$ . Because  $CS^{RR} = CS^{NR}$  in this case, the comparison of  $CS^{RR}$  with  $CS^{RN}$  is the same as part 1 of this proposition. We have  $CS^{RR} \geq CS^{RN}$  when  $0.5 \leq \phi < \frac{2t}{3t - (V_A - V_B)}$  and  $CS^{RR} \leq CS^{RN}$  when  $\frac{2t}{3t - (V_A - V_B)} \leq \phi \leq \frac{3t + V_A - V_B}{4t}$ .

Scenario (b): Condition  $S_1$  is satisfied. In this case, we have  $\phi \leq \min(1, \frac{t + V_B}{3t - V_A + V_B})$ . Then  $CS^{RR} - CS^{RN} = -\frac{(1-\phi)(t + V_A - V_B)}{2} \leq 0$ .

Scenario (c):  $\frac{3t + V_A - V_B}{4t} \leq \phi \leq \frac{3t - V_B}{2t}$  and  $V_A + V_B \leq 3t$ . It is easy to check that  $\frac{t + V_B}{3t - (V_A - V_B)} \leq \frac{3t + V_A - V_B}{4t} \leq \frac{3t - V_B}{2t}$  when  $V_B \leq V_A - 2t + \sqrt{t(9t - 4V_A)}$ ,  $\frac{3t + V_A - V_B}{4t} \leq \frac{t + V_B}{3t - (V_A - V_B)} \leq \frac{3t - V_B}{2t}$  when  $V_A - 2t + \sqrt{t(9t - 4V_A)} \leq V_B \leq \frac{V_A}{2} - t + \frac{\sqrt{32t^2 - 16tV_A + V_A^2}}{2}$  and  $\frac{3t + V_A - V_B}{4t} \leq \frac{3t - V_B}{2t} \leq \frac{t + V_B}{3t - (V_A - V_B)}$  when  $\frac{V_A}{2} - t + \frac{\sqrt{32t^2 - 16tV_A + V_A^2}}{2} \leq V_B \leq 3t - V_A$ . Below we will discuss these cases separately.

Case c.1:  $V_B \leq V_A - 2t + \sqrt{t(9t - 4V_A)}$ . We have  $CS^{RR} - CS^{RN} = \frac{(t + V_A - V_B)^2}{8t} - \frac{[(t + V_A - V_B)\phi + (V_B - t)]^2}{8t(2\phi - 1)}$ . It is easy to show that  $CS^{RR} \leq CS^{RN}$  because  $V_B \geq t$ .

Case c.2:  $V_A - 2t + \sqrt{t(9t - 4V_A)} \leq V_B \leq \frac{V_A}{2} - t + \frac{\sqrt{32t^2 - 16tV_A + V_A^2}}{2}$ . When  $\frac{3t + V_A - V_B}{4t} \leq \phi \leq \frac{t + V_B}{3t - (V_A - V_B)}$ , we have  $CS^{RR} - CS^{RN} = \frac{f_4(\phi)}{8t}$  with  $f_4(\phi) = -8t^2\phi + V_A^2 + 2(t - V_B)V_A + 5t^2 - 2tV_B + V_B^2$ . It is easy to check that  $f_4(\frac{3t + V_A - V_B}{4t}) = (V_A - V_B - t)(V_A - V_B + t) \leq 0$  and thus  $CS^{RR} \leq CS^{RN}$  holds. When  $\frac{t + V_B}{3t - (V_A - V_B)} \leq \phi \leq \frac{3t - V_B}{2t}$ , We have  $CS^{RR} - CS^{RN} = \frac{(t + V_A - V_B)^2}{8t} - \frac{[(t + V_A - V_B)\phi + (V_B - t)]^2}{8t(2\phi - 1)} \leq 0$ , which is the same as Case 1.

Case c.3:  $\frac{V_A}{2} - t + \frac{\sqrt{32t^2 - 16tV_A + V_A^2}}{2} \leq V_B \leq 3t - V_A$ . We have  $CS^{RR} - CS^{RN} = \frac{f_4(\phi)}{8t} \leq 0$ , which is the same as Case c.2.

In all, we have shown that  $CS^{RR} \leq CS^{RN}$  in Scenario (c).

Scenario (d): Condition  $S_2$  is satisfied. We consider two cases, depending on the parameters.

(i)  $V_A + V_B \geq 3t$ ,  $V_A \leq 2t$ , and  $\phi \geq \frac{V_A}{2t}$ . In this case, there are two subcases to consider.

Case d.i.1:  $\frac{V_A}{2t} \leq \phi \leq \frac{t+V_B}{3t-V_A+V_B}$ . We have  $CS^{RR} - CS^{RN} = \frac{f_5(\phi)}{8t}$  with  $f_5(\phi) = (-4t^2)\phi^2 + (4tV_A - 4tV_B + 4t^2)\phi + (-4t^2 + 4V_Bt - 4tV_A + V_A^2)$ , which is decreasing in  $\phi$  when  $\phi \geq \frac{V_A - V_B + t}{2t}$ . Together with  $\frac{V_A}{2t} \geq \frac{V_A - V_B + t}{2t}$  and  $f_5(\frac{V_A}{2t}) = 2(V_A - 2t)(t + V_A - V_B) \leq 0$ , we know that  $CS^{RR} \leq CS^{RN}$  holds.

Case d.i.2:  $\phi \geq \frac{t+V_B}{3t-V_A+V_B}$ . We have  $CS^{RR} - CS^{RN} = \frac{(1-\phi)f_6(\phi)}{8t(2\phi-1)}$  with  $f_6(\phi) = (8t^2)\phi^2 + [V_A^2 - 2(3t + V_B)V_A + (-19t^2 + 6tV_B + V_B^2)]\phi - V_A^2 + 4tV_A + (7t^2 - 2tV_B - V_B^2)$ . It is easy to check that  $f_6(\phi)$  is decreasing in  $\phi$  when  $\phi \leq \frac{-V_A^2 + 2(3t+V_B)V_A + (19t^2 - 6tV_B - V_B^2)}{16t^2}$ . We can prove that  $\frac{-V_A^2 + 2(3t+V_B)V_A + (19t^2 - 6tV_B - V_B^2)}{16t^2} \geq 1$  when  $V_A \geq 3t - V_B$  and  $V_B \leq V_A + t \leq 3t$ . Therefore  $f_6(\phi)$  is decreasing in  $\phi$ . We can check that  $f_6(\frac{t+V_B}{3t-V_A+V_B}) = \frac{(V_A+V_B-t)f_7(V_B)}{(3t-V_A+V_B)^2}$  with  $f_7(V_B) = (2t - V_A)V_B^2 + (4t^2 - 10tV_A + 2V_A^2)V_B + (8tV_A^2 - 9t^2V_A - 14t^3 - V_A^3)$ . It is easy to check that  $f_7(V_B)$  is decreasing in  $V_B$  when  $V_B \leq \frac{-2t^2 + 5tV_A - V_A^2}{2t - V_A}$  and  $\frac{-2t^2 + 5tV_A - V_A^2}{2t - V_A} \geq V_A + t$  holds because  $t \leq V_A \leq 2t$ . Therefore  $f_7(V_B)$  is decreasing in  $V_B$  when  $V_B \leq V_A + t$ . Because  $3t - V_A \leq V_B \leq V_A + t$  and  $f_7(3t - V_A) = -4V_A(2t - V_A)(6t - V_A) - 16t^2(V_A - t) \leq 0$  when  $V_A \leq 2t$ , we know that  $f_7(V_B) \leq 0$  and  $CS^{RR} \leq CS^{RN}$ .

(ii)  $V_A + V_B \leq 3t$  and  $\phi \geq \frac{3t-V_B}{2t}$ . In this case, there are also two subcases to consider.

Case d.ii.1:  $V_B \leq \frac{V_A}{2} - t + \frac{\sqrt{32t^2 - 16tV_A + V_A^2}}{2}$ . We have  $CS^{RR} - CS^{RN} = \frac{(1-\phi)f_6(\phi)}{8t(2\phi-1)}$  with  $f_6(\phi) = (8t^2)\phi^2 + [V_A^2 - 2(3t + V_B)V_A + (-19t^2 + 6tV_B + V_B^2)]\phi - V_A^2 + 4tV_A + (7t^2 - 2tV_B - V_B^2)$ . It is easy to check that  $f_6(\phi)$  is decreasing in  $\phi$  when  $\phi \leq \frac{-V_A^2 + 2(3t+V_B)V_A + (19t^2 - 6tV_B - V_B^2)}{16t^2}$  and increasing in  $\phi$  when  $\phi \geq \frac{-V_A^2 + 2(3t+V_B)V_A + (19t^2 - 6tV_B - V_B^2)}{16t^2}$ . We have  $\frac{-V_A^2 + 2(3t+V_B)V_A + (19t^2 - 6tV_B - V_B^2)}{16t^2} - \frac{3t-V_B}{2t} \geq 0$  because  $V_B \leq V_A + t$ . We observe that  $f_6(\frac{3t-V_B}{2t}) = \frac{f_8(V_A)}{2t}$  with  $f_8(V_A) = -(V_B - t)V_A^2 + 2(V_B^2 - 5t^2)V_A + (V_B - t)(-2tV_B - V_B^2 + 7t^2)$ . It is easy to check that  $f_8(V_A) \leq 0$  because  $f_8(V_B - t) = 4t^2(t - V_B) \leq 0$  and  $f_8(V_A)$  is decreasing in  $V_A$  when  $V_A \geq V_B - t$  and  $V_B \leq 2t$ . Thus  $f_6(\frac{3t-V_B}{2t}) \leq 0$  holds. We also have  $f_6(1) = (4t - 2V_A)V_B - (4t^2 + 2V_At) \leq -2t(3V_A - 2t) \leq 0$  because  $V_B \leq 2t$  and  $V_A \leq 2t$ . With  $f_6(\frac{3t-V_B}{2t}) \leq 0$ ,  $f_6(1) \leq 0$  and  $\frac{-V_A^2 + 2(3t+V_B)V_A + (19t^2 - 6tV_B - V_B^2)}{16t^2} \geq \frac{3t-V_B}{2t}$ , we know that  $CS^{RR} \leq CS^{RN}$  holds.

Case d.ii.2:  $\frac{V_A}{2} - t + \frac{\sqrt{32t^2 - 16tV_A + V_A^2}}{2} \leq V_B \leq 3t - V_A$ . When  $\frac{3t-V_B}{2t} \leq \phi \leq \frac{t+V_B}{3t-(V_A-V_B)}$ , we have  $CS^{RR} - CS^{RN} = \frac{f_5(\phi)}{8t}$ . We know that  $f_5(\phi)$  is decreasing in  $\phi$  when  $\phi \geq \frac{V_A - V_B + t}{2t}$ . We also have  $\frac{3t-V_B}{2t} \geq \frac{V_A - V_B + t}{2t}$  and  $f_5(\frac{3t-V_B}{2t}) = V_B^2 + (2t - 2V_A)V_B + (2tV_A - 7t^2 + V_A^2) \leq 0$

because  $V_B \leq 3t - V_A$ . Therefore  $CS^{RR} \leq CS^{RN}$  holds. When  $\phi \geq \frac{t+V_B}{3t-(V_A-V_B)}$ , we have  $CS^{RR} - CS^{RN} = \frac{(1-\phi)f_6(\phi)}{8t(2\phi-1)}$  and we can prove  $CS^{RR} \leq CS^{RN}$  in the same way as we did in Case d.ii.1.

## 7.9 Proof of Proposition 6

By substituting  $k_A = V_A - tx_A - \tilde{v}$ ,  $k_B = V_B - tx_B - \tilde{v}$ , and  $\tilde{v} = \frac{V_A+V_B-t}{2}$  into (10), the profit function can be rewritten as:  $(V_A - tx_A - \tilde{v})x_A + (V_B - tx_B - \tilde{v})x_B = -t(x_A - \frac{V_A-V_B+t}{4t})^2 - t(x_B - \frac{V_B-V_A+t}{4t})^2 + \frac{(V_A-V_B)^2+t^2}{8t}$ . In this case it is obvious that the unconstrained solution is given by:  $(x_A, x_B) = (\frac{V_A-V_B+t}{4t}, \frac{V_B-V_A+t}{4t})$ . By considering the upper bounds on  $x_A$  and  $x_B$ , it is easy to show that the optimal solution is

$$(x_A^*, x_B^*) = \begin{cases} (\phi, \frac{V_B-V_A+t}{4t}) & \text{if } \phi \leq \frac{V_A-V_B+t}{4t}; \\ (\frac{V_A-V_B+t}{4t}, \frac{V_B-V_A+t}{4t}) & \text{if } \phi \in (\frac{V_A-V_B+t}{4t}, \frac{V_A-V_B+3t}{4t}]; \\ (\frac{V_A-V_B+t}{4t}, 1 - \phi) & \text{if } \phi > \frac{V_A-V_B+3t}{4t}. \end{cases}$$

Through substitution, we can retrieve the corresponding optimal options price  $(k_A^*, k_B^*)$  and the optimal additional profit as stated. Because of assumptions 1, 2, and 3 as stated in Section 3, it is easy to check that the prices and the profit are all positive.

## 7.10 Proof of Proposition 7

Consider two subproblems.

**Subproblem 1:** In the first subproblem, the sales of A, B, and the opaque product are all positive. The firm's additional profit from options pricing is

$$\pi_1^{PI} = k(x_A + x_B)$$

with  $0 \leq x_A \leq \phi$ , and  $0 \leq x_B \leq 1 - \phi$ . The customer located at  $x_A$  is indifferent between buying product A and buying the opaque product. Similarly, the customer located at  $1 - x_B$  is indifferent between buying product B and the opaque product. So,  $k + \tilde{v} = V_A - tx_A = V_B - tx_B$ , and  $\pi_1^{PI}$  is expressed as a function of  $k$ :

$$\pi_1^{PI} = -\frac{2(k - \frac{t}{4})^2}{t} + \frac{t}{8}.$$

The unconstrained optimizer is  $k^* = \frac{t}{4}$  and the constraints are equivalent to  $|t\phi - \frac{V_A-V_B+t}{2}| \leq k \leq \frac{V_B-V_A+t}{2}$ .

When  $V_A - t \leq V_B \leq V_A - \frac{t}{2}$ , the optimal solution is:

$$k = \begin{cases} \frac{V_B-V_A+t}{2} & \text{if } \phi \geq \frac{V_A-V_B}{t}; \\ N.A. & \text{o.w.} \end{cases}$$

and the optimal additional profit is

$$\pi_1^{PI} = \begin{cases} \frac{V_A - V_B}{2} - \frac{(V_A - V_B)^2}{2t} & \text{if } \phi \geq \frac{V_A - V_B}{t}; \\ N.A. & \text{o.w.} \end{cases}$$

When  $V_A - \frac{t}{2} \leq V_B \leq V_A$ , the optimal solution is:

$$k = \begin{cases} \frac{V_A - V_B + t}{2} - t\phi & \text{if } \frac{V_A - V_B}{t} \leq \phi \leq \frac{V_A - V_B}{2t} + \frac{1}{4}; \\ \frac{t}{4} & \text{if } \frac{V_A - V_B}{2t} + \frac{1}{4} \leq \phi \leq \frac{V_A - V_B}{2t} + \frac{3}{4}; \\ t\phi - \frac{V_A - V_B + t}{2} & \text{if } \phi \geq \frac{V_A - V_B}{2t} + \frac{3}{4}; \\ N.A. & \text{o.w.} \end{cases}$$

and the optimal additional profit is

$$\pi_1^{PI} = \begin{cases} -\frac{(2V_A - 2V_B + t - 4t\phi)^2}{8t} + \frac{t}{8} & \text{if } \frac{V_A - V_B}{t} \leq \phi \leq \frac{V_A - V_B}{2t} + \frac{1}{4}; \\ \frac{t}{8} & \text{if } \frac{V_A - V_B}{2t} + \frac{1}{4} \leq \phi \leq \frac{V_A - V_B}{2t} + \frac{3}{4}; \\ -\frac{(4t\phi - 2V_A + 2V_B - 3t)^2}{8t} + \frac{t}{8} & \text{if } \phi \geq \frac{V_A - V_B}{2t} + \frac{3}{4}; \\ N.A. & \text{o.w.} \end{cases}$$

**Subproblem 2:** In this subproblem, the sale of product B is zero. The firm's additional profit is

$$\pi_2^{PI} = kx_A$$

with  $0 \leq x_A \leq \phi$  and  $V_B \leq k + \tilde{v}$ . Here  $k + \tilde{v} = V_A - tx_A$ , and the firm's additional profit is expressed as a function of  $k$ :

$$\pi_2^{PI} = -\frac{(k - \frac{V_A - V_B + t}{4})^2}{t} + \frac{(V_A - V_B + t)^2}{16t}.$$

The unconstrained optimizer is  $k^* = \frac{V_A - V_B + t}{4}$  and the constraints are equivalent to  $\frac{V_A - V_B + t}{2} - t\phi \leq k \leq \frac{V_A - V_B + t}{2}$  and  $k \geq \frac{V_B - V_A + t}{2}$ .

When  $V_A - t \leq V_B \leq V_A - \frac{t}{3}$ , the optimal solution is:

$$k = \begin{cases} \frac{V_A - V_B + t}{4} & \text{if } \phi \geq \frac{V_A - V_B}{4t} + \frac{1}{4}; \\ \frac{V_A - V_B + t}{2} - t\phi & \text{if } \phi \leq \frac{V_A - V_B}{4t} + \frac{1}{4} \end{cases}$$

and the optimal additional profit is

$$\pi_2^{PI} = \begin{cases} \frac{(V_A - V_B + t)^2}{16t} & \text{if } \phi \geq \frac{V_A - V_B}{4t} + \frac{1}{4}; \\ -\frac{(t\phi - \frac{V_A - V_B + t}{4})^2}{t} + \frac{(V_A - V_B + t)^2}{16t} & \text{if } \phi \leq \frac{V_A - V_B}{4t} + \frac{1}{4} \end{cases}$$

When  $V_A - \frac{t}{3} \leq V_B \leq V_A$ , the optimal solution is:

$$k = \begin{cases} \frac{V_B - V_A + t}{2} & \text{if } \phi \geq \frac{V_A - V_B}{t}; \\ \frac{V_A - V_B + t}{2} - t\phi & \text{if } \phi \leq \frac{V_A - V_B}{t} \end{cases}$$

and the optimal additional profit is

$$\pi_2^{PI} = \begin{cases} \frac{V_A - V_B}{2} - \frac{(V_A - V_B)^2}{2t} & \text{if } \phi \geq \frac{V_A - V_B}{t}; \\ -\frac{(t\phi - \frac{V_A - V_B + t}{4})^2}{t} + \frac{(V_A - V_B + t)^2}{16t} & \text{if } \phi \leq \frac{V_A - V_B}{t} \end{cases}$$

**Comparison** We can obtain the maximal solution by comparing the two subproblems. The results are summarized as follows.

When  $V_A - V_B \in [(\sqrt{2} - 1)t, t]$ , the optimal solution is

$$(x_A^*, x_B^*, k^*) = \begin{cases} (\phi, 0, \frac{V_A - V_B + t}{2} - t\phi) & \text{if } \phi \leq \frac{V_A - V_B}{4t} + \frac{1}{4}; \\ (\frac{V_A - V_B + t}{4t}, 0, \frac{V_A - V_B + t}{4}) & \text{if } \phi \geq \frac{V_A - V_B}{4t} + \frac{1}{4}; \end{cases}$$

and the optimal additional profit is

$$\pi^{PI} = \begin{cases} -\frac{(t\phi - \frac{V_A - V_B + t}{4})^2}{t} + \frac{(V_A - V_B + t)^2}{16t} & \text{if } \phi \leq \frac{V_A - V_B}{4t} + \frac{1}{4}; \\ \frac{(V_A - V_B + t)^2}{16t} & \text{if } \phi \geq \frac{V_A - V_B}{4t} + \frac{1}{4}; \end{cases}$$

When  $V_A - V_B \in [(\frac{t}{3}, (\sqrt{2} - 1)t]$ , the optimal solution is

$$(x_A^*, x_B^*, k^*) = \begin{cases} (\phi, 0, \frac{V_A - V_B + t}{2} - t\phi) & \text{if } \phi \leq \frac{V_A - V_B}{4t} + \frac{1}{4}; \\ (\frac{V_A - V_B + t}{4t}, 0, \frac{V_A - V_B + t}{4}) & \text{if } \frac{V_A - V_B}{4t} + \frac{1}{4} \leq \phi \leq \frac{V_A - V_B}{2t} + \frac{1}{4}; \\ (\frac{V_A - V_B}{2t} + \frac{1}{4}, \frac{-V_A + V_B}{2t} + \frac{1}{4}, \frac{t}{4}) & \text{if } \frac{V_A - V_B}{2t} + \frac{1}{4} \leq \phi \leq \frac{V_A - V_B}{2t} + \frac{3}{4}; \\ (\frac{V_A - V_B + t}{t} - \phi, 1 - \phi, t\phi - \frac{V_A - V_B + t}{2}) & \text{if } \frac{V_A - V_B}{2t} + \frac{3}{4} \leq \phi \leq \alpha; \\ (\frac{V_A - V_B + t}{4t}, 0, \frac{V_A - V_B + t}{4}) & \text{if } \phi \geq \alpha \end{cases}$$

where  $\alpha = \frac{4(V_A - V_B) + 6t + \sqrt{-2(V_A - V_B)^2 - 4t(V_A - V_B) + 2t^2}}{8t}$ , and the optimal additional profit is

$$\pi^{PI} = \begin{cases} -\frac{(t\phi - \frac{V_A - V_B + t}{4})^2}{t} + \frac{(V_A - V_B + t)^2}{16t} & \text{if } \phi \leq \frac{V_A - V_B}{4t} + \frac{1}{4}; \\ \frac{(V_A - V_B + t)^2}{16t} & \text{if } \frac{V_A - V_B}{4t} + \frac{1}{4} \leq \phi \leq \frac{V_A - V_B}{2t} + \frac{1}{4}; \\ \frac{t}{8} & \text{if } \frac{V_A - V_B}{2t} + \frac{1}{4} \leq \phi \leq \frac{V_A - V_B}{2t} + \frac{3}{4}; \\ -\frac{(4t\phi - 2V_A + 2V_B - 3t)^2}{8t} + \frac{t}{8} & \text{if } \frac{V_A - V_B}{2t} + \frac{3}{4} \leq \phi \leq \alpha; \\ \frac{(V_A - V_B + t)^2}{16t} & \text{if } \phi \geq \alpha \end{cases}$$

When  $V_A - V_B \in [0, \frac{t}{3}]$ , the optimal solution is

$$(x_A^*, x_B^*, k^*) = \begin{cases} (\phi, 0, \frac{V_A - V_B + t}{2} - t\phi) & \text{if } \phi \leq \frac{V_A - V_B}{t}; \\ (\phi, \phi + \frac{V_B - V_A}{t}, \frac{V_A - V_B + t}{2} - t\phi) & \text{if } \frac{V_A - V_B}{t} \leq \phi \leq \frac{V_A - V_B}{2t} + \frac{1}{4}; \\ (\frac{V_A - V_B}{2t} + \frac{1}{4}, \frac{-V_A + V_B}{2t} + \frac{1}{4}, \frac{t}{4}) & \text{if } \frac{V_A - V_B}{2t} + \frac{1}{4} \leq \phi \leq \frac{V_A - V_B}{2t} + \frac{3}{4}; \\ (\frac{V_A - V_B + t}{t} - \phi, 1 - \phi, t\phi - \frac{V_A - V_B + t}{2}) & \text{if } \phi \geq \frac{V_A - V_B}{2t} + \frac{3}{4} \end{cases}$$

and the optimal additional profit is

$$\pi^{PI} = \begin{cases} -\frac{(t\phi - \frac{V_A - V_B + t}{4})^2}{t} + \frac{(V_A - V_B + t)^2}{16t} & \text{if } \phi \leq \frac{V_A - V_B}{t}; \\ -\frac{(2V_A - 2V_B + t - 4t\phi)^2}{8t} + \frac{t}{8} & \text{if } \frac{V_A - V_B}{t} \leq \phi \leq \frac{V_A - V_B}{2t} + \frac{1}{4}; \\ \frac{t}{8} & \text{if } \frac{V_A - V_B}{2t} + \frac{1}{4} \leq \phi \leq \frac{V_A - V_B}{2t} + \frac{3}{4}; \\ -\frac{(4t\phi - 2V_A + 2V_B - 3t)^2}{8t} + \frac{t}{8} & \text{if } \phi \geq \frac{V_A - V_B}{2t} + \frac{3}{4} \end{cases}$$

It is easy to verify that  $k^*$  and  $\pi^{PI}$  are positive in all cases.