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The valuation of options on coupon bonds

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We derive simple closed-form expressions for European options on coupon bonds using the general equilibrium term-structure framework of Cox, Ingersoll, and Ross. The properties of these options are very different from those implied by the Black–Scholes model. For example, bond call and put values can move in the same direction as the value of the underlying bond changes. This has important implications for hedging interest-rate risk with bond options. Furthermore, bond option values can be decreasing functions of interest-rate volatility as well as their time to expiration. We also examine the properties of American bond options.

1. Introduction

Many recent papers in the financial literature have addressed the important topic of bond option valuation. A partial listing includes Brennan and Schwartz (1977), Rendleman and Bartter (1980), Courtadon (1982), Ball and Torous (1983), Brennan and Schwartz (1983), Cox et al. (1985), Ho (1985), Dietrich-Campbell and Schwartz (1986), Schaefer and Schwartz (1987), Black et al (1988), Brenner and Jarrow (1988), Heath et al. (1988), Buser et al. (1990), Jamshidian (1989), and Hull and White (1990). Of these papers, only Jamshidian (1989) provides a closed-form solution for the value of an option on a coupon bond – the most common type of bond option. However, Jamshidian's model has the drawback of allowing negative interest rates – potentially affecting the validity of the model's pricing and hedging implications.

This paper presents simple closed-form expressions for the values of European calls and puts on coupon bonds using the general equilibrium term-structure framework of Cox, Ingersoll, and Ross [CIR framework; see Cox et al. (1985)]. A key advantage of this framework is that it implies

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nonnegative interest rates. In addition, by allowing yields of all maturities to be stochastic, this framework is able to capture the full effect of a shift in the term structure on coupon bond prices.[1]

These valuation expressions have many important implications for the hedging behavior of coupon bond options. For example, we show that calls on bonds are decreasing functions of the riskless interest rate. An increase in the riskless interest rate, however, has an indeterminate effect on the value of a bond put. Thus, bond call and put prices *can move in the same direction* as the value of the underlying bond changes. In addition, we show that bond option prices can be *decreasing* functions of their time to expiration as well as the volatility of the underlying interest-rate process. Intuitively, the reason for these properties is that changes in the riskless interest rate affect both the value of the underlying asset and the present value of the strike price – the interplay of these two effects leads to option values that are fundamentally different from those implied by the Black–Scholes formula.

We also consider the properties of American bond option prices. We show that early exercise of American calls is not optimal for sufficiently small coupon rates. We provide examples of American option prices and characterize the early exercise boundary for calls and puts. We show that early exercise can be optimal for options that are only slightly in the money.

Section 2 discusses the CIR term-structure framework and derives the closed-form coupon bond option expressions. Section 3 presents comparative statics as well as examples of coupon bond option prices. Section 4 discusses the properties of American bond options. Section 5 summarizes the results and provides concluding remarks.

2. Coupon bond option prices

In deriving expressions for options on coupon bonds, we use the well-known CIR term-structure model as the basic valuation framework. The CIR term-structure model is derived in a general equilibrium setting and has the advantage that the functional form of the market price of interest-rate risk is obtained as part of the equilibrium. This is important because Cox et al. (1985) show that this avoids internal inconsistencies and arbitrage opportunities – something that cannot be guaranteed for other term-structure models in which the functional form of the risk premium is exogenously specified. The CIR term-structure model has been used extensively in the literature in valuing interest-rate-sensitive contingent claims. For example, this model is used by Dunn and McConnell (1981) to value mortgage-backed securities; by Cox et al. (1985) to value discount bond options; by Ramaswamy and

[1]A number of bond option models assume that the instantaneous riskless interest rate is constant. For example, Dietrich-Campbell and Schwartz (1986) apply (and reject) the Black–Scholes model to bond options. Also see Schaefer and Schwartz (1987).

Sundaresan (1985) to value options on futures; by Sundaresan (1991) to value swaps; and by Longstaff (1990) to value options on yields. Finally, the CIR model has the advantage of allowing interest-rate volatility to be conditionally heteroskedastic through its dependence on the level of the interest rate. This is important in light of recent evidence that interest-rate volatility is closely related to interest-rate levels.[2] By capturing the dynamic behavior of interest-rate volatility, the CIR framework has the potential to provide better models for contingent claims such as interest-rate options since their values are closely related to interest-rate volatility.[3]

In the CIR term-structure model, the equilibrium instantaneous riskless interest rate follows a square root process with dynamics given by

$$\mathrm{d}r=(\alpha-\kappa r)\mathrm{d}t+\sigma\sqrt{r}\mathrm{d}Z, \tag{1}$$

where α, κ, and σ are positive constants, and Z is a standard Brownian motion process.[4] Denote the current value of a T-maturity unit discount bond by $D(r, T)$.[5] From CIR

$$D(r,T)=A(T)\exp[-B(T)r], \tag{2}$$

where

$$A(T)=\left[\frac{2\gamma e^{(\beta+\gamma)T/2}}{(\beta+\gamma)(e^{\gamma T}-1)+2\gamma}\right]^{2\alpha/\sigma^2},$$

$$B(T)=\frac{2(e^{\gamma T}-1)}{(\beta+\gamma)(e^{\gamma T}-1)+2\gamma},$$

$$\gamma=\sqrt{\beta^2+2\sigma^2},$$

[2]For example, see Fischer and Zechner (1984), Engle et al. (1990), and Chan et al. (1992).

[3]One drawback of the CIR model is that it may not be possible to fit the entire initial term structure. Hull and White (1990) address this problem by allowing for a time-dependent drift term in their specification of the interest-rate process. It is important to note, however, that since the CIR model inolves three parameters, it is possible to obtain an exact fit to three points on the initial yield curve. In actuality, it may not be desirable to require a model to fit all observed yields exactly since actual data contains measurement errors from sources such as bid–ask spreads, quotation nonsynchronicity, thin trading and the like. By exactly fitting the yield curve – and any inherent measurement error – we introduce the risk of overfitting the model. See the discussion of measurement errors in Stambaugh (1988).

[4]The parameter α corresponds to the term $\kappa\theta$ in CIR equation 17 [Cox et al. (1985)]. We use this simpler notation to conform more closely to Feller (1951) who shows that the behavior of the square root process as it approaches the singularity at zero is governed entirely by the relation between the parameters α and σ^2 and is independent of κ. Also see Capocelli and Ricciardi (1976).

[5]Without loss of generality, we assume that contingent claims are valued as of date zero since the interest-rate process is time homogeneous.

and where β is the sum of κ and the interest-rate risk parameter λ.[6]

Because a coupon bond is just a portfolio of discount bonds of different maturities, the value of any riskless coupon bond can be expressed as a weighted sum of discount bond prices

$$\sum_{i=1}^{N} a_i D(r, T_i), \tag{3}$$

where $T_1, T_2, \ldots, T_N$ represent the N dates on which payments are made, and the $\alpha_i \geqq 0$ terms denote the amount of the payments made. As an example, consider a 20-year 8% bond with a face amount of 1000. In this case, $N=40$ since the bond makes 39 semiannual coupon payments of 40 as well as a final payment of 1040. In addition, $T_1=0.5, T_2=1, \ldots, T_{40}=20$.

The payoff function for a τ-maturity European call on a bond is

$$\max\left(0, \sum_{i=1}^{N} a_i D(r, T_i) - K\right), \tag{4}$$

where K is the strike price of the option. Since $D(r,T)$ is a monotonic function of r for all T_i in the CIR model, there is a critical interest rate r^* such that the call is exercised if $0 \leqq r < r^*$ on its expiration date. This critical interest rate is easily found by solving the following expression for r^*:[7]

$$\sum_{i=1}^{N} a_i D(r^*, T_i) = K \tag{5}$$

Having specified the payoff function for a coupon bond call, the price of the call $C(r, \tau)$ can now be obtained as the solution of the partial differential equation

$$\frac{\sigma^2}{2} r C_{rr} + (\alpha - \beta r) C_r - rC = C_\tau, \tag{6}$$

subject to the condition that $C(r, 0)$ equals the payoff function in (4). A separation of variables, in conjunction with the superposition of solutions

[6] Using our notation, it is clear from CIR equation 22 [Cox et al. (1985)] that interest-rate-dependent contingent claims depend on κ and the market price of interest-rate risk parameter λ only through their sum β. This follows because β, not κ, is the coefficient of r in the drift term of the risk-adjusted dynamics for r. Thus, β need not be decomposed into its component terms for the purpose of pricing interest-rate-dependent contingent claims. An important advantage of this is that the market price of interest-rate risk λ does not need to be estimated as a separate parameter in order to value bond options.

[7] The critical interest rate can be determined by solving (5) numerically, or by expressing the bond prices in a Taylor series expansion and using series techniques to find an analytical solution. The analytical expressions given by Roll (1977) for American calls with dividends and by Jamshidian (1989) for bond options also require determining the critical value of the state variable.

property [see Zauderer (1983)] of linear partial differential equations, leads to the following closed-form expression for the value of the call option $C(r,\tau)$:

$$\sum_{i=1}^{N} a_i D(r,\tau+T_i)\, Q(\mu_i; \nu, \eta_i) - K D(r,\tau)\, Q(\mu_o; \nu, \eta_0), \tag{7}$$

where

$$\mu_0 = \frac{4\gamma(1+\xi e^{\gamma\tau})r^*}{\sigma^2(1+\xi)(e^{\gamma\tau}-1)},$$

$$\mu_i = \frac{4\gamma(1+\xi e^{\gamma(\tau+T_i)})r^*}{\sigma^2(1+\xi e^{\gamma T_i})(e^{\gamma\tau}-1)},$$

$$\eta_0 = \frac{4\gamma e^{\gamma\tau}(1+\xi)r}{\sigma^2(1+\xi e^{\gamma\tau})(e^{\gamma\tau-1})},$$

$$\eta_i = \frac{4\gamma e^{\gamma\tau}(1+\xi e^{\gamma T_i})r}{\sigma^2(1+\xi e^{\gamma(\tau+T_i)})(e^{\gamma\tau-1})},$$

$$\nu = 4\alpha/\sigma^2,$$

$$\xi = (\gamma+\beta)/(\gamma-\beta),$$

and where $Q(\mu;\nu,\eta)$ is the cumulative noncentral chi-square distribution function [see Johnson and Kotz (1970, ch. 28)] with ν degrees of freedom and noncentrality parameter η. In this expression, the call option is an explicit function of the current riskless rate r and maturity of the option τ. In addition, the call price depends parametrically on the payoffs of the bond $a_i, i=1,2,\ldots,N$, the timing of the payoffs as indexed by $T_i, i=1,2,\ldots,N$, and the constants α, β, and σ^2. Observe that each bond price in (7) appears in conjunction with a distinct $Q(\cdot)$ term. This is consistent with Jamshidian (1989) who shows that an option on a coupon bond can be decomposed into a portfolio of options whenever bond prices are monotone functions of the interest rate.[8] It is easily shown that if the underlying bond is a discount bond, (7) reduces to the discount bond call valuation expression given by Cox et al. (1985).[9]

[8]This approach is also used by Hull and White (1990) in computing bond option prices implied by the CIR model.

[9]A similar expression for value of the options in retractable and extendible bonds is independently derived in Barone and Cuocco (1989).

Although the call price in (7) requires evaluating $Q(\mu; v, \eta)$, we note that the call value can be approximated to a high degree of accuracy using Sankaran's (1963) algorithm for the cumulative noncentral chi-square distribution function.[10] In this approach, the value of $Q(\mu; v, \eta)$ is approximated by the cumulative standard normal distribution function $\Phi(d)$, where

$$d = k\left(\left(\frac{\mu}{v+\eta}\right)^h - l\right), \tag{8}$$

$$h = 1 - \frac{2}{3}(v+\eta)(v+3\eta)(v+2\eta)^{-2},$$

$$k = \left(h^2 \frac{2(v+2\eta)}{(v+\eta)^2}[1-(1-h)(1-3h)(v+2\eta)(v+\eta)^{-2}]\right)^{-1/2},$$

$$l = 1 + h(h-1)\frac{v+2\eta}{(v+\eta)^2} - h(h-1)(2-h)(1-3h)\frac{(v+2\eta)^2}{2(v+\eta)^4}.$$

An advantage of this approximation is that call prices can be computed using the same types of programming routines used to compute Black–Scholes option prices [Black and Scholes (1973)].

Coupon bond put prices $P(r, \tau)$ can be obtained directly from the put–call parity relation

$$P(r,\tau) = C(r,\tau) + KD(r,\tau) - \sum_{i=1}^{N} a_i D(r, \tau + T_i). \tag{9}$$

Observe that the underlying asset for coupon bond calls and puts is actually the portfolio of discount bonds maturing after the option expires. However, the value of this portfolio is strictly less than the current price of the coupon bond if the bond makes coupon payments prior to the expiration of the option. For example, the value of the underlying asset for a 5-year option on a 10-year bond is not the current price of a 15-year bond, but the price of a 15-year bond minus the present value of coupon payments to be made during the next 5 years.

3. Comparative statics

The closed-form expressions for coupon bond option prices allow us to

[10]Numerical examples illustrating the accuracy of the Sankaran (1963) approximation are given in Johnson and Kotz (1970, ch. 28). See also Schroder (1989).

examine the comparative statics properties of these contingent claims directly. First, consider the relation between coupon bond option prices and the riskless interest rate. Cox et al. (1985) show that calls on discount bonds are decreasing functions of the riskless interest rate. This contrasts with Black–Scholes call option values which are increasing functions of r. The reason for the difference in behavior is that an increase in r affects bond calls in two ways. Specifically, an increase in r reduces the value of the underlying coupon bond which reduces the expected payoff of the call. However, an increase in r also reduces the present value of the strike price which tends to increase the call price. Cox et al. indicate that the first effect dominates for discount bond calls. Note that only the second of these two effects is relevant for Black–Scholes call option values.[11]

Although Cox et al. focus only on the effects of changes in r on the value of discount bond calls, their results hold true for calls on coupon bonds as well – coupon bond calls are decreasing functions of the riskless interest rate. Surprisingly, however, the opposite is not true for bond puts – the relationship between the value of a bond put and r is indeterminate. The reason for this is again related to the fact that an increase in r affects bond put values in two ways. Unlike bond calls, however, neither of the two effects dominates the other in the case of bond puts. This is illustrated in tables 1 and 2, which present coupon bond call and put prices for different values of r and K. Tables 1 and 2 also report the first and second derivatives of the option prices with respect to the underlying bond price – the 'delta' and 'gamma' of the options. The parameters α, β, and σ^2 used in the tables are chosen to be consistent with the unconditional mean, variance, and first-order autocorrelation of r as measured by the one-month Treasury-bill rate during the 1978 to 1990 period.

As shown, coupon bond put prices can be uniformly increasing (decreasing) functions of r for small (large) values of K. However, they can be both increasing and decreasing in r for intermediate values of K. This property has many important implications for the hedging behavior of interest-rate puts. For example, tables 1 and 2 show that bond call and put values can move in the same direction in response to a change in the underlying bond price. Furthermore, it is possible to hedge a long position in a bond put using a long position in another put. Note that for intermediate values of r and K, the price of a bond put may be unaffected by changes in r – the delta for a bond put may be zero. In general, the deltas for longer-maturity in-the-

[11]Specifically, the Black–Scholes formula assumes that the riskless rate is constant. Consequently, the underlying asset price is not a function of the riskless rate, and the partial derivative of option prices with respect to the riskless rate can be computed while holding the value of the underlying asset fixed. By allowing yields of all maturities to be stochastic, the bond option pricing model of this paper captures the effect that changes in the riskless rate have on underlying bond price.

Table 1

Examples of prices, deltas, and gammas for 5-year call and put options on a 10-year 8% coupon bond with par value 1,000 for different levels of the riskless rate (r) and strike price (K). Delta is the partial derivative of the option price with respect to the underlying bond price. Gamma is 10,000 times the second partial derivative of the option price with respect to the underlying bond price. The option values are computed using parameter values $\alpha=0.06$, $\beta=0.75$, and $\sigma^2=0.014$.

	Call prices			Put prices		
r	$K=960$	$K=980$	$K=1{,}000$	$K=960$	$K=980$	$K=1{,}000$
0.01	22.94	12.34	4.98	3.11	7.22	14.58
0.02	22.51	12.08	4.87	3.12	7.22	14.53
0.03	22.09	11.83	4.76	3.13	7.22	14.49
0.04	21.67	11.59	4.65	3.15	7.23	14.39
0.05	21.26	11.35	4.54	3.16	7.23	14.35
0.06	20.86	11.11	4.44	3.17	7.23	14.30
0.07	20.47	10.89	4.33	3.19	7.23	14.25
0.08	20.08	10.66	4.23	3.20	7.23	14.20
0.09	19.70	10.44	4.14	3.21	7.23	14.15
0.10	19.33	10.22	4.04	3.22	7.23	14.10
0.11	18.96	10.01	3.95	3.23	7.22	14.04
0.12	18.60	9.81	3.86	3.24	7.22	13.99
0.13	18.25	9.60	3.77	3.26	7.21	13.94
0.14	17.90	9.40	3.68	3.27	7.21	13.88
0.15	17.56	9.21	3.60	3.28	7.20	13.83

	Call deltas			Put deltas		
r	$K=960$	$K=980$	$K=1{,}000$	$K=960$	$K=980$	$K=1{,}000$
0.01	0.0453	0.0266	0.0119	−0.0016	−0.0004	0.0046
0.02	0.0450	0.0264	0.0118	−0.0015	−0.0003	0.0048
0.03	0.0447	0.0262	0.0117	−0.0015	−0.0002	0.0049
0.04	0.0446	0.0261	0.0117	−0.0014	−0.0001	0.0051
0.05	0.0442	0.0259	0.0115	−0.0014	−0.0001	0.0053
0.06	0.0441	0.0257	0.0114	−0.0014	0.0000	0.0054
0.07	0.0438	0.0255	0.0113	−0.0014	0.0001	0.0055
0.08	0.0435	0.0253	0.0111	−0.0013	0.0001	0.0057
0.09	0.0434	0.0251	0.0110	−0.0013	0.0002	0.0059
0.10	0.0432	0.0250	0.0109	−0.0013	0.0004	0.0061
0.11	0.0429	0.0247	0.0108	−0.0013	0.0004	0.0062
0.12	0.0427	0.0246	0.0108	−0.0012	0.0005	0.0064
0.13	0.0424	0.0244	0.0107	−0.0012	0.0006	0.0065
0.14	0.0423	0.0242	0.0106	−0.0011	0.0006	0.0067
0.15	0.0419	0.0241	0.0105	−0.0011	0.0008	0.0069

	Call gammas			Put gammas		
r	$K=960$	$K=980$	$K=1{,}000$	$K=960$	$K=980$	$K=1{,}000$
0.01	0.254	0.203	0.122	−0.019	−0.075	−0.159
0.02	0.257	0.204	0.123	−0.021	−0.078	−0.163
0.03	0.260	0.206	0.123	−0.022	−0.080	−0.166
0.04	0.261	0.206	0.123	−0.024	−0.083	−0.170
0.05	0.266	0.209	0.124	−0.024	−0.084	−0.173
0.06	0.267	0.211	0.124	−0.025	−0.086	−0.176
0.07	0.271	0.212	0.124	−0.026	−0.089	−0.179
0.08	0.275	0.214	0.125	−0.028	−0.090	−0.183
0.09	0.275	0.216	0.126	−0.029	−0.094	−0.187
0.10	0.278	0.217	0.126	−0.030	−0.097	−0.193
0.11	0.282	0.219	0.127	−0.031	−0.098	−0.195
0.12	0.283	0.221	0.127	−0.034	−0.102	−0.199
0.13	0.287	0.223	0.128	−0.034	−0.105	−0.201
0.14	0.290	0.225	0.128	−0.037	−0.106	−0.206
0.15	0.294	0.226	0.129	−0.037	−0.109	−0.210

Table 2

Examples of prices, deltas, and gammas for 5-year call and put options on a 10-year 14% coupon bond with par value 1,000 for different levels of the riskless rate (r) and strike price (K). Delta is the partial derivative of the option price with respect to the underlying bond price. Gamma is 10,000 times the second partial derivative of the option price with respect to the underlying bond price. The option values are computed using parameter values $\alpha=0.06$, $\beta=0.75$, and $\sigma^2=0.014$.

	Call prices			Put prices		
r	$K=1{,}340$	$K=1{,}360$	$K=1{,}380$	$K=1{,}340$	$K=1{,}360$	$K=1{,}380$
0.01	37.18	25.31	15.54	3.17	6.11	10.95
0.02	36.50	24.91	15.22	3.19	6.13	10.96
0.03	35.83	24.43	14.90	3.21	6.15	10.96
0.04	35.18	23.95	14.59	3.23	6.16	10.96
0.05	34.53	23.48	14.28	3.25	6.18	10.96
0.06	33.89	23.03	13.98	3.27	6.20	10.95
0.07	33.27	22.58	13.69	3.28	6.21	10.95
0.08	32.66	22.13	13.41	3.30	6.22	10.94
0.09	32.06	21.70	13.12	3.32	6.24	10.94
0.10	31.47	21.28	12.85	3.33	6.25	10.93
0.11	30.89	20.86	12.58	3.35	6.26	10.92
0.12	30.32	20.45	12.32	3.37	6.27	10.91
0.13	29.76	20.05	12.06	3.38	6.28	10.90
0.14	29.21	19.66	11.80	3.40	6.29	10.89
0.15	28.67	19.27	11.56	3.41	6.30	10.88

	Call deltas			Put deltas		
r	$K=1{,}340$	$K=1{,}360$	$K=1{,}380$	$K=1{,}340$	$K=1{,}360$	$K=1{,}380$
0.01	0.0509	0.0369	0.0241	−0.0015	−0.0013	−0.0001
0.02	0.0506	0.0367	0.0240	−0.0014	−0.0013	0.0000
0.03	0.0504	0.0365	0.0238	−0.0014	−0.0013	0.0001
0.04	0.0502	0.0362	0.0237	−0.0014	−0.0012	0.0002
0.05	0.0499	0.0360	0.0234	−0.0014	−0.0012	0.0002
0.06	0.0496	0.0359	0.0233	−0.0014	−0.0012	0.0003
0.07	0.0494	0.0355	0.0231	−0.0014	−0.0011	0.0004
0.08	0.0492	0.0354	0.0229	−0.0014	−0.0011	0.0005
0.09	0.0489	0.0352	0.0228	−0.0013	−0.0010	0.0006
0.10	0.0487	0.0350	0.0227	−0.0013	−0.0010	0.0007
0.11	0.0485	0.0348	0.0225	−0.0013	−0.0009	0.0008
0.12	0.0483	0.0346	0.0222	−0.0013	−0.0009	0.0009
0.13	0.0480	0.0343	0.0221	−0.0013	−0.0009	0.0010
0.14	0.0478	0.0342	0.0219	−0.0012	−0.0008	0.0011
0.15	0.0475	0.0339	0.0217	−0.0012	−0.0008	0.0012

	Call gammas			Put gammas		
r	$K=1{,}340$	$K=1{,}360$	$K=1{,}380$	$K=1{,}340$	$K=1{,}360$	$K=1{,}380$
0.01	0.184	0.167	0.136	−0.007	−0.026	−0.059
0.02	0.187	0.168	0.136	−0.007	−0.027	−0.061
0.03	0.188	0.170	0.138	−0.008	−0.028	−0.062
0.04	0.190	0.171	0.139	−0.009	−0.030	−0.064
0.05	0.193	0.173	0.140	−0.009	−0.031	−0.066
0.06	0.195	0.175	0.141	−0.009	−0.031	−0.067
0.07	0.197	0.177	0.143	−0.010	−0.033	−0.069
0.08	0.199	0.178	0.144	−0.011	−0.034	−0.070
0.09	0.202	0.180	0.144	−0.012	−0.036	−0.073
0.10	0.204	0.182	0.146	−0.012	−0.037	−0.075
0.11	0.206	0.183	0.148	−0.013	−0.038	−0.076
0.12	0.208	0.185	0.149	−0.014	−0.039	−0.078
0.13	0.210	0.187	0.149	−0.014	−0.039	−0.081
0.14	0.213	0.188	0.149	−0.015	−0.041	−0.082
0.15	0.215	0.191	0.150	−0.016	−0.042	−0.085

Table 3

Examples of prices for call and put options on a 10-year 8% coupon bond with par value 1,000 and strike price 1,000 for different levels of the riskless rate (r) and time to expiration (τ). The option prices are computed using parameter values $\alpha=0.06$, $\beta=0.75$, and $\sigma^2=0.014$.

	Call prices			Put prices		
r	$\tau=1$	$\tau=2$	$\tau=20$	$\tau=1$	$\tau=2$	$\tau=20$
0.01	25.46	11.54	1.42	1.43	8.47	4.77
0.02	20.83	10.33	1.40	2.65	9.72	4.70
0.03	16.77	9.22	1.39	4.33	11.03	4.64
0.04	13.30	8.22	1.37	6.48	12.39	4.58
0.05	10.39	7.31	1.35	9.08	13.78	4.52
0.06	7.99	6.49	1.33	12.10	15.21	4.46
0.07	6.06	5.75	1.31	15.46	16.67	4.40
0.08	4.54	5.09	1.30	19.13	18.15	4.35
0.09	3.35	4.49	1.28	23.03	19.65	4.29
0.10	2.44	3.96	1.26	27.11	21.16	4.23
0.11	1.75	3.48	1.25	31.31	22.68	4.18
0.12	1.25	3.06	1.23	35.59	24.20	4.12
0.13	0.87	2.68	1.21	39.91	25.72	4.09
0.14	0.61	2.35	1.20	44.23	27.24	4.02
0.15	0.42	2.05	1.18	48.54	28.74	3.96

money options are much smaller than one. The reason for this is that mean reversion tends to reverse the effects of a small change in the current price of the underlying bond. Thus, a small change in the current bond price has little effect on the value of a European option that expires in five years. These results again contrast with the familiar properties of Black–Scholes option prices and drive home the point that option pricing in a stochastic interest-rate setting is much more complex.

Another interesting feature of coupon bond option prices is their relationship with the length of the option's life τ. Recall that Black–Scholes option prices are increasing functions of τ. It is easily shown, however, that both coupon bond call and put prices can be decreasing functions of τ (note that we are focusing on options on bonds with final maturity date $\tau+T_N$, not on a bond with a specific maturity date). This is illustrated in table 3, which presents numerical examples for calls and puts on a 10-year 8% coupon bond for various values of τ. As shown, calls and puts can be increasing or decreasing in τ for small τ. For large τ, however, the options become uniformly decreasing functions of τ.

The intuition for these results is related to the mean-reversion property of interest rates (and bond prices) in the CIR framework. For small τ, the option values are close to their intrinsic value. As τ increases, however, the future distribution of bond prices converges to a stationary distribution. Thus, the variation in option values in response to changes in r decreases as

Table 4

Examples of prices for two-year call and put options on a ten-year 8% coupon bond with par value 1,000 and strike price 1,000 for different levels of the riskless rate (r) and interest-rate volatility (σ^2). The option prices are computed using parameter values $\alpha=0.06$, $\beta=0.75$, and $\sigma^2=0.014$.

	Call prices			Put prices		
r	$\sigma^2=0.01$	$\sigma^2=0.015$	$\sigma^2=0.02$	$\sigma^2=0.01$	$\sigma^2=0.015$	$\sigma^2=0.02$
0.01	9.20	12.10	14.71	7.63	8.65	9.40
0.02	8.02	10.87	13.46	8.93	9.89	10.61
0.03	6.97	9.75	12.29	10.30	11.19	11.85
0.04	6.04	8.94	11.22	11.74	12.53	13.13
0.05	5.22	7.81	10.23	13.22	13.91	14.44
0.06	4.49	6.97	9.31	14.75	15.32	15.77
0.07	3.86	6.21	8.47	16.32	16.75	17.12
0.08	3.30	5.53	7.70	17.91	18.21	18.48
0.09	2.82	4.91	6.99	19.52	19.69	19.86
0.10	2.40	4.35	6.34	21.15	21.17	21.25
0.11	2.03	3.85	5.74	22.78	22.67	22.64
0.12	1.72	3.41	5.20	24.41	24.16	24.04
0.13	1.45	3.01	4.70	26.04	25.66	25.44
0.14	1.22	2.65	4.24	27.66	27.15	26.83
0.15	1.02	2.33	3.83	29.27	28.64	28.22

τ increases. Furthermore, this convergence means that the expected payoff of an option does not increase with τ sufficiently to offset the effect of an increase in the length of the period over which payoffs are discounted. Consequently, for some sufficiently large τ, coupon bond options are decreasing functions of τ – coupon bond option prices converge to zero at $\tau \to \infty$. In contrast, the upward drift of the risk-neutral process in the Black–Scholes model just cancels the discount factor, leading to option prices that are increasing functions of τ, and, therefore, do not converge to zero as τ increases.[12]

Black–Scholes option prices are increasing functions of the volatility of the underlying asset. Surprisingly, the same is not true for the prices of coupon bond options. This is shown in table 4, which gives the values of call and puts on a 10-year 8% coupon bond for different values of the riskless rate and the volatility parameter σ^2. As shown, for higher values of the riskless rate, put prices become decreasing functions of the volatility of the term structure. The intuition for this result is that bond prices are increasing functions of σ^2 in the CIR framework. Thus, as σ^2 increases, there are two effects on option prices. First, there is the standard convexity effect which enhances the value of calls and puts. In addition, however, there is an effect on the value of the underlying bond. For calls on bonds, the two effects reinforce each other and calls are increasing functions of σ^2. For puts on

[12]We note that similar comparative statics results for coupon bond options may occur in other term-structure frameworks besides CIR.

bonds, however, the second effect can offset the first, resulting in bond put prices that are decreasing functions of σ^2.

Some of the remaining comparative statics can be signed. For example, bond calls are decreasing functions of K while the opposite is true for bond puts. An increase in any a_i increases the value of the underlying asset and has a corresponding effect on option values. Similarly, increasing any of the T_i payment periods defers a cash flow and reduces the underlying asset value. The remaining comparative statics are indeterminate since changes in the interest-rate parameters have complex effects on the relative values of bonds with different maturities.

4. American bond options

Many of the bond options found in financial markets have European exercise provisions. One important example is the implicit call option in callable U.S. Treasury bond prices. These bonds consist of a straight coupon bond and a short call on the coupon bond with a strike price of par. The implicit call option can be exercised by the U.S. Treasury only during the last five years of the bond's life, which is originally as long as thirty years. In addition, the Treasury can only exercise the call option upon four months notice prior to a coupon payment date. Because of these restrictions, these options are much more like European options than American options.[13] The total par value of callable U.S. Treasury bonds currently outstanding is in excess of $95 billion.

Some coupon bond options, however, have American exercise provisions. For example, until recently, the Chicago Board Options Exchange traded American options on specific U.S. Treasury bonds. Although these options no longer trade, it is still worthwhile to consider the general properties of American bond options.

An important feature of coupon bonds is that they do not exhibit price discontinuities on coupon payment dates – in contrast to stock prices on ex dividend dates. The reason for this is that the purchaser of a coupon bond pays the seller any accrued coupon at the time of the purchase. Thus, coupon bonds can be viewed as paying a continuous cash flow proportional to the coupon rate. This convention of accruing coupon payments is also followed when an American option on a coupon bond is exercised. Quoting from the Option Clearing Corporation's (1987) brochure on characteristics of standardized options,

[13]For a discussion of the exercise provisions of callable U.S. Treasury bonds, see Longstaff (1992).

'Exercise prices for Treasury bond options and Treasury note options are expressed in the same way as prices in the cash market for the underlying securities, that is, as a percentage of par value. For example, a large Treasury bond call with an exercise price of 102 would entitle the holder to purchase the underyling bond for \$102,000 (102% of \$100,000) plus accrued interest on the bond from the date of issue or that last interest payment date, whichever is later, through and including the exercise settlement date.'

As shown by Merton (1973), the continuous flow of fixed payments by an underlying asset has important implications for American calls. Merton shows that a sufficient condition for no early exercise of a call option is that the fixed dividend rate be less than rK. Since interest rates are stochastic in the CIR framework, this condition must be modified in order to apply it to coupon bond options. Following an argument similar to that of Merton, it is easily shown that if the present value of all coupons to be accrued prior to the expiration date of the option is less than $(1-D(r,T))K$, then it is never optimal to exercise an American call early.

In general, however, early exercise is optimal and the value of an American option on a coupon bond will exceed that of a European option. In order to make comparisons, table 5 presents American option values for the same coupon bond, riskless rates, and strike prices as in table 1 (results corresponding to table 2 are very similar to those in table 5).[14] As shown, in-the-money American option prices can differ significantly from the corresponding European option values. Note, however, that deep-out-of-the-money American puts have values very similar to those for European puts. In addition, table 5 shows that American calls (puts) are monotone decreasing (increasing) functions of the riskless rate – American calls and puts have positive and negative deltas, respectively. Furthermore, American calls and puts have positive gammas. Thus, American options on coupon bonds behave more like Black–Scholes option prices than do European options on coupon bonds.

Examples of the early exercise boundary for different maturities are given in table 6. Again, the parameters used are chosen to correspond to those in tables 1 and 5. The early exercise boundary is expressed as the ratio of the critical bond price for exercise to the strike price of the option. Hence, the ratio can be interpreted as a measure of the degree to which the option has to be in the money before early exercise is optimal. As shown, the early exercise boundary is a function both of the maturity of the option as well as

[14]American option values are computed using an implicit finite difference algorithm which assumes that the underlying bond pays a continuous coupon. I am grateful to Warren Bailey for providing me with this program.

Table 5

Examples of prices for 5-year American call and put options on a 10-year 8% coupon bond with par value 1,000 for different levels of the riskless rate (r) and strike price (K). Options where immediate early exercise is optimal are denoted by an asterisk. Option values are computed numerically using parameter values $\alpha=0.06$, $\beta=0.75$, and $\sigma^2=0.014$.

	Call prices			Put prices		
r	$K=960$	$K=980$	$K=1{,}000$	$K=960$	$K=980$	$K=1{,}000$
0.01	134.76*	114.76*	94.76*	3.27	7.48	14.84
0.02	121.74*	101.74*	81.74*	3.47	7.86	15.42
0.03	108.87*	88.87*	68.87*	3.71	8.29	16.08
0.04	96.17*	76.17*	56.17*	3.98	8.80	16.87
0.05	83.63*	63.63*	43.90	4.31	9.42	17.81
0.06	72.41	54.02	36.70	4.72	10.17	19.97
0.07	65.53	48.43	32.44	5.23	11.11	20.42
0.08	60.76	44.56	29.51	5.90	12.33	22.29
0.09	57.14	41.64	27.31	6.78	13.96	24.78
0.10	54.20	39.27	25.54	7.99	16.16	28.16
0.11	51.77	37.32	24.09	9.77	19.42	33.14
0.12	49.63	35.61	22.84	12.36	24.12	40.46*
0.13	47.74	34.10	21.73	16.32	31.54	51.13*
0.14	46.04	32.76	20.76	22.99	42.30*	62.30*
0.15	44.50	31.54	19.88	33.33*	53.33*	73.33*

the strike price of the option. The longer the maturity, the more the option needs to be in the money before it is optimal to exercise early. This is intuitive since the cost of early exercise is higher for longer-maturity options. Note that calls generally have to be further in the money before early exercise is optimal Results for options on bonds with different coupon rates are qualitatively similar to those shown in table 6.

5. Conclusion

This paper has provided simple closed-form expressions for European coupon bond options in a general equilibrium setting where interest rates are nonnegative. The resulting option pricing formulas have many important implications for hedging interest-rate risk. For example, we have shown that European bond call and put values can move in the same direction as the value of the underlying bond changes. Furthermore, in-the-money coupon bond puts can have a delta of zero – a long position in a coupon bond put can be a perfectly hedged position by itself.

We have also examined the properties of American options on coupon bonds. Our results suggest that for sufficiently small coupon rates, early exercise of American calls is never optimal. Numerical examples show that

Table 6

Early exercise boundaries for American call and put options on a 10-year 8% coupon bond with par value 1,000 for different option maturities (τ) and strike price (K). The critical early exercise boundary is expressed as the ratio of the critical bond price to the strike price of the option. Option values are computed using parameter values $\alpha=0.06$, $\beta=0.75$, and $\sigma^2=0.014$.

	Call prices			Put prices		
τ	$K=960$	$K=980$	$K=1{,}000$	$K=960$	$K=980$	$K=1{,}000$
0.00	1.000	1.000	1.000	1.000	1.000	1.000
0.25	1.054	1.036	1.024	0.985	0.985	0.978
0.50	1.058	1.042	1.029	0.985	0.981	0.975
0.75	1.062	1.045	1.032	0.985	0.978	0.972
1.00	1.065	1.048	1.033	0.981	0.978	0.969
1.25	1.067	1.050	1.036	0.981	0.978	0.969
1.50	1.069	1.052	1.036	0.981	0.974	0.969
1.75	1.071	1.053	1.037	0.981	0.974	0.966
2.00	1.072	1.055	1.039	0.981	0.974	0.966
2.25	1.073	1.055	1.040	0.981	0.974	0.966
2.50	1.074	1.056	1.040	0.981	0.974	0.966
2.75	1.075	1.058	1.041	0.981	0.974	0.966
3.00	1.076	1.058	1.041	0.981	0.974	0.963
3.25	1.077	1.059	1.041	0.981	0.971	0.963
3.50	1.077	1.059	1.043	0.977	0.971	0.963
3.75	1.077	1.059	1.043	0.977	0.971	0.963
4.00	1.078	1.060	1.044	0.977	0.971	0.963
4.25	1.079	1.061	1.044	0.977	0.971	0.963
4.50	1.079	1.061	1.044	0.977	0.971	0.959
4.75	1.079	1.061	1.044	0.977	0.971	0.959
5.00	1.080	1.062	1.045	0.977	0.971	0.959

American options on coupon bonds behave much more like Black–Scholes options than do European options on coupon bonds.

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