Journal of Financial Economics 32 (1992) 333–344. North Holland

Multiple equilibria and term structure models

Francis A. Longstaff*
Ohio State University, Columbus, OH 43210-1309, USA
University of California–Los Angeles, Los Angeles, CA 90024-1481, USA

Received July 1992, final version received September 1992

We show the Cox, Ingersoll, and Ross term structure framework can allow a variety of alternative equilibrium solutions for discount bond prices. This is important since it allows us additional flexibility in developing models that capture the properties of the term structure. As an example, we solve for the value of a discount bond when the short-term rate is absorbed at zero. We compare the yields implied by this model to those implied by the original Cox, Ingersoll, and Ross model. We also show that alternative equilibria can occur in other term structure models.

1. Introduction

Cox, Ingersoll, and Ross (1985b) present a general equilibrium model of the term structure in which the short-term interest rate is the single factor. Although Cox, Ingersoll, and Ross (CIR) consider only one equilibrium, their framework can allow other interesting equilibria. These alternative equilibria are obtained by imposing boundary conditions on bond prices when the short-term interest rate reaches zero. Each equilibrium corresponds to a different assumption about the behavior of the short-term interest-rate process at zero. This feature of the CIR framework is important since it provides an additional degree of freedom in developing general equilibrium term structure models that capture the actual properties of the term structure.

In section 2, we identify the conditions under which it is possible to have alternative equilibria in the CIR framework. In section 3, we provide an example of an alternative equilibrium by deriving the value of a discount bond when the short-term interest rate is absorbed at zero. We compare the yields implied by

Correspondence to: Francis A. Longstaff, John E. Anderson Graduate School of Management, University of California–Los Angeles, 405 Hilgard Avenue, Los Angeles, CA 90024-1481, USA.

*I am grateful for the comments and suggestions of Warren Bailey, David Beaglehole, Michael Brennan, Peter Carr, George Constantinides, Jerome DeTemple, Wayne Ferson, Michael Hemler, Michael Hogan, John Persons, and Eduardo Schwartz. I am particularly grateful for the many helpful comments of the editor, John Long. All errors are my responsibility.

0304-405X/92/$05.00

this equilibrium to those implied by the original CIR equilibrium and investigate the empirical significance of their differences. In section 4, we show that multiple equilibria can also occur in other general equilibrium term structure models, such as Longstaff (1989). Since the choice among different equilibria must ultimately be made on the basis of empirical evidence, section 5 presents historical statistics about the behavior of interest rates when close to zero. Concluding remarks are made in section 6.

2. Alternative equilibria

In the CIR (1985b) framework, the short-term interest rate r is proportional to an exogenous state variable Y which follows a square root process. This proportionality allows CIR to make a change of variables from Y to the endogenous short-term interest rate r. With this change of variables, the dynamics of r can be expressed as

$$\mathrm{d}r = (\alpha - \kappa r)\,\mathrm{d}t + \sigma\sqrt{r}\,\mathrm{d}Z\,, \tag{1}$$

where $\alpha, \kappa, \sigma > 0$ and r is defined on $(0, \infty)$.[1]

Let $D(r, \tau)$ denote the value of a riskless unit discount bond with maturity τ. Under CIR's assumption of logarithmic utility, the bond's equilibrium risk premium or instantaneous expected excess return equals the instantaneous covariance between the bond's return and the return on the representative investor's portfolio. Since the covariance between changes in r and production rates of return is proportional to r in the CIR model, the covariance between the bond's rate of return and the return on the investor's equilibrium portfolio is $\lambda r D_r/D$, where the constant λ represents the market price of interest-rate risk. Thus, the bond's equilibrium expected rate of return is

$$r + \lambda r D_r/D\,. \tag{2}$$

Equating this expression to Ito's formula for the bond's expected rate of return yields the fundamental valuation equation

$$\frac{\sigma^2}{2} r D_{rr} + (\alpha - \beta r) D_r - rD = D_\tau\,, \tag{3a}$$

[1]The parameter α corresponds to the term $\kappa\theta$ in CIR (1985b, eq. 17). We use this simpler notation to conform more closely to Feller (1951), who shows that the behavior of the square root process as it approaches the singularity at zero is governed entirely by the relation between the parameters α and σ^2 and is independent of κ.

for r and τ in $(0, \infty) \times (0, \infty)$, where $\beta = \kappa + \lambda$. In addition, the discount bond price must satisfy the maturity condition

$$D(r, 0) = 1 . \tag{3b}$$

To complete the specification of the fundamental valuation equation, we need to consider whether boundary conditions can be imposed on $D(r, \tau)$ at $r = 0$ and $r = \infty$. As shown by Karlin and Taylor (1981), this can be determined by examining the behavior of the risk-adjusted stochastic process

$$\mathrm{d}r = (\alpha - \beta r)\,\mathrm{d}t + \sigma\sqrt{r}\,\mathrm{d}Z \tag{4}$$

at the singularities $r = 0$ and $r = \infty$.

Using the boundary classification criteria described in Karlin and Taylor, the boundary at ∞ can be shown to be inaccessible in finite time. This means that no boundary condition can be imposed at $r = \infty$ when $0 < \tau < \infty$, and that the behavior of $D(r, \tau)$ as $r \to \infty$ is implicitly specified by the fundamental valuation equation.

The behavior of eq. (4) at zero is more complicated. As demonstrated by Feller (1951), the nature of the singularity depends on the relative values of α and σ^2. When $0 < \sigma^2 < 2\alpha$, zero is inaccessible and no boundary condition can be imposed there. In this case, the behavior of $D(r, \tau)$ as $r \to 0$ is implicitly specified by the fundamental valuation equation and the solution to eqs. (3a) and (3b) is unique. A sufficient condition for the absence of multiple equilibria is that the state variable Y have no accessible boundaries; see CIR (1985a). From eq. (23) of CIR (1985b), the resulting equilibrium discount bond price is

$$D(r, \tau) = A(\tau)\exp(-B(\tau)r), \tag{5}$$

where

$$A(\tau) = \left(\frac{2\gamma\exp((\beta + \gamma)\tau/2)}{(\beta + \gamma)(\exp(\gamma\tau) - 1) + 2\gamma}\right)^{2\alpha/\sigma^2},$$

$$B(\tau) = \frac{2(\exp(\gamma\tau) - 1)}{(\beta + \gamma)(\exp(\gamma\tau) - 1) + 2\gamma},$$

$$\gamma = \sqrt{\beta^2 + 2\sigma^2}.$$

By construction, the discount bond price in (5) earns the equilibrium expected rate of return given in (2) for all r and τ in $(0, \infty) \times (0, \infty)$. While no explicit boundary condition is imposed at $r = 0$, the discount bond price converges to the well-defined mathematical function $A(\tau)$ as $r \to 0$.

In the alternative case in which $0 < 2\alpha < \sigma^2$, zero is a regular or attainable boundary for the risk-adjusted interest-rate process. In this case, the dynamics in (1) must be supplemented with a description of the behavior of r at $r = 0$ in order to uniquely specify the process beyond its first passage to zero. An important implication is that there are now many possible ways for the interest rate to behave after reaching zero. Karlin and Taylor (1981, p. 239) state: 'For a regular boundary a variety of boundary behavior can be prescribed in a consistent way, including the contingencies of complete absorption or reflecting, elastic or sticky barrier phenomena, and even the possibility of the particle (path), when attaining the boundary point, waiting there for an exponentially distributed duration followed by a jump into the interior of the state space according to a specified probability distribution.'

When $r = 0$ is attainable, there are many possible solutions to (3a) and (3b). To completely specify the fundamental valuation equation and obtain a unique solution, we must impose an additional boundary condition at $r = 0$. The form of the additional boundary condition is determined by the requirement that markets clear and that bonds earn an equilibrium rate of return when $r = 0$, *given* the representative investor's beliefs about future interest-rate behavior. Thus, different assumptions about the behavior of the interest rate at zero map into different equilibrium boundary conditions, which in turn map into different equilibrium solutions to the fundamental valuation equation. To illustrate, consider the following examples:

Example 1: *The Absorbing Equilibrium.* This equilibrium assumes that the short-term interest rate is absorbed at zero if it reaches zero. If absorption occurs, then market clearing requires that discount bonds be priced so that all forward rates are zero. Furthermore, since bonds are no longer stochastic when absorption occurs, absence of arbitrage requires that the expected return for each bond equal zero when $r = 0$. These two economic conditions are satisfied by imposing the boundary condition $D(0, \tau) = 1$.

Example 2: *The Unrestricted Equilibrium.* This equilibrium assumes that the behavior of the process at $r = 0$ is specified by extending the dynamics in (1) to $r = 0$; no other restriction is imposed. Thus, in this equilibrium, the interest-rate process returns immediately to positive values if it reaches zero.[2] When $r = 0$, the interest rate is locally deterministic and has dynamics $\mathrm{d}r = \alpha \mathrm{d}t$. Since the bond is also locally riskless, equilibrium requires that the rate of return on the

[2]This tendency is not necessarily the same as reflection at zero, which often requires placing additional restrictions on derivatives. For example, it may also be consistent with sticky behavior in which the interest rate spends a strictly positive amount of time on the boundary. Karlin and Taylor (1981, pp. 257–258) show that the duration on a sticky boundary has no interval. Hence, sticky behavior involves reaching zero, returning immediately to positive values, and returning to zero infinitely often.

bond be zero when $r = 0$. Assuming that D_r and D_τ are finite at $r = 0$, the boundary condition corresponding to the equilibrium requirement that the rate of return on the bond be zero at $r = 0$ is $\alpha D_r - D_\tau = 0$. Since the functional form for the value of a discount bond given by CIR holds for all values of $\alpha > 0$ and implies finite derivatives D_r and $D_\tau = 0$, at $r = 0$, unrestricted equilibrium discount bond prices are given by (5). Note that (5) can also be viewed as the solution to (3a) and (3b) with the boundary condition that the discount bond price equals $A(\tau)$ at $r = 0$, which implies strictly positive forward rates at $r = 0$.

Example 3: *Empirical Equilibria.* A variety of other solutions can be obtained by directly imposing boundary conditions on the bond price at $r = 0$. As in the previous examples, however, market-clearing and absence-of-arbitrage conditions require that any boundary condition must imply not only nonnegative forward rates but also an equilibrium expected rate of return when $r = 0$ in order to be a valid equilibrium. In some cases, the boundary condition may be chosen in a way that causes the resulting discount bond prices to fit the observed term structure. This feature could provide an additional degree of freedom in developing general equilibrium models that are consistent with the actual properties of the term structure.

3. A specific example

As a specific illustration of an alternative equilibrium in the CIR framework, we derive the value of a unit discount bond when the short-term interest rate is absorbed at zero. As shown above, the value of the discount bond in the absorbing equilibrium is obtained by solving (3a) and (3b) subject to the boundary condition $D(0, \tau) = 1$. The resulting absorbing equilibrium value for the discount bond is

$$D(r,\tau) = A(\tau)\exp(-B(\tau)r)\Bigg[P(1-2\alpha/\sigma^2, C(\tau)r) + \frac{1}{\Gamma(1-2\alpha/\sigma^2)} \times \int_0^\tau \frac{\beta + \sigma^2 B(t) + \sigma^2 C(t)/2}{A(\tau - t)} \exp(-C(t)r)(C(t)r)^{1-2\alpha/\sigma^2}\, dt\Bigg], \tag{6}$$

where $A(\tau)$ and $B(\tau)$ are as defined in (5), $\Gamma(\cdot)$ is the gamma function, $P(\cdot,\cdot)$ is the incomplete gamma function, and

$$C(\tau) = \frac{2\gamma^2 \exp(\gamma\tau) B(\tau)}{\sigma^2(\exp(\gamma\tau) - 1)^2}.$$

Table 1

Yields to maturity implied by the Cox, Ingersoll, and Ross (1985b) unrestricted equilibrium. Yields are expressed in basis points. Maturities are expressed in years. The underlying parameter values are $\alpha = 0.0025$, $\beta = 0.05$, and $\sigma^2 = 0.01$.

	Short-term interest rate				
Maturity	0.005	0.02	0.04	0.06	0.08
1	61.0	207.0	401.8	596.6	791.4
2	71.4	213.2	402.4	591.5	780.6
3	81.2	218.6	401.8	585.0	768.2
4	90.5	223.3	400.3	577.3	754.4
5	99.2	227.3	398.0	568.8	739.5
6	107.4	230.7	395.1	559.5	723.9
7	115.1	233.6	391.7	549.8	707.9
8	122.2	236.1	387.9	539.8	691.6
9	128.9	238.2	383.9	529.6	675.4
10	135.1	240.0	379.7	519.5	659.3
11	140.9	241.4	375.4	509.5	643.5
12	146.3	242.6	371.1	499.6	628.1
13	151.3	243.7	366.9	490.1	613.3
14	155.9	244.5	362.7	480.8	598.9
15	160.3	245.3	358.5	471.9	585.2
16	164.3	245.9	354.6	463.3	572.1
17	168.1	246.4	350.8	455.2	559.6
18	171.6	246.8	347.1	447.4	547.7
19	174.9	247.2	343.6	440.0	536.4
20	178.0	247.5	340.2	432.9	525.6

The gamma and incomplete gamma functions are described in Abramowitz and Stegun (1970, ch. 6). Differentiation shows that (6) satisfies the partial differential equation in (3a). In addition, (6) satisfies the maturity condition $D(r, 0) = 1$, since $P(1 - 2\alpha/\sigma^2, C(\tau)r)$ converges to one and the integral converges to zero as $\tau \to 0$. The integral term in (6) is the expected value of $1/A(\tau - t)$ at the time of the first passage of r to zero, where the expectation is taken with respect to the first-passage density for the risk-neutral process for r described in the separation theorem in Longstaff (1990). The first-passage density is the time derivative of the first-passage distribution function $1 - P(1 - 2\alpha/\sigma^2, C(\tau)r)$ of r to zero. Since this first-passage density converges to a Dirac delta function as $r \to 0$, we define the value of the integral term at zero to equal its limit as $r \to 0$. With this definition, the integral term converges to $1/A(\tau)$ as $r \to 0$. Hence, (6) satisfies the boundary condition $D(0, \tau) = 1$. Finally, $D(r, \tau)$ converges to zero as $r \to \infty$.

To examine the differences between prices implied by the unrestricted equilibrium and the absorbing equilibrium, tables 1 and 2 present yields for the unrestricted and absorbing equilibria, respectively. Table 3 reports the difference between the unrestricted and absorbing yields. The parameter values used in the tables imply a long-run average value of r of 0.05.

Table 2

Yields to maturity implied by the Cox, Ingersoll, and Ross (1985b) absorbing equilibrium. Yields are expressed in basis points. Maturities are expressed in years. The underlying parameter values are $\alpha = 0.0025$, $\beta = 0.05$, and $\sigma^2 = 0.01$.

	Short-term interest rate				
Maturity	0.005	0.02	0.04	0.06	0.08
1	60.6	207.0	401.8	596.6	791.4
2	69.1	213.1	402.4	591.5	780.6
3	75.7	218.0	401.7	585.0	768.2
4	80.9	221.7	400.0	577.3	754.3
5	84.9	224.3	397.3	568.5	739.4
6	88.0	225.8	393.7	559.0	723.7
7	90.3	226.4	389.3	548.8	707.4
8	91.8	226.1	384.2	538.1	690.8
9	92.9	225.2	378.6	527.1	674.0
10	93.4	223.6	372.5	515.8	657.2
11	93.5	221.6	366.1	504.4	640.5
12	93.3	219.1	359.4	492.9	623.9
13	92.7	216.2	352.5	481.4	607.7
14	91.9	213.1	345.5	470.1	591.8
15	90.9	209.7	338.3	458.8	576.2
16	89.8	206.1	331.2	447.8	561.1
17	88.5	202.4	324.0	436.9	546.3
18	87.1	198.6	316.9	426.2	532.0
19	85.6	194.7	309.8	415.8	518.2
20	84.1	190.8	302.8	405.7	504.7

The term structures implied by the two equilibria can be significantly different. When $r = 0.02$, the unrestricted term structure is monotone increasing, while the absorbing term structure has a hump at a maturity of seven years, which is important since an often-cited criticism of the CIR unrestricted model is its inability to generate humps at longer maturities.

Tables 1 and 2 show that the absorbing yields are always less than the unrestricted yields. Intuitively, the economic implication of absorption is that future cash flows are no longer discounted. Hence, absorption has the same effect as accelerating the payment of principal on the discount bond.

Table 3 shows that the difference between the unrestricted and absorbing yields is generally an increasing function of maturity; logically, an increase in maturity implies an increasing probability of absorption. The difference in the yields is a decreasing function of r, because an increase in r lessens the probability of absorption occurring during the life of a discount bond. Note that the magnitudes of the differences in yields can be quite large for small values of r and can be economically significant for values of r that are above the long-run mean value of r. These results drive home the point that by imposing different

Table 4

Summary statistics for three-month Treasury-bill yields and monthly changes in three-month Treasury-bill yields for the period January 1930 to December 1940. The data are taken from table A1 of Cecchetti (1988). Yields are expressed in annualized terms.

	Yields	Changes in yields
Mean	0.00519	− 0.00027
Std. dev.	0.00767	0.00267
Minimum	0.00050	− 0.01200
Median	0.00200	0.00000
Maximum	0.03840	0.01130
ρ_1	0.867	0.157
ρ_2	0.750	0.067
ρ_3	0.636	− 0.045
N	132	132

about the behavior of the short-term rate needs to be made, either implicitly or explicitly, to obtain a specific model. Which assumption about the boundary behavior of the short-term interest rate is most realistic is ultimately an empirical issue.

To provide some evidence on this issue, table 4 reports summary statistics for the three-month Treasury-bill yield during the 1930–1940 period. These yields are taken from Cecchetti (1988, table A1) and are based on the midpoint of bid and ask quotations from the *New York Times.* This timeframe is chosen since it includes the lowest three-month Treasury-bills during this century. In addition, interest on Treasury-bills was not taxable throughout this sample period. Fig. 1 plots the time series of three-month Treasury-bill yields.

As shown, Treasury-bill yields were near zero for extended periods of time. In particular, fig. 1 shows that extremely low levels of yields tend to persist rather than immediately increasing back toward higher levels. This behavior appears to be more consistent with sticky boundary behavior than with reflecting behavior as corroborated by the serial correlation of changes in yields: reflecting behavior would induce mean reversion in yields at low interest-rate levels, but in fact, the first-order serial correlation of monthly changes in 0.157, which is significant at the 0.10 level.

6. Conclusion

By allowing boundaries to be accessible, multiple equilibria can be accommodated in a variety of term structure models, making it possible to generate an entire class of general equilibrium term structure models from a single framework. Furthermore, when the origin is accessible, it may be possible to impose boundary conditions that make the model fit the current yield or forward rate

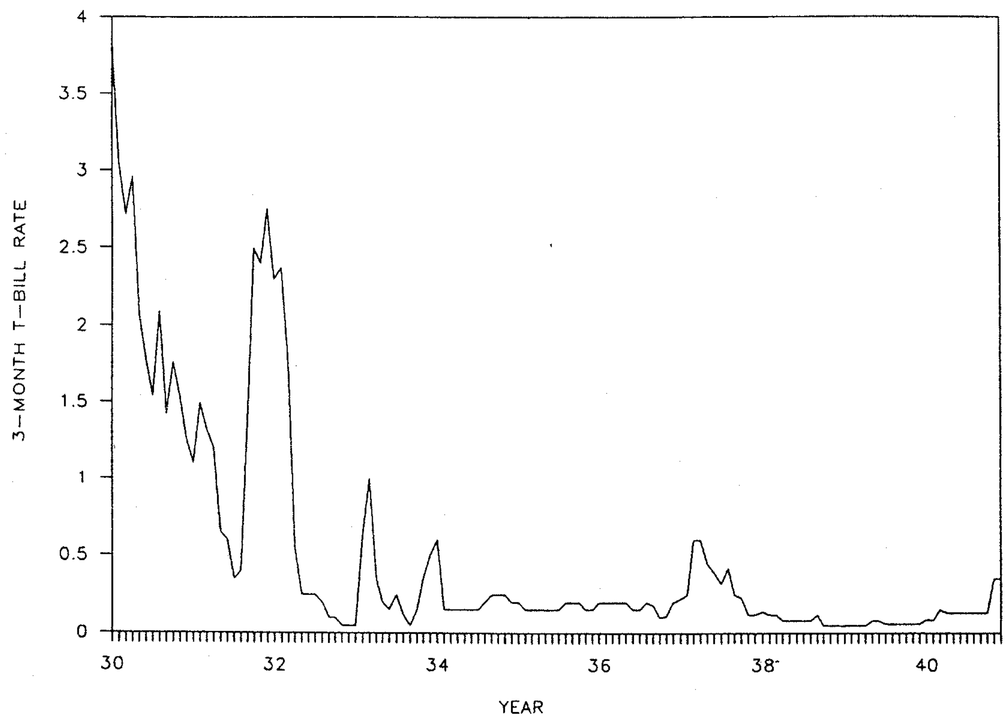

Fig. 1. The three-month Treasury-bill rate during 1930 to 1940.

curve, as in Ho and Lee (1986), Hull and White (1990), Black and Karasinski (1991), and Heath, Jarrow, and Morton (1992).

References

Abramowitz, Milton and Irene A. Stegun, 1970, Handbook of mathematical functions (Dover Publications, New York, NY).

Black, Fischer and Piotr Karasinski, 1991, Bond and option pricing when short rates are lognormal, Financial Analysts Journal 47, 52–59.

Cecchetti, Stephen G., 1988, The case of the negative nominal interest rates: New estimates of the term structure of interest rates during the Great Depression, Journal of Political Economy 96, 1111–1141.

Cox, John C., Jonathan E. Ingersoll, and Stephen A. Ross, 1985a, An intertemporal general equilibrium model of asset prices, Econometrica 53, 363–384.

Cox, John C., Jonathan E. Ingersoll, and Stephen A. Ross, 1985b, A theory of the term structure of interest rates, Econometrica 53, 385–407.

Feller, William F., 1951, Two singular diffusion problems, Annals of Mathematics 54, 173–182.

Karlin, Samuel and Howard M. Taylor, 1981, A second course in stochastic processes (Academic Press, New York, NY).

Heath, David, Robert Jarrow, and Andrew Morton, 1992, Bond pricing and the term structure of interest rates: A new methodology, Econometrica 60, 77–106.

Ho, Thomas S.Y. and S.B. Lee, 1986, Term structure movements and pricing interest rate contingent claims, Journal of Finance 41, 1011–1029.

Hull, John and Alan White, 1990, Pricing interest-rate derivative securities, Review of Financial Studies 3, 573–592.

Longstaff, Francis A., 1989, A nonlinear general equilibrium model of the term structure of interest rates, Journal of Financial Economics 23, 195–224.

Longstaff, Francis A., 1990, The valuation of options on yields, Journal of Financial Economics 26, 97–121.