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Asset Pricing in Markets with Illiquid Assets

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Abstract

Many important classes of assets are illiquid in the sense that they cannot always be traded immediately. Thus, a portfolio position in these types of illiquid investments becomes at least temporarily irreversible. We study the asset-pricing implications of illiquidity in a two-asset exchange economy with heterogeneous agents. In this market, one asset is always liquid. The other asset can be traded initially, but then not again until after a “blackout” period. Illiquidity has a dramatic effect on optimal portfolio decisions. Agents abandon diversification as a strategy and choose highly polarized portfolios instead. The value of liquidity can represent a large portion of the equilibrium price of an asset. We present examples in which a liquid asset can be worth up to 25 percent more than an illiquid asset even though both have identical cash flow dynamics. We also show that the expected return and volatility of an asset can change significantly as the asset becomes relatively more liquid.

**ASSET PRICING IN MARKETS
WITH ILLIQUID ASSETS**

Francis A. Longstaff*

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*Allstate Professor of Insurance and Finance, Anderson School at UCLA and the NBER. This paper incorporates and significantly extends results previously circulated in a paper titled Financial Claustrophobia: Asset Pricing in Illiquid Markets. I am grateful for valuable comments from Tony Bernardo, Duke Bristow, John Campbell, John Cochrane, Darrell Duffie, John Heaton, Harrison Hong, Ravi Jagannathan, Robert Jarrow, Martin Lettau, Jun Liu, Rajnish Mehra, Bruno Miranda, Robert Novy-Marx, Lasse Pedersen, Monika Piazzesi, Matt Pritzker, Pedro Santa-Clara, Avanidhar Subrahmanyam, and Dimitri Vayanos, and from seminar participants at UC Berkeley, Cornell, the Federal Deposit Insurance Corporation, the NBER Asset Pricing Summer Conference, NYU, Northwestern, Maryland, Princeton, Simon Fraser University, Stanford, UC Irvine, and UCLA. All errors are my responsibility.

ABSTRACT

Many important classes of assets are illiquid in the sense that they cannot always be traded immediately. Thus, a portfolio position in these types of illiquid investments becomes at least temporarily irreversible. We study the asset-pricing implications of illiquidity in a two-asset exchange economy with heterogeneous agents. In this market, one asset is always liquid. The other asset can be traded initially, but then not again until after a “blackout” period. Illiquidity has a dramatic effect on optimal portfolio decisions. Agents abandon diversification as a strategy and choose highly polarized portfolios instead. The value of liquidity can represent a large portion of the equilibrium price of an asset. We present examples in which a liquid asset can be worth up to 25 percent more than an illiquid asset even though both have identical cash flow dynamics. We also show that the expected return and volatility of an asset can change significantly as the asset becomes relatively more liquid.

1. INTRODUCTION

One of the cornerstones of traditional asset-pricing theory is the assumption that all assets are liquid and readily tradable by economic agents. In reality, however, many important classes of assets are not completely liquid, and agents often cannot buy and sell them immediately. For example, a large percentage of the wealth of the typical household is held in the form of illiquid assets such as human capital, sole proprietorships, partnerships, and equity in other closely-held firms, deferred compensation, pension plans, tax-deferred retirement accounts, savings bonds, annuities, trusts, inheritances, and residential real estate. On the institutional side, an increasing amount of wealth is being allocated to illiquid asset classes such as private equity, emerging markets, venture capital, commercial real estate, and the rapidly-growing hedge fund sector. In each of these examples, an investor might have to wait months, years, or even decades before being able to unwind a position.¹

Asset illiquidity has major implications for asset pricing since it changes the economics of portfolio choice in a fundamental way. When there are assets that cannot be bought or resold immediately, portfolio decisions take on an important dimension of permanence or irreversibility. In this sense, illiquidity in financial investment parallels the role that irreversibility of physical investment plays in the real options literature. Asset illiquidity raises a number of key asset-pricing issues. For example, how does illiquidity affect optimal portfolio decisions? What risks are created by asset illiquidity? How does the liquidity of an asset affect its own price and the prices of other assets?

To address these issues, this paper examines the asset-pricing implications of illiquidity within a continuous-time exchange economy with multiple assets and heterogeneous agents. In this framework, agents have identical logarithmic preferences, but differ in their subjective time discount factors. Thus, we can characterize the agents as either patient or impatient. Liquid assets can always be traded. Illiquid assets can be traded initially, but then cannot be traded again until after a fixed horizon or trading “blackout” period. Although simple, this approach to modeling illiquidity is consistent with many actual examples of illiquidity observed in the markets such as IPO lockups, Rule 144 restrictions on security transfers, trading blackouts around earnings announcements, etc. Also, this approach has the advantage of capturing the intuitive notion of illiquidity as the absence of immediacy.

¹A familiar example is the decision of a student to invest in his or her human capital by getting an MBA degree. This investment (which typically requires a large financial commitment and hopefully pays dividends throughout the student’s life) is completely illiquid and cannot be unwound or reversed.

As a benchmark for comparison, we first present results for the traditional case where both assets are fully liquid. In this case, we obtain the standard result that all agents choose to hold the market portfolio. Furthermore, assets with identical cash flow dynamics have identical prices. With an illiquid asset, however, asset pricing becomes dramatically different. For example, the logarithmic agents cease being myopic and no longer choose to hold the market portfolio. Rather, the impatient agent tilts his portfolio towards the liquid asset and holds less of the illiquid asset. In some cases, the agents may almost completely abandon diversification as a strategy. The portfolio for the impatient agent may consist almost entirely of the liquid asset, while the opposite is true for the patient agent.

Intuitively, the reason for this polarization in portfolio choice stems from the interplay of two key factors. First, the impatient agent has a strong incentive to accelerate his consumption by selling his portfolio over time to the patient agent. To do so, however, the impatient agent needs to hold more of the liquid asset than is optimal in the fully-liquid case. Second, illiquidity creates a new need for intertemporal risk sharing not present in traditional asset pricing models with fully-liquid assets. In particular, even though the agents have identical attitudes towards instantaneous risk, the impatient agent is better able to bear the portfolio-irreversibility risk induced by illiquidity. This is because his higher subjective discount rate dampens the effect of noninstantaneous risk on his utility. In equilibrium, prices adjust to give the impatient agent an incentive to hold riskier assets. When the liquid asset is riskier, these two factors combine to make the optimal portfolio much more polarized. When the illiquid asset is riskier, these two factors tend to offset and the optimal portfolio is less polarized.

In addition to the effects on portfolio choice, asset illiquidity has major implications for asset prices. In general, the more polarized the optimal portfolios, the greater are the valuation effects. Assets with identical cash flow dynamics, that otherwise would have the same price, can differ in value by as much as 25 percent in the presence of illiquidity. As might be expected, illiquidity generally has the effect of making the liquid asset more valuable, and the illiquid asset less valuable, relative to what prices would be in the fully-liquid case. Surprisingly, however, the opposite can be true. This situation occurs primarily when the impatient agent's incentive to hold the liquid asset conflicts with his incentive to hold the riskier asset.

Illiquidity affects not only current prices, but also the distribution of future returns for the liquid asset. We show that it is possible for both the initial price and the expected return of the liquid asset to be higher when there is an illiquid asset in the market. The intuition for this is that the portfolio effects resulting from illiquidity have long-term implications for the distribution of wealth among agents. Thus, even after the illiquidity horizon lapses and assets become fully liquid again, their prices need not revert back to what they would have been otherwise. Because of this, the presence of illiquidity can induce long-term effects and path dependency in asset returns. We also show that asset illiquidity can have a large dampening effect on the volatility of the liquid asset's returns.

To provide additional insight into the asset-pricing effects of illiquidity, we also consider the scenario in which neither asset can be traded during the trading “blackout” period. Thus, the entire market becomes illiquid in this scenario. This extreme illiquidity has parallels with actual markets in which search or adverse selection costs become so high that trading cannot always occur. In this illiquid-market scenario, there is no liquid asset for the agents to hold. Thus, the intertemporal risk-sharing effects predominate and initial asset prices adjust so that the impatient agent is willing to hold a portfolio consisting almost entirely of the risky asset. We show that asset-pricing effects can again be very large. Interestingly, riskier assets can become more valuable when agents face this type of market illiquidity, and vice versa. The reason for this apparent breakdown in the usual risk-return relation stems from the extreme polarization of the agents’ portfolios.

In summary, this paper makes four key points about asset pricing and illiquidity. First, illiquidity affects optimal portfolio choice profoundly. In particular, agents with identical attitudes toward risk may rationally choose to hold very different types of portfolios. These results may help explain a number of well-known puzzles about the way actual agents choose to invest (for example, Mankiw and Zeldes (1991) and Benartzi and Thaler (2001)). Second, the type of illiquidity we consider in this paper can have first-order effects on equilibrium asset prices. This contrasts sharply with the transactions-cost or trading-friction literature in which trading costs have only second-order effects on asset prices (for example, Constantinides (1986)). Third, these results demonstrate that the value of a liquid asset can be greater than the “simple” present value of its cash flows. Specifically, we show that a liquid asset can be worth significantly more than an illiquid asset with the identical cash flow dynamics. Thus, in asset pricing, it is not directly the dividend cash flows that matter, but rather the consumption stream that asset ownership generates. Finally, these results illustrate that differences in patience across investors can have major asset-pricing effects. Thus, heterogeneity in patience may provide an important (but underresearched) channel for understanding financial markets and resolving asset-pricing puzzles. Furthermore, differences in patience may map well into the familiar notions of short-horizon financial institutions and longer-horizon investors often found in the literature.

This paper contributes to the growing literature on the asset-pricing effects of illiquidity. Important empirical work in this area includes Amihud and Mendelson (1986, 1991), Cornell and Shapiro (1990), Boudoukh and Whitelaw (1991), Silber (1992), Daves and Ehrhardt (1993), Kamara (1994), Grinblatt and Longstaff (2000), Brenner, Eldor, and Hauser (2001), Chordia, Roll, and Subrahmanyam (2001), Krishnamurthy (2002), Pastor and Stambaugh (2003), Wang (2003), Longstaff (2004), and many others. Similarly, important theoretical contributions include Mayers (1972, 1973, 1976), Lippman and McCall (1986), Constantinides (1986), Amihud and Mendelson (1986), Grossman and Laroque (1990), Boudoukh and Whitelaw (1993), Longstaff (1995, 2001), Vayanos (1998, 2003), Hölmstrom and Tirole (2001), Weill (2002), Kahl, Liu, and Longstaff (2003), Acharya and Pedersen (2003), Huang (2003), Duffie, Gârleanu, and Pedersen (2003a, 2003b), O’Hara (2003), Vayanos and

Wang (2003), Eisfeldt (2004), and many others. Finally, the approach taken in this paper of modeling illiquidity as portfolio irreversibility has a number of parallels in the literature on asset pricing and uninsurable labor income. Important papers in this literature include Aiyagari and Gertler (1991), Heaton and Lucas (1992, 1996), Telmer (1993), Duffie and Constantinides (1996), and others.

The remainder of this paper is organized as follows. Section 2 describes the basic modeling framework used throughout the paper. Section 3 presents the fully-liquid benchmark case. Section 4 presents the illiquid-asset case. Section 5 compares the two cases and discusses the asset-pricing implications of asset illiquidity. Section 6 extends the basic framework to the illiquid-market case. Section 7 summarizes the results and make concluding remarks.

2. THE MODEL

The key challenge in exploring the asset-pricing implications of illiquidity is developing a modeling framework in which agents endogenously want to trade, and, therefore, care about whether they can buy or sell individual assets. This task is complicated by the fact that many standard paradigms in asset pricing, such as the familiar single-asset representative-agent model, imply that there is no trading in equilibrium. Thus, to capture the incentives that agents have for trading assets over time, we need to move beyond the usual types of asset-pricing models. To this end, we develop a simple two-asset version of the standard Lucas (1978) pure exchange economy in which there are two heterogeneous agents. When both assets can be traded, the market is effectively dynamically complete. When one asset is illiquid, however, the market becomes dynamically incomplete. All models are stylized to some degree, and the model we present is no exception to the rule. Despite this, however, the model does capture the economics of asset illiquidity in an intuitive way and is consistent with many actual forms of asset illiquidity observed in the markets.

The basic structure of the model can be viewed as the extension of Cochrane, Longstaff, and Santa-Clara (2004) to a heterogeneous agent economy. There are two assets or “trees” in this economy. Each asset produces a stream of dividends in the form of the single consumption good. Let X_t and Y_t denote the dividends generated by the assets. The dividends follow simple i.i.d. geometric Brownian motions

$$\frac{dX}{X} = \mu_X dt + \sigma_X dZ_X, \tag{1}$$

$$\frac{dY}{Y} = \mu_Y dt + \sigma_Y dZ_Y, \tag{2}$$

where the correlation between dZ_X and dZ_Y is ρdt . Although we refer to the assets as risky throughout the discussion, nothing prevents one of the assets from being a riskless bond since σ_X or σ_Y can equal zero.² We normalize the number of shares of each asset in the economy to be one. To keep notation as simple as possible, expectations and variables without time subscripts (such as X and Y) will denote initial or time-zero values.

There are two agents in this model. The first agent is endowed with w shares of the first and second assets, respectively. Thus, the second agent is endowed with $1 - w$ shares of the two assets. Denote the agents' consumption streams by C_t and D_t , respectively. Since the effect of illiquidity in this model is that agents may have to wait to rebalance their portfolios, intuition suggests that attitudes about waiting may play a central role in the economics of illiquidity. Accordingly, we allow for heterogeneity in the agents' level of patience. Specifically, we assume that preferences at time zero are given by

$$\ln(C) + E \left[\int_0^\infty e^{-\beta t} \ln(C_t) dt \right], \quad (3)$$

$$\ln(D) + E \left[\int_0^\infty e^{-\delta t} \ln(D_t) dt \right], \quad (4)$$

where the subjective time discount rates β and δ may differ, implying that one agent is less patient than the other. For concreteness, assume that the first agent is less patient than the second, $\beta > \delta$. We refer to the first and second agents as the impatient and patient agents, respectively.³

3. THE FULLY-LIQUID CASE

As a preliminary, we present results for the case where both assets are fully liquid. These results will be used as a benchmark for comparison to the illiquid-asset case in later sections.

²More specifically, one of the assets can be a riskless consol bond in positive net supply by setting either σ_X or σ_Y equal to zero.

³These preferences imply that consumption at time zero takes the form of a discrete “gulp,” while consumption at all future times occurs as a flow. Thus, X and Y represent discrete dividends while X_t and Y_t represent dividend flows. This feature simplifies the exposition of the model, but has no effect on any of the results. In particular, the model could easily be modified to allow only continuous consumption, or only discrete consumption, without any loss of generality.

Let P_t and Q_t denote the equilibrium prices for the first and second assets. The first-order conditions for the first agent imply

$$P_t = E_t \left[\int_0^\infty e^{-\beta s} \left(\frac{C_t}{C_{t+s}} \right) X_{t+s} ds \right], \quad (5)$$

$$Q_t = E_t \left[\int_0^\infty e^{-\beta s} \left(\frac{C_t}{C_{t+s}} \right) Y_{t+s} ds \right]. \quad (6)$$

Similarly, the first-order conditions for the second agent imply

$$P_t = E_t \left[\int_0^\infty e^{-\delta s} \left(\frac{X_t + Y_t - C_t}{X_{t+s} + Y_{t+s} - C_{t+s}} \right) X_{t+s} ds \right], \quad (7)$$

$$Q_t = E_t \left[\int_0^\infty e^{-\delta s} \left(\frac{X_t + Y_t - C_t}{X_{t+s} + Y_{t+s} - C_{t+s}} \right) Y_{t+s} ds \right], \quad (8)$$

after substituting in the market clearing condition $D_t = X_t + Y_t - C_t$.

In this setting, the first agent chooses his consumption C_t , and a portfolio consisting of N_t and M_t shares of the two assets, respectively, and similarly for the second agent. The Appendix shows that equilibrium consumption of the first agent is given by

$$C_t = \frac{e^{-\beta t} \beta (1 + \delta) w}{e^{-\beta t} \beta (1 + \delta) w + e^{-\delta t} \delta (1 + \beta) (1 - w)} (X_t + Y_t). \quad (9)$$

From Equation (9), optimal consumption depends directly on the distribution of wealth in the economy as measured by the fraction w of total assets initially held by the first agent.

Similarly, the optimal portfolio for the first agent is given by

$$N_t = M_t = \frac{e^{-\beta t} (1 + \delta) w}{e^{-\beta t} (1 + \delta) w + e^{-\delta t} (1 + \beta) (1 - w)}. \quad (10)$$

This optimal portfolio rule has several important aspects. First, each agent holds a portfolio that has the same number of shares of each of the two assets. Since there are equal numbers of shares of the two assets in the economy, however, this means that the optimal portfolio for each agent is simply the market portfolio. Thus, a one-fund separation result holds;

agents would be indifferent between trading the assets individually or trading the shares of a stock index fund. Second, trading occurs in equilibrium since the number of shares held by the two agents changes over time. In particular, since the first agent is less patient than the second, the first agent systematically sells his portfolio to the second agent over time. This enables the first agent to consume more than the total dividends he receives initially. Over time, however, the share of dividends consumed by the first agent declines while the share of dividends consumed by the second agent increases. Third, the portfolio rule is a deterministic function of time; the portfolio rule does not vary with changes in the state variables X_t and Y_t . Finally, the optimal portfolio rule is also affected by the initial distribution of wealth as measured by w .

Asset prices can now be obtained by substituting the optimal consumption process into the first-order equations and evaluating the expectations. The Appendix shows that the prices for the two assets are given by

$$P_t = C_t A(\beta, X_t, Y_t) + (X_t + Y_t - C_t) A(\delta, X_t, Y_t), \quad (11)$$

$$Q_t = C_t B(\beta, X_t, Y_t) + (X_t + Y_t - C_t) B(\delta, X_t, Y_t), \quad (12)$$

where we define the functions $A(\cdot, X_t, Y_t)$ and $B(\cdot, X_t, Y_t)$ as

$$A(\cdot, X_t, Y_t) = k_1 \left(\frac{X_t}{Y_t} \right) F \left(1, 1 - \gamma; 2 - \gamma; -\frac{X_t}{Y_t} \right) + k_2 F \left(1, \theta; 1 + \theta; -\frac{Y_t}{X_t} \right), \quad (13)$$

$$B(\cdot, X_t, Y_t) = k_3 \left(\frac{Y_t}{X_t} \right) F \left(1, 1 + \theta; 2 + \theta; -\frac{Y_t}{X_t} \right) - k_4 F \left(1, -\gamma; 1 - \gamma; -\frac{X_t}{Y_t} \right), \quad (14)$$

and where $k_1, k_2, k_3, k_4, \gamma$, and θ are constants defined in the Appendix. The function $F(a, b; c; z)$ is the standard hypergeometric function (see Abramowitz and Stegun (1970) Chapter 15). The hypergeometric function is defined by the power series

$$F(a, b; c; z) = 1 + \frac{a \cdot b}{c \cdot 1} z + \frac{a(a+1) \cdot b(b+1)}{c(c+1) \cdot 1 \cdot 2} z^2 + \frac{a(a+1)(a+2) \cdot b(b+1)(b+2)}{c(c+1)(c+2) \cdot 1 \cdot 2 \cdot 3} z^3 + \dots \quad (15)$$

The hypergeometric function has an integral representation, which can be used for numerical evaluation and as an analytic continuation beyond $\|z\| < 1$,

$$F(a, b; c; z) = \frac{\Gamma(c)}{\Gamma(b)\Gamma(c-b)} \int_0^1 w^{b-1} (1-w)^{c-b-1} (1-wz)^{-a} dw, \quad (16)$$

where $\text{Re}(c) > \text{Re}(b) > 0$.

Interestingly, the functions $A(\cdot, X_t, Y_t)$ and $B(\cdot, X_t, Y_t)$ have intuitive interpretations as the equilibrium prices for the first and second assets (normalized by total consumption) that would exist in a representative-agent version of this economy where the subjective time discount rate for the representative agent equals the first argument. Thus, the price functions P_t and Q_t in Equations (11) and (12) are simple consumption-weighted (or, equivalently, wealth-weighted) averages of the prices in a representative-agent economy with subjective discount rates of β and δ , respectively.

4. THE ILLIQUID-ASSET CASE

In this section, we relax the assumption that both assets are always liquid. Specifically, we assume that the first asset can always be traded. Thus, there is always a way for agents to smooth their intertemporal consumption by trading in the first asset. Note that the first asset can play the role of a riskless asset in this framework since the volatility of its dividends can be set equal to zero and the agents are allowed to take whatever long or short position in this asset they choose.⁴ The second asset can be traded at time zero, but then becomes illiquid and cannot be traded again until time T . After time T , the second asset reverts back to being fully liquid. Thus, this approach models the illiquidity of the second asset as a trading “blackout” period. By contrasting this case with the benchmark fully-liquid case, we can examine directly how asset prices and portfolio choice are affected by illiquid assets in the market.

In this setting, agents choose an initial consumption level C and a portfolio of assets N and M at time zero. Once the portfolio is chosen, however, the number of shares of the second asset cannot be changed until time T . Thus, $M_t = M$ for $t < T$. Each agent’s initial consumption C consists of their endowment less the value of the portfolio they choose,

⁴The fact that the asset is in positive net supply rather than in zero net supply has little effect on the results. In equilibrium, market clearing requires that the total amount of the first asset must be held by market participants. Thus, the existence of this market clearing condition is important in determining the qualitative nature of the equilibrium. Whether the market clears at a level of zero, one, two, etc. shares is qualitatively less important than the fact that some market clearing condition is imposed.

$$C = w(P + X + Q + Y) - NP - MQ. \quad (17)$$

Subsequent consumption equals the dividends on their portfolio net of changes in their holdings of the first asset,

$$C_t = N_t X_t + M Y_t - H_t P_t, \quad (18)$$

for $0 < t < T$, where H_t is the rate at which shares of the first asset are sold. By definition,

$$N_t = N + \int_0^t H_s ds. \quad (19)$$

At time T , both assets become fully liquid again and the framework reverts back to the benchmark fully-liquid case, although with one slight difference. Specifically, the first agent arrives at time T with N_T (before rebalancing the portfolio) and M shares of the two assets, respectively, rather than with w shares of each asset. With this modification, the Appendix shows that optimal consumption at time T is given by,

$$C_T = \frac{(N_T A(\delta, X_T, Y_T) + M B(\delta, X_T, Y_T)) (X_T + Y_T)}{\frac{1}{\beta} + N_T (A(\delta, X_T, Y_T) - A(\beta, X_T, Y_T)) + M (B(\delta, X_T, Y_T) - B(\beta, X_T, Y_T))}. \quad (20)$$

Similarly, the prices of the two assets at time T are again given by

$$P_T = C_T A(\beta, X_T, Y_T) + (X_T + Y_T - C_T) A(\delta, X_T, Y_T), \quad (21)$$

$$Q_T = C_T B(\beta, X_T, Y_T) + (X_T + Y_T - C_T) B(\delta, X_T, Y_T). \quad (22)$$

but where C_T is defined as in Equation (20).

Solving for the values of the assets at time zero in this illiquid-asset case requires an recursive approach in which we first solve for the value of the liquid asset and the optimal number of shares of the first asset to hold at times $T - \Delta t$, $T - 2\Delta t$, $T - 3\Delta t$, ..., Δt , conditional on the corresponding state variables. In doing this, we make sequential use of the first-order conditions implied by the agents' optimal portfolio and consumption choices

$$P_t = E_t \left[\int_0^{T-t} e^{-\beta s} \left(\frac{C_t}{C_{t+s}} \right) X_{t+s} ds + e^{-\beta(T-t)} \left(\frac{C_t}{C_T} \right) P_T \right], \quad (23)$$

$$P_t = E_t \left[\int_0^{T-t} e^{-\delta s} \left(\frac{X_t + Y_t - C_t}{X_{t+s} + Y_{t+s} - C_{t+s}} \right) X_{t+s} ds \right. \\ \left. + e^{-\delta(T-t)} \left(\frac{X_t + Y_t - C_t}{X_T + Y_T - C_T} \right) P_T \right]. \quad (24)$$

The Appendix describes the recursive numerical approach that is used to solve these pairs of equations for the values of P_t and N_t .

Once this recursive process is complete, we then need to solve for the time-zero values N , M , P , and Q . At time zero, the first-order conditions for the first agent imply

$$P = E \left[\int_0^T e^{-\beta t} \left(\frac{C}{C_t} \right) X_t dt + e^{-\beta T} \left(\frac{C}{C_T} \right) P_T \right], \quad (25)$$

$$Q = E \left[\int_0^T e^{-\beta t} \left(\frac{C}{C_t} \right) Y_t dt + e^{-\beta T} \left(\frac{C}{C_T} \right) Q_T \right], \quad (26)$$

where consumption values are given as above. Similarly, the first-order conditions for the second agent imply

$$P = E \left[\int_0^T e^{-\delta t} \left(\frac{X + Y - C}{X_t + Y_t - C_t} \right) X_t dt + e^{-\delta T} \left(\frac{X + Y - C}{X_T + Y_T - C_T} \right) P_T \right], \quad (27)$$

$$Q = E \left[\int_0^T e^{-\delta t} \left(\frac{X + Y - C}{X_t + Y_t - C_t} \right) Y_t dt + e^{-\delta T} \left(\frac{X + Y - C}{X_T + Y_T - C_T} \right) Q_T \right]. \quad (28)$$

The Appendix shows how the equilibrium values of N , M , P , and Q can be determined from these four equations numerically.⁵ A key difference between the equilibrium trading rules in the fully-liquid case and in this illiquid-asset case is that trading becomes state dependent

⁵As shown by Diamond (1967), Geanakoplos and Polemarchakis (1986), Geanakoplos, Mag-

when only the liquid asset can be traded. In fact, illiquidity can have large effects on the amount of trading that occurs in equilibrium in this framework.

5. ASSET-PRICING IMPLICATIONS

In this section, we explore the asset-pricing implications of illiquidity. To make these as clear as possible, we present numerical results for scenarios that highlight key asset-pricing effects such as the how illiquidity changes optimal portfolio decisions, equilibrium prices, and return distributions.

To make the results easier to compare, we use a common format and set of parameters in reporting the results for these scenarios. Specifically, in each scenario, we provide results for the benchmark fully-liquid case (in which $T = 0$), and then report results for illiquidity horizons of 2, 5, 10, 20, and 30 years. These values reflect degrees of asset illiquidity ranging from private equity, IPO, or hedge-fund lockups all the way to the extended illiquidity of human capital, pensions, or other retirement assets. In each scenario, the parameter values μ_X , μ_Y , and ρ are fixed at zero, and the subjective discount rates for the two agents are fixed at $\beta = 0.20$, and $\delta = 0.01$. Unless otherwise denoted, the initial values of the dividends are assumed to be the same, $X = Y = 0.50$, the volatilities of the two assets are assumed to be the same, $\sigma_X = \sigma_Y = 0.50$, and the two agents are assumed to have equal initial wealth, $w = 0.50$.⁶

5.1 Identical Assets

We first consider the scenario in which the two assets are identical in terms of the distribution of their future cash flows. This scenario provides a natural starting point for our analysis since, in the absence of liquidity restrictions, the two assets would have identical prices and return distributions.

ill, Quizii, and Dreze (1990) and others, an unconstrained Pareto optimal equilibrium need not exist in an incomplete market. This occurs in our framework when one of the agents is at a corner by choosing zero shares of an asset. In this situation, the agent's first-order condition for that asset is not required to hold. The Appendix shows how the equilibrium portfolio holdings for the other asset and the prices can be determined from the remaining first-order conditions in this situation.

⁶Although these asset volatilities appear larger than the typical volatility of reported dividends, they are actually consistent with the volatility of dividends based on broader definitions. For example, reported dividends typically do not include share repurchases. For a discussion of the volatility of economic (nonsmoothed) dividends imputed from corporate earnings, see Longstaff and Piazzesi (2004).

Table 1 reports key asset-pricing results for the cases in which both dividend volatilities are set at 20 percent, and in which both are set at 50 percent. As mentioned above, the length of the illiquidity horizon ranges from zero (the benchmark fully-liquid case) to 30 years.

Table 1 shows that when both assets are liquid, each agent chooses to hold the market portfolio. In particular, the impatient agent chooses a portfolio consisting of 0.458 shares of each of the two assets (implying that the patient agent chooses a portfolio consisting of 0.542 shares of each of the two assets). When both assets are liquid, Equation (10) implies that the optimal portfolio does not depend on dividend volatility. This is evident in Table 1 since the optimal portfolio $N = M = 0.458$ is the same for each set of volatility assumptions.

When one asset is illiquid, however, optimal portfolio behavior becomes strikingly different. Instead of holding the market portfolio, the impatient first agent now begins to tilt his portfolio towards the liquid asset and away from the illiquid asset. This effect becomes more pronounced as the length of the illiquid horizon increases. Thus, with illiquidity, the logarithmic agents no longer behave myopically as is the case in the standard Merton (1969, 1971) portfolio-choice paradigm. For example, when the illiquidity horizon is 30 years, the first agent's optimal portfolio consists of 0.843 and 0.030 shares of the two assets, or 0.740 and 0.107 shares, depending on the dividend volatility parameterization. Thus, the first agent abandons portfolio diversification as a strategy, and chooses a highly polarized portfolio instead.⁷

Although these results differ dramatically from those that hold when assets are fully liquid, the intuition behind them is not hard to understand. In this economy, the impatient agent wants to accelerate his consumption. His only mechanism for doing so is to systematically sell off his portfolio to the more-patient agent over time. When one of the assets is illiquid, however, the impatient agent needs to hold more of the liquid asset to be able to do so. The longer the illiquidity horizon, the larger is the position in the liquid asset that the first agent chooses.

By tilting his portfolio towards the liquid asset, however, the impatient first agent suffers the welfare costs of holding an undiversified portfolio. Ultimately, the optimal portfolio represents a tradeoff between the benefits of being able to sell assets to accelerate consumption and the costs of being undiversified. The effects of this tradeoff can be seen in Table 1 which shows that the first agent deviates less from the market portfolio when the dividend volatility (and, consequently, the cost of being undiversified) is higher.

⁷These results, however, depend on the assumption that there are no trading costs. If there were costs to rebalancing a portfolio, then the degree of polarization in optimal portfolios would likely decrease. For an in-depth analysis of the effects of transaction costs in incomplete markets with heterogeneous agents, see Heaton and Lucas (1996).

The second panel of results in Table 1 shows the valuation effects induced by the illiquidity of the second asset. In particular, the table reports the percentage difference between the first asset and its liquid-benchmark value, the percentage difference between the second asset and its liquid-benchmark value, and the net percentage change in the relative values of the two assets. Note that since the two assets have identical values in the liquid benchmark case, this latter measure is simply the percentage difference in the price of the two assets.⁸

Table 1 shows that when the second asset becomes illiquid, the prices of the two assets are no longer equal. In particular, as the length of the illiquidity horizon increases, the liquid asset becomes more valuable while the illiquid asset becomes less valuable. Thus, the values of both assets are affected by the illiquidity of the second asset.

The magnitude of the valuation effects varies significantly with the length of the illiquidity horizon. In these examples, the net valuation effect is much smaller than one percent when the illiquid horizon is only two years. The valuation effects, however, increase rapidly with the length of the illiquidity horizon and can be very large. For example, if the illiquidity horizon is 30 years, the prices of the two assets differ by 11.39 and 24.88 percent. Recall that in the absence of illiquidity, the two assets have identical values since they have identical cash flow dynamics. Thus, liquidity introduces an additional dimension into asset pricing that extends beyond the simple present value of an asset's cash flows. Intuitively, this is because the cash flows an agent receives from an asset are not simply the dividends it pays, but rather the dividends plus the market value of shares bought and sold over time. Thus, the consumption stream from a liquid asset can be very different from that generated by an illiquid asset even when both assets have identical dividend dynamics. In asset pricing, it is the actual consumption stream generated from asset ownership that matters.

Finally, these results indicate that the value of liquidity can represent a large portion of the price of an asset. This contrasts with earlier results such as Constantinides (1986) that show that liquidity only has a second-order effect on asset prices. The key difference, however, is that the earlier literature focuses only on a narrow definition of liquidity as transaction costs or other types of market frictions. In our setting, illiquidity is not merely a friction, but rather something that changes the very nature of investing.

At first glance, the magnitude of the valuation differences between otherwise identical assets implied by the model may seem implausibly large. These valuation differences, however, are consistent with the extensive recent empirical evidence about the effects of illiquidity on otherwise identical securities. For example, Silber (1992) finds that the differ-

⁸The net percentage difference is defined as the percentage difference in the ratio of P/Q in the illiquid-asset case to P/Q in the fully-liquid case. Consequently, the net percentage difference will generally not equal the sum of the percentage differences for the individual assets.

ence in value between privately placed illiquid stock and otherwise identical publicly traded stock averages 35 percent. Wruck (1989) documents valuation differences of about 15 percent for the illiquid privately-placed shares of major NYSE firms. Boudoukh and Whitelaw (1991) show that illiquid Japanese Government bonds traded at as much as a five percent discount to virtually identical liquid benchmark Japanese Government bonds. Amihud and Mendelson (1991), Kamara (1994), Krishnamurthy (2002), Longstaff (2004), and others find similar large differences in the values of otherwise identical liquid and illiquid Treasury bonds. Brenner, Eldor, and Hauser (2001) show that the valuation difference between liquid and illiquid foreign currency options can be as large as 30 percent. Thus, this model suggests that these huge observed illiquidity-related differences in the values of securities could potentially be reconciled within the context of a rational general equilibrium model.⁹

In this model, both assets trade at time zero. After time zero, however, the first asset continues to trade even though the second asset does not. By solving numerically for the value of the first asset after time zero, we can directly examine how the distribution of its returns is affected by the illiquidity of the second asset.

The third panel of Table 1 reports the expected return and standard deviation of the first asset's returns over the next one-year horizon. As shown, the illiquidity of the second asset can have surprising effects on the distribution of the liquid asset's returns. For example, Table 1 shows that not only does the time-zero value of the liquid asset increase as the illiquidity horizon increases, the expected return of the liquid asset can also increase. This result is somewhat counterintuitive since after the illiquidity horizon lapses, the model returns to the liquid-benchmark case. Thus, intuition suggests that the prices of the two assets should again converge once the illiquidity horizon lapses. What this intuition misses, however, is that illiquidity has real effects on the economy over time. In particular, the distribution of wealth between the agents over time is affected in fundamental ways by the liquidity restrictions. Thus, the economy that emerges when the illiquidity restriction lapses can differ significantly from what the economy would have been in the absence of liquidity restrictions. Because of these differences, expected returns for the liquid asset can increase, decrease, or oscillate as the illiquidity horizon increases.

The third panel also shows that the illiquidity of the second asset can have a major effect on the volatility of the liquid first asset's returns. For example, when the illiquidity horizon is 30 years, the volatility of the liquid asset's returns may only be 70 to 80 percent of what it would be in the absence of the liquidity restriction. Thus, the liquidity restriction can have the effect of significantly dampening the volatility of the liquid asset.

5.2 Assets with Different Volatilities

In the previous section, we considered the effects of illiquidity on assets with identical cash

⁹Partial equilibrium models include Longstaff (2001), and Kahl, Liu, and Longstaff (2003).

flow distributions. In this section, we explore asset-pricing effects for the case in which one asset is riskier than the other. Specifically, the top part of Table 2 presents results for the case where the liquid asset is less risky than the illiquid asset; the bottom part of Table 2 presents results for the case where the liquid asset is riskier than the illiquid asset. All other parameters are the same as in Table 1.

Table 2 shows that the implications of illiquidity depend heavily on the relative volatility of the two assets. When the liquid asset is less risky, the optimal portfolios for the agents deviate far less from the market portfolio than when the two assets have the same volatility. In contrast, the optimal portfolio tends to be much more polarized when the liquid asset is riskier than the illiquid asset.

The intuition for these results can best be understood by recognizing that there are two major factors at work in determining the agents' optimal portfolios. The first is the strong incentive that the impatient agent has to hold the liquid asset. As discussed previously, this allows the impatient agent to accelerate his consumption by selling his portfolio over time. The second factor is more subtle and has to do with intertemporal risk sharing. The two agents have the same attitude towards instantaneous risk because they both have logarithmic preferences. When illiquidity is introduced, however, patience emerges as an important additional factor driving portfolio choice. In particular, the impatient agent is better suited to bear the risk of holding the riskier asset during the illiquid period since his future utility is more heavily discounted and, consequently, less sensitive to noninstantaneous risk. This aspect introduces a strong demand for intertemporal risk sharing that is not present when markets are fully liquid. Thus, the impatient agent has strong incentives to tilt his portfolio heavily towards the risky asset when the market is illiquid.

The intuition for the optimal portfolio results in Table 2 can now be understood in light of these two factors. When the liquid asset is less risky than the illiquid asset, the impatient agent's incentives to hold more of the liquid asset are partially offset by his incentives to hold more of the riskier asset. Thus, the optimal portfolio does not diverge as far from the market portfolio. In contrast, when the liquid asset is riskier, the two factors reinforce each other and the optimal portfolio is much more tilted towards the risky liquid asset.

These results have many important implications for financial economics since they may help explain a number of well-known puzzles about the way actual agents choose investment portfolios. For example, Mankiw and Zeldes (1991) find evidence that equity market participation of the typical U.S. household is far less than predicted by classical models of portfolio choice (Merton (1969, 1971)). Our results suggest that if households view equity markets as riskier and more liquid than markets for other assets such as real estate, it may be rational for them to hold less stock than short-horizon institutional investors.

The intuition above also helps explain the valuation effects shown in the second panel of Table 2. For example, the top portion of the table shows that when the liquid asset is less

risky, its value can actually be less than it would be in the absence of liquidity restrictions for short illiquidity horizon. Similarly, the value of the illiquid asset can also be greater than it would be in the absence of liquidity restrictions. These counterintuitive valuation effects happen precisely when the two factors offset each other, at least for illiquidity horizons of two to five years.¹⁰ In contrast, the bottom panel of Table 2 shows that when these two factors reinforce each other (since the liquid asset is also the riskier asset), the valuation effects are much stronger and go in a more intuitive direction. For example, when the illiquidity horizon is 30 years, the difference in the relative valuation between the two assets can be as much as 43 percent.

Which situation is more realistic? There is certainly a case to be made that illiquid assets such as human capital and real estate tend to have less volatile cash flows than highly liquid assets such as stocks. If so, then the results in Table 2 suggest that illiquidity has the potential to have huge effects on the relative valuation of assets in the economy.

The third panel of Table 2 again illustrates that the expected return and volatility of the liquid asset can either increase or decrease with the length of the liquidity restriction. These effects on the distribution of returns can also be large in magnitude. For example, the presence of a illiquid less-risky asset in the economy can reduce the volatility of the liquid asset from 46.35 to 33.69 percent when the illiquidity horizon is 30 years. Effects of this size clearly have major asset-pricing implications.

5.3 Assets of Different Sizes

In this section, we examine scenarios in which the size of the assets differs (size defined as the proportion of total dividends an asset currently generates). Specifically, we focus on the cases where $X = 0.25$ and $Y = 0.75$, and where $X = 0.75$ and $Y = 0.25$. Other than these initial values, the two assets have identical dividend dynamics.

Table 3 reports the results for these two scenarios. As shown, the impatient agent tends to hold more of the liquid asset when it accounts for 25 percent of total dividends than when it accounts for 75 percent of total dividends. For example, when the illiquidity horizon is 30 years, the impatient agent's optimal portfolio includes 0.902 and 0.644 shares of the liquid asset for the two scenarios, respectively.

The intuition for this result is easily explained. The impatient agent has strong incentives to sell his portfolio over time to accelerate his consumption. To do this, he needs to hold some of the liquid asset. If the liquid asset is small, then he needs to sell more of it to generate the same amount of accelerated consumption. In turn, he needs to hold more of it to be able to sell more of it. In essence, he needs a larger slice of a "smaller pie."

¹⁰This finding parallels Vayanos (1998) who finds that the price of an asset can be an increasing function of its transaction costs.

One common theme throughout all of the examples we have presented so far is that the valuation effects tend to increase as the optimal portfolio becomes more polarized. This tendency can be seen in the second panel of Table 3. When the liquid asset is small, the valuation effects are again very large. In fact, when the illiquidity horizon is 30 years, the net valuation effect is 33.72 percent.

Finally, as in previous tables, the effects of illiquidity on the liquid asset’s return distribution can be large. Furthermore, these effects need not even be monotonic in the length of the illiquidity horizon.

5.4 Agents with Different Wealth

As a final set of examples, we consider the cases where the impatient agent starts with either 25 percent or 75 percent of the wealth in the economy. These examples allow us to examine directly how asset pricing is affected by changes in the (wealth weighted) average level of patience in the economy. Table 4 reports key statistics for these examples.

The primary result that emerges from Table 4 is that valuation effects become much larger as the average market participant becomes less patient. For example, when the illiquidity horizon is 30 years, the net valuation effect is 9.69 percent when the impatient agent’s initial wealth is 25 percent of the total, and 25.42 percent when the impatient agent’s initial wealth is 75 percent of the total.

6. THE ILLIQUID-MARKET CASE

The primary focus of this paper is on the asset-pricing implications when one asset is not fully liquid. To provide some additional perspective on this issue, however, it is also worthwhile to consider the extended case where both assets are illiquid. In essence, this is the case where the entire market becomes illiquid. While this situation is clearly less common, there are instances where actual markets can approach this level of illiquidity. Examples may include emerging equity and debt markets as well as other economies without well-functioning capital markets.

6.1 Extending the Model

In this extended setting, agents choose an initial consumption level and portfolio of assets at time zero. Once the portfolio is chosen, however, it cannot be rebalanced again until time T . Thus, $N_t = N$ and $M_t = M$ for $0 < t < T$.

The first-order conditions for the agents at time zero are the same as in Equations (25) through (28). What is different, however, is since the portfolio cannot be rebalanced during the trading “blackout” period, the first agent’s consumption during this period equals the

amount of dividends he receives. Thus, consumption is given by

$$C = w(P + X + Q + Y) - NP - MQ, \quad (29)$$

$$C_t = NX_t + MY_t, \quad (30)$$

$$C_T = \frac{(NA(\delta, X_T, Y_T) + MB(\delta, X_T, Y_T)) (X_T + Y_T)}{\frac{1}{\beta} + N(A(\delta, X_T, Y_T) - A(\beta, X_T, Y_T)) + M(B(\delta, X_T, Y_T) - B(\beta, X_T, Y_T))}, \quad (31)$$

where $0 < t < T$. The expressions for P_T and Q_T also the same as in Equations (21) and (22), but where Equation (31) is substituted in for C_T .

Substituting the expressions for P_T , Q_T , C , C_t , and C_T into the first-order conditions results in a system of four equations in P , Q , N , and M . Since C is linear in P and Q , the model can again be solved explicitly for P and Q , leaving a system of two equations in N and M to solve. These equations in N and M are easily solved numerically. Substituting the values N and M into the explicit expressions for P and Q completes the solution. When there is only a constrained equilibrium, we follow the same procedure as in Section 4.

6.2 Asset-Pricing Results

Table 5 reports numerical results for the illiquid-market case. In these examples, the first asset is less risky with dividend volatility of 20 percent. The second asset is riskier with dividend volatility of 50 percent. In the top panel, the first asset's dividend is 25 percent of aggregate dividends; in the second panel, the first asset's dividend is 75 percent of aggregate dividends. The remaining parameters are the same as in the previous examples.

As shown, optimal portfolios become even more polarized when both assets are illiquid. Since there is no liquid asset to hold, the impatient agent's portfolio choice is driven entirely by the demand for intertemporal risk-sharing. In both the top and bottom panels of Table 5, the impatient agent holds virtually all of the risky asset in the economy, even though both agents are endowed with the same initial wealth. Thus, the patient second agent ends up holding virtually only the less-risky asset. The key lesson here is that illiquidity fundamentally changes the way agents behave towards risk. Agents who behave in one way when decisions have only short-term reversible consequences for risk can behave in a completely different way when consequences are longer lasting.

Table 5 also shows that there are large effects on asset values at time zero resulting from the illiquidity of the market. The magnitude of these effects, however, is similar to that seen in earlier tables where one asset remained liquid. Thus, while market illiquidity makes the valuation effects larger, asset-specific illiquidity plays the central role in determining these effects.

One counterintuitive feature of the valuation results is that the riskier asset is worth more when agents face illiquid markets, while the opposite is true for the less-risky asset. This result is particularly surprising given that one might expect the riskier asset to trade at a lower price when investors may have to bear its risk for extended periods of time. To understand these results, observe that when portfolios are highly polarized, the consumption streams of the two agents in these examples are essentially uncorrelated. Thus, the marginal utilities of the two agents are also nearly uncorrelated. Recall that first-order conditions can be rearranged to express an asset’s price in terms of the covariance (consumption beta) of its cash flows with marginal utility. In a representative-agent or complete-markets setting, an asset need only satisfy a single “consumption CAPM” expression. For the first-order conditions of both agents to be satisfied in this illiquid market, however, the price of an asset has to simultaneously satisfy two conflicting “consumption CAPMs,” with betas based on uncorrelated marginal utilities. To resolve the tension between these conflicting conditions, equilibrium prices must be largely unrelated to the riskiness of the assets’ cash flows.

In a liquid market, the second asset’s value is heavily discounted because of the riskiness of its cash flows, and conversely for the first asset. In an illiquid market, however, conventional risk and return relations break down, and the price of the risky asset is not as heavily discounted as in a liquid market. Thus, as shown in Table 5, the price of the risky asset is generally higher when markets are illiquid, and vice versa for the less-risky asset.

7. CONCLUSION

We study the asset-pricing implications of relaxing the standard assumption that assets can always be traded whenever investors would like to. To do this, we develop a two-asset heterogeneous agent model in which one asset can always be traded. The other asset, however, can be traded initially, but not again until after a trading “blackout” period. Thus, during the “blackout” period, only the liquid asset can be traded by the agents.

Because portfolio decisions become temporarily irreversible in this setting, the economics of portfolio choice are much more complex than in the traditional portfolio-choice framework. We show that when one of the assets is illiquid, the agents no longer find it optimal to hold the market portfolio. Instead, they abandon diversification as a strategy and tend to hold highly polarized portfolios. The reason for this stems from effects of two factors not present in the traditional portfolio choice problem. First, because the impatient agent needs to sell assets over time to accelerate his consumption, he needs to hold more of the liquid asset during the trading “blackout” period. Second, illiquidity introduces an additional demand for intertemporal risk sharing which induces the impatient agent to hold more of the riskier asset. These two factors can reinforce or offset each other. In general, the impatient agent tilts his portfolio significantly towards the liquid asset.

The illiquidity-induced changes in the optimal portfolios have major effects on asset prices. We show that assets with identical cash flow distributions, that would otherwise have the same value, can differ by as much as 25 percent in their prices when one is liquid and the other is not. Thus, the model can produce large discounts for illiquidity similar in magnitude to those documented in many empirical studies.

Finally, our results indicate that heterogeneity in investor patience can have large asset-pricing implications. Heterogeneity in patience has been largely unexplored as a channel for resolving asset-pricing puzzles. Our results indicate that this may be a promising direction well worth pursuing in future research.

APPENDIX

1. The Fully-Liquid Case.

Setting the expressions for P_t in Equations (5) and (7) equal to each other, and similarly for the expressions for Q_t in Equations (6) and (8), implies the following pair of equations:

$$0 = E_t \left[\int_0^\infty \left\{ e^{-\beta s} \left(\frac{C_t}{C_{t+s}} \right) - e^{-\delta s} \left(\frac{X_t + Y_t - C_t}{X_{t+s} + Y_{t+s} - C_{t+s}} \right) \right\} X_{t+s} ds \right], \quad (\text{A1})$$

$$0 = E_t \left[\int_0^\infty \left\{ e^{-\beta s} \left(\frac{C_t}{C_{t+s}} \right) - e^{-\delta s} \left(\frac{X_t + Y_t - C_t}{X_{t+s} + Y_{t+s} - C_{t+s}} \right) \right\} Y_{t+s} ds \right]. \quad (\text{A2})$$

Both of the above expressions are satisfied by requiring that

$$e^{-\beta s} \left(\frac{C_t}{C_{t+s}} \right) = e^{-\delta s} \left(\frac{X_t + Y_t - C_t}{X_{t+s} + Y_{t+s} - C_{t+s}} \right), \quad (\text{A3})$$

hold for all s , X_{t+s} , and Y_{t+s} . Solving this expression for C_{t+s} implies

$$C_{t+s} = \frac{e^{-\beta s} C_t}{e^{-\beta s} C_t + e^{-\delta s} (X_t + Y_t - C_t)} (X_{t+s} + Y_{t+s}). \quad (\text{A4})$$

Thus, C_{t+s} can be expressed in terms of C_t .

Similarly, satisfying the first-order conditions as of time zero requires that

$$C_t = \frac{e^{-\beta t} C}{e^{-\beta t} C + e^{-\delta t} (X + Y - C)} (X_t + Y_t), \quad (\text{A5})$$

$$C_{t+s} = \frac{e^{-\beta(t+s)} C}{e^{-\beta(t+s)} C + e^{-\delta(t+s)} (X + Y - C)} (X_{t+s} + Y_{t+s}). \quad (\text{A6})$$

Substituting the expression for C_t in Equation (A5) into Equation (A4), however, reduces Equation (A4) to Equation (A6). Thus, requiring that Equation (A5) hold for all t is sufficient for the first-order conditions in Equations (5) through (8) to be satisfied for all t .

Dividing C_t by the expression for C_{t+s} in Equation (A4) and rearranging gives

$$\left(\frac{C_t}{C_{t+s}}\right) = \frac{C_t + (X_t + Y_t - C_t) e^{(\beta-\delta)s}}{X_{t+s} + Y_{t+s}}. \quad (\text{A7})$$

Substituting this into Equations (5) and (6) and rearranging yields

$$P_t = C_t E_t \left[\int_0^\infty e^{-\beta s} \left(\frac{X_{t+s}}{X_{t+s} + Y_{t+s}} \right) ds \right] + (X_t + Y_t - C_t) E_t \left[\int_0^\infty e^{-\delta s} \left(\frac{X_{t+s}}{X_{t+s} + Y_{t+s}} \right) ds \right], \quad (\text{A8})$$

$$Q_t = C_t E \left[\int_0^\infty e^{-\beta s} \left(\frac{Y_{t+s}}{X_{t+s} + Y_{t+s}} \right) ds \right] + (X_t + Y_t - C_t) E_t \left[\int_0^\infty e^{-\delta s} \left(\frac{Y_{t+s}}{X_{t+s} + Y_{t+s}} \right) ds \right]. \quad (\text{A9})$$

Section 3 of this Appendix shows that these equations can be reexpressed as

$$P_t = C_t A(\beta, X_t, Y_t) + (X_t + Y_t - C_t) A(\delta, X_t, Y_t), \quad (\text{A10})$$

$$Q_t = C_t B(\beta, X_t, Y_t) + (X_t + Y_t - C_t) B(\delta, X_t, Y_t), \quad (\text{A11})$$

which are Equations (11) and (12).

To solve for C , note that after consuming at time zero, the first agent's wealth equals $w(P + X + Q + Y) - C$, where the first term represents the value of the agent's endowment (with dividends). Setting the value of the agent's wealth equal to the present value of his future consumption stream gives

$$w(P + X + Q + Y) - C = E \left[\int_0^\infty e^{-\beta t} \left(\frac{C}{C_t} \right) C_t dt \right], \quad (\text{A12})$$

$$= C/\beta, \quad (\text{A13})$$

which implies

$$C = \frac{w(P + X + Q + Y)}{1 + \frac{1}{\beta}}. \quad (\text{A14})$$

Substituting in the expressions for P and Q from Equations (A8) and (A9), and solving for C gives

$$C = \frac{w \beta (1 + \delta)}{\delta (1 + \beta) + w(\beta - \delta)}(X + Y). \quad (\text{A15})$$

Substituting this expression into Equation (A5) gives Equation (9).

From Equation (9), optimal consumption is homogeneous of degree one in total dividends $X_t + Y_t$. Based on this, we conjecture (and later verify) that the dynamic portfolio strategy that generates C_t consists of equal numbers of shares of the two assets, $N_t = M_t$, where N_t is a differentiable function of time. By definition, consumption equals the sum of dividends received minus net purchases of assets. Thus,

$$C_t = N_t(X_t + Y_t) - (P_t + Q_t)N'_t, \quad (\text{A16})$$

where N'_t denotes a derivative. From Equations (A8) and (A9) it follows that

$$P_t + Q_t = \frac{C_t}{\beta} + \frac{X_t + Y_t - C_t}{\delta}. \quad (\text{A17})$$

Substituting this and the expression for C_t in Equation (A5) into Equation (A16) gives the ordinary differential equation

$$N'_t - \left(\frac{\beta\delta(Ce^{\delta t} + (X + Y - C)e^{\beta t})}{C\delta e^{\delta t} + \beta(X + Y - C)e^{\beta t}} \right) N_t = \frac{-C\beta\delta e^{\delta t}}{C\delta e^{\delta t} + \beta(X + Y - C)e^{\beta t}}. \quad (\text{A18})$$

This is a standard first-order linear differential equation which can be solved directly by an integration. The initial value of N is determined by imposing the condition that $N(P + Q)$ equals the first agent's initial wealth after time-zero consumption. From Spiegel (1967), the solution to this differential equation is the expression given in Equation (10). This verifies the conjecture and also establishes that the consumption strategy identified in Equation (A5) is feasible.

In solving the illiquid-asset problem, it is also necessary to solve for C_T when the first agent has w shares of the first asset and v shares of the second asset at time T (instead of w shares of each). Since the model reverts back to the fully-liquid case at time T (with the exception that there is no consumption gulp at time T), the expressions for P_T and Q_T are as given from Equations (11) and (12). To solve for C_T in this more general case, we set the value of the first agent's wealth at time T equal to the present value of his remaining consumption stream

$$wP_T + vQ_T = E_T \left[\int_0^\infty e^{-\beta s} \left(\frac{C_T}{C_{T+s}} \right) C_{T+s} ds \right], \quad (\text{A19})$$

$$= C_T/\beta, \quad (\text{A20})$$

which implies

$$C_T = \beta(wP_T + vQ_T). \quad (\text{A21})$$

Substituting in the expressions for P_T and Q_T in Equations (A10) and (A11), and then solving for C_T gives,

$$C_T = \frac{(wA(\delta, X_T, Y_T) + vB(\delta, X_T, Y_T))(X_T + Y_T)}{\frac{1}{\beta} + w(A(\delta, X_T, Y_T) - A(\beta, X_T, Y_T)) + v(B(\delta, X_T, Y_T) - B(\beta, X_T, Y_T))}. \quad (\text{A22})$$

in this general case. In the illiquid-asset case, C_T is given by substituting in $w = N_T$ and $v = M$ into the above equation. In the illiquid-market case, C_T is given by substituting in $w = N$ and $v = M$ into the above equation.

2. The Illiquid-Asset Case.

To solve the illiquid-asset case numerically, we first discretize the problem by approximating the dividend dynamics using binomial processes. Let Δt denote the discretization step. Then,

$$X_{t+\Delta t} = X_t \exp \left((\mu_X - \sigma_X^2/2)\Delta t \pm \sigma_X \sqrt{\Delta t} \right), \quad (\text{A23})$$

$$Y_{t+\Delta t} = Y_t \exp \left((\mu_Y - \sigma_Y^2/2)\Delta t \pm \sigma_Y \sqrt{\Delta t} \right), \quad (\text{A24})$$

where the probability of both processes increasing or decreasing together is $p/2$, the probability of the first process increasing and the second decreasing is $(1-p)/2$, the probability of the first process decreasing and the second increasing is $(1-p)/2$, and where $p = (1+\rho)/2$. This assumption insures that the correlation between the two processes is ρ . Note that we use this binomial process merely as an approximation to the continuous-time process.

To simplify the exposition, we make the assumption that $\Delta t = 1$. The numerical procedure, however, is applicable to any positive value of Δt . Note that at time t , where $0 < t < T$, the first agent's consumption is

$$C_t = N_{t-1} X_t + M Y_t - P_t(N_t - N_{t-1}). \quad (\text{A25})$$

The recursive approach begins by first solving the problem at time $T-1$, conditional on knowing the functional forms of P_T and C_T . Since the model reverts back to the fully-liquid model at time T , however, these functional forms are those for the fully-liquid model. At time $T-1$, the first-order conditions of optimality for the two agents are

$$\frac{P_{T-1}}{C_{T-1}} = E_{T-1} \left[\frac{e^{-\beta}(X_T + P_T)}{C_T} \right], \quad (\text{A26})$$

$$\frac{P_{T-1}}{X_{T-1} + Y_{T-1} - C_{T-1}} = E_{T-1} \left[\frac{e^{-\delta}(X_T + P_T)}{X_T + Y_T - C_T} \right]. \quad (\text{A27})$$

Substituting in the expression for C_{T-1} from Equation (A25) into Equations (A26) and (A27) and solving for P_{T-1} gives the following pair of equations,

$$P_{T-1} = \frac{(N_{T-2} X_{T-1} + M Y_{T-1}) E_{T-1} \left[\frac{e^{-\beta}(X_T + P_T)}{C_T} \right]}{1 + (N_{T-1} - N_{T-2}) E_{T-1} \left[\frac{e^{-\beta}(X_T + P_T)}{C_T} \right]}, \quad (\text{A28})$$

$$P_{T-1} = \frac{((1 - N_{T-2}) X_{T-1} + (1 - M) Y_{T-1}) E_{T-1} \left[\frac{e^{-\delta}(X_T + P_T)}{X_T + Y_T - C_T} \right]}{1 - (N_{T-1} - N_{T-2}) E_{T-1} \left[\frac{e^{-\delta}(X_T + P_T)}{X_T + Y_T - C_T} \right]}. \quad (\text{A29})$$

For a given choice of M , and recognizing that both P_T and C_T can be expressed as explicit functions of N_{T-1} , M , X_T , and Y_T (using the generalized solution for consumption in the fully-liquid case given in Equation (A22)), these two equations can be set equal to each other to provide a single equation in the unknown N_{T-1} . This single equation is easily

solved numerically for N_{T-1} , which, in turn, can be substituted back into Equation (A28) or Equation (A29) to give P_{T-1} .

To provide a full solution at time $T - 1$, we solve this problem repeated for every combination of X_{T-1} and Y_{T-1} on the two-dimensional binomial tree and for every value of M and N_{T-2} ranging from zero to one, (in steps of 0.001). Although computationally intensive, this procedure gives us a lookup table from which we can interpolate the equilibrium values of N_{T-1} and P_{T-1} for any value of X_{T-1} , Y_{T-1} , M and N_{T-2} .

The algorithm now rolls back to time $T - 2$. Exactly as before, the agents first-order conditions as of time $T - 2$ can be rearranged to give the pair of equations,

$$P_{T-2} = \frac{(N_{T-3} X_{T-2} + M Y_{T-2}) E_{T-2} \left[\frac{e^{-\beta}(X_{T-1} + P_{T-1})}{C_{T-1}} \right]}{1 + (N_{T-2} - N_{T-3}) E_{T-2} \left[\frac{e^{-\beta}(X_{T-1} + P_{T-1})}{C_{T-1}} \right]}, \quad (\text{A30})$$

$$P_{T-2} = \frac{((1 - N_{T-3}) X_{T-2} + (1 - M) Y_{T-2}) E_{T-2} \left[\frac{e^{-\delta}(X_{T-1} + P_{T-1})}{X_{T-1} + Y_{T-1} - C_{T-1}} \right]}{1 - (N_{T-2} - N_{T-3}) E_{T-2} \left[\frac{e^{-\delta}(X_{T-1} + P_{T-1})}{X_{T-1} + Y_{T-1} - C_{T-1}} \right]}. \quad (\text{A31})$$

These can again be set equal and solved for N_{T-2} (and P_{T-2}) for each set of values for X_{T-2} , Y_{T-2} , M , and N_{T-3} , given the functional forms for P_{T-1} and C_{T-1} . These functional forms, however, can be approximated directly using the lookup table constructed in the previous step (the value of C_{T-1} follows from the value of N_{T-1} because of Equation (A25)). Once again, we solve the problem at time $T - 2$ repeatedly and construct a new lookup table for N_{T-2} and P_{T-2} . This process iterates back until we have constructed a lookup table for time one.

At time zero, the first-order conditions for the agents are

$$\frac{P}{C} = E \left[\frac{e^{-\beta}(X_1 + P_1)}{C_1} \right], \quad (\text{A32})$$

$$\frac{P}{X + Y - C} = E \left[\frac{e^{-\delta}(X_1 + P_1)}{X_1 + Y_1 - C_1} \right], \quad (\text{A33})$$

$$\frac{Q}{C} = E \left[\sum_{i=1}^T e^{-\beta i} \frac{Y_i}{C_i} + e^{-\beta T} \frac{Q_T}{C_T} \right], \quad (\text{A34})$$

$$\frac{Q}{X + Y - C} = E \left[\sum_{i=1}^T e^{-\delta i} \frac{Y_i}{X_i + Y_i - C_i} + e^{-\delta T} \frac{Q_T}{X_T + Y_T - C_T} \right]. \quad (\text{A35})$$

Since both assets can be traded at time zero,

$$C = w(X + Y + P + Q) - PN - QM. \quad (\text{A36})$$

By substituting this expression into Equations (A32) through (A35), the values P and Q can be eliminated, resulting in a system of two equations in the two unknowns M and N . Again, this system is straightforward to solve numerically. Once N and M are determined, the values of P and Q are given directly by substituting N and M back into the first-order conditions.

To minimize the computational burden, we make a number of simplifying assumptions in implementing the algorithm. First, we set Δt equal to one year. Second, we model the binomial process and value the cash flows out only to 100 years. Third, rather than using the closed-form solutions for the fully-liquid case, we solve for these numerically using the same binomial grid. This allows us to compare the fully-liquid case directly with the illiquid-asset case without introducing small discretization errors. Finally, rather than evaluating the expectations on the right hand side of Equations (A34) and (A35) directly as of time zero, we use a similar type of lookup table approach to solve for the values of the expectations sequentially from time $T - 1$ to time zero (using the double expectation theorem repeatedly).

Finally, since an unconstrained equilibrium may not exist, it is important to consider how the algorithm should be modified in this situation. First, it is important to recognize that agents will never take a short position in the liquid asset at time $0 < t < T$. The reason for this is that the dividends for the illiquid asset could decline to the point where an agent's consumption became negative, and therefore, his utility became negative infinity. Thus, zero and one are upper and lower bounds for N_t . Now consider the situation where the first agent has a lower valuation for the liquid asset even when $N_t = 0$. Since the first agent cannot sell any more shares to the second agent, his first-order conditions cannot be satisfied. In this situation, $N_t = 0$, and P_t is given from the second agent's first-order conditions. A similar argument holds for the second agent.

Now consider the situation at time zero. Similar reasoning to the above shows that agents will not take short positions in either asset at time zero. Thus, zero and one become upper and lower bounds for N and M . If one agent has a lower valuation for an asset even when his holdings of the asset are zero, then that agent's first-order condition for the asset cannot be satisfied. In this situation, we use the first-order condition for the other agent in determining the equilibrium values for the other portfolio weight and for the asset prices at time zero.

3. The $A(\cdot, X_t, Y_t)$ and $B(\cdot, X_t, Y_t)$ Functions.

We define $A(c, X_t, Y_t)$ as the expectation

$$A(c, X_t, Y_t) = E_t \left[\int_0^\infty e^{-cs} \left(\frac{X_{t+s}}{X_{t+s} + Y_{t+s}} \right) ds \right], \quad (\text{A37})$$

that appears in various forms in the agents' first-order conditions. This can be rewritten as

$$E_t \left[\int_0^\infty e^{-cs} \left(\frac{1}{1 + qe^u} \right) ds \right], \quad (\text{A38})$$

where $q = \frac{Y_t}{X_t}$, and u is a normally distributed random variable with mean μs and variance $\sigma^2 s$, where

$$\mu = \mu_Y - \mu_X - \sigma_Y^2/2 + \sigma_X^2/2, \quad (\text{A39})$$

$$\sigma^2 = \sigma_X^2 + \sigma_Y^2 - 2\rho\sigma_X\sigma_Y. \quad (\text{A40})$$

Introducing the density for u into the above expectation gives

$$\int_0^\infty \int_{-\infty}^\infty e^{-cs} \frac{1}{\sqrt{2\pi\sigma^2 s}} \frac{1}{1 + qe^u} \exp\left(\frac{-(u - \mu s)^2}{2\sigma^2 s}\right) du ds. \quad (\text{A41})$$

Interchanging the order of integration and collecting terms in s gives,

$$\int_{-\infty}^\infty \frac{1}{\sqrt{2\pi\sigma^2}} \frac{1}{1 + qe^u} \exp\left(\frac{\mu u}{\sigma^2}\right) \int_0^\infty s^{-1/2} \exp\left(-\frac{u^2}{2\sigma^2} \frac{1}{s} - \frac{\mu^2 + 2c\sigma^2}{2\sigma^2} s\right) ds du. \quad (\text{A42})$$

From Equation (3.471.9) of Gradshteyn and Ryzhik (2000), this expression becomes,

$$\int_{-\infty}^\infty \frac{2}{\sqrt{2\pi\sigma^2}} \frac{1}{1 + qe^u} \exp\left(\frac{\mu u}{\sigma^2}\right) \left(\frac{u^2}{\mu^2 + 2c\sigma^2}\right)^{1/4} K_{1/2}\left(2\sqrt{\frac{u^2(\mu^2 + 2c\sigma^2)}{4\sigma^4}}\right) du, \quad (\text{A43})$$

where $K_{1/2}(\cdot)$ is the modified Bessel function of order 1/2 (see Abramowitz and Stegun (1970) Chapter 9). From the identity relations for Bessel functions of order equal to an integer plus one half given in Gradshteyn and Ryzhik Equation (8.469.3), however, the above expression can be expressed as,

$$\frac{1}{\psi} \int_{-\infty}^{\infty} \frac{1}{1+qe^u} \exp\left(\frac{\mu u}{\sigma^2}\right) \exp\left(-\frac{\psi}{\sigma^2} |u|\right) du, \quad (\text{A44})$$

where

$$\psi = \sqrt{\mu^2 + 2c\sigma^2}.$$

In turn, Equation (A44) can be written

$$\frac{1}{\psi} \int_0^{\infty} \frac{1}{1+qe^u} \exp(\gamma u) du + \frac{1}{\psi} \int_{-\infty}^0 \frac{1}{1+qe^u} \exp(\theta u) du, \quad (\text{A45})$$

where

$$\gamma = \frac{\mu - \psi}{\sigma^2}, \quad \theta = \frac{\mu + \psi}{\sigma^2}.$$

Define $y = e^{-u}$. By a change of variables Equation (A37) can be written

$$\frac{1}{q\psi} \int_0^1 \frac{1}{1+y/q} y^{-\gamma} dy + \frac{1}{\psi} \int_0^1 \frac{1}{1+qy} y^{\theta-1} dy. \quad (\text{A46})$$

From Abramowitz and Stegun Equation (15.3.1), this expression becomes

$$\frac{1}{q\psi(1-\gamma)} F(1, 1-\gamma; 2-\gamma; -1/q) + \frac{1}{\psi\theta} F(1, \theta; 1+\theta; -q). \quad (\text{A47})$$

Substituting in for q gives the expression for $A(c, X_t, Y_t)$

$$A(c, X_t, Y_t) = k_1 (X_t/Y_t) F\left(1, 1-\gamma; 2-\gamma; -\frac{X_t}{Y_t}\right) + k_2 F\left(1, \theta; 1+\theta; -\frac{Y_t}{X_t}\right), \quad (\text{A48})$$

where

$$k_1 = \frac{1}{\psi(1-\gamma)}, \quad k_2 = \frac{1}{\psi\theta}.$$

The function $B(c, X_t, Y_t)$ is defined as the expectation,

$$B(c, X_t, Y_t) = E_t \left[\int_0^\infty e^{-cs} \left(\frac{Y_{t+s}}{X_{t+s} + Y_{t+s}} \right) ds \right]. \quad (A49)$$

The evaluation of this expectation is omitted since it is virtually the same as that for $A(c, X_t, Y_t)$ above. The resulting expression for $B(c, X_t, Y_t)$ is

$$B(c, X_t, Y_t) = k_3 (Y_t/X_t) F \left(1, 1 + \theta; 2 + \theta; -\frac{Y_t}{X_t} \right) - k_4 F \left(1, -\gamma; 1 - \gamma; -\frac{X_t}{Y_t} \right), \quad (A50)$$

where

$$k_3 = \frac{1}{\psi(1 + \theta)}, \quad k_4 = \frac{1}{\psi\gamma}.$$

Substituting the solutions for $A(c, X_t, Y_t)$ and $B(c, X_t, Y_t)$ into Equations (A8) and (A9) gives the expressions for P_t and Q_t in Equations (11) and (12).

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Table 1

Summary Statistics for Assets with Identical Dividend Dynamics. This table reports results for scenarios in which the two assets have the identical dividend dynamics. The first two columns show the volatilities of the assets' dividend dynamics. Horizon denotes the number of years that the illiquid asset cannot be traded. The Portfolio columns report the optimal holdings of the two assets for the impatient agent for the indicated illiquidity horizons. The Valuation Effect columns report the percentage difference (relative to the fully-liquid or zero-horizon case) in the prices of the liquid and illiquid assets as well as the ratio of the liquid to illiquid asset prices (the net percentage). The Returns columns report the expected return and standard deviation (Vol.) of the liquid asset over a one-year horizon.

Volatility Liquid	Volatility		Horizon	Portfolio		Valuation Effect			Returns	
	Illiquid	Horizon		Liq.	Illiq.	Liq.	Illiq.	Net	Expected	Vol.
20	20	0	0	0.458	0.458	0.00	0.00	0.00	8.05	19.11
		2	2	0.486	0.430	0.03	-0.02	0.04	8.11	18.71
		5	5	0.539	0.378	0.24	-0.22	0.46	8.13	17.91
		10	10	0.682	0.225	2.13	-2.09	4.32	7.95	15.91
		20	20	0.813	0.069	4.73	-4.60	9.78	8.02	15.26
		30	30	0.843	0.030	5.58	-5.52	11.39	8.02	15.19
50	50	0	0	0.458	0.458	0.00	0.00	0.00	7.54	40.03
		2	2	0.484	0.432	0.08	-0.07	0.16	7.66	39.57
		5	5	0.539	0.376	1.08	-1.06	2.17	7.50	38.13
		10	10	0.640	0.256	5.33	-5.28	11.21	7.45	36.14
		20	20	0.723	0.135	10.21	-10.01	22.47	7.59	35.65
		30	30	0.740	0.107	11.29	-10.88	24.88	7.56	35.59

Table 2

Summary Statistics for Assets with Different Dividend Volatilities. This table reports results for scenarios in which the two assets have different dividend volatilities. The first two columns show the volatilities of the assets' dividend dynamics. Horizon denotes the number of years that the illiquid asset cannot be traded. The Portfolio columns report the optimal holdings of the two assets for the impatient agent for the indicated illiquidity horizons. The Valuation Effect columns report the percentage difference (relative to the fully-liquid or zero-horizon case) in the prices of the liquid and illiquid assets as well as the ratio of the liquid to illiquid asset prices (the net percentage). The Returns columns report the expected return and standard deviation (Vol.) of the liquid asset over a one-year horizon.

Volatility Liquid	Volatility		Horizon	Portfolio		Valuation Effect			Returns	
	Illiquid	Illiquid		Liq.	Illiq.	Liq.	Illiq.	Net	Expected	Vol.
20	50	0	0.458	0.458	0.00	0.00	0.00	0.00	8.83	22.05
			0.469	0.431	-0.25	0.67	-0.92	9.24	22.43	
			0.485	0.390	-0.16	0.47	-0.63	9.41	22.92	
	20	10	0.508	0.330	0.19	-0.33	0.52	9.62	23.70	
			0.548	0.216	1.47	-3.26	4.89	10.01	25.27	
			0.557	0.187	2.03	-3.85	6.13	10.12	25.64	
50	20	0	0.458	0.458	0.00	0.00	0.00	0.00	5.11	46.35
			0.548	0.423	1.13	-0.43	1.57	4.11	44.61	
			0.803	0.311	7.86	-3.02	11.22	2.41	36.05	
	10	20	0.972	0.209	17.90	-6.87	26.60	2.53	33.27	
			1.000	0.163	27.29	-10.47	42.17	1.81	33.53	
			1.000	0.161	27.78	-10.64	43.00	1.39	33.69	

Table 3

Summary Statistics for Assets of Different Sizes. This table reports results for scenarios in which the dividends of the two assets represent different proportions of the total dividends in the economy. The first two columns show the initial dividends of the assets. Horizon denotes the number of years that the illiquid asset cannot be traded. The Portfolio columns report the optimal holdings of the two assets for the impatient agent for the indicated illiquidity horizons. The Valuation Effect columns report the percentage difference (relative to the fully-liquid or zero-horizon case) in the prices of the liquid and illiquid assets as well as the ratio of the liquid to illiquid asset prices (the net percentage). The Returns columns report the expected return and standard deviation (Vol.) of the liquid asset over a one-year horizon.

Dividend		Horizon	Portfolio		Valuation Effect			Returns	
Liquid	Illiquid		Liq.	Illiq.	Liq.	Illiq.	Net	Expected	Vol.
0.25	0.75	0	0.458	0.458	0.00	0.00	0.00	6.01	36.06
		2	0.502	0.433	1.91	-1.07	3.01	4.70	35.41
		5	0.623	0.359	4.74	-2.64	7.59	5.55	34.99
		10	0.780	0.244	11.50	-6.41	19.14	7.41	36.64
		20	0.884	0.145	18.27	-10.10	31.56	8.18	37.71
		30	0.902	0.126	19.47	-10.65	33.72	8.20	37.81
0.75	0.25	0	0.458	0.458	0.00	0.00	0.00	8.40	46.65
		2	0.475	0.429	-0.96	1.74	-2.65	9.36	46.57
		5	0.504	0.379	-0.69	1.27	-1.94	9.02	45.51
		10	0.566	0.259	1.38	-2.38	3.85	7.89	42.67
		20	0.629	0.112	4.54	-7.82	13.42	7.14	40.64
		30	0.644	0.074	5.45	-9.06	15.97	6.97	40.26

Table 4

Summary Statistics for Scenarios in which the Impatient and Patient Agents have Different Initial Endowments. This table reports results for scenarios in which the two agents differ in their initial endowments. The first two columns show the initial endowments. Horizon denotes the number of years that the illiquid asset cannot be traded. The Portfolio columns report the optimal holdings of the two assets for the impatient agent for the indicated illiquidity horizons. The Valuation Effect columns report the percentage difference (relative to the fully-liquid or zero-horizon case) in the prices of the liquid and illiquid assets as well as the ratio of the liquid to illiquid asset prices (the net percentage). The Returns columns report the expected return and standard deviation (Vol.) of the liquid asset over a one-year horizon.

Impatient Share	Patient Share	Horizon	Portfolio		Valuation Effect			Returns	
			Liq.	Illiq.	Liq.	Illiq.	Net	Expected	Vol.
0.25	0.75	0	0.220	0.220	0.00	0.00	0.00	9.07	39.45
		2	0.239	0.201	0.06	-0.04	0.10	9.19	39.02
		5	0.286	0.153	0.77	-0.70	1.48	8.90	37.62
		10	0.352	0.081	2.67	-2.51	5.31	8.57	36.38
		20	0.397	0.027	4.73	-4.04	9.14	8.38	36.00
		30	0.403	0.020	5.22	-4.08	9.69	8.34	35.96
0.75	0.25	0	0.717	0.717	0.00	0.00	0.00	4.01	40.18
		2	0.737	0.698	0.07	-0.07	0.13	4.09	39.84
		5	0.768	0.666	0.57	-0.56	1.13	4.15	39.13
		10	0.825	0.600	4.27	-4.25	8.89	4.26	37.07
		20	0.887	0.509	10.30	-10.22	22.86	5.15	36.17
		30	0.903	0.485	11.37	-11.20	25.42	5.23	36.08

Table 5

Summary Statistics for the Illiquid-Market Case. This table reports results for the illiquid-market case in which neither asset can be traded again until the end of the illiquidity horizon. The first two columns show the initial values of the dividends for the two assets. Horizon denotes the number of years that the two assets cannot be traded. The Portfolio columns report the optimal holdings of the two assets for the impatient agent for the indicated illiquidity horizons. The Valuation Effect columns report the percentage difference (relative to the fully-liquid or zero-horizon case) in the prices of the liquid and illiquid assets as well as the ratio of the liquid to illiquid asset prices (the net percentage).

Less-Risky Dividend	Risky Dividend	Horizon	Portfolio		Valuation Effect		
			Less-Risky	Risky	Less-Risky	Risky	Net
0.25	0.75	0	0.458	0.458	0.00	0.00	0.00
		2	0.111	0.986	-2.05	3.32	-5.20
		5	0.088	0.982	-4.82	7.77	-11.68
		10	0.065	0.974	-7.90	12.74	-18.31
		20	0.051	0.960	-10.30	16.62	-23.09
		30	0.051	0.952	-10.99	17.72	-24.38
0.75	0.25	0	0.458	0.458	0.00	0.00	0.00
		2	0.356	1.000	-2.28	14.35	-14.55
		5	0.343	1.000	-4.18	26.25	-24.10
		10	0.335	1.000	-5.42	34.09	-29.47
		20	0.331	0.998	-5.92	37.19	-31.42
		30	0.333	0.988	-6.02	37.84	-31.82