

Optimal portfolio liquidation with distress risk*

David B. Brown[†]

Bruce Ian Carlin[‡]

Miguel Sousa Lobo[§]

February 17, 2010

Abstract

We analyze the problem of an investor who needs to unwind a portfolio in the face of recurring and uncertain liquidity needs, with a model that accounts for both permanent and temporary price impact of trading. We first show that a risk-neutral investor who myopically deleverages his position to meet an immediate need for cash always prefers to sell more liquid assets. If the investor faces the possibility of a downstream shock, however, the solution differs in several important ways. If the ensuing shock is sufficiently large, the non-myopic investor unwinds positions more than immediately necessary and, all else being equal, prefers to retain more of the assets with low temporary price impact in order to hedge against the possible distress. More generally, optimal liquidation involves selling strictly more of the assets with a lower ratio of permanent to temporary impact, even if these assets are relatively illiquid. The results suggest that properly accounting for the possibility of future distress should play an important role in managing large portfolios.

*We would like to thank Tobias Adrian, Darrell Duffie, Paul Glasserman, Bengt Holmstrom, Albert ‘Pete’ Kyle, Massimo Massa, Lasse Pedersen, Tano Santos, S. ‘Vish’ Viswanathan, and the participants at a session of INFORMS in Washington DC, USA and at a seminar at Abu Dhabi Investment Authority, UAE for their comments and suggestions.

[†]Fuqua School of Business, Duke University, One Towerview Drive, Durham, NC 27708, USA, dbbrown@duke.edu, +1 919 660-7968.

[‡]Anderson School of Management, University of California, Los Angeles, 110 Westwood Plaza Suite C519, Los Angeles, CA 90995, USA, bruce.carlin@anderson.ucla.edu, +1 310 825-7246.

[§]INSEAD, Middle East Campus, Box 48049, Abu Dhabi, UAE, miguel.lobos@insead.edu, +971 2 446-0808.

1 Introduction

Following the crisis that surrounded the downfall of Long-Term Capital Management (LTCM) in 1998, Myron Scholes (2000) raised the following problem: *how should an investment manager unwind a portfolio when faced with present and possible future liquidity needs?* This problem is ubiquitous during unstable financial periods. For example, in the recent financial crisis, banks incurred large losses during the forced contraction of their balance sheets as access to short-term financing through repo markets dried up (*e.g.*, Adrian and Shin, 2009; Brunnermeier, 2009). A systemic deleveraging process propagated through the banking sector, in which careful liquidation became crucial to preserving wealth and surviving the crisis.

The key question here is which assets should be sold to meet short-term obligations, keeping in mind the potential for liquidity needs in the future. This problem is distinct from a related problem that has been extensively analyzed in the past. Previous work has focused on the optimal way to liquidate a single asset, either as a monopolist (*e.g.*, Bertsimas and Lo, 1998; Huberman and Stanzl, 2005; Schied and Schöneborn, 2009) or against selling pressure (*e.g.*, Brunnermeier and Pedersen, 2005; Carlin, Lobo, and Viswanathan, 2007).¹ In these papers, a trader needs to sell a given asset for exogenous reasons. In the problem addressed in this paper, the trader needs to generate cash or reduce leverage, and chooses which assets to sell. The solution to this problem has fundamentally different economic implications.

We develop a two-period model where, in each period, the net cost of trading and the price impact of trading on the market value of the assets is based on a continuous-time market. In keeping with the optimal liquidation literature, we take the price impact of trading the assets as a primitive rather than derive a general equilibrium in the market.² A single, risk-neutral investor holds a portfolio of assets, where the market price of each asset is impacted by trading depending on its permanent and temporary components of liquidity. The permanent component of liquidity is the change in the asset's price that depends on the cumulative amount traded, and is independent of the rate at which the asset is traded. The temporary component of liquidity measures the

¹See also Vayanos, 1998; Almgren and Chriss, 1999; Almgren, 2000; Fedyk, 2001; DeMarzo and Uroevic, 2004; Oehmke, 2008; and Chu, Lehnert, and Passmore, 2009.

²It is unclear whether an equilibrium model is either feasible or appropriate here. The solution to such a model would depend on the holdings of every other investor subject to the potential shock, and this information is not publicly available. Note, however, that the price impact parameters in our model can be interpreted as the summary beliefs of the investor regarding the market conditions given the actions of every other investor in the event of a shock.

instantaneous, reversible price pressure that results from trading. Partitioning liquidity into these components has been shown to be empirically robust and important (*e.g.*, Sadka, 2005; see Carlin, Lobo, and Viswanathan (2007) for further discussion).

We begin by analyzing a single-period version of the problem in which the trader does not have to consider future needs for liquidity - that is, there is no possibility of needing to liquidate assets in the second period. The investor faces a limit on leverage (a margin constraint), and experiences an urgent need for liquidity. We call this scenario *myopic deleveraging*. We find that, for a given portfolio and price impact parameters, the investor optimally sells assets that are more liquid to meet pending obligations.

This result changes when we consider a two-period problem as follows. In period one, the investor is required to unwind part of the portfolio to reduce leverage. Subsequently, with some probability, the investor may experience another liquidity shock and be required to further unwind the portfolio in the second period. If no further distress occurs, trading ends. However, if the investor suffers further distress, the problem faced in the second period is similar in nature to the single-period case. The probability of the future need for liquidity is known to the investor, as is the size of the potential shock.

A central question of interest is *how do the liquidity characteristics of the assets affect the trader's ability to hedge distress risk?* A tradeoff arises in the first period of the two-period problem. Selling the more liquid assets first will limit the immediate loss in value; however, the resulting portfolio will be more vulnerable to a continued shock in future periods. Selling the less liquid assets first will result in a portfolio that is more robust to a continued adverse environment; however, this can result in possibly unnecessary loss in value if there is no subsequent shock.

The solution in the two-period model is qualitatively different from the myopic deleveraging case in several ways. In the case of myopic deleveraging, the investor will only trade just enough to meet the margin constraint; since trading is costly, there is no benefit to trading any more than necessary. This is not the case in the two-period model. When the expected second-period shock is large enough, the investor will always want to trade away from the margin in the first period. In doing so, the investor retains cash to protect against a future shock.

We also find that, for large shocks, the amount sold of each asset is monotonically increasing (across assets) in the ratio of permanent to temporary price impact. This leads to a key difference relative to the single-period case, which suggests that the temporary component of liquidity is central to risk-management behavior in the two-period model. If the expected need for liquidity is small, the investor behaves in a similar way as in the one-period problem. However, when the

expected need for liquidity is large, the investor holds on to assets with a low temporary impact of trading and, all else being equal, sells assets that are less liquid on this dimension. The same does not hold for the permanent price impact. No matter how large the size of the second-period shock, all else equal, the investor always favors selling more of assets with a low permanent price impact of trading in the first period. This sheds light on the nature of the practical solution to the portfolio liquidation problem. Assets with concentrated ownership or those with a high degree of asymmetric information (*i.e.*, those with high permanent price-impact) will not be prioritized for liquidation when an investor experiences a recurrent need for liquidity. Assets that are heavily traded, where there are many opportunities to access counterparties, may or may not be liquidated early. If the expected need for liquidity is small, the investor optimally sells these securities to meet early obligations. However, if the expected need for liquidity is large, the investor will hold onto these assets, preserving the option to sell them in the future.

Our predictions are given validation by the empirical analysis of Manconi, Massa, and Yasuda (2010), who study the behavior of institutional investors holding asset-backed securities and corporate bonds in the financial crisis of 2007-2008. They find that at the onset of the crisis, investors liquidated assets with lower permanent price impact (such as corporate bonds). They also find that investors with a shorter investment horizon (who are therefore more severely constrained) sold more heavily at the onset of the crisis. Finally, investors with a shorter investment horizon are also found to have liquidated proportionally more assets with high temporary price impact (such as lower-rated securitized bonds).³

The analysis in this paper adds to a rather large literature on optimal liquidation, which has focused on the case of a single asset. One exception is Duffie and Ziegler (2003), who numerically investigate the trade-off between selling off an illiquid asset to keep a “cushion of liquid assets,” and selling a liquid asset to maximize short-term portfolio value. Illiquidity is modeled as linear transaction costs, and permanent price impact of trading is not considered (they note, however, that this may be a central concern for large investors). Our paper considers both temporary and permanent price impact and, albeit with a considerably simpler model of uncertainty, provides an analytical derivation of structural properties of the optimal solution for a portfolio with any number of assets.

We present the one-period model in Section 2, the two-period model in Section 3, and conclude in Section 4. All proofs are in appendix.

³Note that this finding in Manconi, Massa and Yassuda (2010) was contrary to the authors’ prior expectations (personal communication).

2 One Period: Myopic Deleveraging

2.1 Price and Trading Model

Consider a single risk-neutral investor who trades a portfolio of n assets in continuous-time over a finite horizon. At any time $t \in [0, \tau]$, $Y_t \in \mathbb{R}^n$ is the rate at which the investor trades the assets. The investor's holdings are denoted by $X_t \in \mathbb{R}^n$, where $X_t = x_0 + \int_0^t Y_s ds$. We will generally assume $x_0 > 0$. We assume that Y_t is an L^2 -function.

The prices of the assets at time t are given by $P_t \in \mathbb{R}^n$, which is determined by

$$P_t = q + \Gamma X_t + \Lambda Y_t. \tag{1}$$

This is a multi-dimensional version of the pricing equation used in Carlin, Lobo, and Viswanathan (2007). A similar pricing relationship was derived in equilibrium by Genotte and Kyle (1991), and also by Vayanos (1998). Pritsker (2004) and Huberman and Stanzl (2004) obtain similar relationships. We characterize an investor's optimal trading behavior as a function of current and possible future liquidity needs, taking equation (1) as given for the price equilibrium in a market in which there is price impact of trading (and otherwise independent of the investor's optimal strategy).

The expression in (1) has three parts. The first term $q \in \mathbb{R}^n$ specifies the intercept of the linear model, that is, the equilibrium prices that arise when the investor does not hold any assets and is not trading.⁴ The second and third terms partition the price impact of trading into permanent and temporary components. The permanent component measures the change in price that is independent of the rate at which any of the assets are traded. This impact is likely to be high when the amount of asymmetric information associated with an asset is high or ownership of the asset in the market is concentrated. The temporary component measures the instantaneous, reversible price pressure that results from trading, which can also be interpreted as a transaction cost. This component is likely to be high when the asset is thinly traded or there is a paucity of readily-available counterparties in the market. Both $\Gamma \in \mathbb{R}^{n \times n}$ and $\Lambda \in \mathbb{R}^{n \times n}$ are diagonal matrices in which diagonal entry i is the price impact of asset i . We assume that these price impact parameters are positive, implying that the matrices Λ and Γ are both positive definite ($x' \Gamma x > 0$ and $x' \Lambda x > 0$

⁴We keep the within-period price path deterministic to be able to focus on the consequences of uncertainty about future liquidity shocks, while keeping the problem analytically tractable. For small τ and when considering the possibility of a large shock, we believe this to be a reasonable assumption. Further, as will be noted below, our model captures the case of a uniform relative price movement over all assets.

for all $x \in \mathbb{R}^n$ such that $x \neq 0$).⁵

Partitioning the price impact of trading into permanent and temporary components is justified on empirical grounds. Kraus and Stoll (1972), Holthausen, Leftwich, and Mayers (1990), Cheng and Madhavan (1997) find large permanent and temporary effects for block trades on the NYSE. Cheng and Madhavan (1997) estimate these effects for block trades exceeding 10,000 shares in the “downstairs” and “upstairs” markets at the NYSE. Sadka (2005) finds that the correlation between the temporary and permanent components of price impact to be approximately 0.28, which suggests a significant variation in the ratio of permanent to temporary price impact. For a complete discussion, see Carlin, Lobo, and Viswanathan (2007).

We denote the initial and final positions by $x_0 = X_0$ and $x_1 = X_\tau$, and the cumulative trade by $y_1 = x_1 - x_0$. Prior to trading, the asset prices are $p_0 = P_{0-} = q + \Gamma x_0$. After trading is complete, the price is

$$p_1 = P_{\tau+} = q + \Gamma x_1 = q + \Gamma(x_0 + y_1) = p_0 + \Gamma y_1.$$

Using the notation r' to denote the transpose of a vector r , the end-of-period assets are

$$a_1 = p_1' x_1 = (p_0 + \Gamma y_1)' (x_0 + y_1) = a_0 + (p_0 + \Gamma x_0)' y_1 + y_1' \Gamma y_1,$$

a quadratic function of y_1 . The cash that is generated from trading over $[0, \tau]$ is

$$\kappa_1 = \int_0^\tau -P_t' Y_t dt = \int_0^\tau - \left(p_0 + \Gamma \int_0^t Y_s ds + \Lambda Y_t \right)' Y_t dt.$$

We assume that cash is counted directly against liabilities for risk-management purposes or for the satisfaction of margin constraints. Denoting the initial liabilities by l_0 , the liabilities at time τ can then be written as $l_1 = l_0 - \kappa_1$. Consider an investor who wishes to maximize some function $f(a_1, l_1)$, increasing in a_1 and decreasing in l_1 . The optimal execution schedule is a constant trading rate, that is

$$Y_t^* = \frac{1}{\tau} y_1, \quad t \in [0, \tau].$$

For any given set of trades y_1 and resulting final prices p_1 and assets a_1 , the concavity of the integrand in κ_1 leads the trader to smooth trades over time to minimize the transaction costs due to the temporary price impact of trading. This allows us to simply focus on how much to liquidate

⁵One can easily incorporate off-diagonal entries in Λ and Γ and still solve numerically the resulting optimization problems that we will formulate. Moreover, much of the qualitative behavior that we establish here, such as preemptive deleveraging, still holds. However, analytical expressions for the optimal trades are no longer feasible.

each asset. Since trading occurs at a constant rate for each asset, the end-of-period liabilities may be computed as

$$l_1 = l_0 + \left(p_0 + \Lambda y_1 + \frac{1}{2} \Gamma y_1 \right)' y_1 = l_0 + p_0' y_1 + y_1' \left(\Lambda + \frac{1}{2} \Gamma \right) y_1.$$

By simple accounting, e_1 , the investor's equity at the end of trading, equals $a_1 - l_1$. We can use this to express the equity as:

$$e_1 = a_1 - l_1 = e_0 + x_0' \Gamma y_1 - y_1' \left(\Lambda - \frac{1}{2} \Gamma \right) y_1. \quad (2)$$

The end-of-period equity is strictly concave in the trade vector y_1 if and only if $\Lambda - \frac{1}{2} \Gamma$ is positive definite. We assume this to be the case, that is

$$\Lambda \succ \frac{1}{2} \Gamma, \quad (3)$$

ensuring that the trader's problem is well-posed. This is a reasonable condition to ensure economic soundness of the model, as without this restriction the trader may embark on trades of infinite size and, in doing so, obtain arbitrarily large equity. From a computational standpoint, this ensures that the optimization problem involves maximization of a strictly concave objective over a bounded, convex set.⁶

2.2 Margin Constraint and Forced Deleveraging

Due to either margin requirements imposed by lenders or to regulatory or risk-management constraints there is, under normal circumstances, a limit on the financial leverage that an investor can incur. Different ratios quantifying the degree to which an investor is leveraged can be found in the literature. Three commonly-used ratios are liabilities over assets, assets over equity, and liabilities over equity. All three ratios are increasing in the degree of financial leverage, and are readily related to each other by $l/a = \frac{l/e}{l/e+1}$ and $a/e = l/e + 1$. We specify limits on financial leverage via a bound ρ on the ratio of debt to equity, that is

$$\frac{l_1}{e_1} \leq \rho.$$

This inequality can be written as a quadratic constraint on y_1 ,

$$\rho e_0 - l_0 + (\rho \Gamma x_0 - p_0)' y_1 - y_1' \left(\rho \left(\Lambda - \frac{1}{2} \Gamma \right) + \Lambda + \frac{1}{2} \Gamma \right) y_1 \geq 0. \quad (4)$$

⁶In practice, an investor may of course want to incorporate a number of other constraints into the problem. From the point of view of computational tractability, any modification that preserves the convexity of the problem can be easily handled. This includes constraints on position size or on trade size or any number of risk constraints in a mean-variance framework (see, *e.g.*, Lobo, Fazel and Boyd (2007)).

If no leverage is allowed after the trading period ($\rho = 0$), this constraint is $-p'_0 y_1 - y'_1 (\Lambda + \frac{1}{2}\Gamma) y_1 \geq l_0$, which states that, after accounting for transaction costs, the trades must generate enough cash to cover all liabilities. If arbitrarily large leverage is permitted ($\rho \rightarrow +\infty$), the constraint becomes the solvency constraint $e_1 \geq 0$.⁷

We are interested in modeling liquidity shocks which force an investor to quickly sell assets to reduce leverage, or equivalently to generate cash to meet liabilities. These shocks may arise from a number of reasons, such as a decrease in the value of assets, unexpectedly large investor withdrawals, or margin calls due to a transition to a more risk-averse environment (which in our model translates to requiring a lower ρ). In this model of a fire sale to mitigate risk, the investor is restricted from increasing positions and from short-selling, which corresponds to the box constraints

$$-x_{0,i} \leq y_{1,i} \leq 0.$$

In what follows, we consider the problem in which the investor maximizes the value of his equity subject to both a margin constraint and the box constraints. The problem of maximizing equity subject to the box constraints and a requirement to generate a fixed amount of cash yields qualitatively similar solutions. We remind the reader that, in everything that follows, we will be considering sales of assets, which correspond to negative values in the trades vector. Thus, $y_{1,i} < y_{1,j}$ should be interpreted as meaning we liquidate *more* of asset i than asset j .

A recurring assumption in our analysis in both the single-period and two-period models is that, for all i , $\rho \gamma_i x_{0,i} \leq p_{0,i}$. If this condition is violated for a particular asset, selling any amount of it hurts both net equity and the leverage ratio. Such an asset can therefore be excluded from the problem without loss of generality.

Result 1. *Consider the single-period deleveraging problem*

$$\begin{aligned} & \text{maximize} && e_1 \\ & \text{subject to} && l_1 \leq \rho e_1 \\ & && -x_0 \leq y_1 \leq 0, \end{aligned}$$

⁷The condition for the constraint on leverage to be convex and bounded is that $\Lambda \succ \frac{\rho-1}{2(\rho+1)}\Gamma$. (Note that if $\rho \leq 1$ the constraint is convex for any Λ and Γ such that $\Lambda \succ 0$ and $\Gamma \succ 0$.) This condition is implied by (3) and is therefore automatically ensured in our framework. It is a less restrictive assumption than (3) so that the problem may be bounded for some objective functions that are not concave. Though maximization of a non-concave function over a convex set is in general an intractable problem, in such a case we can still solve the problem by using a result from convex analysis known as the *S-lemma* (see, e.g., Pólik, Terlaky (2007)), a quadratic analog to the Farkas lemma. If condition (3) is not satisfied, however, an unrestricted trader may still improve equity arbitrarily by repeated trading.

where deleveraging is required ($\frac{l_0}{e_0} > \rho$). The optimal solution satisfies $\frac{l_1}{e_1} = \rho$ and there exists a $z > 0$ such that the optimal trades are given by

$$y_{1,i}^* = \max \left(-x_{0,i}, \min \left(0, \frac{1}{2} \cdot \frac{(1+z\rho)\gamma_i x_{0,i} - zp_{0,i}}{(1+z\rho)(\lambda_i - \frac{1}{2}\gamma_i) + z(\lambda_i + \frac{1}{2}\gamma_i)} \right) \right).$$

The optimal trade of asset i , $y_{1,i}^*$, is increasing in ρ and decreasing in l_0 , but is not, in general, monotonic in $x_{0,i}$, in λ_i , nor in γ_i .

Not surprisingly, the trader will sell off to the point where the margin constraint binds. This maximizes value in the single period case but, as we will see in the next section, may not hold in the multi-period setting. Trades are non-monotonic in the price-impact parameters due to two opposing effects, which can be appreciated as follows. Consider an asset for which the price impact of trading increases, and how this changes the optimal trades. On the one hand it will be comparatively more costly to deleverage, requiring the investor to liquidate a larger share of the portfolio. On the other hand, the investor will prefer to sell less of this particular asset, and more of others. Which effect dominates as to the amount that is liquidated of the asset in question is determined by how quickly the optimal trades shift away from that asset, versus how quickly the fraction of the portfolio that needs to be liquidated increases. Simple examples can be constructed where, over a reasonable range for the price impact parameters, $y_{1,i}^*$ exhibits non-monotonic behavior.

The next result establishes a cross-asset ordering for myopic distressed sales.

Result 2. *For the myopic deleveraging problem, the following hold:*

- (a) *Assets with low price impact are prioritized for liquidation. If two assets i and j are such that $p_{0,i} = p_{0,j}$, $\gamma_i \leq \gamma_j$, $\lambda_i \leq \lambda_j$ and $x_{0,i} = x_{0,j}$, then $y_{1,i}^* \leq y_{1,j}^*$.*
- (b) *Assets with smaller holdings are prioritized for liquidation. If two assets i and j are such that $p_{0,i} = p_{0,j}$, $\gamma_i = \gamma_j$, $\lambda_i = \lambda_j$, and $x_{0,i} < x_{0,j}$, then $y_{1,i}^* \leq y_{1,j}^*$ (unless asset i is fully liquidated, such that $y_{1,i}^* = -x_{0,i} > y_{1,j}^*$).*

Over assets that are otherwise identical and for which the investor has similar holdings, the more liquid assets are sold first. Likewise, over assets that are equally liquid, assets of which the investor has smaller holdings are sold first. Note that, while the proofs of the monotonicity with respect to λ_i and $x_{0,i}$ are trivial from the partial derivatives of $y_{1,i}^*$ and hold without the distressed-deleveraging box constraints, this is not the case for γ_i , for which the monotonicity does not hold without the no-shorting constraint. The monotonicities in the price impact parameters imply that

in distressed sales due to short-lived shocks, traders should deleverage by selling off their most liquid holdings to generate cash or decrease their liabilities. We illustrate this with a simple four-asset example that we will later revisit.

Example 1. Consider an investor with four assets, each with price \$5, and holding 1 million units of each, for current holdings of $a_0 = \$20$ M. Liabilities are \$19 M, and the maximum allowable leverage is $\rho = 18$, forcing the investor partially unwind the positions.

The assets have temporary impacts of $\lambda_1 = .06$, $\lambda_2 = .055$, $\lambda_3 = .02$, $\lambda_4 = .015$, and permanent impacts of $\gamma_1 = .016$, $\gamma_2 = .015$, $\gamma_3 = .006$, and $\gamma_4 = .006$. Thus, according to both temporary and permanent price impacts, the assets are in order, from 1 to 4, of increasing liquidity, with assets 1 and 2 considerably more illiquid than assets 3 and 4.

Solving the myopic deleveraging problem, we find the optimal trade is $y_{1,1}^* = 0$, $y_{1,2}^* = 0$, $y_{1,3}^* = -.0851$ M, and $y_{1,4}^* = -.1205$ M. Consistent with Result 2 and with intuition, most of the sale is of the most liquid assets (assets 3 and 4), and, in fact, the optimal solution involves selling nothing of the least liquid assets (assets 1 and 2). Asset 3 has temporary impact that is 33% higher than that of asset 4, so it is also not surprising that $y_{1,4}^* < y_{1,3}^*$. These sales result in $c_1^* = \$.9985$ M, so that liquidation costs the investor .15% of net equity.

3 Deleveraging with Risk of Future Distress

We now consider a situation in which the investor faces the possibility of future shocks. We first discuss the two-period model that we study and then provide some results on the nature of the optimal solution.

3.1 Model

Consider again a single investor who trades in n assets over two periods. Each period is a discrete amount of time $[0, \tau]$, in which trading occurs continuously as before. Prices arise from the process in (1) and the investor is restricted to satisfy constraints on leverage at the end of each time period $j \in \{1, 2\}$,

$$\frac{l_j}{e_j} \leq \rho.$$

Since we wish to study policies regarding deleveraging under distress, we restrict the investor's trades to be reductions in positions and disallow shorting.⁸

The key difference now is that there is uncertainty during period one about whether the investor will face the need for further liquidity during period two. This uncertainty may arise because of unforeseen equity withdrawals, higher cash requirements to fund other areas of the business, less favorable funding conditions (*e.g.*, tighter margin constraints), or a uniform drop in asset prices. The uncertainty, which we call *distress risk*, is resolved between the periods. We model the shock as an early equity withdrawal, which generalizes to all the situations mentioned. Specifically, the amount withdrawn is a Bernoulli random variable Δ such that

$$\Delta = \begin{cases} \delta, & \text{with probability } \pi \\ 0, & \text{with probability } 1 - \pi. \end{cases} \quad (5)$$

If there is a second-period shock, liabilities increase by δ , resulting in a more leveraged balance sheet. The investor is then required to liquidate assets in the second trading period to deleverage to within allowed limits. Although more general shock distributions can be handled in the framework of the optimization problem, our goal is to obtain structural insights on the interaction between the magnitude of future distress and optimal liquidation strategies. To this end, we present the analysis of a Bernoulli shock, varying the shock magnitude δ .

Before moving forward with full details on the optimization problem, we discuss some technical assumptions on the underlying problem parameters. In everything that follows, we will assume the following holds.

Assumptions. The following conditions hold:

1. The price impact matrices satisfy:

$$\Lambda \succ \max\left(\frac{1 + \sqrt{\pi}}{2}, \frac{\rho - 1}{\rho + 1}\right) \Gamma, \quad (6)$$

$$p_0 \geq ((1 + \rho) \Lambda + \Gamma) x_0. \quad (7)$$

2. The trade $y_1 = -x_0/2$ generates sufficient cash to meet the first-period margin constraint.

Condition (6) is the analog of the condition $\Lambda - \Gamma/2 \succ 0$ from the myopic problem and is only slightly more restrictive. From an optimization standpoint, it ensures that the objective function is

⁸We believe this is the appropriate restriction since, around a crisis event for the investor, or around a period of heightened market uncertainty, more strict risk management would most likely prevent the investor from quickly increasing exposure.

concave in the decision variables (this is justified in the appendix). Economically, it simply means that investors cannot arbitrarily improve wealth simply through the act of trading and, as such, seems to be a reasonable requirement.⁹

Condition (7) is an upper bound on the value of the price impact parameters. It says that a full fire-sale of a particular asset will not result in a price movement more than a $1/(1 + \rho)$ fraction of the current price. If this condition is violated, trading costs are so prohibitive that some of the structural results that follow may no longer hold.

Finally, we assume that a liquidation of one-half of the portfolio holdings generates sufficient cash to meet the first-period margin requirement. Given that such a trade is extreme, this assumption is weak. If this is not true, then the investor is so constrained by the initial fire-sale that some of the tradeoffs we discuss shortly for this two-period model may not occur.

Following the notation of the previous section, the equilibrium price after the second period is

$$\begin{aligned} p_2 &= p_1 + \Gamma x_2 \\ &= p_0 + \Gamma (y_1 + y_2), \end{aligned}$$

and the assets at the end of the second period are

$$\begin{aligned} a_2 &= p_2' x_2 \\ &= (p_0 + \Gamma (y_1 + y_2))' (x_0 + y_1 + y_2) \\ &= a_0 + \begin{bmatrix} p_0 + \Gamma x_0 \\ p_0 + \Gamma x_0 \end{bmatrix}' \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} + \begin{bmatrix} y_1 \\ y_2 \end{bmatrix}' \begin{bmatrix} \Gamma & \Gamma \\ \Gamma & \Gamma \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \end{bmatrix}. \end{aligned}$$

Using the same price model as for the single-period problem, the investor trades y_2 for an average price of $p_1 + \Lambda y_2 + \frac{1}{2}\Gamma y_2$. After withdrawals Δ , the investor is then left with liabilities

$$\begin{aligned} l_2 &= l_1 + \Delta + \left(p_1 + \Lambda y_2 + \frac{1}{2}\Gamma y_2 \right)' y_2 \\ &= l_0 + \Delta + \begin{bmatrix} p_0 \\ p_0 \end{bmatrix}' \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} + \begin{bmatrix} y_1 \\ y_2 \end{bmatrix}' \begin{bmatrix} \Lambda + \frac{1}{2}\Gamma & \frac{1}{2}\Gamma \\ \frac{1}{2}\Gamma & \Lambda + \frac{1}{2}\Gamma \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \end{bmatrix}, \end{aligned}$$

⁹Note, however, that the two-period problem is guaranteed to be bounded using only the weaker assumption that we introduced for the single-period problem, $\Lambda \succ \frac{\rho-1}{2(\rho+1)}\Gamma$. This assumption ensures strict convexity of the first-period leverage constraint, and therefore boundedness of y_1 . With y_1 bounded by the first-period constraint on leverage, we only need convexity in y_2 of the second-period constraint on leverage to ensure boundedness of y_2 , which is also ensured by $\Lambda \succ \frac{\rho-1}{2(\rho+1)}\Gamma$. This assumption is therefore sufficient to ensure that the optimal trades are finite. The same caveats as in the single-period case apply for this weaker constraint.

and the second-period equity, $e_2 = a_2 - l_2$, is

$$e_2 = e_0 - \Delta + \begin{bmatrix} \Gamma x_0 \\ \Gamma x_0 \end{bmatrix}' \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} - \begin{bmatrix} y_1 \\ y_2 \end{bmatrix}' \begin{bmatrix} \Lambda - \frac{1}{2}\Gamma & -\frac{1}{2}\Gamma \\ -\frac{1}{2}\Gamma & \Lambda - \frac{1}{2}\Gamma \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \end{bmatrix}.$$

The second period leverage constraint

$$\frac{l_2}{e_2} \leq \rho$$

can be written as a quadratic constraint on the vector of first- and second-period trades,

$$\begin{aligned} & \rho e_0 - l_0 - (1 + \rho)\Delta + \begin{bmatrix} \rho\Gamma x_0 - p_0 \\ \rho\Gamma x_0 - p_0 \end{bmatrix}' \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} \\ & - \begin{bmatrix} y_1 \\ y_2 \end{bmatrix}' \begin{bmatrix} \rho(\Lambda - \frac{1}{2}\Gamma) + \Lambda + \frac{1}{2}\Gamma & \frac{1}{2}(1 - \rho)\Gamma \\ \frac{1}{2}(1 - \rho)\Gamma & \rho(\Lambda - \frac{1}{2}\Gamma) + \Lambda + \frac{1}{2}\Gamma \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} \geq 0. \end{aligned} \quad (8)$$

In order to avoid the possibility of bankruptcy, the investor must choose their first-period trades in such a manner that the second-period constraint on leverage can be met under any realization of Δ . The two-period problem of the expected-equity maximizing investor is then

$$\begin{aligned} & \text{maximize} && \mathbf{E}_\Delta e_2 \\ & \text{subject to} && l_1 \leq \rho e_1, \quad l_2 \leq \rho e_2, \quad \forall \Delta \\ & && -x_0 \leq y_1 \leq 0, \quad -x_1 \leq y_2 \leq 0, \quad \forall \Delta, \end{aligned}$$

where the optimization is over y_1 and y_2 , where y_1 is in \mathbb{R}^n and y_2 is $\{0, \delta\} \mapsto \mathbb{R}^n$ (or, equivalently, a random variable in \mathbb{R}^n measurable in the sigma-algebra generated by Δ).

We can simplify the problem by noting that, when $\Delta = 0$, the optimal second-period trade is $y_2 = 0$. This can be seen by computing the gradient of the objective with respect to y_2 at $y_2 = 0$, which is Γx_1 . Under assumptions of no shorting and $\gamma_{i,j} \geq 0$, all entries of the gradient are non-negative. This, together with the concavity of the objective in y_2 (guaranteed by $\Lambda \succ \frac{1}{2}\Gamma$) and the convexity of the box constraints, ensures that, if the constraint on leverage is not binding, the optimum is achieved at $y_2 = 0$. The first-period leverage constraint and first-period trades in turn ensure that, absent a second-period shock, the second-period leverage constraint is not binding. Intuitively, if there is no shock, the portfolio still meets the margin obligation and, since selling assets is costly, there is no need to further rebalance.

We therefore use $y_2 \in \mathbb{R}^n$ to denote the second period trades associated with the realization $\Delta = \delta$. Likewise, we refer to l_2 and e_2 as the liabilities and equity when $\Delta = \delta$. Noting that when

$\Delta = 0$, the optimal second-period equity is $e_2 = e_1$, we can now write the investor's expected-equity-maximization problem as

$$\begin{aligned} & \text{maximize} && (1 - \pi)e_1 + \pi e_2 \\ & \text{subject to} && l_1 \leq \rho e_1, \quad l_2 \leq \rho e_2 \\ & && -x_0 \leq y_1 \leq 0, \quad -x_1 \leq y_2 \leq 0. \end{aligned} \tag{9}$$

The program variables are $y_1 \in \mathbb{R}^n$ and $y_2 \in \mathbb{R}^n$, and the objective is a quadratic function in \mathbb{R}^{2n} ,

$$\begin{aligned} \mathbf{E}_\Delta e_2 &= (1 - \pi)e_1 + \pi e_2 \\ &= e_0 - \pi\delta + \begin{bmatrix} \Gamma x_0 \\ \pi \Gamma x_0 \end{bmatrix}' \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} - \begin{bmatrix} y_1 \\ y_2 \end{bmatrix}' \begin{bmatrix} \Lambda - \frac{1}{2}\Gamma & -\pi\frac{1}{2}\Gamma \\ -\pi\frac{1}{2}\Gamma & \pi(\Lambda - \frac{1}{2}\Gamma) \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \end{bmatrix}. \end{aligned}$$

To understand the nature of the optimal solution to this problem, it is helpful to consider the problem with the two leverage constraints dualized:

$$\begin{aligned} & \text{maximize} && (1 - \pi)e_1 + \pi e_2 + z_1(\rho e_1 - l_1) + z_2(\rho e_2 - l_2) \\ & \text{subject to} && -x_0 \leq y_1 \leq 0, \quad -x_1 \leq y_2 \leq 0. \end{aligned} \tag{10}$$

If we fix the values of z_1 and z_2 , the problem can be decoupled in the assets in that the optimal solution can be obtained by the independent maximization of $(y_{1,i}, y_{2,i})$ for each asset i . One can then independently derive the solution for each asset i as a function of the first- and second-period shadow prices. The objective can be written as a sum of terms associated with each asset,

$$\mathbf{E}_\Delta e_2 = c + \sum_{i=1}^n (b'_i y_i - y'_i A_i y_i),$$

where $y_i = \begin{bmatrix} y_{1,i} \\ y_{2,i} \end{bmatrix}$. The constant term c and the linear and quadratic terms b_i and A_i depend on the first- and second-period shadow prices z_1 and z_2 , and are as follows,

$$\begin{aligned} c &= e_0 - \pi\delta + (\rho e_0 - l_0)(z_1 + z_2) - (1 + \rho)\delta z_2 \\ b_i &= \begin{bmatrix} \rho\gamma_i x_{0,i} + (\rho\gamma_i x_{0,i} - p_{0,i})(z_1 + z_2) \\ \pi\rho\gamma_i x_{0,i} + (\rho\gamma_i x_{0,i} - p_{0,i})z_2 \end{bmatrix} \\ A_i &= \begin{bmatrix} \rho(\lambda_i - \frac{1}{2}\gamma_i) + ((\rho + 1)\lambda_i - \frac{1}{2}(\rho - 1)\gamma_i)(z_1 + z_2) & -\frac{1}{2}\pi\rho\gamma_i - \frac{1}{2}(\rho - 1)\gamma_i z_2 \\ -\frac{1}{2}\pi\rho\gamma_i - \frac{1}{2}(\rho - 1)\gamma_i z_2 & \pi\rho(\lambda_i - \frac{1}{2}\gamma_i) + ((\rho + 1)\lambda_i - \frac{1}{2}(\rho - 1)\gamma_i)z_2 \end{bmatrix}. \end{aligned}$$

The constraints disallowing position increases and short sales can be equivalently stated as

$$y_{1,i} \leq 0, \quad y_{2,i} \leq 0, \quad \text{and} \quad y_{1,i} + y_{2,i} \geq -x_{0,i}.$$

This defines a triangular feasible set for the first- and second-period trades in each asset. In short, for the optimal values of the shadow prices z_1 and z_2 of the margin constraints in both periods, the problem decouples into n optimization problems (one per asset), with each problem maximization of a concave quadratic objective over three linear constraints. We derive the solution in detail in the appendix.

3.2 Preemptive Deleveraging and Distress Management

We now proceed to characterize optimal trading in this setting that allows for a downstream shock. The first important question that we address is whether the leverage constraint

$$\frac{l_1}{e_1} \leq \rho$$

binds in the first period. That is, we consider whether the investor deleverages preemptively (*i.e.*, more than is immediately required) in the first period when there is a potential need for liquidity in the future. The following result addresses this question.

Result 3. *Suppose that the investor's initial holdings are such that $l_0/e_0 > \rho$, *i.e.*, immediate deleveraging is required. Then there exists a threshold shock level $\hat{\delta} \geq 0$ such that the optimal two-period solution satisfies $l_1/e_1 = \rho$ for all $\delta \in [0, \hat{\delta}]$ and $l_1/e_1 < \rho$ for all feasible $\delta > \hat{\delta}$.*

Result 3 says that the optimal two-period liquidator may in fact preemptively deleverage beyond what is required in the first period. When the potential need for future liquidity is large, the margin constraint does not bind in the first period. Due to the convexity of the penalty incurred in a large fire sale, the investor manages future liquidity risk by over-liquidating the portfolio early on. Note that this result arises even though the investor is risk neutral - the effect would be more pronounced for risk-averse investors. Recall that in the case of myopic deleveraging as in Section 2.2, such preemptive deleveraging will never occur.

Another interpretation of Result 3 is that, if the future need for liquidity is high enough, the investor substitutes liquid assets for illiquid ones. If we interpret cash to be the $(n + 1)^{th}$ asset in the portfolio, the investor overweights this (most liquid) asset to the detriment of other securities when the future need for liquidity is sufficiently large.

We now investigate how the investor should prioritize assets for liquidation given the possibility of downstream distress. The first finding we report is that, unlike the case of myopic deleveraging, it is no longer the case that the investor simply sells assets from most liquid to least. Moreover, all else equal, the temporary and permanent components of price impact have, in some sense, opposing effects on the optimal solution.

Result 4. *Let i and j be any two assets with equal initial prices and equal initial holdings. Then for any $\pi \in (0, 1)$, $\rho > 0$, the following hold:*

- (a) *If $\lambda_i = \lambda_j$ and $\gamma_i < \gamma_j$, then, regardless of δ , lower permanent price impact assets are prioritized for immediate liquidation, i.e., $y_{1,i}^* < y_{1,j}^*$.*
- (b) *If $\gamma_i = \gamma_j$ and $\lambda_i > \lambda_j$, then for sufficiently large δ , higher temporary price impact assets are prioritized for immediate liquidation, i.e., $y_{1,i}^* < y_{1,j}^*$.*

The first statement says that, all else equal, the investor will always want to trade assets that have a low permanent price impact, no matter how great the need for future liquidity may be. However, the second statement says that this does not hold for the temporary component of liquidity: if the future shock is sufficiently large, then the investor optimally retains assets that have a small temporary component of liquidity in the first period in preparation for the possibility of future distress.

In short, optimal liquidation may, all else equal, involve retaining assets with low temporary price impact as a hedge against a crisis. This is not the case for assets with low permanent impact: the investor prefers to sell more of such assets. Such liquidation behavior represents a striking departure from the case of myopic deleveraging.

Result 4 looks at two special cases where one of the components of price impact is constant across assets. It is a special case of the following, more general result that shows the interplay between permanent and temporary price impact when liquidating under risk of distress.

Result 5. *Let i and j be any two assets with equal initial prices and equal initial holdings. Then for any $\pi \in (0, 1)$, $\rho > 0$, the following hold:*

- (a) *If asset i is more liquid than asset j ($\lambda_i < \lambda_j$, $\gamma_i < \gamma_j$) and, in addition, $\gamma_i/\lambda_i < \gamma_j/\lambda_j$, then, regardless of δ , asset i is prioritized for immediate liquidation, i.e., $y_{1,i}^* < y_{1,j}^*$.*
- (b) *If $\gamma_i/\lambda_i > \gamma_j/\lambda_j$, then for sufficiently large δ , asset j is prioritized for immediate liquidation, i.e., $y_{1,j}^* < y_{1,i}^*$.*

Part (a) of Result 5 aligns with what we have found for the optimal liquidation behavior described for the case of myopic deleveraging (Section 2.2). Namely, even accounting for the possibility of future distress, it is optimal for the investor to sell more of assets that are more liquid, no matter how extreme the future shock may be. For this result to hold, however, note that an extra condition is required: the more liquid asset needs to have a lower ratio of permanent to temporary price impacts.

Part (b) of the result shows that this ratio condition is not merely sufficient, but it is also *necessary* to ensure prioritization of one asset over another in the optimal trades when facing the prospect of a large shock. According to this part of the result, for a large enough future shock, the ratio of permanent to temporary price impact drives the investor's optimal trades more than the price impacts considered individually. In this case, the investor may favor selling less liquid assets in the first period to hedge against the future need for liquidity.

It is not obvious that a strategy that immediately sells more of illiquid assets could ever be optimal. Intuition may suggest that the investor, as a hedge against future distress, would increase the *relative* proportion of illiquid to liquid assets sold in period one, yet still sell a greater absolute amount of the more liquid assets. This may reasonably be expected, as the future shock is not sure to occur, whereas first-period trades result in immediate costs that are certain. Yet this alone is not necessarily optimal: for sufficiently large shocks, an optimal liquidator chooses to actively unload a strictly larger quantity of assets with a low ratio of permanent to temporary price impact, even if these assets are less liquid across both components of price impact.

It is also somewhat surprising that a fairly simple index – the ratio of permanent to temporary price impact – emerges from the analysis as a central quantity describing the structure of optimal liquidation behavior. Some intuition for why this ratio is crucial can be gained as follows. If the shock does not occur, the problem is essentially like the one faced by the myopic liquidator: in this case, the investor wants to sell as little as possible to keep trading costs low and to avoid depressing, as much as possible, the price of assets currently held. On the other hand, if a shock does occur, deleveraging may be expensive, primarily due to the dominant effect of temporary price impact. In this case, the investor would prefer to smooth trades as much as possible over the two periods to avoid large and costly downstream rebalancing.

The ratio of permanent to temporary impacts represents, in some sense, a normalized measure of efficiency in balancing between these two competing effects: namely, if the ratio of permanent to temporary price impacts is low, the investor can smooth trades to hedge against future distress in a manner that is relatively cheap in comparison to the resulting depression of prices. When the

future shock is sufficiently large, even if it is unlikely, the marginal cost of future distress becomes prohibitive enough to tilt the investor towards favoring sales of assets that are, in this sense, efficient. The fact that the ratio of price impacts drives the liquidation strategy more generally underscores the finding from Result 4: in the presence of future distress that is potentially severe, permanent and temporary price impact influence optimal liquidation strategies in opposite ways.

To illustrate our findings more concretely, we now revisit the example from earlier.

Example 2. Consider the same parameters as in Example 1. Recall that the assets are in order of increasing liquidity (from 1 to 4) in both temporary and permanent impact. On the other hand, we have

$$\frac{\gamma_1}{\lambda_1} < \frac{\gamma_2}{\lambda_2} < \frac{\gamma_3}{\lambda_3} < \frac{\gamma_4}{\lambda_4}.$$

Because of this ordering of the ratios, Result 5(b) shows that, no matter how likely future distress may be, there is always a sufficiently large shock such that a larger amount of asset i will be liquidated than asset $j > i$, even though asset j is more liquid than asset i . Recall that the optimal myopic solution is $y_{1,1}^* = 0$, $y_{1,2}^* = 0$, $y_{1,3}^* = -.0851$ M, and $y_{1,4}^* = -.1205$ M.

Figure 1 shows the optimal first period trades y_1^* versus the second period shock magnitude (δ) for the case when the shock is believed to be fairly unlikely ($\pi = .01$). One can see that if the possible shock is large enough, the optimal solution involves selling more of the relatively illiquid assets - in the zoomed portion on the right, it is apparent that for $\delta > \$.85$ M, less of the most liquid asset (asset 4) is sold, and, in fact, it is optimal to sell assets in amounts that decrease as liquidity improves.

Also note from Figure 1 that the amounts of each asset sold need not be nondecreasing in δ (though the total trade size, summing across all assets, will be). In particular, we see that around $\delta = \$.83$ M, the optimal liquidation strategy involves selling *less* of the most liquid asset as δ increases: over this range, it actually becomes optimal to “hedge” the post-trade position by selling more of the relatively illiquid assets and retaining a larger stake in the liquid asset.

It is notable that the optimal trades may be so different than the myopic trades, and these differences have significant implications in terms of future ruin. For instance, for $\delta = \$.85$ M, the optimal first-period trade is $y_{1,1}^* = -.0849$ M, $y_{1,2}^* = -.0829$ M, $y_{1,3}^* = -.0736$ M, and $y_{1,4}^* = -.0379$ M. The net equity after the first-period is now $e_1^* = \$.9860$ M, which is $\$.0125$ M worse than the the first period equity following the optimal myopic strategy (or 1.25% of the original net equity). However, in the event of a shock, an investor following the myopic strategy goes bankrupt. This is

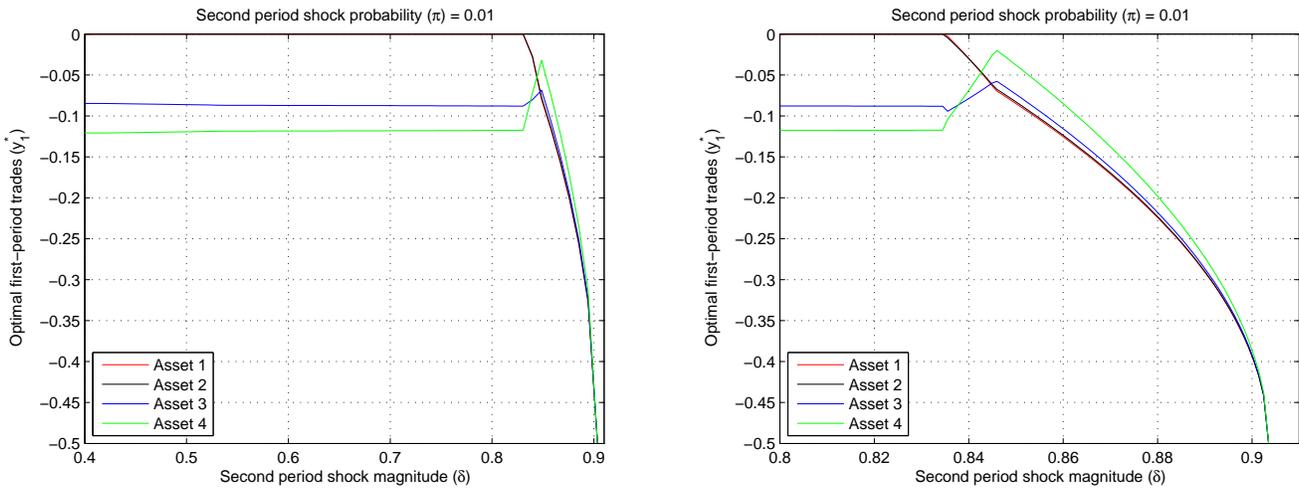


Figure 1: Optimal trades for Example 2 with future distress relatively unlikely ($\pi = .01$) (right zoomed scale).

not the case for the anticipative liquidator, who adjusts positions in a manner that incurs slightly higher immediate costs but retains a more liquid portfolio.

Figure 2 shows similar illustrations in the case where future distress is considerably more likely ($\pi = .25$ or $\pi = .5$). Most of the phenomena discussed above are still present in this case but are even more pronounced. For instance, the crossover δ when more of the illiquid asset is sold relative to the intermediate asset is now much smaller. Moreover, because distress is considerably more likely, the investor is now much more willing to sell off amounts of the two, highly illiquid assets (assets 1 and 2). Comparing the results for the cases $\pi = .01$, $\pi = .25$, and $\pi = .5$, we see that the investor's beliefs about the likelihood of future distress drastically affect the structure of the trades: this underscores the fact that the modeling of future shocks plays a crucial role in optimal liquidation.

Finally, Figure 3 shows the leverage ratio of the optimal liquidator's position following first-period trade as δ grows, which illustrates Result 3. For large enough δ , it is optimal for the investor to trade more than required by the margin constraint. Not surprisingly, this happens for a smaller δ when the shock is more likely, as the investor will be more cautious as π grows. Note that the kink in sales of the liquid asset in Figure 1 near $\delta = \$0.85$ M corresponds to the point where the first-period margin constraint is no longer binding.

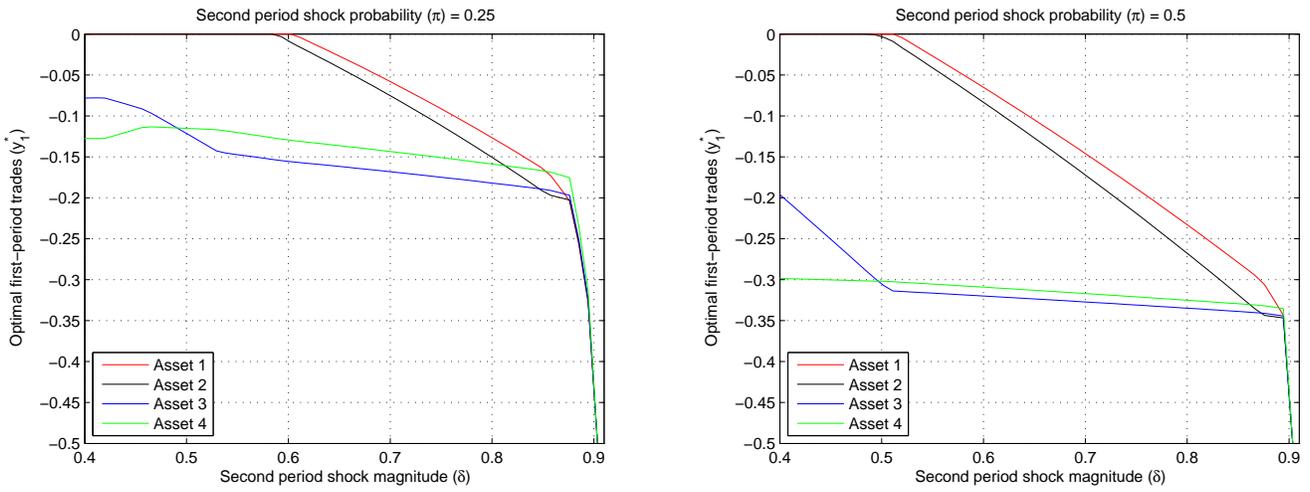


Figure 2: Optimal trades for Example 2 with future distress probabilities $\pi = .25$ (L) and $\pi = .5$ (R).

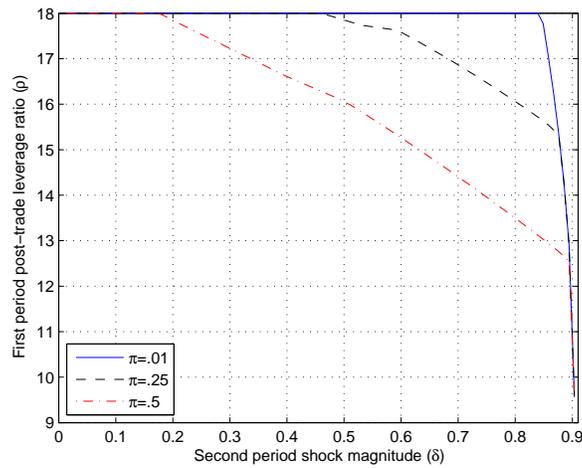


Figure 3: First period post-trade leverage ratios for Example 2.

4 Conclusion

Hedging future distress risk is crucial for portfolio management, especially during financial crises. This requires an understanding of the tradeoffs between, on the one hand, the immediate cost of selling illiquid assets and, on the other, the risk of retaining an illiquid portfolio. As noted by Myron Scholes (2000), “since it is not possible to know the extent of the unfolding crisis, holding and not selling the less liquid instruments is similar to buying an option to hold a position.” We have studied how the liquidity characteristics of the assets in a portfolio affect optimal liquidation in the presence of financial pressures by analyzing a two period problem in which an investor faces present and possibly future liquidity needs. The assets in the investor’s portfolio differ along two liquidity dimensions: their temporary and permanent price impacts of trading. The investor’s goal is to maximize portfolio value while meeting present liquidity constraints and planning for downstream shocks. We find that, when the risk of future distress is high, it is optimal to sell more of assets with a lower ratio of permanent to temporary price impact.

It is unclear to what extent large investors already give consideration to these questions. At least for institutional investors at the onset of the 2007-2008 crisis, our predictions are given empirical support by Manconi, Massa, and Yasuda (2010). This includes the counter-intuitive finding regarding increased sales of assets with higher temporary price impact when the shock severity is higher. However, from anecdotal evidence and from the sparsity of published work, it is plausible that insufficient attention is given to these issues outside of periods of crisis. Our work suggests that investors would benefit from including the cost of forced deleveraging of a large portfolio as a consideration in portfolio selection, as well as from maintaining contingency plans for its partial unwinding. More work on these questions would be worthwhile, both from an empirical perspective as well as on computational strategies for realistic model formulations that include more complex uncertainty and multiple time periods.

A Appendix: Proofs

Proof of Result 1.

Proof. Consider the problem with the leverage constraint dualized,

$$\begin{aligned} & \text{maximize} && e_1 + z(l_1 - \rho e_1) \\ & \text{subject to} && -x_0 \leq y_1 \leq 0, \end{aligned}$$

With diagonal structure the assets are decoupled, in that the solution can be obtained by separately optimizing the trades in each asset. If the box constraint in asset i is not active, its solution is obtained from the first-order condition which, from problem convexity, is sufficient. Increasing ρ (or decreasing l_0) relaxes the margin constraint, which implies a smaller shadow price z , from which the monotonicities follow by simple algebra. (The non-monotonicities can be verified by simple counter-examples.) \square

Proof of Result 2.

Proof. We show the results by arguing that the partial derivative of $y_{1,i}^*$ as λ_i , γ_i , or $x_{0,i}$ varies in the appropriate direction, noting that the shadow price z will be constant, as we are analyzing a fixed trade. The claims for λ_i and $x_{0,i}$ are immediate from the expression for $y_{1,i}^*$. Consider now γ_i . Since the expression only involves asset i (the constancy of z ensures that the assets remain decoupled), we can without loss of generality normalize $p_{0,i} = 1$ and $x_{0,i} = 1$. The numerator of the partial derivative $\frac{\partial y_{1,i}^*}{\partial \gamma_i}$ can then be seen to simplify to

$$2\lambda_i + (2\lambda_i - 1)z + z^2.$$

Therefore, the partial derivative is positive for any $z < z_0$ where

$$z_0 = \frac{1}{2} \left(1 - 2\lambda_i - \sqrt{4\lambda_i^2 - 12\lambda_i + 1} \right),$$

if $\lambda_i \leq 3/2 - \sqrt{2}$, and for all z otherwise. From the constraint on short sales $y_{1,i}^* \geq -1$, the dual variable z must satisfy

$$z \leq \frac{2\lambda_i}{1 - 2\lambda_i - \gamma_i} \leq \frac{2\lambda_i}{1 - 4\lambda_i}.$$

The proof is completed by showing that this is less than z_0 for any $\lambda_i > 0$. The inequality can be written as

$$8\lambda_i^2 - 10\lambda_i + 1 > (1 - 4\lambda_i) \sqrt{4\lambda_i^2 - 12\lambda_i + 1}.$$

From inspection of the roots of the different factors, both sides are positive for $\lambda_i \leq 3/2 - \sqrt{2}$, and we can therefore square both sides. Collecting terms, the inequality then simplifies to $64\lambda_i^3 > 0$.

The two claims in the result are then immediate. \square

Proof of Condition (6).

Proof. Consider two matrices A and B in $\mathbb{R}^{n \times n}$ such that $A = A'$, $B = B'$, $A \succ 0$, and $B \succ 0$. We show that the matrix

$$M = \begin{bmatrix} A & -B \\ -B & A \end{bmatrix}$$

is positive definite if and only if $A \succ B$. Since $A \succ 0$, M is positive definite if and only if its Schur complement is positive definite: $A - BA^{-1}B \succ 0$. Since $A \succ 0$, by change of coordinates this is equivalent to $I - (A^{-1/2}BA^{-1/2})^2 \succ 0$. Since $A^{-1/2}BA^{-1/2}$ is symmetrical and therefore has identical left and right eigenvectors, the condition is true if and only if all the eigenvalues of $A^{-1/2}BA^{-1/2}$ satisfy $\lambda^2 < 1$. Since $A^{-1/2}BA^{-1/2}$ is positive definite, this is equivalent to $\lambda < 1$. We conclude that we can write the condition on the Schur complement as $I - A^{-1/2}BA^{-1/2} \succ 0$. By change of coordinates, this is equivalent to $A - B \succ 0$.

Applying this result to the matrix in the quadratic form in (8) leads to the condition

$$\Lambda \succ \frac{\rho - 1}{\rho + 1} \Gamma.$$

By change of coordinates, the matrix in the quadratic form in $\mathbf{E}_\Delta e_2$ is positive definite if

$$\begin{bmatrix} \Lambda - \frac{1}{2}\Gamma & -\sqrt{\pi}\frac{1}{2}\Gamma \\ -\sqrt{\pi}\frac{1}{2}\Gamma & \Lambda - \frac{1}{2}\Gamma \end{bmatrix}$$

is positive definite. Applying the result above to this matrix leads to the condition

$$\Lambda \succ \frac{1 + \sqrt{\pi}}{2} \Gamma.$$

\square

Proof of Result 3.

Proof. Notice that for $\delta = 0$, the two-period problem is identical to the myopic deleveraging problem. Thus, at $\delta = 0$ it must be the case for the optimal solution to satisfy $l_1/e_l = \rho$. Suppose not, *i.e.*, suppose at $\delta = 0$, we have $l_1/e_1 < \rho$. Then the shadow price of the first-period margin

constraint is zero, and the problem is equivalent to a problem that maximizes the net equity subject to $-x_0 \leq y_1 \leq 0$. Since we have $y_1 \leq 0$, $\Gamma \geq 0$ and $x_0 \geq 0$, it follows that

$$\begin{aligned} e_1 &= e_0 + x'_0 \Gamma y_1 - y_1 \left(\Lambda - \frac{1}{2} \Gamma \right) y_1 \\ &\leq e_0 - y_1 \left(\Lambda - \frac{1}{2} \Gamma \right) y_1 \\ &\leq e_0, \end{aligned}$$

with equality strict for any nonzero y_1 due to $\Lambda - \Gamma/2 \succ 0$. This means $y_1 = 0$ must be optimal, which is a contradiction, as the trader initially satisfied $l_0/e_0 > \rho$.

Now consider the maximum possible δ such that the two-period problem is still feasible. We will consider the problem of finding the maximum such δ and show that at this value, it must be that $l_l/e_l < \rho$. Denote this maximum feasible δ by $\bar{\delta}$; one can see that $\bar{\delta}$ is given by

$$\bar{\delta} = \frac{\rho a_0 - (\rho + 1)l_0 + \rho v^*}{\rho + 1},$$

where v^* is the optimal value of the (convex) problem

$$\begin{aligned} &\text{maximize}_{y_1, y_2} && -(p_0 - \rho \Gamma x_0)(y_1 + y_2) - [y'_1 \ y'_2] \begin{bmatrix} (\rho + 1) \left(\Lambda + \frac{1}{2} \Gamma \right) - \rho \Gamma & \frac{1-\rho}{2} \Gamma \\ \frac{1-\rho}{2} \Gamma & (\rho + 1) \left(\Lambda + \frac{1}{2} \Gamma \right) - \rho \Gamma \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} \\ &\text{subject to} && \rho a_0 - (\rho + 1)l_0 - (p_0 - \rho \Gamma x_0)' y_1 - y'_1 \left((\rho + 1) \left(\Lambda + \frac{1}{2} \Gamma \right) - \rho \Gamma \right) y_1 \geq 0 \\ &&& -x_0 \leq y_1 \leq 0 \\ &&& -x_0 - y_1 \leq y_2 \leq 0. \end{aligned}$$

We will show that the optimal solution will strictly satisfy the first-period margin constraint, so we will omit this constraint for now as we consider computation of $\bar{\delta}$. Define the following for ease of notation:

$$\begin{aligned} d &\triangleq p_0 - \rho \Gamma x_0 \\ D &\triangleq \begin{bmatrix} A & B \\ B & A \end{bmatrix} = \begin{bmatrix} (\rho + 1) \left(\Lambda + \frac{1}{2} \Gamma \right) - \rho \Gamma & \frac{1-\rho}{2} \Gamma \\ \frac{1-\rho}{2} \Gamma & (\rho + 1) \left(\Lambda + \frac{1}{2} \Gamma \right) - \rho \Gamma \end{bmatrix}. \end{aligned}$$

Since $d \geq 0$ (by assumption; recall that for any asset i for which $d_i \leq 0$, we can remove it from the original problem without loss of generality) and $D \succ 0$ (strict convexity), it can never be optimal to have $y_1 > 0$ or $y_2 > 0$ above, and therefore we can ignore the non-positivity constraints.

We thus focus on finding the optimal value to the problem

$$\begin{aligned} &\text{maximize}_{y_1, y_2} && -d'(y_1 + y_2) - [y'_1 \ y'_2] D \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} \\ &\text{subject to} && y_1 + y_2 + x_0 \geq 0. \end{aligned}$$

Using Lagrange multipliers $\nu \geq 0$ for the inequality constraints, the Lagrangian to this problem is given by

$$\mathcal{L}(y_1, y_2, \nu) = x'_0 \nu + (\nu - d)'(y_1 + y_2) - [y'_1 \ y'_2] D \begin{bmatrix} y_1 \\ y_2 \end{bmatrix},$$

and the optimal solution for any ν is given by

$$\begin{bmatrix} y_1(\nu) \\ y_2(\nu) \end{bmatrix} = \frac{1}{2} D^{-1} \begin{bmatrix} \nu - d \\ \nu - d \end{bmatrix}.$$

If we can find a $\nu \geq 0$ such that the corresponding solution is also feasible, then it must be optimal.

Let $H = A + B$ and $\nu = d - Hx_0$. We have

$$\begin{aligned} \nu &= d - (A + B)x_0 \\ &= p_0 - \rho\Gamma x_0 - ((\rho + 1)\Lambda + (1 - \rho)\Gamma) x_0 \\ &\geq p_0 - \rho\Gamma x_0 - (p_0 - \Gamma x_0 + (1 - \rho)\Gamma x_0) \\ &= 0, \end{aligned}$$

where in the inequality we are using the condition that the temporary impact component be sufficiently small, *i.e.*, $(\rho + 1)\Lambda x_0 \leq p_0 - \Gamma x_0$. Thus, $\nu \geq 0$ for this choice. Moreover,

$$\begin{aligned} \begin{bmatrix} y_1(\nu) \\ y_2(\nu) \end{bmatrix} &= \frac{1}{2} D^{-1} \begin{bmatrix} \nu - d \\ \nu - d \end{bmatrix} \\ &= -\frac{1}{2} D^{-1} \begin{bmatrix} H^{-1} x_0 \\ H^{-1} x_0 \end{bmatrix} \\ &= -\frac{1}{2} \begin{bmatrix} x_0 \\ x_0 \end{bmatrix}. \end{aligned}$$

Clearly, splitting up the trade in half over the two periods satisfies the no-short sales constraints. Moreover, by assumption, it strictly satisfies the first-period margin constraint. Thus, this solution is feasible and therefore optimal to the above problem for finding v^* , and hence the maximum level $\bar{\delta}$.

Notice that strict convexity of the objective implies that this is the only optimal solution to this problem; hence, at $\delta = \bar{\delta}$, the trade $(y_1, y_2) = (-x_0/2, -x_0/2)$ is the only feasible solution; since it satisfies the first-period margin constraint strictly, we have found a large enough δ such that the optimal solution to the two-period problem satisfies $l_1/e_l < \rho$.

Thus, at $\delta = 0$ we have $l_1/e_1 = \rho$ and at $\delta = \bar{\delta}$ we have $l_1/e_1 < \rho$; the shadow price z_1 associated with the first-period margin constraint is a continuous and nonincreasing function of δ . This implies that at some $\delta \in (0, \bar{\delta})$, the shadow price goes to zero, proving the threshold property that was claimed. \square

Discussion of Form of 2-period Optimal Solution

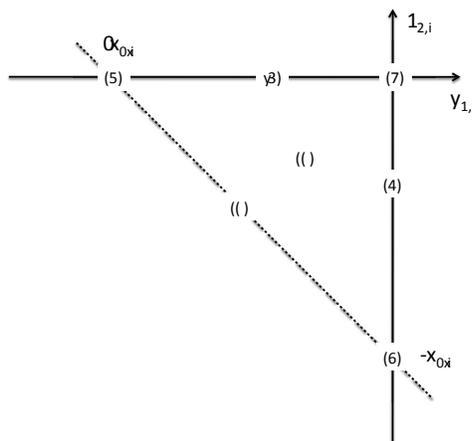


Figure 4: Enumeration of cases for the linear constraints on each asset.

As noted, feasible set for the first- and second-period trades in each asset is triangular. We graph the possible cases in Figure 4 and derive their optimal trades in Table 1. Depending on which constraints are active and inactive, there are seven different cases to consider. Which case occurs depends on the amount of deleveraging required immediately, the size and likelihood of a potential shock in the second period, and the liquidity parameters and holdings of a particular asset.

The first two cases are the most interesting, as the remaining five correspond to more extreme situations. In Case 1, the optimal solution is strictly in the interior of the triangle. If an asset is in this region, it has favorable enough liquidity properties that is optimal to sell some of it in the first period, and some of it in the second period if a subsequent need for liquidity arises. However, even if a shock does occur in the second period, the investor will not fully liquidate the asset. Case 2 is similar to Case 1, except that if the shock occurs, the investor is forced to liquidate the entire stake in the asset. The investor would prefer to sell more of the asset, but the no-short-sales constraint

Case	$y_{1,i}^*$	$y_{2,i}^*$
1	$\frac{A_{i,2,2}b_{i,1}-A_{i,1,2}b_{i,2}}{2(A_{i,1,1}A_{i,2,2}-A_{i,1,2}^2)}$	$\frac{-A_{i,1,2}b_{i,1}+A_{i,1,1}b_{i,2}}{2(A_{i,1,1}A_{i,2,2}-A_{i,1,2}^2)}$
2	$\frac{b_{i,1}-b_{i,2}+2x_{0,i}(A_{i,1,2}-A_{i,2,2})}{2(A_{i,1,1}+A_{i,2,2}-2A_{i,1,2})}$	$\frac{b_{i,2}-b_{i,1}+2x_{0,i}(A_{i,1,2}-A_{i,1,1})}{2(A_{i,1,1}+A_{i,2,2}-2A_{i,1,2})}$
3	$\frac{b_{i,1}}{2A_{i,1,1}}$	0
4	0	$\frac{b_{i,2}}{2A_{i,2,2}}$
5	$-x_{0,i}$	0
6	0	$-x_{0,i}$
7	0	0

Table 1: Enumeration of cases for the linear constraints on each asset.

binds and changes the character of the optimal solution.

In Case 3, the investor chooses to liquidate some of the asset in period one but nothing further in period two if a shock occurs. This may occur, for instance, if a large amount of deleveraging is required immediately but the subsequent size of Δ is small. Case 5 is similar except that the investor liquidates all of the asset immediately. This may occur if the first-period deleveraging is large and the asset is very liquid.

In Case 4, the investor does not sell the asset in the first period, but does sell some of it in the second period. This might occur if the asset is relatively illiquid but, due to limitations on positions of the other assets, the investor has no choice but to sell some of it if a second-period shock occurs. Case 6 is similar to Case 4 in that the asset is not sold in the first period, but is completely liquidated in the second period. This might occur if Δ is very large. Finally, in case 7, the investor does not liquidate any of the asset. This might happen if the asset is highly illiquid or if Δ is very small.

Proof of Result 4.

Proof. Both parts are corollaries of Result 5. □

Proof of Result 5.

Proof. Part (a): We prove the result for the interesting Cases in Table 1 (Cases 1 and 2); the proof for Cases 3 to 7 follows in similar fashion.

We will show that $y_{1,i}^* < y_{1,j}^*$ holds for any such δ and π in the three situations below, which cover all possibilities at the optimal solution:

1. The no-short sales constraints are not binding for either asset i or asset j (i.e., both are in case 1).
2. The no-short sales constraint is binding for asset i but not asset j (asset i is in case 2 and asset j is in case 1; we will argue the intuitive fact that the reverse situation cannot occur).
3. The no-short sales constraints are binding for both assets i and j (both assets are in case 2).

Situation 1: Assume that at the optimal solution, the no-short sales constraints are not binding for either asset i or asset j , and assume that $y_{1,i}^* \geq y_{1,j}^*$. We will show that such a solution cannot be optimal. To this end, we will show there exists a feasible direction from (y_1^*, y_2^*) in which we can head and strictly improve the objective. This implies that the solution cannot be optimal.

In particular, for some $\epsilon > 0$, let ϵ_1 be the vector with zeros everywhere but $-\epsilon$ in entry i and $+\epsilon$ in entry j . Notice that, for a sufficiently small ϵ , the solution $(y_1^* + \epsilon_1, y_2^*)$ still satisfies the no-short sales constraints.

Now we examine the gradients of the liability amount and the net equity in each period. For the first period, we have

$$\begin{aligned}\nabla l_1(y_1) &= p_0 + (2\Lambda + \Gamma)y_1 \\ \nabla e_1(y_1) &= \Gamma x_0 - (2\Lambda - \Gamma)y_1,\end{aligned}$$

and thus, since we have $0 \geq y_{1,i}^* \geq y_{1,j}^*$, $\lambda_i < \lambda_j$, and $\gamma_i < \gamma_j$:

$$\epsilon_1' \nabla l_1(y_1^*)/\epsilon = -(2\lambda_i + \gamma_i)y_{1,i}^* + (2\lambda_j + \gamma_j)y_{1,j}^* \leq 0,$$

and (recalling that convexity requires $2\lambda_i - \gamma_i \geq 0$ for all i):

$$\begin{aligned}\epsilon_1' \nabla e_1(y_1^*)/\epsilon &= (\gamma_j - \gamma_i)x_0 - (2\lambda_j - \gamma_j)y_{1,j}^* + (2\lambda_i - \gamma_i)y_{1,i}^* \\ &\geq (\gamma_j - \gamma_i)x_0 - (2(\lambda_j - \lambda_i) - (\gamma_j - \gamma_i))y_{1,j}^* \\ &\geq \begin{cases} (\gamma_j - \gamma_i)x_0 & \text{if } 2(\lambda_j - \lambda_i) - (\gamma_j - \gamma_i) \geq 0 \\ 2(\lambda_j - \lambda_i)x_0 & \text{otherwise,} \end{cases} \\ &> 0.\end{aligned}$$

We also note that

$$\begin{aligned}\nabla l_2(y_1, y_2) &= \begin{bmatrix} p_0 + (2\Lambda + \Gamma)y_1 + \Gamma y_2 \\ p_0 + \Gamma y_1 + (2\Lambda + \Gamma)y_2 \end{bmatrix} \\ \nabla e_2(y_1, y_2) &= \begin{bmatrix} \Gamma x_0 - (2\Lambda - \Gamma)y_1 + \Gamma y_2 \\ \Gamma x_0 + \Gamma y_1 - (2\Lambda - \Gamma)y_2 \end{bmatrix}\end{aligned}$$

We now distinguish several sub-scenarios, and show how we can find an ϵ_2 in each case such that $[\epsilon'_1 \ \epsilon'_2] \nabla l_2(y_1^*, y_2^*) \leq 0$ and $[\epsilon'_1 \ \epsilon'_2] \nabla e_2(y_1^*, y_2^*) \geq 0$ in each case.

First, if $\epsilon'_1 \Gamma(y_1^* + y_2^*) \leq 0$ and $\epsilon'_1 \Gamma(x_0 + y_1^* + y_2^*) \geq 0$, then $[\epsilon'_1 \ 0] \nabla l_2(y_1^*, y_2^*) \leq 0$ and $[\epsilon'_1 \ 0] \nabla e_2(y_1^*, y_2^*) \geq 0$.

Second, if $\epsilon'_1 \Gamma(y_1^* + y_2^*) \leq 0$ but $\epsilon'_1 \Gamma(x_0 + y_1^* + y_2^*) < 0$, then let $\epsilon_2 = \epsilon_1$. Note that $\epsilon'_1 \Gamma(x_0 + y_1^* + y_2^*) < 0$ requires $y_{1,j}^* + y_{2,j}^* < y_{1,i}^* + y_{2,i}^*$, so $\epsilon'_1 \Lambda(y_1^* + y_2^*) \leq 0$ and $\epsilon'_1 \Gamma(y_1^* + y_2^*) \leq 0$, and hence $[\epsilon'_1 \ \epsilon'_2] \nabla l_2(y_1^*, y_2^*) \leq 0$. In addition, we have

$$\begin{aligned}
[\epsilon'_1 \ \epsilon'_2] \nabla e_2(y_1^*, y_2^*) / \epsilon &= 2(\epsilon'_1 \Gamma x_0 - \epsilon'_1 (2\Lambda - \Gamma)(y_1^* + y_2^*)) / \epsilon \\
&= 2((\gamma_j - \gamma_i)x_0 - (2\lambda_j - \gamma_j)(y_{1,j}^* + y_{2,j}^*) + (2\lambda_i - \gamma_i)(y_{1,i}^* + y_{2,i}^*)) \\
&\geq 2((\gamma_j - \gamma_i)x_0 - ((2\lambda_j - \lambda_i) - (\gamma_j - \gamma_i))(y_{1,j}^* + y_{2,j}^*)) \\
&\geq \begin{cases} 2(\gamma_j - \gamma_i)x_0 & \text{if } 2(\lambda_j - \lambda_i) - (\gamma_j - \gamma_i) \geq 0 \\ 4(\lambda_j - \lambda_i)x_0 & \text{otherwise,} \end{cases} \\
&> 0.
\end{aligned}$$

Finally, consider the case $\epsilon'_1 \Gamma(y_1^* + y_2^*) \geq 0$. Then let $\epsilon_2 = -\epsilon_1$ and note that

$$\begin{aligned}
[\epsilon'_1 \ \epsilon'_2] \nabla l_2(y_1^*, y_2^*) / \epsilon &= 2\epsilon'_1 \Lambda(y_1^* + y_2^*) / \epsilon \\
[\epsilon'_1 \ \epsilon'_2] \nabla e_2(y_1^*, y_2^*) / \epsilon &= -2\epsilon_1 \Lambda(y_1^* + y_2^*) / \epsilon,
\end{aligned}$$

and we therefore need to show that $\epsilon'_1 \Lambda(y_1^* + y_2^*) \leq 0$. We claim this is always true under the given conditions. Note that $\epsilon'_1 \Gamma(y_1^* + y_2^*) \geq 0$ requires $y_{2,i}^* \leq (\gamma_j / \gamma_i) y_{2,j}^*$. Therefore,

$$\begin{aligned}
\epsilon'_1 \Lambda(y_1^* + y_2^*) / \epsilon &= (\lambda_j y_{1,j}^* - \lambda_i y_{1,i}^*) + (\lambda_j y_{2,j}^* - \lambda_i y_{2,i}^*) \\
&\leq \lambda_j y_{2,j}^* - \lambda_i y_{2,i}^* \\
&\leq \lambda_j y_{2,j}^* - \frac{\lambda_i \gamma_j}{\gamma_i} y_{2,j}^* \\
&< \lambda_j y_{2,j}^* - \lambda_j y_{2,j}^* \\
&= 0,
\end{aligned}$$

where in the first inequality, we use $0 \geq y_{1,i}^* \geq y_{1,j}^*$ and $\lambda_j > \lambda_i$, in the second inequality we use $y_{2,i}^* \leq (\gamma_j / \gamma_i) y_{2,j}^*$ and in the third inequality we use $\lambda_i / \gamma_i > \lambda_j / \gamma_j$.

Therefore, in each circumstance we have constructed a feasible direction for which l_1 and l_2 are no larger, e_2 is no smaller, and e_1 is strictly larger. Since $\pi < 1$, this means we have found a direction that still satisfies all the problem constraints (no-short sales and margin constraints) with

strictly larger objective, contradicting the optimality of (y_1^*, y_2^*) .

Situation 2: Assume that at the optimal solution, the no-short sales constraints are binding for asset i but not binding for asset j (using an argument similar to the one above, we can argue that $y_{1,i}^* + y_{2,i}^* < y_{1,j}^* + y_{2,j}^*$ must hold, meaning it can never be the case that the constraints are binding in the reverse direction). Note that the no-short sales constraint is binding for asset i , so the above argument no longer applies (as we cannot sell any more of asset i).

For the next two cases, the following will be of use.

Lemma 1. Let $r_{z_1, z_2} : \mathbb{R}_+^2 \rightarrow \mathbb{R}$ be the family of functions

$$r_{z_1, z_2}(\lambda, \gamma) \triangleq \frac{\rho(1 - \pi + z_1)\gamma - 2(\pi\rho + (1 + \rho)z_2)\lambda - z_1 \frac{p_0}{x_0}}{(\rho(\pi - 1) + z_1(1 - \rho))\gamma + 2(\rho(1 + \pi) + (z_1 + 2z_2)(1 + \rho))\lambda},$$

parameterized by $(z_1, z_2) \in \mathbb{R}_+^2$, and where $\pi \in [0, 1]$, $\rho \geq 0$, and the denominator is strictly positive. If $\lambda_i < \lambda_j$, $\gamma_i < \gamma_j$, and $\gamma_i/\lambda_i < \gamma_j/\lambda_j$, then for any $0 \leq z_1 < \infty$, $0 \leq z_2 < \infty$, $r_{z_1, z_2}(\lambda_i, \gamma_i) < r_{z_1, z_2}(\lambda_j, \gamma_j)$.

Proof. We will fix a λ and a γ as well as all parameters in the function. The claim holds if and only if $\epsilon' \nabla r_{z_1, z_2}(\lambda, \gamma) > 0$ for all vectors $\epsilon = [\epsilon_\lambda \ \epsilon_\gamma] \in \mathbb{R}_+^2$ with $\epsilon_\lambda \leq (\lambda/\gamma)\epsilon_\gamma$ (i.e., moving in any directions such that λ and γ do not decrease, nor does their ratio γ/λ , must increase $r_{z_1, z_2}(\lambda, \gamma)$). After some algebra, we arrive at

$$\nabla r_{z_1, z_2}(\lambda, \gamma) \propto \begin{bmatrix} BD\gamma - A \left(C\gamma - \frac{z_1 p_0}{x_0} \right) \\ AC\lambda - D \left(B\lambda - \frac{z_1 p_0}{x_0} \right) \end{bmatrix},$$

where

$$\begin{aligned} A &\triangleq \rho(1 + \pi) + (z_1 + 2z_2)(1 + \rho) \\ B &\triangleq \rho\pi + (1 + \rho)z_2 \\ C &\triangleq \rho(1 - \pi + z_1) \\ D &\triangleq \rho(1 - \pi) + z_1(1 - \rho). \end{aligned}$$

To verify that the gradient condition holds over all such ϵ , we need only check the extreme rays of the set, which are given by $\epsilon_1 = [0 \ 1]'$ and $\epsilon_2 = [(\lambda/\gamma) \ 1]'$. This is clearly true for ϵ_1 , as $A > 0$,

$C > 0$, $A\lambda \geq B\lambda - z_1 p_0 / (x_0)$, and $C \geq D$. For ϵ_2 , we have

$$\begin{aligned} \epsilon_2' \nabla r_{z_1, z_2}(\lambda, \gamma) &= \frac{\lambda}{\gamma} \left(DB\gamma - A \left(C\gamma - \frac{z_1 p_0}{x_0} \right) \right) + \left(AC\lambda - D \left(B\lambda - \frac{z_1 p_0}{x_0} \right) \right) \\ &= \frac{\lambda}{\gamma} (BD - AC)\gamma + (AC - BD)\lambda + (A\lambda + B\gamma) \frac{z_1 p_0}{x_0} \\ &= (A\lambda + B\gamma) \frac{z_1 p_0}{x_0} \\ &\geq 0, \end{aligned}$$

where we are using $A \geq 0$, $B \geq 0$. □

Now note that in this case, $y_{1,i}^* = r_{z_1, z_2}(\lambda_i, \gamma_i)$, with (z_1, z_2) the optimal Lagrange multipliers, because at the optimal solution the no-short sales constraint is active for asset i ($r_{z_1, z_2}(\lambda_i, \gamma_i)$ is the form of the optimal solution, as discussed earlier, for asset i in this case). We will argue that for asset j , which satisfies $y_{1,j}^* + y_{2,j}^* > -x_0$ by assumption, we have $y_{1,j}^* > r_{z_1, z_2}(\lambda_j, \gamma_j)$. Since Lemma 1 implies that $r_{z_1, z_2}(\lambda_j, \gamma_j) > r_{z_1, z_2}(\lambda_i, \gamma_i) = y_{1,i}^*$, this will establish the result.

We can interpret $r_{z_1, z_2}(\lambda_j, \gamma_j)$ as the optimal first-period trade for asset j (with the Lagrange multipliers fixed at (z_1, z_2)) if we are forcing asset j to satisfy the no-short sales constraint tightly (*i.e.*, forcing $y_{1,j} + y_{2,j} = -x_0$). Put another way, $r_{z_1, z_2}(\lambda_j, \gamma_j)$ is the optimal solution y to the problem

$$\begin{aligned} \text{maximize} \quad & - \begin{bmatrix} y - y_{1,j}^* \\ -x - y_{2,j}^* - y \end{bmatrix}' \begin{bmatrix} f_j & h_j \\ h_j & g_j \end{bmatrix} \begin{bmatrix} y - y_{1,j}^* \\ -x - y_{2,j}^* - y \end{bmatrix} \\ \text{subject to} \quad & -x \leq y \leq 0, \end{aligned}$$

where $f_j > g_j \geq h_j$. Assuming the inequalities are inactive at the optimal solution (it is easy to verify that $y = 0$ cannot be optimal, and if $-x_0$ is optimal, then $y = r_{z_1, z_2}(\lambda_j, \gamma_j) = -x_0 < y_{1,j}^*$, so we are done), the optimal solution y must satisfy the first-order condition:

$$\begin{aligned} (f_j - h_j)(y - y_{1,j}^*) - (g_j - h_j)(-x - y_{2,j}^* - y) &= 0 \\ \Downarrow \\ (f_j + g_j - 2h_j)y &= (f_j - h_j)y_{1,j}^* + (g_j - h_j)(-x - y_{2,j}^*). \end{aligned}$$

Since $y_{1,j}^* + y_{2,j}^* > -x_0$ by assumption in this case, we have

$$(f_j + g_j - 2h_j)y < (f_j + g_j - 2h_j)y_{1,j}^*,$$

and since $f_j > g_j \geq h_j$, this implies the optimal y must satisfy $y < y_{1,j}^*$, which gives us the result.

Situation 3: Assume that at the optimal solution, the no-short sales constraints are binding for both assets i and j . In this case, we have $y_{1,i}^* = r_{z_1, z_2}(\lambda_i, \gamma_i)$ and $y_{1,j}^* = r_{z_1, z_2}(\lambda_j, \gamma_j)$. The result now follows by Lemma 1.

(b) Throughout the proof for this result, we will use the notation (δ) to denote that a parameter in question (*e.g.*, optimal solution, shadow price, etc.) is a function of the second-period shock size, $\delta \geq 0$, which will be varying. We will show that under the given conditions we can find a $\hat{\delta}$ with $z_1(\hat{\delta}) = 0$, the no-short sales constraints active for both asset i and asset j and $z_2(\hat{\delta})$ finite but arbitrarily large. We start with the following lemma.

Lemma 2. *Let $r_{z_1, z_2} : \mathbb{R}_+^2 \rightarrow \mathbb{R}$ be the family of functions as described in Lemma 1. Then if $\gamma_i/\lambda_i < \gamma_j/\lambda_j$, then there exists a $0 \leq z_2 < \infty$ such that $r_{0, z_2}(\lambda_i, \gamma_i) < r_{0, z_2}(\lambda_j, \gamma_j)$.*

Proof. Note that when $z_1 = 0$, we can express the function in question as

$$r_{0, z_2}(\lambda, \gamma) = \frac{\rho \left(1 - \pi - 2\pi \left(\frac{\lambda}{\gamma} \right) \right) - 2(1 + \rho) \left(\frac{\lambda}{\gamma} \right) z_2}{\rho \left(\pi - 1 + 2(1 + \pi) \left(\frac{\lambda}{\gamma} \right) \right) + 4(1 + \rho) \left(\frac{\lambda}{\gamma} \right) z_2}.$$

Denoting λ/γ by σ , note that we can write the functions as

$$\begin{aligned} r_{0, z_2}(\lambda_i, \gamma_i) &= \frac{a_i - b_i z_2}{c_i + 2b_i z_2} \\ r_{0, z_2}(\lambda_j, \gamma_j) &= \frac{a_j - b_j z_2}{c_j + 2b_j z_2}, \end{aligned}$$

where

$$\begin{aligned} a_i &= \rho(1 - \pi - 2\pi\sigma_i) \\ b_i &= 2(1 + \rho)\sigma_i \\ c_i &= \rho(\pi - 1 + 2(1 + \pi)\sigma_i), \end{aligned}$$

and analogously for (a_j, b_j, c_j) . Since the denominators are both positive, one can verify that there exists a $z_2 \geq 0$ such that $r_{0, z_2}(\lambda_i, \gamma_i) < r_{0, z_2}(\lambda_j, \gamma_j)$ if

$$2(a_i b_j - a_j b_i) - (b_i c_j - b_j c_i) < 0$$

holds. Some simple algebra shows that

$$2(a_i b_j - a_j b_i) - (b_i c_j - b_j c_i) = 2\rho(1 + \rho)(1 - \pi)(\sigma_j - \sigma_i),$$

and since $\rho > 0$, $\pi \in [0, 1)$, $\sigma_j < \sigma_i$ implies the result. \square

Now consider the problem of finding the maximum possible $\bar{\delta} > 0$ such that the two-period problem is still feasible. Following the proof of Result 3, we find that such a $\bar{\delta}$ corresponds to the trade $y_1 = -x_0/2$, $y_2 = -x_0/2$, i.e., splitting up all assets equally across the two periods. Since the objective function for computing this $\bar{\delta}$ is strictly convex, the solution $y_1 = -x_0/2$ and $y_2 = -x_0/2$ can be the only solution that satisfies the second-period margin constraint at $\bar{\delta}$, and it must be that $\bar{\delta} > 0$ (since the trade $y_1 = -x_0/2$, $y_2 = 0$ satisfies the constraints for the problem $\delta = 0$ and the objective function for computing $\bar{\delta}$ is strictly convex). So, the feasible set to the original problem with $\delta = \bar{\delta}$ is a singleton at which the no-short sales constraints are obviously tight. By assumption, the first-period margin constraint is strictly satisfied, and hence $z_1(\bar{\delta}) = 0$. Moreover, for any $\delta > \bar{\delta}$, the problem is infeasible, and therefore we must have $z_2(\bar{\delta}) = +\infty$.

To complete the proof, we need to argue that we can find a small enough perturbation, $\epsilon > 0$, such that at $\bar{\delta} - \epsilon$, it is still optimal to have both box constraints active, $z_1(\bar{\delta} - \epsilon) = 0$, and $z_2(\bar{\delta} - \epsilon)$ is finite but arbitrarily large. If we can do this, we will have found a δ for which the first-period margin constraint is inactive, assets i and j are both tight on the no-short sales constraint, and $z_2(\delta)$ can be made as large as desired; since $y_{1,i}^*(\delta) = r_{0,z_2(\delta)}(\lambda_i, \gamma_i)$ and $y_{1,j}^*(\delta) = r_{0,z_2(\delta)}(\lambda_j, \gamma_j)$, the result will then follow by Lemma 2.

First, notice that the feasible set at $\delta = \bar{\delta}$ is a singleton, as argued above. For any $\epsilon > 0$, we are enlarging a single ellipsoid, and the feasible set must therefore still be compact (closed and bounded). The singleton at $\bar{\delta}$ is strictly contained inside the first-period margin ellipsoid. Since the feasible set shrinks to a singleton strictly contained in the first-period margin ellipsoid as $\epsilon \rightarrow 0$, we can find a sufficiently small $\epsilon_1 > 0$ such that the feasible set is still strictly contained inside the first-period margin constraint for all $\epsilon \in [0, \epsilon_1]$.

Now consider how z_2 varies with δ . z_2 is an optimal Lagrange multiplier; it is obtained by minimizing a rational function and therefore $z_2(\delta)$ is continuous, and $z_2(\bar{\delta}) = \infty$. Since it is a continuous function, we can find an $\epsilon_2 > 0$ such that $z_2(\bar{\delta} - \epsilon_2)$ is finite but arbitrarily large. Since it can be arbitrarily large, we can make it large enough such that the no-short sales constraints for both assets i and j must still be active for any $\epsilon \in [0, \epsilon_2]$.

Now take $\epsilon = \min(\epsilon_1, \epsilon_2) > 0$. We have $z_1(\bar{\delta} - \epsilon) = 0$, $z_2(\bar{\delta} - \epsilon) < \infty$, and both no-short sales constraints active at $\bar{\delta} - \epsilon$. The proof is complete. \square

References

- Adrian, Tobias and Hyun Song Shin. 2008. Financial Intermediary Leverage and Value-at-Risk. Federal Reserve Bank of New York Staff Reports. Staff Report No. 338.
- Almgren, R., and Neil Chriss. 1999. "Value Under Liquidation." *Risk*, 12, 61-63.
- Almgren, Robert. 2000. "Optimal Execution of portfolio transactions." *Journal of Risk*, 3, 5-39.
- Bertsimas, Dimitris, and Andrew Lo. 1998. "Optimal Control of Execution Costs." *Journal of Financial Markets*, 1, 1-50.
- Brunnermeier, Markus. 2009. Deciphering the Liquidity and Credit Crunch 2007-2008. *Journal of Economic Perspectives*, 23, 77-100.
- Brunnermeier, Markus, and Lasse Pedersen. 2005. Predatory Trading. *Journal of Finance* 60, 1825-1863.
- Carlin, Bruce Ian, Miguel Sousa Lobo, and S. Viswanathan. 2007. Episodic Liquidity Crises: Cooperative and Predatory Trading. *Journal of Finance*, 62, 2235-2274.
- Cheng, Minder, and Ananth Madhavan. 1997. "In Search of Liquidity: Block Trades in the Upstairs and Downstairs Markets." *Review of Financial Studies*, 10, 175-203.
- Chu, Sean, Andreas Lehnert, and Wayne Passmore. 2009. Strategic Trading in Multiple Assets and the Effects on Market Volatility. Forthcoming, *International Journal of Central Banking*.
- DeMarzo, P., and Branko Uroevic. 2009. "Optimal Trading by a Large Shareholder." Forthcoming in *Journal of Political Economy*.
- Duffie, Darrell, and Alexandre Ziegler. 2003. "Liquidity Risk." *Financial Analysts Journal*, May-June, 42-51.
- Fedyk, Yuriy. 2001. "Large Investor Optimization Problem with Direct Effect of Trades on Prices." Working paper, Olin School of Business, Washington University of St. Louis.
- Genotte, Gerard., and Albert Kyle. 1991. "Intertemporal Insider Trading with a Smooth Order Flow." Working Paper, Haas School of Business Administration, UC Berkeley.

- Holthausen, Robert, Richard Leftwich, and David Mayers. 1990. "Large Block Transactions, the Speed of Response, and Temporary and Permanent Stock-Price Effects." *Journal of Financial Economics*, 26, 71-95.
- Huberman, Gur, Werner Stanzl. 2004. Arbitrage-Free Price Update and Price-Impact Functions. *Econometrica* 72, 1247-1275.
- Huberman, Gur, and Werner Stanzl. 2005. "Optimal Liquidity Trading." *Review of Finance*, 9, 165-200.
- Lobo M. S., M. Fazel and S. Boyd (2007). "Portfolio optimization with linear and fixed transaction costs." *Annals of Operations Research*, 152 (1), 341-365.
- Kraus, Alan., and Hans Stoll. 1972. "Price Impacts of Block Trading on the New York Stock Exchange." *Journal of Finance*, 27, 569-588.
- Manconi, Alberto, Massimo Massa, and Ayako Yasuda. 2010. "The Behavior of Intoxicated Investors: The Role of Institutional Investors in Propagating the Financial Crisis of 2007-2008." Working paper, INSEAD.
- Oehmke, Martin. 2008. "Liquidating Illiquid Collateral." Working Paper, Columbia University.
- Pólik, I. and T. Terlaky. 2007. "A survey of the S-Lemma." *SIAM Review*, 49(3), 371-418.
- Pritsker, Matthew. 2004. Large Investors: Implications for Equilibrium Asset Returns, Shock Absorption, and Liquidity. Working paper, Federal Reserve, Washington D.C.
- Sadka, Ronnie. 2005. "Momentum and Post-Earnings Drift Anomalies: The Role of Liquidity Risk." *Journal of Financial Economics*, 80, 309-349.
- Schied, Alexander, and Torsten Schöneborn. 2009. "Risk aversion and the dynamics of optimal liquidation strategies in illiquid markets." *Finance and Stochastics*, 13.
- Scholes, Myron. 2000. Crisis and Risk Management. *American Economic Review*. 90, 17-21.
- Vayanos, Dimitri. 1998. "Transaction Costs and Asset Prices." *Review of Financial Studies*, 11, 1-58.