Aggregate Leverage and Preemptive Selling by Individual Financial Institutions

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Abstract

Our paper studies an economy in which each financial institution takes into account that if it has to sell its assets *after* others have already sold, the price will be lower. This causes preemptive selling, driven not by actual margin calls, but by the fear of future margin calls. Financial institutions cannot determine their optimal capitalizations in isolation, but need to know the aggregate capitalization. The resulting equilibrium is fragile: Small changes in model parameters can cause large changes in the equilibrium allocation of risk. Our model is a natural complement to Allen and Gale (2004).

JEL Codes: **G2** (Financial Institutions). **G31** (Capital Budgeting, Investment Policy).

Our paper examines how financial institutions may choose to liquidate assets pre-emptively in order to avoid selling behind others. Our model describes the aggregate consequences of this pre-emptive selling behavior and shows that the "fear of worse execution later" can become pervasive when the overall financial sector is not well-capitalized. For example, a decline in asset values may cause investors to sell in advance of others who may be forced to liquidate to avoid future margin calls. This early selling, however, lowers asset prices, exacerbates margin constraints, and may contribute to even more pre-emptive selling. Small changes in fundamentals or leverage can cause most of the financial sector to liquidate assets preemptively. Ultimately, this is not due to margin calls *today*, but due to the fear of potential margin calls *in the future* after others have already sold.

In our model, financial institutions are special in two ways. First, they are the more efficient holders of risky assets. This could be due to a variety of factors. For example, financial institutions such as banks may be able to monitor or select loans better than non-financial firms (as in Diamond and Rajan (2005)). We model the financial sector's advantage for holding risky assets by assuming it is less risk-averse than the non-financial sector. Thus, the market price of the risky asset will be lower when the non-financial sector holds more of it. Second, each financial investor may be exposed to a random liquidity shock which will force it to liquidate the risky asset before maturity. There does not need to be aggregate uncertainty about how many financial institutions will be forced to liquidate, as long as each one does not know whether it will have to be among the sellers. For convenience, we will often call our financial investors and institutions simply "banks," but the model should be interpreted as applying more broadly to the entire financial sector with its myriad investor types.

Crucially, our banks are competitive and cannot coordinate their behavior. In equilibrium, each bank takes into account that if it sells its assets when other banks are selling or after other banks have already sold, the price will be lower. Thus, each bank faces a fundamental tradeoff: either sell today (possibly with other banks) at a discounted price below the fundamental value of the asset; or hold on and hope that it can wait until the asset matures. If a bank does hold on and happens to face a liquidity shock in the future, it would then have to sell *after* other banks have already sold, which would be at a steeper discount than if it had sold preemptively.

Our model works out the equilibrium behavior of the financial sector. The first-best coordinated outcome would have every bank sell only when (and if) it faces a margin call so that the financial sector would (efficiently) hold *all* of the risky asset until forced to sell. Yet, this is not an equilibrium in our model: the first bank that sells preemptively can do so at almost no discount to the asset's fundamental (risk-neutral) value. This is because the risk-averse non-financial sector would pay close to the risk-neutral value if only asked to absorb this one bank's tiny share of the risky asset. The exact fraction of the risky asset that is held in-equilibrium by the two sectors is determined by various parameters, including the depth of the non-financial sector, uncertainty about the asset's value, exogenous drops in the asset's fundamentals, the distribution of leverage ratios among the banks, and regulatory restrictions on the maximum leverage ratios allowed by banks. We show how small parameter changes translate into changes in the asset holdings of the two sectors. The resulting equilibrium allocation of risk can be not only fragile—where small changes in parameters can cause large but smooth changes in holdings—but even discontinuous—where small changes in parameters can cause a discontinuous change in holdings.

An important parameter in our model is the aggregate capitalization of the financial sector itself. We show that progressively more conservative banks decrease their leverage and liquidate preemptively in response to adverse changes in aggregate capitalization ratios—such as those that would be caused by the revelation of new fundamental information that changes the value of risky assets. We also show that banks cannot determine their optimal capital ratios in isolation. When the rest of the financial sector is conservative, an individual bank can be more aggressive because its fears of selling behind other banks would be modest. Some legislative proposals under consideration would impose higher capital constraints on some (larger) financial institutions. Our analysis suggests that this will increase the likelihood of forced liquidation and encourage more pre-emptive selling in the short-run. Moreover, it may encourage other smaller financial institutions, not subject to these constraints, to take on more leverage.

Our own paper fits into the modern literature of financial fragility and bank runs that was spawned by Diamond and Dybvig (1983). It is in the descendant branch that has developed explanations of system-wide contagion and a resulting fragility of the entire financial system. Our own paper highlights the central role of fear of future margin constraints and the role of the distribution of bank leverage in the economy.

Without discounting important factors modelled in other papers, our own model is intentionally simple in order to illustrate the consequences of the preemptive selling motive. We do not require two assumptions first introduced in Diamond and Dybvig (1983): inefficient liquidation of real assets and actual margin calls. Our substitutes are, respectively, an assumption that the non-financial sector is reluctant to absorb most of the risky assets in the economy and sets prices sequentially, and the fear of possible future margin calls. We have also handicapped our base model by not requiring endogenous margin calls that trigger *because* the price has declined, or positive correlation among institutions interim liquidity needs.[1] In the base model, the probability that individual financial institutions require *interim* liquidity is random and exogenous—and even perfectly known in the aggregate in advance.

Our model is about the precautionary choice of some financial institutions not to hold risky assets. In this sense, it relates to Allen and Gale (2004), Gorton and Huang (2004), and Acharya, Shin, and Yorulmazer (2009). These papers model situations in which some investors decide to "stand by" and hold liquid, safe, but low-yielding assets. This will be profitable if the regular holders of risky assets (banks) were to run into trouble. In this case, the stand-by investors can purchase assets at fire sale prices. Their high-liquidity low-yield holdings are thus strategic. Our own model suggests that the rest of the financial sector does not stand by idly in response to the possibility of a future liquidity crisis, either. Instead, some regular investors may divest their own optimal holdings preemptively, too. In all models, some or all financial investors end up holding lower-risk assets than they otherwise would. In this sense, our paper also helps resolve the puzzle that motivated Allen and Gale (2004): why would some investors choose to hold low-yield securities in equilibrium? Their answer was that these investors want to take advantage of a crisis if it were to develop. Our answer is that some investors

[1]The only such mechanism in the base model is an *exogenous* price drop in fundamentals that affects all investors. This does of course induce a correlation in bank capitalization and in turn in the preemptive selling mechanism of our model. (Banks begin to fear future liquidation more.) However, the base model specifically ignores that the *endogenous* future price drop can trigger further margin calls. In Appendix A, we allow endogenous value declines to trigger margin calls. This opens up a second contagion channel and strengthens the model.

do not want to take the chance to be caught up in one. Although the models are not embedded, they are natural complements. One can think of strategic investors "lurking" for crises as a contributory determinant of the depth of the non-financial sector in our model. And one can think of the preemptive asset sales of crisis-exposed investors as a contributory determinant to how strong the incentives for "lurking" are in the aforementioned models. The presence of competing investors in both models can in turn dampen the impact of financial crises.

The equilibria in these models can show great fragility, just like those in our own. For example, in Allen and Gale (2000) (where interbank loans can cause fragility) and Allen and Gale (2004), small shocks to the demand for liquidity can cause high asset price volatility or bank defaults or both. It is this price volatility that provides incentives for at least some banks to hold low yielding risky assets in equilibrium.

Diamond and Rajan (2005) focus on the endogenous timing of banks' asset payoffs. These assets can sometimes deliver late. Thus, banks may need more interim liquidity and decide to charge a lower interest rate to short-term borrowers. They can also call in loans from long-term borrowers ahead of time (although this is inefficient). The fear of depositors to be left out at the interim date adds a further feedback loop, leading to banks having even more demand for short-term liquidity. The end effect is that when one bank liquidates its assets, it reduces liquidity for everyone else, leading to contagion. Similarly, in Acharya and Yorulmazer (2008), financial institutions are heterogeneous. Banks can provide liquidity or consume liquidity. As more banks liquidate, the supply of cheap risky assets in the market increases, but the number of institutions that can take advantage thereof, decreases. This is again an additional feedback mechanism. In Acharya, Gale, and Yorulmazer (2009), small liquidation costs can dramatically increase rollover risk.

Two recent papers by Bolton, Santos, and Scheinkman (2009) and Diamond and Rajan (2009) also examine the *voluntary* timing of asset sales (i.e., before divestment is forced by hard margin constraints). In Bolton, Santos, and Scheinkman (2009), potential asset sellers have inside information that grows over time. This can create a tradeoff between liquidating early in a crisis, vs. trying to wait it out but face more adverse selection. In Diamond and Rajan (2009), banks hold onto their assets instead of selling them at low prices today, because they are "underwater." The bank managers care about the bank shareholders, and thus prefer delaying the sale in order to maximize the shareholders' option value. In contrast, in our own model, financial institutions compete to sell early, not late.

Our paper is also closely related to Bernardo and Welch (2004), in which equity investors try to front-run other equity investors when sequential execution breaks down. Our current paper suggests that the preemptive selling motive can also play a role in the context of financial institutions which are subject to margin constraints. Importantly, the financial institutions in our current paper vary in their liquidity needs. This allows us to explore the effects of aggregate and cross-sectional differences in leverage in equilibrium, to develop comparative statics, and to derive both fragile and discontinuous equilibria. (In our earlier paper, a shock impacted all stock investors equally.) Moreover, our current paper is also exploring the role of regulatory margin constraints, endogenous liquidation constraints, and the presence of some arbitrageurs.

Our paper now proceeds as follows: Section 1 lays out the assumptions of our model. Section 2 solves for the equilibrium asset holdings by our two sectors. Section 3 describes the comparative statics of the model. It also discusses fragilities and discontinuities that arise in our model, the externalities that banks impose on one another, and price deviations from fundamentals and the consequent model-induced excess volatility. Section 4 offers an analysis of the welfare costs of our equilibrium compared to the first-best coordinated outcome, and discusses the possible role government intervention and more arbitrageurs could play. And Section 5 concludes.

1 Model Setup

Our model is designed to illustrate how fear of selling behind other institutions determines the equilibrium allocation of risk and the pricing of risky assets. We are particularly interested in how leverage in the financial sector influences individual banks' decisions to liquidate stock *before* any margin calls. For the reader's convenience, Table 1 summarizes our variables and conditions. **[Tbl 1 here]**

There are two assets in the model. The first is a bond in infinitely elastic supply with a normalized risk-free interest rate of zero. The second is a risky asset (or "stock") with a supply normalized to 1. Figure 1 shows the model's timeline. The stock pays off a random amount $\tilde{Z}$ at the final time, $t = 2$, which is normally distributed with mean μ and variance σ^2 ($\tilde{Z} \sim N(\mu, \sigma^2)$). The parameters μ and σ become public knowledge at time 0. The stock trades at $t = 0$ and $t = 1$. **[Fig 1 here]** $\boxed{\tilde{Z}}$ $\boxed{\mu}$ $\boxed{\sigma^2}$

There are two types of traders in the model. The financial sector (banks) consists of a unit mass of atomistic, risk-neutral banks. Each bank holds a levered position in the stock and may be forced to liquidate its position if it cannot meet margin requirements. In contrast, the non-financial sector has sufficient capital to purchase the stock without any fear of liquidation, but it is risk averse in aggregate. Thus, this sector provides liquidity in the stock market, but at a cost.

Earlier, at $t = -1$, banks had purchased stock at a common price $p_{\text{-1}}$, which is exogenous to our model. We assume $p_{\text{-1}} > \mu$, which is sufficient to ensure that the price at $t = 0$, denoted p_0, has dropped below the purchase price $p_{\text{-1}}$. This is the interesting case in our model, because it means that all banks' margin constraints have just tightened, and some banks may even receive margin calls which force them to liquidate. $\boxed{p_{\text{-1}}}$ $\boxed{p_0}$

At time 0, banks decide simultaneously whether or not to sell its stock to the risk-averse non-financial sector. By selling at time 0 (i.e., before banks selling at time 1), a bank may be able to obtain better pricing from the risk-averse non-financial sector than it could at time 1. However, if a bank is not selling at time 0, it may still be able to obtain sufficient capital at time 1 to meet any margin calls and thus hold the stock until maturity. We assume that this income $\tilde{\mathcal{I}}_i$ is uniformly distributed over the interval $[0, 1]$ and independent across banks and independent of any stock price movements. $\boxed{\tilde{\mathcal{I}}_i}$

We assume that at time -1, each bank i had financed its stock purchases (at price $p_{\text{-}1}$) with its own debt level DT_i. It is the presence of this debt financing that introduces the possibility that a bank will not be able to meet its margin call if it wants to hold onto the stock until maturity. Thus, the existing leverage ratio for a bank of type i is $m_i \equiv \mathrm{DT}_i / p_{\text{-}1}$. We assume that the distribution of banks' leverage ratios, m_i, is uniform from M_L to M_H, with density $f(m) = 1/\mathcal{M}$, where $\mathcal{M} \equiv M_H - M_L$ and $0 \le M_L < M_H \le 1$:

$\boxed{\mathrm{DT}_i}$ $\boxed{m_i}$ $\boxed{M_L}$ $\boxed{M_H}$ $\boxed{\mathcal{M}}$

$$\text{Bank Types:} \qquad m_i \;\sim\; U[M_L\,,\, M_H]\,.$$

M_L is the most conservative bank and M_H is the most aggressive bank. A financial sector with a higher M_L or a higher M_H is more heavily levered (populated by more aggressive types). It is not important in the model whether each bank's leverage ratio is privately known or common knowledge. Because banks are atomistic, their behavior is the same in either case.

The maximum leverage ratio allowed in the economy is $\overline{M}$. It may be determined, e.g., by regulatory restrictions on the banks or by mutual convention in the financial sector. A bank cannot exceed $\overline{M}$ if it wants to hold on to the shares until time 2.[2] We assume that $\overline{M} \ge M_H$, so the banks with the highest leverage ratio M_H have not triggered this constraint at time -1. $\boxed{\overline{M}}$

Together our assumptions imply that a bank will be forced to liquidate its shares under the following condition:

$$\textbf{Forced Liquidation if:} \qquad \frac{\mathrm{DT}_i - \tilde{\mathcal{I}}_i}{p_0} \;>\; \overline{M} \qquad \Leftrightarrow \qquad p_{\text{-}1} \cdot m_i - p_0 \cdot \overline{M} \;>\; \tilde{\mathcal{I}}_i\,. \qquad (1)$$

For example, if a bank of type $m_i = 0.4$ had purchased the risky asset at $p_{\text{-}1} = \$100$, then it starts with a debt of $\mathrm{DT} = m_i \cdot \$100 = \$40$. If the maximum leverage ratio is $\overline{M} = 50\%$, and the asset price p_0 falls to \$60, then bank i must have $\tilde{\mathcal{I}}_i = \$10$ in external income to avoid liquidation.[3]

[2] One could view Bolton, Santos, and Scheinkman (2009) and Diamond and Rajan (2009) as modeling the attempts of the banks to delay liquidation in the interim period in order to reach the final period.

[3] We assume the asset is indivisible so that a bank must liquidate its entire holdings of the stock if it cannot meet its margin call. Our model is isomorphic to one in which each bank can reduce its risky holdings. It would still be in the interest of more levered banks to cut back more on their holdings, and this would still cause the selling externality. We also assume that there are no conflicts of interests arising from bankruptcy frictions. Our banks maximize the proceeds from the sale of stock just as if they had no debt. That is, they do not maximize just shareholder wealth, but bank wealth.

Our assumption that margin requirements are based on the price of the stock at time 0, and not the price of the stock at time 1, allows us to focus on the motive of preemptive selling in the absence of a second contagion channel, wherein banks fears having to sell even more because the future prevailing price (p_1) further tightens their margin constraints. By making the margin call at time 1 dependent only on lagged price (p_0), banks do not fear the lower endogenous price at time 1 and are concerned only with their own idiosyncratic wealth shocks $\tilde{\mathcal{I}}_i$. Thus, this assumption dampens the full effect of the fear of margin calls on pre-emptive selling by the banks. Appendix A shows that the main results of our model are stronger if margin calls depend on p_1 instead of p_0.[4]

The equilibrium stock price at time 1 and time 2 depends on the willingness of the non-financial sector to bear risk. In contrast to the risk-neutral financial sector, the non-financial sector is risk averse in aggregate but can absorb risky assets without fear of liquidity events. It consists of many small competitive investors, whose aggregate utility is described by a negative exponential utility function of the form

$$u(w) = -e^{-\gamma \cdot w} ,$$

where γ is the coefficient of absolute risk aversion. Operationally, our price-setting $\boxed{\gamma}$
mechanism has banks submit sell orders at each trading date that are batched and bought by the liquidity-providing non-financial sector at a single average price that earns zero expected utility at *each* trading date.

This pricing mechanism is based on three assumptions that govern the behavior of the non-financial sector.

First, we assume this sector is not infinitely deep ($\gamma > 0$). Thus, it is less efficient in its ability to hold risky assets than the risk-neutral financial sector. The resulting downward slope in the demand curve is reasonable, because we are concerned about

[4] In the real world, the timing of margin calls are often discretionary. A time lag between call and satisfaction is not uncommon. For example, Lawrence Macdonald's "A Colossal Failure of Common Sense" (page 313f) describes that JPMorganChase had been requesting \$5 billion in collateral from Lehman Bros in July. This was only satisfied in August—though in structured securities which were probably worth less than the required \$5 billion. On September 4, JPMC demanded \$5 billion in cash collateral, which Lehman again failed to satisfy. On September 9, they demanded the \$5 billion again, but with the threat of freezing all Lehman accounts. (This time, Lehman provided \$3 billion, scrounged together from many sources. The fact that this was less than asked did little to soothe JPMC.) In sum, one can defend margin calls as being based either on past or current prices.

the entire non-financial sector's ability to absorb the entire financial sector holdings. The kind of divestments our model is considering are large aggregate flows of risk in the economy from one sector to the other.

Second, we assume non-financial firms do not coordinate and thus will not act like a monopolist to extract surplus. Prices offered by the non-financial sector are therefore determined by a zero net-utility condition.[5] This assumption is not critical to the results but is made for computational simplicity.

Third, we assume the non-financial sector ignores future trading opportunities when setting prices (i.e., behaves myopically). Thus, the non-financial sector's willingness-to-pay decreases in its inventory of the risky stock. We believe such myopia is a more reasonable assumption in markets with few precedents, such as the financial crisis of 2008. The non-financial sector would have had little confidence in its ability to predict future trading opportunities. Thus, it would have initially traded with the best possible information given its risk tolerance, and without *full* regard to their future trading opportunities if more banks were hit by margin constraints later. Once deeper into the crisis, non-financial investors would have had again to make the best decision possible, albeit now with a greater inventory of assets, which would have resulted in a lower price (or, equivalently, less willingness to trade and absorb more risk). One could also justify our assumption with banks optimizing their holdings with respect to additional information that is not yet available to the non-financial sector.

Although our third assumption is not innocuous, it is also not novel. Limited capacity to absorb assets is also present in Morris and Shin (2004), Bernardo and Welch (2004), Brunnermeier and Pedersen (2005), Schnabel and Shin (2002), Allen and Gale (2004), Gorton and Huang (2004), and others. (An equivalent assumption is also common in the market microstructure literature.) Nevertheless, this assumption downplays two important counterveiling economic forces. First, there could be entry of new liquidity providers at time 1. Given that we view the entire non-financial sector as the liquidity provider, an assumption of limited entry by new investors is only modestly objectionable. Of course, over a sufficiently long time span, other investors (such as

[5] If one wishes to view the non-financial sector as one agent, then we have to appeal to the argument in Baumol, Panzar, and Willig (1982)). It shows that in a contestable market, a monopolist may find it profit-maximizing to set prices competitively in order to deter entry.

foreign investors, government bond holders, or the government itself) would enter the liquidity provision market. Second, there could be buyers in this sector who withhold participation at time 0 in order to take advantage of the lower price at time 1. This occurs, e.g., in Allen and Gale (2004), where strategic investors remain on the sidelines in order to arbitrage the price in a liquidity crisis. Appendix B shows that as long as exit and entry of investors is not enough to *fully* arbitrage away any time variation in the price of risk—the dependence of price on inventory—the implications of our model hold.

Descriptively, the financial crisis of 2008 seems to support our assumptions. Prices slipped rapidly, but did not do so fully instantly. There was a noted absence of traditional financial institutions in the market for even modestly risky asset classes. Many financial instruments, even risk-free government issued GNMA bonds, remained illiquid for an extended period of time, which meant forced sellers had to accept steep discounts. It is thus plausible that investors at the beginning of the crisis should have wanted to sell out preemptively in fear of having to sell behind others in the future, which in turn would have deepened the crisis.

1.1 Price Policies of the Non-Financial Sector

Given our assumptions, the price of the risky asset is a function of the stock holdings of the non-financial sector (the "liquidity provider"). Let α_0 denote the proportion of banks selling shares to the liquidity provider at time 0 and recall that p_0 denotes the time 0 price of the stock. If the liquidity provider has initial wealth $\mathcal{W}_0$ and ignores future trading opportunities, i.e., behaves myopically, then his random wealth at time 2 is $\tilde{\mathcal{W}}_2 = \mathcal{W}_0 + \alpha_0 \cdot (\tilde{Z} - p_0)$. The share price $p_0(\alpha_0)$ that makes him indifferent between buying α_0 shares at time 0 and maintaining zero inventory of shares is determined by

$\boxed{\alpha_0}$ $\boxed{\mathcal{W}_0}$

$$
\begin{aligned}
& \mathcal{E}[\,-e^{-\gamma\cdot\tilde{\mathcal{W}}_2}\,] &&= \mathcal{E}[\,-e^{-\gamma\cdot\mathcal{W}_0}\,] \\
\Longrightarrow\quad & \mathcal{E}[\,\mathcal{W}_0 + \alpha_0\cdot(\tilde{Z}-p_0)\,] \;-\; \gamma/2\cdot\mathcal{V}ar[\,\mathcal{W}_0 + \alpha_0\cdot(\tilde{Z}-p_0)\,] &&= \mathcal{W}_0\,, \\
\Longrightarrow\quad & p_0(\,\alpha_0\,) \;=\; \mu - \kappa\cdot\alpha_0\,, && \qquad (2)
\end{aligned}
$$

where $\kappa \equiv \gamma\cdot\sigma^2/2$ is a factor determining the risk discount required by the liquidity provider to hold the proportion α_0 shares at time 0. Assume $\mu > \kappa$, so that if all banks $\boxed{\kappa}$

sold, the price would still be positive. The risk discount is increasing when the liquidity provider is more risk averse (γ is higher), the stock is more risky (σ is higher), and his inventory is larger (α_0 is higher).

The share price at time 1 depends on the proportion of banks having sold their stock at time 0, α_0, and the proportion of banks forced to liquidate at time 1, denoted α_1. $\boxed{\alpha_1}$
The liquidity provider already holds α_0 shares prior to trading at time 1, so the price $p_1(\,\alpha_1;\alpha_0\,)$ must make him indifferent between buying α_1 new shares at time 1 and $\boxed{p_1}$
maintaining an inventory of α_0 shares,

$$\mathcal{E}[\,\tilde{\mathcal{W}}_2 + \alpha_1 \cdot (\tilde{Z} - p_1)\,] \;-\; \gamma/2 \cdot \mathcal{V}ar[\,\tilde{\mathcal{W}}_2 + \alpha_1 \cdot (\tilde{Z} - p_1)]\,] \;=\; \mathcal{E}[\,\tilde{\mathcal{W}}_2\,] \;-\; \gamma/2 \cdot \mathcal{V}ar[\,\tilde{\mathcal{W}}_2\,] \;,$$

$$\Longrightarrow \qquad p_1(\,\alpha_1;\alpha_0\,) \;=\; \mu \;-\; 2 \cdot \kappa \cdot \alpha_0 \;-\; \kappa \cdot \alpha_1 \;=\; p_0(\,\alpha_0\,) \;-\; \kappa \cdot (\alpha_0 + \alpha_1)\,. \tag{3}$$

The cost to banks of being forced to liquidate at time 1 compared to selling at time 0 is increasing in *both* the proportions of banks that sell at time 0 and at time 1. Just as at time 0, the discount is steeper when the liquidity provider has to to absorb more shares (α_1). However, in addition, the liquidity provider already holds some inventory (α_0). This produces the critical aspect for our model: $p_1 < p_0$. Some non-myopic behavior by the liquidity providers would not invalidate this, as long as it is not fully arbitraging the price difference away.

1.2 The Benefit of Preemptive Selling at Time 0

Each bank's decision to sell or hold the stock at time 0 is a function of the prevailing price. Consider a bank with leverage ratio m_i which conjectures that proportion α_0 of banks will sell their shares at time 0 and proportion α_1 of banks will sell their shares at time 1. If this bank chooses to sell at time 0, it will receive the price $p_0(\,\alpha_0\,)$ for its shares. However, if the bank chooses not to sell at time 0, it will receive the price $p_1(\,\alpha_1;\alpha_0\,)$ at time 1 if forced to liquidate or receive the expected payoff μ at time 2 if not forced to liquidate. The bank will be forced to liquidate its shares at time 1 if it cannot meet its margin call, i.e., if its external income at $t = 1$, $\tilde{\mathcal{I}}_i$ is less than its (price-lagged) cash requirement, $m_i \cdot p_{\text{-}1} - \overline{M} \cdot p_0$, as noted in (1).

Because $\tilde{I}_i$ is uniformly distributed on the interval $[0, 1]$, the probability that the bank of type i will be forced to liquidate at time 1 is equal to zero if $m_i \cdot p_{\text{-}1} - \overline{M} \cdot p_0 < 0$, equal to $m_i \cdot p_{\text{-}1} - \overline{M} \cdot p_0$ if $0 \le m_i \cdot p_{\text{-}1} - \overline{M} \cdot p_0 \le 1$, and equal to one if $m_i \cdot p_{\text{-}1} - \overline{M} \cdot p_0 > 1$.

Let the net benefit of bank i for selling its shares at time 0 be S_i. For a bank with $0 \le m_i \cdot p_{\text{-}1} - \overline{M} \cdot p_0 \le 1$, $\boxed{S_i}$

$$\begin{aligned}
S_i(\alpha_0, \alpha_1) \;&\equiv\; p_0 \;-\; \Big\{ \overbrace{\big[m_i \cdot p_{\text{-}1} - \overline{M} \cdot p_0\big]}^{\substack{\text{probability}\\ \text{forced sale}}} \cdot p_1 \;+\; \overbrace{\big[1 - (m_i \cdot p_{\text{-}1} - \overline{M} \cdot p_0)\big]}^{\substack{\text{probability}\\ \text{not forced}}} \cdot \mu \Big\} \\
&= \mu - \kappa \cdot \alpha_0 \;-\; \mu - (2 \cdot \alpha_0 + \alpha_1) \cdot \big[m_i \cdot p_{\text{-}1} - \overline{M} \cdot (\mu - \kappa \cdot \alpha_0)\big] \\
&= \kappa \cdot \big\{\big[m_i \cdot p_{\text{-}1} - \overline{M} \cdot (\mu - \kappa \cdot \alpha_0)\big] \cdot (2 \cdot \alpha_0 + \alpha_1) - \alpha_0\big\} \;,
\end{aligned} \tag{4}$$

where we substituted the pricing policies in the second line and simplified down to the third line. For a bank i with $m_i \cdot p_{\text{-}1} - \overline{M} \cdot p_0 < 0$, i.e., one that will always be able to satisfy any margin call with its external income,

$$S_i(\alpha_0, \alpha_1) \;\equiv\; p_0 \;-\; \mu \,.$$

This is always (weakly) less than 0; thus, it should never sell at time 0. For a bank i with $m_i \cdot p_{\text{-}1} - \overline{M} \cdot p_0 > 1$, i.e., one that is certain to have to divest at time 1,

$$S_i(\alpha_0, \alpha_1) \;\equiv\; p_0 \;-\; p_1 \,.$$

This is always (weakly) greater than 0; thus, it should always sell at time 0.

2 Equilibria

Equilibrium in our model is a pair (α_0^*, α_1^*), such that, given the price policies $p_0(\alpha_0^*)$ and $p_1(\alpha_1^*; \alpha_0^*)$,

- it is optimal for a proportion α_0^* of banks to sell shares at time 0;
- a proportion α_1^* of banks are forced to liquidate at time 1 (there are no voluntary liquidations at time 1, because μ is always at least as great as $p_1(\alpha_1^*; \alpha_0^*)$);
- $p_0(\alpha_0^*)$ and $p_1(\alpha_1^*; \alpha_0^*)$ are set by the liquidity provider according to equations (2) and (3).

2.1 The Types of Preemptive Sellers at Time 0

The following result characterizes which banks choose to sell at time 0 in any equilibrium.

Lemma 1 *In equilibrium, if α_0^* banks sell stock at time 0, these are only the banks with the highest leverage ratios $m_i \in [M^*, M_H]$, where M^* is* $\boxed{M^*}$

$$M^* \equiv M_H - \mathcal{M} \cdot \alpha_0^* . \tag{5}$$

Proof:

In Section 1.2, we showed that it is always optimal for banks with leverage ratios $m_i \cdot p_{-1} - \overline{M} \cdot p_0 > 1$ to sell at time 0, because they will be forced to divest at a lower price at time 1 with certainty. For banks with leverage ratios $0 \leq m_i \cdot p_{-1} - \overline{M} \cdot p_0 \leq 1$, the net benefit of selling at time 0 rather than time 1, given arbitrary α_0 and α_1, appeared in eq. (4) as

$$S_i \equiv \kappa \cdot \{ [m_i \cdot p_{-1} - \overline{M} \cdot (\mu - \kappa \cdot \alpha_0)] \cdot (2 \cdot \alpha_0 + \alpha_1) - \alpha_0 \} .$$

Both κ and p_{-1} are positive, so the net benefit of selling at time 0 is increasing in the bank's leverage ratio m_i. Hence, if α_0^* proportion of banks find it optimal to sell at time 0, then it must be the banks with the highest leverage ratios.

Q.E.D.

Lemma 1 is intuitive because the most highly levered banks face the greatest likelihood of having to liquidate at time 1 at a lower price than they can obtain at time 0; thus, they have the greatest incentive to sell preemptively.

2.2 Corner Equilibria

There are two possible corner equilibria in our model: one in which no bank sells preemptively and one in which every bank sells preemptively. We can characterize the complete condition for the existence of a corner equilibrium in which no bank sells at time 0.

Theorem 1 *There is no corner equilibrium in which no bank sells at time 0 ($\alpha_0^* = 0$) if and only if*

> ***A1** (the most aggressive bank is not safe): $M_H > (\mu/p_{-1}) \cdot \overline{M}$.*

Proof:

By Lemma 1, a corner equilibrium in which no bank sells at time 0 will occur if and only if $S_{i=M_H}(\alpha_0 = 0) \leq 0$. If no bank sells at time 0 ($\alpha_0 = 0$), the net benefit of selling for an individual bank is

$$
\begin{aligned}
S_i(\alpha_0 = 0) &= \kappa \cdot \{[m_i \cdot p_{-1} - \overline{M} \cdot (\mu - \kappa \cdot \alpha_0)] \cdot (2 \cdot \alpha_0 + \alpha_1) - \alpha_0\} \\
&= \kappa \cdot \{[m_i \cdot p_{-1} - \overline{M} \cdot \mu] \cdot \alpha_1\} \ .
\end{aligned}
$$

For the most aggressive bank, the expression in parentheses is $M_H \cdot p_{-1} - \overline{M} \cdot \mu$. Under **A1**, this is positive. Hence, this bank would choose to sell at time 0. This implies that $\alpha_0 = 0$ can not be an equilibrium.

Q.E.D.

The intuition is that if $\alpha_0 = 0$ then the most highly levered bank could sell its shares at the price $p_0 = \mu$ and receive the full expected value of the stock's payoff. This avoids the (positive) probability of liquidating at a lower price at time 1, which means that it would be better off selling immediately.

We can also characterize the complete condition for the existence of a second corner equilibrium in which every bank sells at time 0.

Theorem 2 *There is no corner equilibrium in which every bank sells at time 0 ($\alpha_0^* = 1$) if and only if*

> ***A2** (the probability of liquidation for the most conservative bank is sufficiently small): $M_L < [1/2 + \overline{M} \cdot (\mu - \kappa)]/p_{-1}$.*

Proof:

By Lemma 1, a corner equilibrium in which every bank sells at time 0 exists if and only if the most conservative bank prefers to sell when all other banks choose to sell, $S_{i=M_L}(\alpha_0^* = 1) \geq 0$. If every bank sells at time 0 ($\alpha_0 = 1$), the net benefit of selling for the most conservative bank is

$$S_{i=M_L}(1) \quad = \quad \kappa \cdot \left[-1 + 2 \cdot (M_L \cdot p_{\text{-}1} - \overline{M} \cdot \mu + \overline{M} \cdot \kappa)\right] \ .$$

The result follows immediately.

Q.E.D.

The expression ($M_L \cdot p_{\text{-}1} - \overline{M} \cdot \mu + \overline{M} \cdot \kappa$) in **A2** is the probability of forced liquidation at time 1 for the most conservative bank, given that all other banks have sold their shares at time 0 (i.e., $\alpha_0 = 1$). If this probability is sufficiently small, then the most conservative bank is better off not selling its shares at time 0, in which case it can not be an equilibrium for *all* banks to sell at time 0.

2.3 Forced Selling at Time 1

To prepare the ground for proving the existence of an interior equilibrium in which α_0^* banks sell stock at time 0 and the rest hold on to their stock, we first simplify the derivation of our equilibrium from a search over four quantities (α_0^*, α_1^*, $p_0(\alpha_0^*)$, $p_1(\alpha_1^*; \alpha_0^*)$) into a search over a single quantity (α_0^*).

The assumption that there is a unit mass of atomistic banks allows all traders to compute the exact proportion of banks that will not obtain sufficient income to meet their margin calls (by the Law of Large Numbers). These banks will be forced to liquidate at time 1. Although each bank knows this aggregate proportion, a bank that is subject to potential liquidation does not know whether it itself will be among the liquidating banks. (This will depend on its *own* future realization of $\tilde{\mathcal{I}}_i$.) Lemma 2 gives the exact proportion of banks that will be forced to liquidate their stock at time 1 as a function of the proportion of banks that have sold at time 0.

Lemma 2 *If only the most aggressive banks have liquidated at time 0 (incl. all banks that will face liquidation at time 1 with certainty), then the proportion of banks forced to liquidate their stock at time 1, α_1, is the (deterministic) function of the proportion of banks having sold preemptively at time 0, α_0,*

$$\alpha_1(\alpha_0) = \begin{cases} \dfrac{(1-\alpha_0)\cdot[(M_L+\alpha_0\cdot M_L+M_H-\alpha_0\cdot M_H)\cdot p_{\text{-}1}-2\cdot\overline{M}\cdot(\mu-\kappa\cdot\alpha_0)]}{2} & \text{if } M_L\cdot p_{\text{-}1}>\overline{M}\cdot p_0 \\[2ex] \dfrac{\left[p_{\text{-}1}\cdot(M_H-\mathcal{M}\cdot\alpha_0)-\overline{M}\cdot(\mu-\kappa\cdot\alpha_0)\right]^2}{2\cdot\mathcal{M}\cdot p_{\text{-}1}} & \text{if } M_L\cdot p_{\text{-}1}\le\overline{M}\cdot p_0 \end{cases}.$$

Proof:

The lemma posits that only banks with leverage ratios above $M_H-\mathcal{M}\cdot\alpha_0$ have already sold their stock at time 0, and that it includes all banks with leverage ratios $m_i\cdot p_{\text{-}1}-\overline{M}\cdot p_0>1$. (This will be equilibrium behavior due to Lemma 1.)

If $M_L\cdot p_{\text{-}1}>\overline{M}\cdot p_0$, all banks that did not sell at time 0 (including those with the most conservative leverage ratio M_L) face the possibility of forced liquidation at time 1. If $M_L\cdot p_{\text{-}1}\le\overline{M}\cdot p_0$, some banks will never be forced to liquidate at time 1. (Only banks with leverage ratios greater than $\overline{M}\cdot p_0/p_{\text{-}1}$ face any chance of forced liquidation.) Therefore, the proportion of banks forced to liquidate at time 1 is the double integral

$$\begin{aligned}\alpha_1(\alpha_0) &= \overbrace{\int_{\max(M_L,\overline{M}\cdot p_0/p_{\text{-}1})}^{M_H-\mathcal{M}\cdot\alpha_0}}^{\substack{\text{mass of banks}\\ \text{facing possible margin call}}} \overbrace{\int_0^{m\cdot p_{\text{-}1}-\overline{M}\cdot p_0}}^{\substack{\text{not enough}\\ \text{external income}}} \left(\frac{1}{\mathcal{M}}\right)\,dw\,dm \\ &= \left(\frac{1}{\mathcal{M}}\right)\cdot\int_{\max(M_L,\overline{M}\cdot p_0/p_{\text{-}1})}^{M_H-\mathcal{M}\cdot\alpha_0}(m\cdot p_{\text{-}1}-\overline{M}\cdot p_0)\,dm\,.\end{aligned} \tag{6}$$

The outer integral represents the set of banks exposed to possible forced liquidation at time 1. The inner integral represents the set of banks who will not obtain sufficient income at time 1 to meet their margin call and who will therefore be forced to liquidate at time 1. If $M_L > \overline{M} \cdot p_0 / p_{\text{-}1}$, all banks are subject to liquidation, M_L is the lower bound of the outer integral, and

$$\begin{aligned}\alpha_1(\alpha_0) &= \left(\frac{p_{\text{-}1}}{2 \cdot \mathcal{M}}\right) \cdot \{[M_H \cdot (1-\alpha_0) + M_L \cdot \alpha_0]^2 - M_L^2\} - \overline{M} \cdot p_0 \cdot (1-\alpha_0) \\ &= \frac{(1-\alpha_0) \cdot [(M_L + \alpha_0 \cdot M_L + M_H - \alpha_0 \cdot M_H) \cdot p_{\text{-}1} - 2 \cdot \overline{M} \cdot (\mu - \kappa \cdot \alpha_0)]}{2} .\end{aligned} \tag{7}$$

If $M_L \leq \overline{M} \cdot p_0 / p_{\text{-}1}$, some banks will never be forced to liquidate at time 1, $\overline{M} \cdot p_0 / p_{\text{-}1}$ is the lower bound of the outer integral, and

$$\alpha_1(\alpha_0) = \frac{[\, p_{\text{-}1} \cdot (M_H - \mathcal{M} \cdot \alpha_0) - \overline{M} \cdot (\mu - \kappa \cdot \alpha_0)\,]^2}{2 \cdot \mathcal{M} \cdot p_{\text{-}1}} . \tag{8}$$

Q.E.D.

Because α_1 is a deterministic function of α_0, the price policy $p_1(\alpha_1; \alpha_0)$ in eq. (3) can also be expressed as a function only of α_0, $p_1(\alpha_0)$. This completes the prerequisite derivations necessary to express the interior equilibrium as a function of one variable, α_0.

2.4 The Interior Equilibrium

In an interior equilibrium, i.e., $\alpha_0^* \in (0,1)$, it must be true that the net benefit of selling at time 0 for some marginal bank $M^* = M_H - \mathcal{M} \cdot \alpha_0^*$ is identically zero. By Lemma 1, all banks with leverage ratios greater than M^* (including banks that will surely be forced to liquidate at time 1) will sell at time 0, and all banks with leverage ratios less than M^* (including banks that will surely not be forced to liquidate at time 1) will not sell at time 0. Consequently, in any interior equilibrium, it must be true that forced liquidation at time 1 for the marginal bank $M^* \cdot p_{\text{-}1} - \overline{M} \cdot p_0$ is probabilistic. To determine the equilibrium proportion of banks that are selling at time 0, α_0^*, we substitute into S_i (in eq. 4) the net benefit for bank $m_i = M_H - \mathcal{M} \cdot \alpha_0$ (from eq. 5) and α_1 (from eq. 8). This yields the following operational definition for an interior equilibrium:

Interior Equilibrium: The net benefit of selling at time 0 for a bank with leverage ratio $m_i = M_H - \mathcal{M} \cdot \alpha_0$, if fraction α_0 of banks sell at time 0, is

$$S_i(\,\alpha_0) \equiv \kappa \cdot \{[m_i \cdot p_{-1} - \overline{M} \cdot p_0] \cdot (2 \cdot \alpha_0 + \alpha_1) - \alpha_0\} \;,$$

where

$$\alpha_1 \quad \equiv \quad \begin{cases} \dfrac{(1-\alpha_0) \cdot [(M_L + \alpha_0 \cdot M_L + M_H - \alpha_0 \cdot M_H) \cdot p_{-1} - 2 \cdot \overline{M} \cdot p_0]}{2} & \text{if } M_L \cdot p_{-1} > \overline{M} \cdot p_0 \\[2ex] \dfrac{[\,p_{-1} \cdot (M_H - \mathcal{M} \cdot \alpha_0) - \overline{M} \cdot p_0\,]^2}{2 \cdot \mathcal{M} \cdot p_{-1}} & \text{if } M_L \cdot p_{-1} \leq \overline{M} \cdot p_0 \;, \end{cases}$$

$$p_0 \;\equiv\; \mu - \kappa \cdot \alpha_0 \;,$$

$$m_i \quad \equiv \quad M_H - \mathcal{M} \cdot \alpha_0 \;.$$

An α_0^* is an interior equilibrium if and only if $S_i(\,\alpha_0^*) = 0$ for an interior alpha.

Substituting the equations into the S_i function yields a cubic polynomial in α_0. For clarity, we denote this equilibrium function $S_*(\alpha_0)$. It measures the benefit of selling $\boxed{S_*}$
relevant for a bank with leverage ratio $m_i = M_H - \mathcal{M} \cdot \alpha_0$ if a fraction α_0 of banks choose to sell at time 0. A value above 0 means that this bank would be better off selling, and thus it would not be the marginal bank. A value below 0 now means that this bank would be better off buying, and thus it would again not be the marginal bank. In equilibrium, this marginal bank must be exactly indifferent, i.e., $S_i(\,\alpha_0) = 0$.

If Assumptions **A1** and **A2** hold, i.e., if no corner equilibrium is feasible, then at least one interior equilibrium exists. However, it is still possible that there is more than one such equilibrium, because S_* is a cubic polynomial in α_0. Thus, we now impose conditions that are both interpretable and sufficient to guarantee that the interior equilibrium is *unique*.

Theorem 3 *There exists a unique, interior equilibrium, $\alpha_0^* \in (0,1)$, if*

A1 *(at least one unsafe bank): $M_H > (\mu/p_{-1}) \cdot \overline{M}$,*

A2a *(modest fundamental deterioration): $\mu > (M_H \cdot p_{-1} - 1/2)/\overline{M}$,*

A2b *(enough non-financial demand) $\kappa < (\mathcal{M} \cdot p_{-1})/\overline{M}$.*

Proof:

First, Theorem 1 showed that when Assumption **A1** holds, $S_*(0) > 0$ and $\alpha_0 = 0$ is not an equilibrium. Second, Theorem 2 showed that when Assumption **A2** holds, $S_*(1) < 0$ and $\alpha_0 = 1$ is not an equilibrium.

A2a can be restated equivalently as $\overline{M} \cdot \mu - \overline{M} \cdot \kappa \;\geq\; M_H \cdot p_{\text{-}1} - \overline{M} \cdot \kappa - 1/2$. **A2b** guarantees that

$$M_H \cdot p_{\text{-}1} - \overline{M} \cdot \kappa - 1/2 \quad > \quad M_H \cdot p_{\text{-}1} - \mathcal{M} \cdot p_{\text{-}1} - 1/2 \quad = M_L \cdot p_{\text{-}1} - 1/2 \;.$$

Thus, together, **A2a** and **A2b** imply **A2** ($\overline{M} \cdot \mu - \overline{M} \cdot \kappa > M_L \cdot p_{\text{-}1} - 1/2$).

If $S_*{}'(\alpha_0) < 0$ over the entire domain $(0,1)$, then the interior equilibrium is unique. Rewrite the S_* function as

$$S_*(\alpha_0) \quad = \quad \kappa \cdot \left\{ q(\alpha_0) \cdot [2 \cdot \alpha_0 + \alpha_1(\alpha_0)] - \alpha_0 \right\} \;,$$

where

$$q(\alpha_0) \quad \equiv \quad (M_H - \mathcal{M} \cdot \alpha_0) \cdot p_{\text{-}1} - \overline{M} \cdot (\mu - \kappa \cdot \alpha_0)$$

is the probability of liquidation for a bank with leverage ratio $M_H - \mathcal{M} \cdot \alpha_0$ when α_0 proportion of banks sell at time 0. Thus,

$$S_*{}'(\alpha_0) \quad = \quad \kappa \cdot \left\{ q(\alpha_0) \cdot \left[2 + \frac{\partial \alpha_1(\alpha_0)}{\partial \alpha_0} \right] - 1 \;+ \left[\frac{\partial q(\alpha_0)}{\partial \alpha_0} \right] \cdot [2 \cdot \alpha_0 + \alpha_1(\alpha_0)] \right\} \;. \tag{9}$$

q is less than 1/2, because

$$\begin{aligned} q(\alpha_0) \quad &= \quad (M_H - \mathcal{M} \cdot \alpha_0) \cdot p_{\text{-}1} - \overline{M} \cdot (\mu - \kappa \cdot \alpha_0) \\ &= \quad (M_H \cdot p_{\text{-}1} - \overline{M} \cdot \mu) \;-\; \alpha_0 \cdot (\mathcal{M} \cdot p_{\text{-}1} - \overline{M} \cdot \kappa) \\ &< \quad (M_H \cdot p_{\text{-}1} - \overline{M} \cdot \mu) && \text{(by } \textbf{A2b}\text{)} \\ &< \quad 0.5 \;. && \text{(by } \textbf{A2a}\text{)} \end{aligned}$$

Moreover, the first derivative of q with respect to α_0 is $\partial q(\alpha_0)/\partial \alpha_0 = \overline{M} \cdot \kappa - \mathcal{M} \cdot p_{\text{-}1}$, which is negative by **A2b.**

Substituting for $q(\alpha_0)$ and $\partial q(\alpha_0)/\partial \alpha_0$ in (9) shows that $S_*{}'(\alpha_0) < 0$ if $\partial \alpha_1/\partial \alpha_0 \leq 0$. We now prove that $\partial \alpha_1/\partial \alpha_0 \leq 0$ by considering three different parameter regions:

Region 1 ($M_L > \overline{M} \cdot \mu / p_{\text{-}1}$): In this region, even the most conservative bank may face liquidation for all values of α_0. Thus, the applicable α_1 function is

$$\alpha_1(\alpha_0) \quad = \quad \frac{(1 - \alpha_0) \cdot \{[(1 - \alpha_0) \cdot M_H + (1 + \alpha_0) \cdot M_L] \cdot p_{\text{-}1} - 2 \cdot \overline{M} \cdot (\mu - \kappa \cdot \alpha_0)\}}{2} \;.$$

The derivative of α_1 with respect to α_0 is negative in this region, because:

$$\begin{aligned}
\frac{\partial \alpha_1}{\partial \alpha_0} &= -q(\alpha_0) + \overline{M} \cdot \kappa \cdot (1 - \alpha_0) \\
&< -(M_H - \mathcal{M} \cdot \alpha_0) \cdot p_{\text{-}1} + \overline{M} \cdot (\mu - \kappa \cdot \alpha_0) + \mathcal{M} \cdot p_{\text{-}1} \cdot (1 - \alpha_0) \qquad \text{(by } \mathbf{A2b}\text{)} \\
&= -M_H \cdot p_{\text{-}1} + \overline{M} \cdot p_0 + \mathcal{M} \cdot p_{\text{-}1} \\
&< -M_H \cdot p_{\text{-}1} + M_L \cdot p_{\text{-}1} + \mathcal{M} \cdot p_{\text{-}1} \\
&= 0\,,
\end{aligned}$$

where the last inequality follows from the fact that $\bar{M} \cdot p_0 \leq \bar{M} \cdot \mu$ and $\bar{M} \cdot \mu < M_L \cdot p_{\text{-}1}$ in Region 1.[6]

Region 2 ($M_L < \overline{M} \cdot (\mu - \kappa)/p_{\text{-}1}$): In this region, the most conservative bank never faces liquidation, and the applicable α_1 function is

$$\alpha_1(\alpha_0) = \frac{\left[\, p_{\text{-}1} \cdot (M_H - \mathcal{M} \cdot \alpha_0) - \overline{M} \cdot (\mu - \kappa \cdot \alpha_0) \,\right]^2}{2 \cdot \mathcal{M} \cdot p_{\text{-}1}}\,.$$

Therefore
$$\frac{\partial \alpha_1}{\partial \alpha_0} = \frac{(\overline{M} \cdot \kappa - \mathcal{M} \cdot p_{\text{-}1}) \cdot q(\alpha_0)}{\mathcal{M} \cdot p_{\text{-}1}}\,,$$
which is negative under **A2b**.

Region 3 ($\overline{M} \cdot (\mu - \kappa)/p_{\text{-}1} \leq M_L \leq \overline{M} \cdot \mu/p_{\text{-}1}$): In this region, the most conservative bank may or may not face liquidation, depending on α_0.

- If $\alpha_0 \leq (\overline{M} \cdot \mu - M_L \cdot p_{\text{-}1})/(\overline{M} \cdot \kappa)$, then the most conservative bank will not face liquidation at time 1. In this case, $\alpha_1(\alpha_0)$ is of the same form as in Region 2, where we showed that $\partial \alpha_1/\partial \alpha_0 \leq 0$.
- If $\alpha_0 > (\overline{M} \cdot \mu - M_L \cdot p_{\text{-}1})/(\overline{M} \cdot \kappa)$ then it follows that $M_L \cdot p_{\text{-}1} > \bar{M} p_0$ (the most conservative bank faces a positive probability of liquidation at time 1). We showed in the proof for Region 1 above that **A2b** and the condition $M_L \cdot p_{\text{-}1} > \bar{M} p_0$ are sufficient to ensure that $\partial \alpha_1/\partial \alpha_0 \leq 0$.

[6] There are two forces that determine whether α_1 is increasing or decreasing in α_0. On the one hand, as α_0 increases, there are fewer banks left holding the stock. This can reduce the proportion α_1 forced to liquidate at time 1. On the other hand, as α_0 increases, the price of the stock falls. This can increase the likelihood of a margin call and, hence, forced liquidation at time 1. Assumption **A2b** is sufficient to ensure that the price does not fall so much that the first effect dominates (i.e., that α_1 decreases in α_0).

- Finally, $\alpha_1(\alpha_0)$ evaluated at the break point $\alpha_0 = (\overline{M} \cdot \mu - M_L \cdot p_{\text{-}1}) / (\overline{M} \cdot \kappa)$ is identical in the two cases, given in equations (7) and (8).
- Because $S_*(\alpha_0)$ is continuous in α_0 and $S_*'(\alpha_0) < 0$, there is a unique, interior equilibrium in Region 3.

Q.E.D.

The two new assumptions that replace Assumption **A2** from Theorem 2 are more restrictive, but carry similar intuition. **A2a** implies that the decline in fundamentals from time -1 to time 0 is not too large. **A2b** implies that liquidity provision is sufficiently deep for prices not to decline too quickly as more banks sell at time 0. If these assumptions are violated, it is possible to get an equilibrium in which all banks sell preemptively. The reason is that each individual bank would then have an increasing incentive to sell early when other banks sell early (the likelihood of forced liquidation would be too high and the price the bank would expect to receive in the event of forced liquidation at time 1 would be too low).

2.5 Graphical Illustration

To convey some more intuition for the equilibrium, Figure 2 plots $S_*(\alpha_0^*)$ for different κ, holding fixed parameters of $M_H = 0.9$, $\mathcal{M} = 0.2$, $\overline{M} = 0.9$, $p_{\text{-}1} = 0.9$. **[Fig 2 here]**

In the upper figure, $\mu = 0.475$. For small κ, the non-financial sector has sufficient risk absorption capacity to take on any selling activity of the financial sector. Thus, the S_* function is relatively flat and intersects at a modest α_0. This intersection is the equilibrium: a fraction α_0^*, at which the marginal bank M^* is indifferent between selling and not selling. For $\kappa = 0.03$, this occurs around $\alpha_0^* = 23.8\%$. In the figure, this is marked by a dotted line reaching the x-axis. Increases in κ both steepen the S_* function (making optimal behavior by banks more value-sensitive) and shift the location where the S_* function intersects the zero axis (α_0^*) towards the right *and* at an increasing rate. Eventually, at some critical κ, the function no longer intersects the axis in the (0,1) domain. At this point, the equilibrium is the corner where every bank divests immediately ($\alpha_0^* = 1$).

In the lower figure, $\mu = 0.6$. Again, increases in κ steepen the S_* function and shift the zero-axis intersection (and thus the optimal α_0) towards the right. However, the S_* function now slopes back up early enough that it can intersect the x axis twice. For example, for $\kappa = 0.5$, these intersections occur at around 17% and 69%. The left α_0^* intersection is just like the interior equilibria that are unique. The right α_0^* intersection is an unstable equilibrium. However, its presence implies that there is also a corner equilibrium that is stable, in which all banks are divesting all their holdings immediately. Finally, further increases in κ cause the S_* function never to drop to zero, and the corner equilibrium becomes the only equilibrium. The conditions in Theorem 3 were sufficient to exclude these multiple and corner equilibria.

3 Analysis

3.1 Comparative Statics

We focus only on the comparative statics for the interior equilibrium, which can be signed unambiguously.[7]

Theorem 4 *Under the assumptions in Theorem 3, the unique, interior equilibrium* $\alpha_0^* \in (0,1)$ *is increasing in* κ, p_{-1}, M_L, *and* M_H; *and decreasing in* μ *and* $\overline{M}$.

[7]Because we are focusing only on the interior equilibrium, we do not need to be concerned with equilibrium jumps and corner equilibria. The assumptions in Theorem 3 are sufficient to guarantee that the equilibrium is unique. However, the assumptions are not necessary. Our comparative statics also hold for the (stable) interior equilibria even beyond the region specified in these assumptions: they hold for the stable interior equilibrium when the corner equilibrium is feasible, and they hold for the switch from a unique interior stable equilibrium to a unique corner equilibrium. (Figure 2 shows that increases in parameters (such as κ) move the S_* functions towards the north-east in a smooth fashion until an interior equilibrium is no longer feasible.)

Proof:

Recall that $S_*(\alpha_0) = \kappa \cdot \{q(\alpha_0) \cdot [2 \cdot \alpha_0 + \alpha_1(\alpha_0)] - \alpha_0\}$. First, the proof of Theorem 3 showed that $\partial S_* / \partial \alpha_0 \leq 0$ for all α_0 under the Assumptions **A1**, **A2a**, and **A2b.** Thus, the Implicit Function Theorem implies that the sign of the derivative of α_0^* with respect to any parameter is the sign of the derivative of $S_*(\alpha_0)$ with respect to that parameter.

Once again, define $q(\alpha_0^*) \equiv (M_H - \mathcal{M} \cdot \alpha_0^*) \cdot p_{\text{-}1} - \overline{M} \cdot (\mu - \kappa \cdot \alpha_0^*) > 0$ where $q(\alpha_0^*)$ is the probability of forced liquidation for the marginal bank, which means that it is positive.

Recall that

$$\alpha_1^*(\alpha_0^*) = \begin{cases} \dfrac{(1-\alpha_0^*) \cdot [(M_L + \alpha_0^* \cdot M_L + M_H - \alpha_0^* \cdot M_H) \cdot p_{\text{-}1} - 2 \cdot \overline{M} \cdot (\mu - \kappa \cdot \alpha_0^*)]}{2} & \text{if } M_L \cdot p_{\text{-}1} > \overline{M} \cdot p_0^* \\[2ex] \dfrac{[\, p_{\text{-}1} \cdot (M_H - \mathcal{M} \cdot \alpha_0^*) - \overline{M} \cdot (\mu - \kappa \cdot \alpha_0^*)\,]^2}{2 \cdot \mathcal{M} \cdot p_{\text{-}1}} & \text{if } M_L \cdot p_{\text{-}1} \leq \overline{M} \cdot p_0^* \end{cases}$$

If $M_L \cdot p_{\text{-}1} > \overline{M} \cdot p_0^*$, then $\partial \alpha_1^* / \partial \kappa \geq 0$, $\partial \alpha_1^* / \partial M_L \geq 0$, $\partial \alpha_1^* / \partial M_H \geq 0$, $\partial \alpha_1^* / \partial p_{\text{-}1} \geq 0$, $\partial \alpha_1^* / \partial \mu \leq 0$, and $\partial \alpha_1^* / \partial \overline{M} \leq 0$.

If $M_L \cdot p_{\text{-}1} \leq \overline{M} \cdot p_0^*$, then $\partial \alpha_1^* / \partial \kappa \geq 0$, $\partial \alpha_1^* / \partial M_L \geq 0$, $\partial \alpha_1^* / \partial \mu \leq 0$, and $\partial \alpha_1^* / \partial \overline{M} \leq 0$. For M_H,

$$\begin{aligned} \frac{\partial \alpha_1^*}{\partial M_H} &= \frac{q(\alpha_0^*)}{\mathcal{M}} \cdot \left[(1-\alpha_0^*) - \frac{M_H \cdot (1-\alpha_0^*) \cdot p_{\text{-}1} + M_L \cdot \alpha_0^* \cdot p_{\text{-}1} - \overline{M} \cdot p_0^*}{2 \cdot p_{\text{-}1} \cdot \mathcal{M}} \right] \\ &\geq \frac{q(\alpha_0^*)}{\mathcal{M}} \cdot \left[(1-\alpha_0^*) - \frac{M_H \cdot (1-\alpha_0^*) \cdot p_{\text{-}1} + M_L \cdot \alpha_0^* \cdot p_{\text{-}1} - M_L \cdot p_{\text{-}1}}{2 \cdot p_{\text{-}1} \cdot \mathcal{M}} \right] \qquad \text{by } M_L \cdot p_{\text{-}1} < \overline{M} \cdot \\ &= \frac{q(\alpha_0^*) \cdot (1-\alpha_0^*)}{\mathcal{M}} \cdot (1/2) \qquad \geq 0\,. \end{aligned}$$

For $p_{\text{-}1}$,

$$\begin{aligned} \frac{\partial \alpha_1^*}{\partial p_{\text{-}1}} &= \frac{q(\alpha_0^*)}{\mathcal{M} \cdot p_{\text{-}1}} \cdot \left[M_H \cdot (1-\alpha_0^*) + M_L \cdot \alpha_0^* - \frac{M_H \cdot (1-\alpha_0^*) \cdot p_{\text{-}1} + M_L \cdot p_{\text{-}1} \cdot \alpha_0^* - \overline{M} \cdot p_0^*}{2 \cdot p_{\text{-}1}} \right] \\ &= \frac{q(\alpha_0^*)}{\mathcal{M} \cdot p_{\text{-}1}} \cdot \left[\frac{M_H \cdot (1-\alpha_0^*)}{2} + \frac{M_L \cdot \alpha_0^*}{2} + \frac{\overline{M} \cdot p_0^*}{2 \cdot p_{\text{-}1}} \right] \qquad \geq 0\,. \end{aligned}$$

Therefore, it follows that

- $\frac{\partial S_*}{\partial \kappa} = \overline{M} \cdot \alpha_0^* \cdot (2 \cdot \alpha_0^* + \alpha_1^*) + q(\alpha_0^*) \cdot \partial \alpha_1^* / \partial \kappa \geq 0$, because $\partial \alpha_1^* / \partial \kappa \geq 0$.
- $\frac{\partial S_*}{\partial M_L} = \kappa \cdot [p_{-1} \cdot \alpha_0^* \cdot (2 \cdot \alpha_0^* + \alpha_1^*) + q(\alpha_0^*) \cdot \partial \alpha_1^* / \partial M_L] \geq 0$, because $\partial \alpha_1^* / \partial M_L \geq 0$.
- $\frac{\partial S_*}{\partial M_H} = \kappa \cdot [p_{-1} \cdot (1 - \alpha_0^*) \cdot (2 \cdot \alpha_0^* + \alpha_1^*) + q(\alpha_0^*) \cdot \partial \alpha_1^* / \partial M_H] \geq 0$, because $\partial \alpha_1^* / \partial M_H \geq 0$.
- $\frac{\partial S_*}{\partial \mu} = \kappa \cdot [-\overline{M} \cdot (2\alpha_0^* + \alpha_1^*) + q(\alpha_0^*) \cdot \partial \alpha_1^* / \partial \mu] \leq 0$, because $\partial \alpha_1^* / \partial \mu \leq 0$.
- $\frac{\partial S_*}{\partial \overline{M}} = \kappa \cdot [-(\mu - \kappa \cdot \alpha_0^*) \cdot (2 \cdot \alpha_0^* + \alpha_1^*) + q(\alpha_0^*) \cdot \partial \alpha_1^* / \partial \overline{M}] \leq 0$, because $\partial \alpha_1^* / \partial \overline{M} \leq 0$.
- $\frac{\partial S_*}{\partial p_{-1}} = \kappa \cdot \{[M_H \cdot (1 - \alpha_0^*) + M_L \cdot \alpha_0^*] \cdot (2 \cdot \alpha_0^* + \alpha_1^*) + q(\alpha_0^*) \cdot \partial \alpha_1^* / \partial p_{-1}\} \geq 0$, because $\partial \alpha_1^* / \partial p_{-1} \geq 0$.

Applying $\partial \alpha_0^* / \partial x = -(\partial S_* / \partial x) / (\partial S_* / \partial \alpha_0^*)$ now yields the comparative statics in the statement of the theorem.

Q.E.D.

The intuition for the comparative statics is straightforward. When κ increases, there is less liquidity in the market. Because prices decline more quickly as inventory builds up in the non-financial sector, there is a greater advantage to selling preemptively. When M_L and M_H increase, the average leverage ratio in the economy is greater. Therefore, more banks will be subject to forced liquidation at time 1 and a greater proportion of banks sell preemptively. When p_{-1} increases or μ decreases, banks are more likely to face margin calls and forced liquidation at time 1, so more banks will sell preemptively. Finally, when $\overline{M}$ increases, banks are less likely to face forced liquidation at time 1, which reduces the incentive to sell preemptively.

3.2 Fragility and Discontinuity

Allen and Gale (2004) find that small changes in parameters can lead to large changes in equilibrium outcomes. They call this fragility. Figure 3 plots the equilibrium holdings **[Fig 3 here]** of the non-financial sector, α_0^*, as a function of one of our parameters, the risk aversion of the non-financial sector, κ. (Similar insights obtain if we change the model's other parameters.) These two plots correspond to the two plots in Figure 2. In the upper plot, for small κ, the financial market is deep. Therefore, small changes in market depth lead to only modest changes in the relative holdings of the two sectors. However, when κ is large, the non-financial sector is reluctant to absorb risk. In this case, even small changes in parameters can cause large changes in the willingness of banks to hold risky assets. This convexity of α_0^* in the parameter κ is economic fragility. The lower plot shows that we can have a second, even stronger form of fragility. A small change in a parameter can cause not only a smooth change (however large) in the equilibrium holdings of the risky assets, but also a discontinuous change.

3.3 Financial Sector Externalities

We are particularly interested in the externality that the presence of aggressive banks (i.e., the aggregate financial sector capitalization) has on the behavior of more conservative banks.

Figure 4 plots the equilibrium bank that is selling its shares as a function of the quality of the financial sector. Recall that M_L is the most conservative bank and M_H is the most aggressive bank. The figure shows how the marginal bank that liquidates preemptively depends on the capitalization of the overall financial sector—its peers. In the upper plot, we are varying the presence of aggressive banks (M_H). In the lower plot, we are varying the presence of conservative banks (M_L). In both cases, there is very little impact on which bank becomes the marginal bank over a wide range of parameters. However, when few "good" peers and many "bad" peers remain, it can "suddenly" become in the interest of even very conservative banks to liquidate preemptively. The fear of poorer execution later can then overwhelm the capital cushion that these conservative bank are holding. In sum, as the financial sector loses its conservative banks, i.e., as the financial sector overall becomes more aggressive, the negative externality across banks increases. and more and more banks find themselves better off liquidating preemptively. **[Fig 4 here]**

3.4 Deviations from Fundamentals and Volatility

Another interesting question concerns the effect of changes in fundamentals on asset prices. In a setting without banks' fears of liquidation, changes in the underlying fundamentals, specifically μ, would have a one-to-one effect on price. In contrast, in our model, a shock to the expected value of the risky asset would have a further effect on prices through the preemptive liquidation channel. Specifically,

$$\frac{\partial p_0^*}{\partial \mu} = \frac{\partial(\mu - \kappa \cdot \alpha_0^*)}{\partial \mu} = 1 - \kappa \cdot \frac{\partial \alpha_0^*}{\partial \mu} > 1 ,$$

because $\partial \alpha_0^* / \partial \mu < 0$ from Theorem 4. Figure 5 shows this effect. The top panel shows the holdings of the non-financial sector at time 0 as a function of μ. The bottom panel shows the equilibrium price p_0^* at time 0 as a function of μ. When μ is small, the non-financial sector holds all the risky asset and equilibrium prices move one-for-one with fundamentals. When μ is high, there is relatively little fear of forced liquidation and, again, equilibrium prices move closely with fundamentals. Between these two extremes, however, changes in fundamentals can have an additional impact on banks' preemptive selling behavior and equilibrium prices can move substantially with even small changes in fundamentals. **[Fig 5 here]**

In our model, p_{-1} is given exogenously, however, if we consider our model to apply to sequential periods, it would itself be the outcome of the model a period earlier. In a model without preemptive selling, changes in fundamentals (μ) would translate one-to-one into pricing. In a model with preemptive selling, our comparative statics on μ suggest that changes in fundamentals would induce additional price volatility over time.

The model also has implications for the transmission of uncertainty in the economy. Aggregate uncertainty (σ^2) increases the effective risk aversion (κ) of the non-financial sector. We have already shown that this encourages preemptive selling and worsens the externality.

4 Welfare Analysis and The First-Best Solution

In our model, the risk-neutral banks are the most efficient bearers of risk. Therefore it is Pareto optimal for the banks to hold on to the stock unless they are forced to liquidate at time 1. In the first-best coordinated equilibrium, no bank would sell stock at time 0 and only banks unable to meet their margin calls would be forced to liquidate at time 1. Thus, the first-best solution is

$$\begin{aligned}
\alpha_0^{**} &= 0\,, \\
\alpha_1^{**} &= [(M_H + M_L)\cdot p_{\text{-}1} - 2\cdot \overline{M}\cdot \mu]/2\,, \\
p_0^{**} &= \mu\,, \\
p_1^{**} &= \mu - \kappa\cdot[(M_H + M_L)\cdot p_{\text{-}1} - 2\cdot \overline{M}\cdot \mu]/2\,.
\end{aligned}$$

We can compare this first-best social welfare to the welfare in the interior competitive equilibrium in which even the most conservative banks face the possibility of liquidation at time 1 ($M_L \cdot p_{\text{-}1} - \bar{M} \cdot p_0 > 0$).

By assumption, the non-financial sector sets prices at date 0 and date 1 so that its expected utility is unchanged after trading. Therefore, this sector is indifferent between the equilibrium (α_0^*, α_1^*) and the first-best outcome $(\alpha_0^{**}, \alpha_1^{**})$, and the welfare implications of our equilibrium can be derived by consideration of the welfare of the risk-neutral banks alone.

In the first-best coordinated equilibrium, a bank i will only sell at date 1 if forced to liquidate. In the assumed interior equilibrium case, this occurs with probability $q_i^{**} = m_i \cdot p_{\text{-}1} - \overline{M} \cdot \mu$. Bank i receives the price p_1^{**} if forced to liquidate at time 1, and μ otherwise. Therefore, the average utility (across all banks) in the benchmark case is

$$\begin{aligned} EU^{**} &= \frac{1}{\mathcal{M}} \cdot \int_{M_L}^{M_H} \left[q_i^{**} \cdot p_1^{**} + (1 - q_i^{**}) \cdot \mu\right] dm_i \\ &= \mu + \mu \cdot \overline{M} \cdot \kappa \cdot \alpha_1^{**} - \frac{p_{\text{-}1} \cdot \kappa \cdot \alpha_1^{**}}{\mathcal{M}} \cdot \int_{M_L}^{M_H} m_i \, dm_i \\ &= \mu - \frac{\kappa \cdot \alpha_1^{**}}{2} \cdot \left[(M_H + M_L) \cdot p_{\text{-}1} - 2 \cdot \overline{M} \cdot \mu\right] \\ &= \mu - \kappa \cdot \alpha_1^{**2}. \end{aligned}$$

In contrast, in the competitive equilibrium, a proportion α_0^* of banks with the highest leverage ($m_i \geq M^* = M_H - \mathcal{M} \cdot \alpha_0^*$) sell preemptively at date 0 for a price p_0^*. Among the remaining banks (those with the lowest leverage), a bank i will be forced to liquidate at time 1 with probability $q_i^* = m_i \cdot p_{\text{-}1} - \overline{M} \cdot (\mu - \kappa \cdot \alpha_0^*)$, again in the assumed interior equilibrium case. If this occurs, it receives the price p_1^*, and μ otherwise. Consequently, the average utility (across all banks) in our equilibrium is

$$\begin{aligned} EU^* &= \frac{1}{\mathcal{M}} \cdot \int_{M_H - \mathcal{M} \cdot \alpha_0^*}^{M_H} p_0^* \, dm_i + \frac{1}{\mathcal{M}} \cdot \int_{M_L}^{M_H - \mathcal{M} \cdot \alpha_0^*} \left[q_i^* \cdot p_1^* + (1 - q_i^*) \cdot \mu\right] dm_i \\ &= \mu - \kappa \cdot \alpha_0^{*2} - \frac{\kappa \cdot (2 \cdot \alpha_0^* + \alpha_1^*)}{\mathcal{M}} \cdot \int_{M_L}^{M_H - \mathcal{M} \cdot \alpha_0^*} \left[m_i \cdot p_{\text{-}1} - \overline{M} \cdot (\mu - \kappa \cdot \alpha_0^*)\right] dm_i \\ &= \mu - \kappa \cdot \alpha_0^{*2} - \kappa \cdot (2 \cdot \alpha_0^* + \alpha_1^*) \\ &\qquad \cdot \left\{ \frac{p_{\text{-}1}}{2 \cdot \mathcal{M}} \cdot \left[(M_H - \Delta \cdot \alpha_0^*)^2 - M_L^2\right] - \overline{M} \cdot (\mu - \kappa \cdot \alpha_0^*) \cdot (1 - \alpha_0^*) \right\} \\ &= \mu - \kappa \cdot (\alpha_0^* + \alpha_1^*)^2. \end{aligned}$$

The following theorem summarizes our welfare results.

Theorem 5 *Under the assumptions of Theorem 3, and in the case in which the most conservative bank faces liquidation with non-zero probability, bank welfare is lower in the preemptive-selling competitive equilibrium than it is in the first-best coordinated equilibrium (in which no bank sells at time 0). The welfare loss in equilibrium is increasing in κ.*

Proof:

The difference in average utility (across all banks) between the first-best equilibrium and the preemptive-selling equilibrium is

$$EU^{**} - EU^{*} \quad = \quad \kappa \cdot (\alpha_0^* + \alpha_1^*)^2 - \kappa \cdot \alpha_1^{**2} \quad = \quad \kappa \cdot (\alpha_0^* + \alpha_1^* + \alpha_1^{**}) \cdot (\alpha_0^* + \alpha_1^* - \alpha_1^{**}) \ .$$

This is non-negative if and only if $\Pi_1 \equiv \alpha_0^* + \alpha_1^* - \alpha_1^{**} \geq 0$. Substituting the formulae for α_1^* and α_1^{**} yields

$$\begin{aligned}
\Pi_1 \quad &= \quad \alpha_0^* \cdot (1 + \overline{M} \cdot \kappa + \overline{M} \cdot \mu - M_H \cdot p_{\text{-}1}) \ + \ 0.5 \cdot \alpha_0^{*2} \cdot (M_H \cdot p_{\text{-}1} - M_L \cdot p_{\text{-}1} - 2 \cdot \overline{M} \cdot \kappa) \\
&= \quad \alpha_0^* \cdot (1 + \overline{M} \cdot \mu - M_H \cdot p_{\text{-}1}) \ + \ 0.5 \cdot \alpha_0^{*2} \cdot \mathcal{M} \cdot p_{\text{-}1} \ + \ \overline{M} \cdot \kappa \cdot \alpha_0^* \cdot (1 - \alpha_0^*) \ .
\end{aligned}$$

This is positive, because $(1 + \overline{M} \cdot \mu - M_H \cdot p_{\text{-}1}) \geq 0$ (by Assumption **A2b**).

For the comparative static result with respect to κ, let $\Pi_2 = \alpha_0^* + \alpha_1^* + \alpha_1^{**}$ so that the welfare loss $EU^{**} - EU^{*} \equiv \kappa \cdot \Pi_1 \cdot \Pi_2$. Because $\partial\alpha_0^*/\partial\kappa \geq 0$ (from Theorem 4), $\partial\Pi_1/\partial\kappa \geq 0$ and $\partial\Pi_2/\partial\kappa \geq 0$. Thus, the welfare loss is increasing in κ.

Q.E.D.

4.1 Government Intervention

There are two roles that a benevolent government could play.

First, the government may have influence over $\overline{M}$, the maximum leverage that is allowed in the economy. By tightening the margin requirements, the regulator induces banks to sell preemptively. Conversely, during a crisis, relaxing the margin constraints would alleviate the pressure to sell preemptively. Of course, although we did not model this explicitly, tighter leverage constraints in ordinary times might be beneficial, because they can constrain the *ex-ante* leverage choices of banks (DT_i), which we assumed to be exogenous.

Second, the government can intervene as a liquidity provider in a beneficial way. Specifically, it can try to negate the cost disadvantage of selling behind other financial institutions. In this sense, the government could play a similar role as an arbitrageur (see Appendix B)—and this could even be profitable. However, there are some caveats against too quick an argument in favor of government intervention. First, the benefit of this intervention has to be weighed against the unmodelled costs of government intervention (such as lack of information, manipulability, and corruption). Second, in our simple model, the government's standby role would not actually be needed in equilibrium. By negating the disadvantage of later selling, there would be no advantage to preemptive selling. Yet, in the 2008 crisis, not only did such intervention occur, but so did preemptive sales. In real life, there could be cases in which the government would end up with some risky assets. Yet, government ownership of risk is likely inefficient, in the same way that the holding of risky assets by the non-financial sector is inefficient. (As noted earlier, although we modeled the costs as due to risk aversion, it is a stand-in for more factors, such as the less efficient administration of credit.)

5 Conclusion

Although our model has been quite minimalist, it could still produce the equivalent of financial crises. They appear because of the preemptive selling behavior of the financial sector when the risk absorption capacity of the non-financial sector is limited. Fear of poorer execution in the future leads to socially inefficient divestment of risky assets from the financial sector. The preemptive liquidation channel also amplifies the transmission of fundamental uncertainty into prices. We view our model as a natural complement to models in which only some strategic investors hold safe assets in order to take advantage of forced divestment during crises, as in Allen and Gale (2004). The natural next step would be to produce a model in which both forces act at the same time. Although we speculate in Appendix B that the main implications of both models will remain, exact solutions will be difficult to obtain. Thus, we leave this to future research.

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Figure 1: Time Line in Base Model

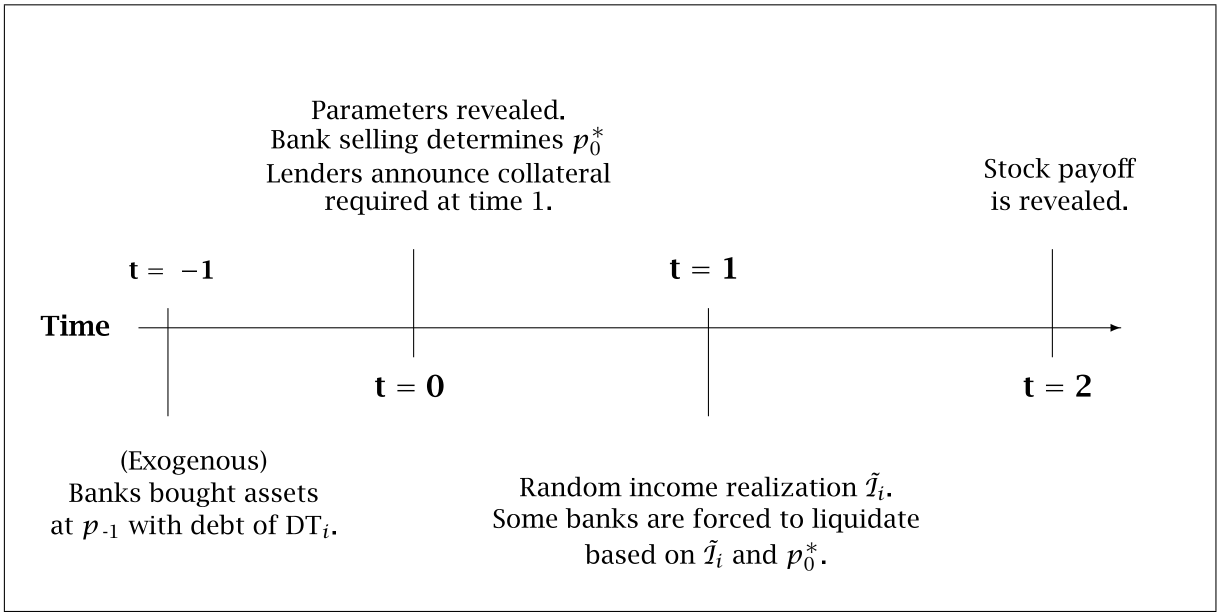
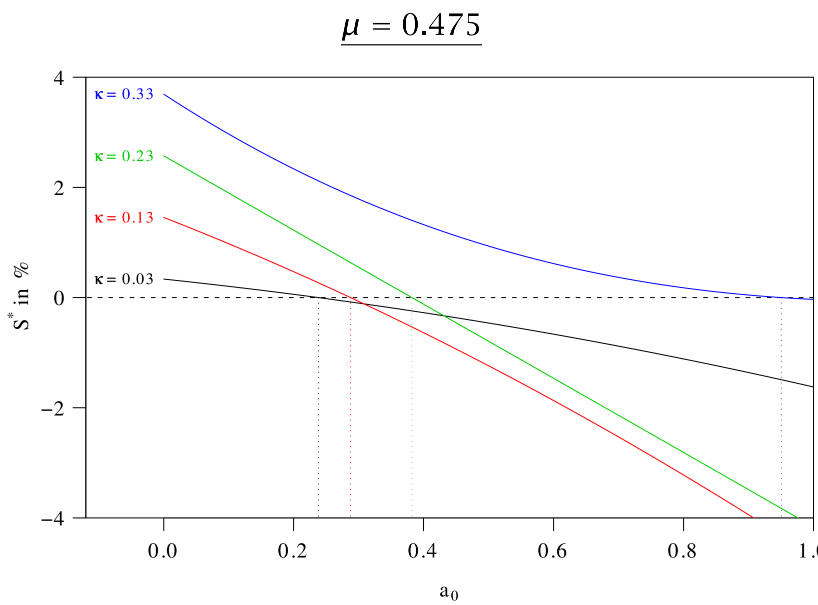
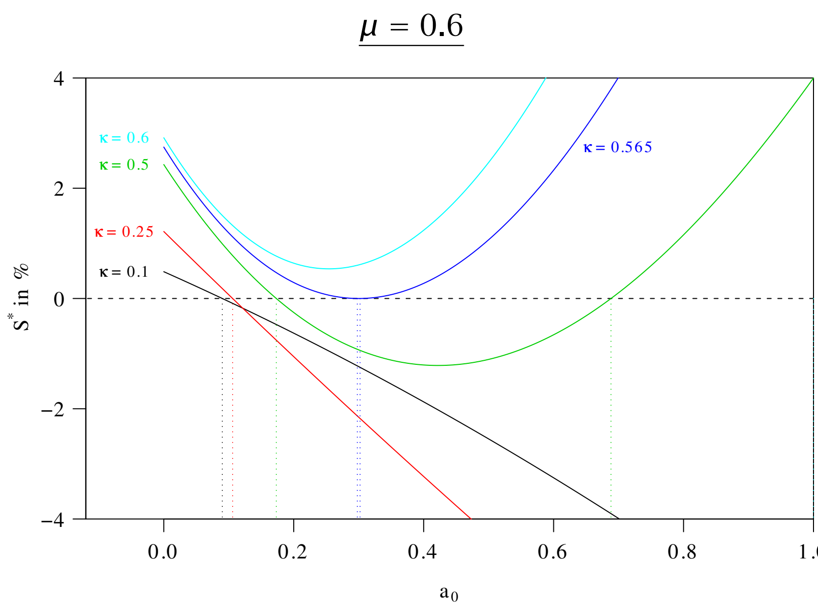


Table 1: Glossary of Parameters, Variables, and Restrictions

	Exogenous Parameters		
p6	Z	$\sim N(\mu, \sigma)$	– asset's final payoff at date 2
p6	μ	positive	– final payoff mean
p6	σ	positive	– final payoff standard deviation
p6	p_{-1}	positive	– (incoming, previously prevailing acquisition) price; less than μ
p7	DT_i	positive	– (incoming) bank debt
p7	m_i	$\sim U[M_L, M_H]$	– bank type
p7	M_L	fraction	– most conservative bank type
p7	M_H	fraction	– most aggressive bank type, $> M_L$
p7	$\overline{M}$	fraction	– maximum allowed leverage, $> M_H$
p6	$\tilde{\mathcal{I}}_i$	$\sim U[0, 1]$	– external source of income of bank i at time 2
p8	γ	positive	– risk-aversion of non-financial sector
p10	$\mathcal{W}_t$	positive	– market-making sector wealth at time t
p10	κ	$\equiv \gamma \cdot \sigma^2/2$	– derived risk-discount of non-financial sector, $< \mu$
p12	$\mathcal{S}_i(\alpha_0, \alpha_1)$	function	– net benefit of bank i selling at time 0 instead of time 1
p18	$\mathcal{S}_*(\alpha_0)$	function	– net benefit to marginal bank i of selling at time 0 instead of time 1
	Endogenous Variables		
p10	α_0	fraction	– holdings of financial sector at time 0
p11	α_1	fraction	– holdings of financial sector at time 1
p6	p_0	positive	– price of risky asset at time 0
p11	p_1	positive	– price of risky asset at time 1
p13	M^*	$\in [M_L, M_H]$	– marginal bank, indifferent about selling in equilibrium

Figure 2: The Selling Benefit Function (S_*) as a Function of α_0

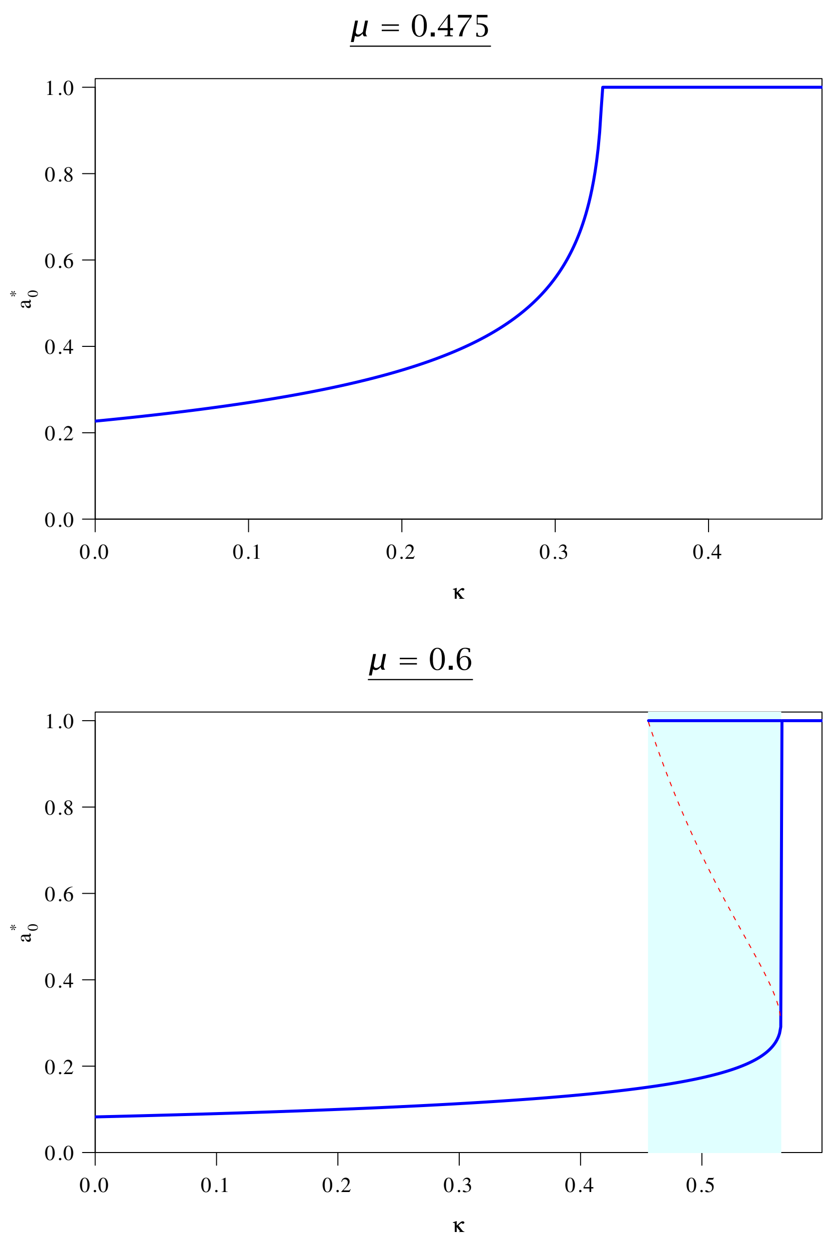




Explanation: These figures plot $S_*(\alpha_0)$ under different parameters. μ is described in the title. κ is annotated. The common parameters are $\overline{M} = 0.9$, $M_H = 0.9$, $\mathcal{M} = 0.2$ (thus $M_L = 0.7$), and $p_{\text{-}1} = 0.9$.

The interior stable equilibrium is where the function intersects the 0-axis from above. A (stable) corner equilibrium at α_0^* is viable if $S_*(1)$ is above 0. Unstable interior equilibria appear only in the lower graph for intermediate κ. When an unstable interior equilibrium exists, so does a stable corner equilibrium. For κ above $\approx$0.565, only the corner equilibrium $\alpha_0^* = 1$ is feasible.

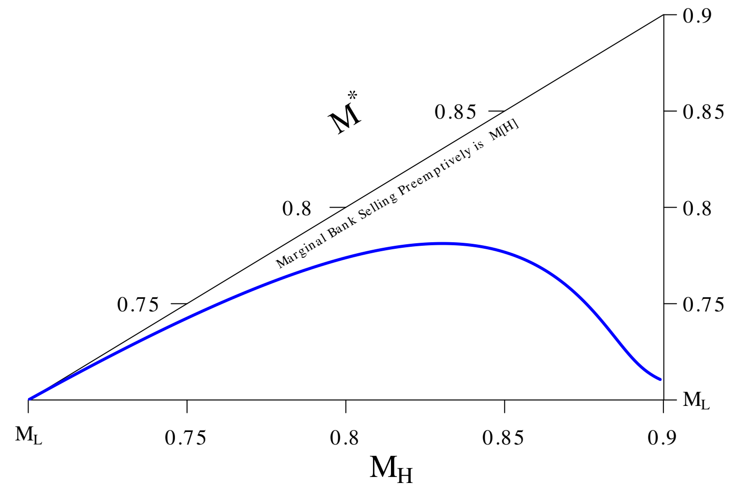
Figure 3: Equilibria as Function of Non-Financial Sector Depth (κ)

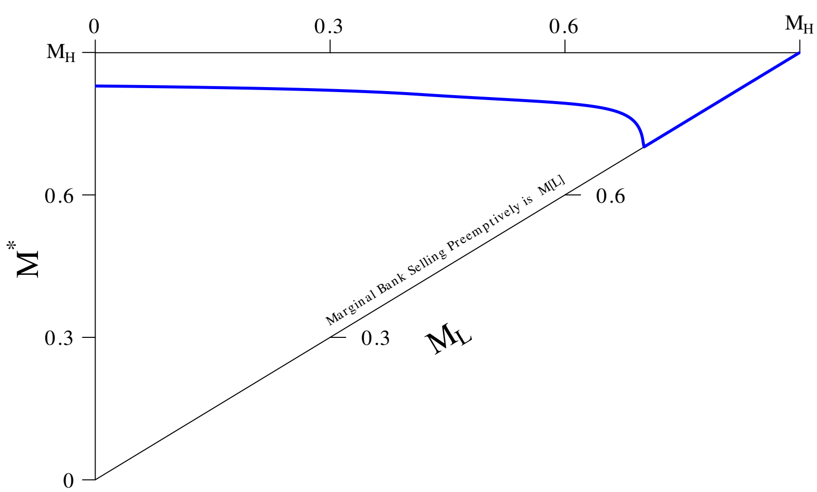


Explanation: These figures plot the equilibrium holdings of the financial sector α_0^* as a function of the depth of the non-financial sector (κ). μ is described in the title. The common parameters are again $\overline{M} = 0.9$, $M_H = 0.9$, $\mathcal{M} = 0.2$ (thus $M_L = 0.7$), and $p_{-1} = 0.9$.

Stable equilibria are plotted in blue. Unstable (interior) equilibria are plotted in dashed red. The shaded region is the area where three equilibria (one interior stable equilibrium, one interior unstable equilibrium, and one corner stable equilibria) are all simultaneously feasible.

Figure 4: Marginal Liquidating Institution (M^*) as Function of The Capitalization of the Financial Sector (M_L and M_H)

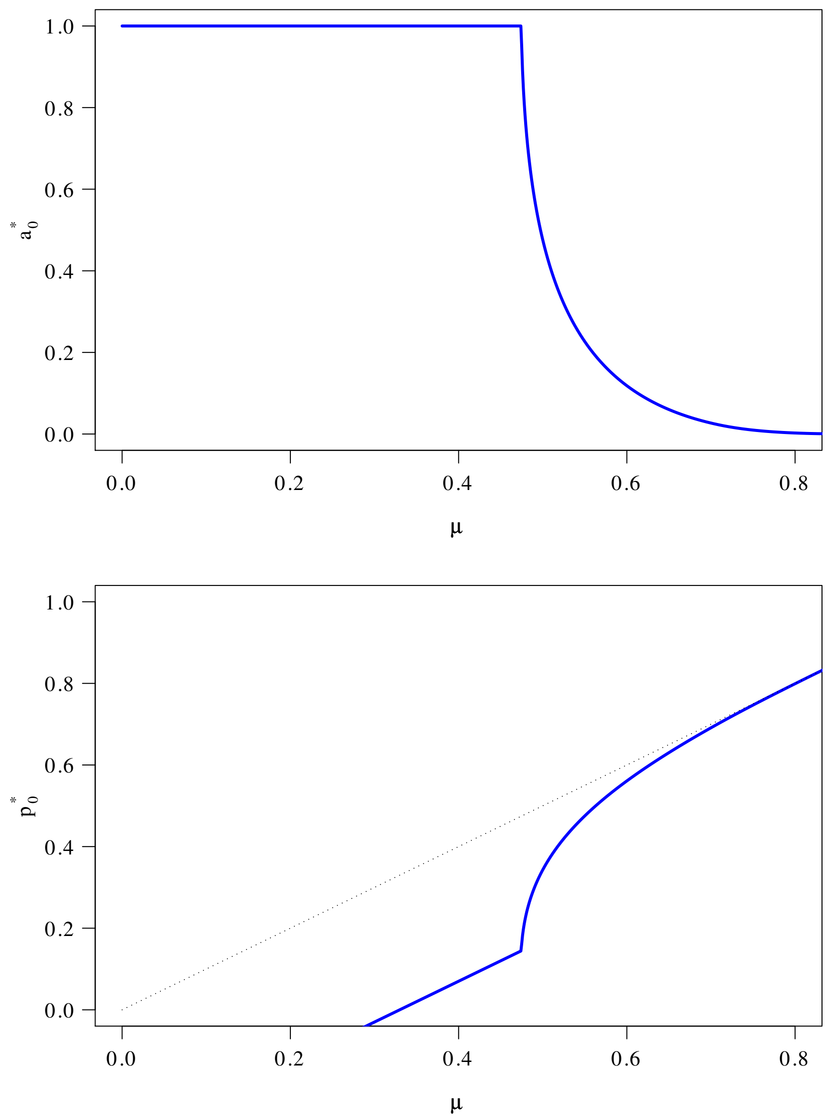




Explanation: These figures plot the equilibrium marginal bank as a function of the quality of the financial sector. The common parameters are $\mu = 0.475$, $p_{-1} = 0.9$, $\kappa = 0.33$, $\overline{M} = 0.9$. In the upper figure, M_L (the most conservative bank) is held constant at 0.7. In the lower figure, M_H (the most aggressive bank) is held constant at 0.9.

The figures show how the marginal bank that liquidates is a decreasing function of the aggressiveness of its peers. When there are relatively more aggressive and relatively fewer conservative banks, even the most conservative banks will decide to sell preemptively, too.

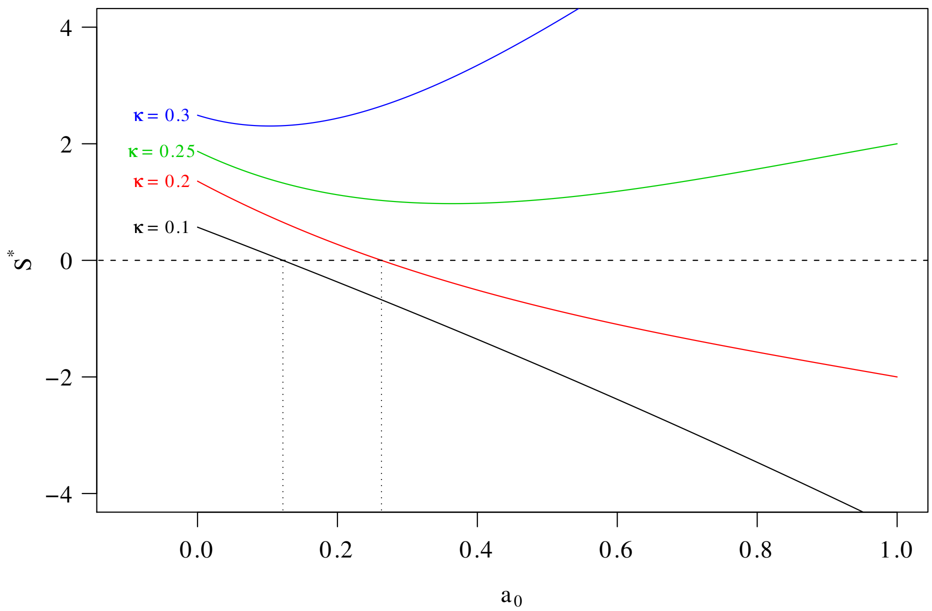
Figure 5: Amplification of Fundamentals (μ)



Explanation: These figures plot the equilibria holdings of the non-financial sector (α_0^*) and the equilibrium price (p_0^*) as a function of the underlying fundamental value (μ). The common parameters are $\overline{M} = 0.9$, $M_H = 0.9$, $p_{-1} = 0.9$, $\kappa = 0.4$, and $M_L = 0.7$. The diagonal dotted line in the lower graph has a slope of 45 degrees. When the slope is 45 degrees, a change in μ has a one-to-one impact on the equilibrium price, p_0^*.

The lower figure shows that when μ is very small or very large, changes in μ translate one-to-one into price changes. For intermediate values of μ, a change in μ has a greater than one-to-one effect on price, because of the additional preemptive liquidation channel.

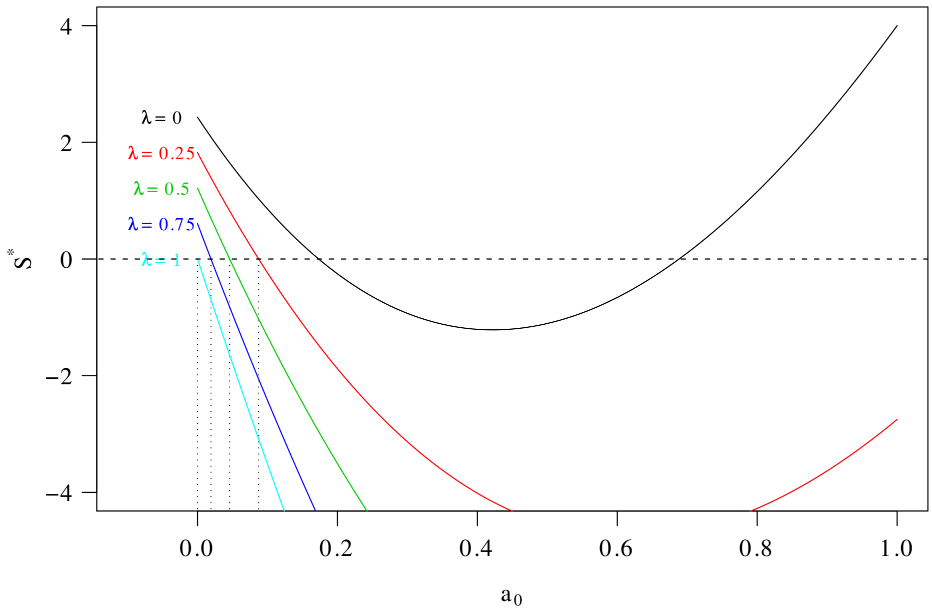
Figure 6: Appendix: The Selling Benefit Function (S_*) as a Function of α_0 when Liquidation depends on Future Price p_1



Explanation: Like Figure 2, this figure plots $S_*(\alpha_0)$ under different parameters. (The figure differs from the upper plot in Figure 2 in that liquidation is based on p_1^* instead of p_0^*.) κ is annotated. The common parameters are $\mu = 0.6$, $M_H = 0.9$, $\overline{M} = 0.9$, $\mathcal{M} = 0.2$ (thus $M_L = 0.7$), and $p_{-1} = 0.9$.

The figure suggests that the model implications stay qualitatively the same.

Figure 7: Appendix: Equilibrium In The Presence of Limited Arbitrage



Explanation: Like Figure 2, the upper figure plots $S_*(\alpha_0)$ under different parameters. (The figure differs from Figure 2 in that partial arbitrage now reduces the price differential between p_0^* and p_0^*.) The common parameters are $\overline{M} = 0.9$, $\mu = 0.6$, $M_H = 0.9$, $\overline{M} = 0.9$, $\mathcal{M} = 0.2$ (thus $M_L = 0.7$), and $p_{\text{-}1} = 0.9$. In addition, κ is 0.5. Lambda is annotated. A lambda of 0 is our base model. A lambda of 1 is perfect arbitrage.

The figure suggests that the model implications stay qualitatively the same.

A Fear of Forced Liquidation Based on Future Price

In our base model, the margin constraint is based on the price at time 0, p_0. However, the margin constraint could also be based on the time 1 price p_1. On page 8, we noted that we would sketch how such a model would be different.

If the margin constraint is based on p_1 the forced liquidation constraint in (1) would change to

$$\textbf{Forced Liquidation if:} \quad \frac{\mathrm{DT}_i - \tilde{\mathcal{I}}_i}{p_1} \; > \; \overline{M} \qquad \Leftrightarrow \qquad p_{\text{-}1} \cdot m_i - p_1 \cdot \overline{M} \; > \; \tilde{\mathcal{I}}_i \, .$$

Eq. 7 derived α_1^*, the fraction of financial institutions at time 1 selling as a function of the fraction of financial institutions selling at time 0, α_0^*. This is now also a function of p_1 rather than of p_0. In the region in which the most conservative bank fears liquidation

$$\alpha_1(\,\alpha_0^*, p_1\,) \;\; = \;\; \frac{(1-\alpha_0^*)\cdot[(M_L+\alpha_0^*\cdot M_L + M_H - \alpha_0^*\cdot M_H)\cdot p_{\text{-}1} - 2\cdot\overline{M}\cdot p_1]}{2} \, .$$

With p_1 instead of p_0 in the integral, we must jointly solve $\alpha_1(\,\alpha_0^*, p_1^*(\alpha_0,\alpha_1)\,)$ and $p_1(\alpha_1^*, \alpha_0^*)$ for α_1^* and p_1^*:

$$\alpha_1^* \;\; = \;\; \frac{(1-\alpha_0^*)\,[[M_L\cdot(1+\alpha_0^*) + M_H\cdot(1-\alpha_0^*)]\cdot p_{\text{-}1} - 2\cdot\overline{M}\cdot(\mu - 2\cdot\kappa\cdot\alpha_0^*)]}{2\cdot[1-\overline{M}\cdot\kappa\cdot(1-\alpha_0^*)]} \, ,$$

$$p_1^* \;\; = \;\; \mu - 2\cdot\kappa\cdot\alpha_0^* - \kappa\cdot\alpha_1^* \, .$$

The S_* function, again expressed in terms of α_0^*, is now

$$\begin{aligned} S_*(\,\alpha_0^*\,) \;\; &= \;\; p_0^* \; - \; [(M_H - \mathcal{M}\cdot\alpha_0)\cdot p_{\text{-}1} - \overline{M}\cdot p_1]\cdot p_1 - [1-(M_H - \mathcal{M}\cdot\alpha_0)\cdot p_{\text{-}1} + \overline{M}\cdot p_1]\cdot\mu \\ &= \;\; \frac{-\,k\cdot A\cdot B}{4\cdot[(1-\alpha_0)\cdot k\cdot M_H - 1]^2} \; - \; \alpha_0\cdot k \, , \end{aligned}$$

where

$$\begin{aligned} A \;\; &\equiv \;\; \alpha_0^2\cdot\mathcal{M}\cdot p_{\text{-}1} \; + \; 2\cdot\alpha_0\cdot[M_H\cdot(\mu - p_{\text{-}1}) + 2] \; + \; 2\cdot M_H\cdot(p_{\text{-}1} - \mu) \; - \; \mathcal{M}\cdot p_{\text{-}1} \\ B \;\; &\equiv \;\; \alpha_0^2\cdot k\cdot M_H\cdot\mathcal{M}\cdot p_{\text{-}1} \; - \; 2\cdot\alpha_0\cdot[k\cdot M_H\cdot(\mathcal{M}\cdot p_{\text{-}1} + 2) - \mathcal{M}\cdot p_{\text{-}1}] \\ &\quad\; + M_H\cdot[p_{\text{-}1}\cdot(k\cdot\mathcal{M} - 2) + 2\cdot\mu] \, . \end{aligned}$$

This is a sixth-order polynomial. Figure 6 plots the S_* function. The main features of the equilibrium remain: for small κ, there is an interior equilibrium. For larger κ, the function shifts towards the north-east and begins to bend more. **[Fig 6 here]**

When liquidation at time 1 depends on the then-prevailing future price (p_1), as in this appendix, rather than on the current price (p_0), as in our main model, then there is a second contagion channel. In this case, banks fear a widespread liquidation crisis even more, thus sell more shares preemptively, and the equilibrium becomes more fragile.

We can easily confirm this by comparing Figure 6 and Figure 2, because both used the same parameters. When $\kappa = 0.1$, and if liquidation depends on p_1, the optimal α_0^* is 12.2%, instead of the lower 9.0% if liquidation depends on p_0. A higher $\kappa = 0.2$ parameter more than doubles the preemptively sold assets to 26.2% in this model, instead of barely moving it (to 10.0%) in our main model. A κ of 0.3 already yields the corner equilibrium of 100%, instead of moving it just a little further to 11.3%.

B The Effect of Presence of Arbitrageurs

We have ignored the role of arbitrageurs in our economy. Specifically, this assumed away strategic investors who did not participate in order to wait for fire sales, as in Allen and Gale (2004). If these investors can enter the market at time 1, the liquidity supply would be time-varying. The price at time 1 would be higher (closer to that at time 0) than our base model assumed. We now sketch a revised model in which the presence of some arbitrageurs allows for limited arbitrage.

If there is a significant mass of arbitrageurs in the model, they would be able to reduce the price differential between time 0 and time 1. We express this by adding an exogenous parameter, $\lambda \in [0, 1]$ that measures their presence. Perfect arbitrage would drive the stock price at time 1 to be equal to that at time 0, while absence of such arbitrage would be the equivalent of our base model. A good example of a reduced-form price function capturing the existence of arbitrageurs would be

$$p_1(\alpha_1; \alpha_0) = p_0(\alpha_0) - (1 - \lambda) \cdot \kappa \cdot (\alpha_0 + \alpha_1), \tag{10}$$

where $\lambda = 1$ corresponds to the case of perfect arbitrage and $\lambda = 0$ corresponds to the absence of arbitrageurs.

In our model, the probability of liquidation depends on the price at time 0 (and not time 1). Thus, the fraction of banks forced to liquidate at time 1, given in (6) remains the same. The revised S_* function becomes

$$S_*(\alpha_0^*) = \kappa \cdot \{[(M_H - \mathcal{M} \cdot \alpha_0^*) \cdot p_{\text{-}1} - \overline{M} \cdot (\mu - \kappa \cdot \alpha_0^*)] \cdot [\alpha_0^* + (1-\lambda) \cdot \alpha_0^* + (1-\lambda) \cdot \alpha_1^*] - \alpha_0^*\} \ .$$

Under suitable parameter restrictions, there are two results of interest. First, there exists an interior equilibrium $\alpha_0^* \in (0,1)$ if $\lambda > 0$. Second, an increased presence of arbitrageurs decreases preemptive selling,

$$\frac{d\alpha_0^*}{d\lambda} < 0 \ .$$

Because $\partial S_* / \partial \alpha_0^* < 0$ for the interior equilibrium, $\partial \alpha_0^* / \partial \lambda < 0$ if $\partial S_* / \partial \lambda < 0$. Differentiating S_* with respect to λ yields

$$S_{*\lambda} \ = \ -[\kappa \cdot (\alpha_0^* + \alpha_1^*)] \cdot [(M_H - \mathcal{M} \cdot \alpha_0^*) \cdot p_{\text{-}1} - \overline{M} \cdot p_0^*] < 0.$$

Both results are illustrated in Figure 7. It plots the S_* function, again under the same common parameters as Figure 2. When $\lambda = 0$, our base solution obtains. When $\lambda = 1$, the no externality solution of $\alpha_0^* = 0$ applies. In between, the equilibrium is interior. Higher λ shift the S_* function towards the south-west resulting in a lower α_0^*. With the specific parameters, a lambda of one-half reduces preemptive selling to about one-quarter of what it is in the absence of arbitrage.

In sum, unless there is perfect arbitrage ($\lambda = 1$), the results of our model continue to hold: the price at time 1 will still be lower than the price at time 0, and the preemptive selling motive will again lead to an inferior allocation of risk in the economy (although in a region that is smaller than it was in our model in the text). The inefficient equilibrium holdings of the non-financial sector will be lower when there is more arbitrage. Thus, arbitrage is also socially beneficial.