

Market Closure, Portfolio Selection, and Liquidity Premia *

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Journal of Economic Literature Classification Numbers: D11, D91, G11, C61.

Keywords: Market Closure, Portfolio Selection, Liquidity Premia, Optimal Investment.

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Abstract

Constantinides (1986) finds that transaction cost has only a second order effect on liquidity premia. In this paper, we show that simply incorporating the well-established time-varying return dynamics across trading and nontrading periods generates a first order effect that is much greater than that found by the existing literature and comparable to empirical evidence. Surprisingly, the higher liquidity premium is Not from higher trading frequency, but mainly from the substantially suboptimal (relative to the no transaction case) trading strategy chosen to control transaction costs. In addition, we show that adopting strategies prescribed by standard models that assume a continuously open market and constant return dynamics can result in significant utility loss. Furthermore, our model predicts that trading volume is greater at market close and market open than the rest of trading times.

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Market closures during nights, weekends, and holidays are implemented in almost all financial markets.¹ An extensive literature on stock return dynamics across trading and nontrading periods finds that while expected returns do not vary significantly across these periods, volatilities do (e.g., French and Roll (1986), Stoll and Whaley (1990), Tsiakas (2008), see Figure 2). For example, French and Roll (1986) and Stoll and Whaley (1990) find that volatility during trading periods is more than four times the volatility during non-trading periods on a per-hour basis.² However, most of the existing portfolio selection models assume that market is continuously open and stock return dynamics is constant across trading and nontrading periods.³ One of the important implications of this assumption is that transaction costs only have a second-order effect for asset pricing, as shown in Constantinides (1986).

Specifically, most portfolio selection models (e.g., Constantinides (1986)) conclude that the liquidity premium (i.e., the maximum expected return an investor is willing to exchange for zero transaction cost) is an order of magnitude smaller than transaction cost. For example, Constantinides (1986) finds that the liquidity premium to transaction cost (LPTC) ratio is only about 0.14 with a proportion transaction cost of 1%. The main intuition behind this conclusion is that with constant return dynamics, investors do not need to trade a large amount and thus the loss from transaction costs is small. However, this finding is in sharp contrast with many empirical studies that suggest the importance of transaction costs in influencing the cross-sectional pat-

¹Although there are overnight markets, the cost of trading in these markets is significant for most investors and therefore most investors do not trade overnight.

²French and Roll (1986) conclude that the principle factor behind high trading-time variances is the private information revealed by informed trades during trading hours, although mispricing also contributes to it.

³See, for example, Merton (1987), Constantinides (1986), Vayanos (1998), Liu and Loewenstein (2002), and Liu (2004).

terns of expected returns.⁴ For example, Amihud and Mendelson (1986) find that the LPTC ratio is about 2.4 for NYSE stocks, while Eleswarapu (1997) finds it is about 0.9 for NASDAQ stocks. Assuming that return dynamics varies across bull and bear economic regimes, Jang, Koo, Liu and Loewenstein (2007) show that transaction costs can have a significantly larger effect on liquidity premia because of the need to trade more frequently. However, Jang et. al. (2007) still assume that in a given regime, return dynamics remains the same across trading and nontrading periods. Since bull and bear regimes switch infrequently and volatilities across these regimes do not differ much, the LPTC ratio found by Jang et. al. (2007) given reasonable calibration is about 0.5, which is still small relative to what is suggested by empirical evidence.

In this paper, we show that incorporating the well-established time-varying return dynamics across trading and nontrading periods *alone* can generate a first order effect that is much greater than that found by the existing literature including Jang et. al. (2007), and comparable to empirical evidence. Surprisingly, the higher liquidity premium in our model is Not from higher trading frequency, but mainly from the substantially suboptimal (relative to the optimal strategy in the absence of transaction costs) trading strategy chosen to control transaction costs.

Specifically, we consider a continuous-time optimal portfolio selection problem of an investor with a finite horizon who can trade a riskfree asset and a risky stock. He faces proportional transaction costs in trading the stock. Different from the standard literature and consistent with empirical evidence, we assume market closes periodically and stock return volatilities differ across trading and nontrading periods. We show the existence, uniqueness, and smoothness of the associated value function. We

⁴See, for example, Amihud and Mendelson (1986), Eleswarapu (1997), and Brennan, Chordia, and Subrahmanyam (1998)).

also explicitly characterize the solution to the investor’s problem and derive certain helpful comparative statics on the optimal trading strategies.⁵ Our extensive numerical analysis, using parameter estimates used by Constantinides (1986), demonstrates that in contrast to the standard conclusion that transaction costs only have a *second-order* effect, transaction costs can have a *first-order* effect that is comparable to empirical findings, if one takes into account the time varying volatilities across trading and nontrading periods. In particular, the LPTC ratio could be well above one in our model. Indeed, the LPTC ratio can be more than 20 times higher than what Constantinides finds for reasonable parameter values. For example, with a proportional transaction cost rate of 0.5%, the LPTC ratio is as high as 3.5, whereas it is 0.16 in Constantinides (1986).⁶

A typical explanation for a higher liquidity premium in the presence of time-varying return dynamics is that when return dynamics varies across time, investors trade more often to adapt stock position to new return distributions and thus increase transaction cost payment (e.g., Jang et. al. (2007)). Surprisingly, we show that investors in our model trade much *less* often than those in Constantinedes’ model (only about 1/30). Since liquidity premium is determined by the difference between the value function in the absence of transaction costs and the value function in the presence of transaction costs, the effect of transaction cost on liquidity premium comes

⁵One equivalent interpretation of Constantinides (1986) model is that, like in our model, the investor derives utility only from the final consumption which occurs at T . The difference with respect to horizon is that in our model T is a constant while in Constantinides (1986), T is the first jump time of an independent Poisson process with intensity equal to the time discount rate ρ . However, we also show in Appendix C that our main results are robust to the introduction of intertemporal consumption as in Constantinides (1986)’s model.

⁶Since trading-period volatility is more than 4 times higher than nontrading period volatility and the fraction of wealth invested in the stock is proportional to the inverse of the variance, even when we use a much lower expected return for the nontrading period the LPTC is still comparable to empirical findings.

from two sources. One is the direct transaction cost payment incurred for trading. The other is the adoption of a trading strategy that is suboptimal in the absence of transaction costs.⁷ We find that the main source for the significantly greater liquidity premium is Not the greater transaction cost payment but the adoption of a significantly more suboptimal trading strategy. Indeed, only about 5% of the liquidity premium comes from direct transaction cost payment.⁸ Intuitively, with the large discrepancy between volatilities across trading and non-trading periods, investors are “forced” to significantly widen the no-transaction region to avoid paying high transaction costs from trading frequently and consequently their stock position is much further from the allocation that is optimal in the absence of transaction costs.⁹ It is essentially this substantial suboptimality of the trading strategy that produces the high liquidity premium in our model.¹⁰

In addition, we find that the LPTC ratio becomes even greater as the difference in volatilities increases, as risk aversion coefficient rises, and as investment horizon decreases. The larger the difference in volatilities, the more suboptimal the trading strategy becomes. As risk aversion increases, the investor invests less in the stock and thus is willing to give up more risk premium for the elimination of transaction costs. As investment horizon decreases, the investor needs to liquidate his stock portfolio

⁷Of course the trading strategy is optimal when the investor is subject to transaction costs.

⁸In contrast, more than 85% of the liquidity premium in Constantinedes’ model is from transaction cost payment.

⁹Indeed, even with the much lower trading frequency, investors in our model still pay more than double the transaction costs than those in Constantinedes’ model. This is because of the larger average trading size caused by the difference in the no-transaction regions across trading and nontrading periods.

¹⁰These findings suggest that the main source for liquidity premium (transaction costs or suboptimal strategy) depends on the nature of return dynamics. When return dynamics does not vary frequently or does not vary much (e.g., Constantinedes (1986), Jang et. al. (2007)), the main source tends to be transaction cost payment. Otherwise, the suboptimality of the trading strategy becomes the main driving force.

and pay the liquidation costs sooner and therefore avoiding transaction costs is more valuable to him.

We also show that the optimal trading strategy prescribed by the standard portfolio selection literature can result in large utility loss. For example, under the assumption of constant relative risk aversion (CRRA) preferences and constant investment opportunity set, the optimal trading strategy is to keep a constant fraction of wealth in the stock in the absence of transaction costs. We show that implementing this “optimal” strategy in a market with time-varying volatilities and zero transaction cost can cost as much as 12.29% of initial wealth for an investor with a risk aversion coefficient of 2 and an investment horizon of 10 years. Intuitively, assuming a constant volatility results in overinvestment or underinvestment almost all the time, which causes substantial utility loss. Finally, periodic market closure and volatility difference across trading and nontrading periods imply that trading volumes at market open and at market close can be much higher than other trading times. This is because the investor needs to adjust his position before market closes and his position may have drifted away from the optimal one just before market opens. This U-shaped trading volume pattern is strongly supported by empirical evidence.

Our analysis also reveals it is not the market closure *per se* that generates high liquidity premium. Specifically, given the same volatility dynamics, market closure only has a small effect. However, this does not imply that market closure is not the driving force for the higher liquidity premium. As shown by French and Roll (1986), the principle factor behind the large difference in volatilities is the private information revealed by informed trades during trading hours. It seems likely that this difference in volatilities across trading and nontrading periods is largely driven by market closure which hinders information flow. This suggests that even though

market closure does not directly generates high liquidity premium, it is likely the root cause for this greater liquidity premium.

The closest work to this model is Jang et. al. (2007). Although the main direct driving force in both models is time varying volatility, there are several important differences. First, this model can generate liquidity premium that is comparable to empirical findings. Since Jang et. al. (2007) rely on regime switching between bear and bull markets and the historical switching frequency is low, for a reasonable calibration of their model they can only generate a LPTC ratio of 0.5, which is significantly lower than those found by empirical studies. Second, the main factor that drives higher liquidity premium is different. In Jang et. al. (2007) (and in many other related models), it is the increased trading frequency. In contrast, in our model, the main factor is the significantly suboptimal trading boundaries. Third, since in Jang et. al. (2007), regimes switch at random times, one needs a good estimation of the switching frequency and estimation error may affect the accuracy of the liquidity premium. In contrast, in this model, market closes and opens at known deterministic times and therefore there is no estimation error.

The rest of the paper is organized as follows. Section I presents the model with transaction cost, market closure, and different return dynamics across trading and nontrading periods. Section II solves the case without transaction costs as a benchmark for later comparison. Section III provides characterizations of the solution and some comparative statics for the optimal trading strategy. Numerical and graphical analysis is presented in Section IV. Section V closes the paper. All of the proofs are in the Appendix A.

I. The model

We consider an investor who maximizes his constant relative risk averse (CRRA) utility from terminal liquidation wealth at $T \in (0, \infty)$. The investor can invest in two financial assets. The first asset (“bond”) is riskless, growing at a continuously compounded, constant rate r . The second one is risky (“stock”). Different from the standard literature, we assume that the stock market closes periodically. Specifically, the investment horizon is partitioned into $0 = t_0 < \dots < \dots < t_{2N+1} = T$. Market is open in time intervals $[t_{2i}, t_{2i+1}]$ (“day”) and trading is allowed; while the market is closed and no trading takes place in $(t_{2i+1}, t_{2i+2}), \forall i = 0, 1, \dots, N$ (“night”).¹¹ When market is open, the investor can buy the stock at the ask price $S_t^A = (1 + \theta)S_t$ and sell the stock at the bid price $S_t^B = (1 - \alpha)S_t$, where $\theta \geq 0$ and $0 \leq \alpha < 1$ represent the proportional transaction cost rates and S_t evolves continuously across day and night as

$$\frac{dS_t}{S_t} = \mu(t)dt + \sigma(t)d\mathcal{B}_t, \quad (1)$$

with

$$\mu(t) = \begin{cases} \mu_d, & \text{day} \\ \mu_n, & \text{night} \end{cases} \quad \text{and} \quad \sigma(t) = \begin{cases} \sigma_d, & \text{day} \\ \sigma_n, & \text{night}, \end{cases}$$

where $\mu_d > r, \mu_n > r, \sigma_d > 0, \sigma_n > 0$ are assumed to be constants and $\{\mathcal{B}_t; t \geq 0\}$ is a one-dimensional Brownian motion on a filtered probability space $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t \geq 0}, P)$ with $\mathcal{B}_0 = 0$ almost surely. We assume $\mathcal{F} = \mathcal{F}_\infty$, the filtration $\{\mathcal{F}_t\}_{t \geq 0}$ is right-continuous and each \mathcal{F}_t contains all null sets of \mathcal{F}_∞ .

When $\alpha + \theta > 0$, the above model gives rise to equations governing the evolution of the dollar amount invested in the bond, x_t , and the dollar amount invested in the

¹¹Of course these intervals can be of different length, and thus can deal with closure on weekends and holidays.

stock, y_t :

$$dx_t = rx_t dt - (1 + \theta)dI_t + (1 - \alpha)dD_t, \quad (2)$$

$$dy_t = \mu(t)y_t dt + \sigma(t)y_t d\mathcal{B}_t + dI_t - dD_t, \quad (3)$$

where the cumulative stock sales and purchases processes D and I are adapted, nondecreasing, and right continuous with $dI_t = 0$ and $dD_t = 0$ during night and $D(0) = I(0) = 0$.

Let x_0 and y_0 be the given initial positions in the bond and the stock respectively. We let $\mathcal{A}(x_0, y_0)$ denote the set of admissible trading strategies (D, I) such that (2), (3), and the investor is always solvent, i.e.,

$$W_t \geq 0, \quad \forall t \geq 0, \quad (4)$$

where

$$W_t = x_t + (1 - \alpha)y_t^+ - (1 + \theta)y_t^- \quad (5)$$

is the time t wealth after liquidation. Because the investor cannot trade when market is closed and the stock price can get arbitrarily close to 0 and is unbounded above, solvency constraint (4) implies that the investor cannot borrow or shortsell at market close.

The investor's problem is then

$$\sup_{(D, I) \in \mathcal{A}(x_0, y_0)} E[u(W_T)], \quad (6)$$

where the utility function is given by

$$u(W) = \frac{W^{1-\gamma} - 1}{1 - \gamma}$$

and $\gamma > 0$ is the constant relative risk aversion coefficient.¹²

¹²This specification allows us to obtain the corresponding results for the log utility case by letting γ approach 1.

II. Optimal trading without transaction costs

For purpose of comparison, let us first consider the case without transaction costs (i.e., $\alpha = \theta = 0$). In this case, let $W_s = x_s + y_s$ be the wealth at time s . Then when market is open, the investor's problem at time t becomes

$$J(W, t) \equiv \sup_{\{\pi\}} E_t [u(W_T) | W_t = W], \quad (7)$$

subject to the self financing condition

$$dW_s = rW_s ds + \pi_s W_s (\mu_d - r) ds + \pi_s W_s \sigma_d d\mathcal{B}_s, \quad \forall s \geq t, \quad (8)$$

and the solvency constraint (4), where π_s represents the fraction of wealth invested in the stock. During night, the investor cannot trade. The basic idea for solving the investor's problem is to solve it backward iteratively for the last day, then the last night, and then the second-to-last day, so on and so forth.

Let π_M ("Merton line") be the optimal fraction of wealth invested in the stock without market closure or transaction costs. Then it can be shown that

$$\pi_M = \frac{\mu(t) - r}{\gamma \sigma(t)^2}. \quad (9)$$

We summarize the main result for this case of no transaction costs in the following theorem (with the convention that $t_{-1} = 0$).

Theorem 1 *Suppose that $\alpha = \theta = 0$. Then the value function for $t \in [t_{2i-1}, t_{2i+1})$, $i = 0, 1, \dots, N$ is given by*

$$J(x, y, t) = \begin{cases} \frac{(x+y)^{1-\gamma}}{1-\gamma} e^{(1-\gamma)\eta(t)} \left(\prod_{k=i+1}^N G_k^* \right) - \frac{1}{1-\gamma}, & t \in [t_{2i}, t_{2i+1}); \\ \frac{(x+y)^{1-\gamma}}{1-\gamma} e^{(1-\gamma)\eta(t)} \left(\prod_{k=i+1}^N G_k^* \right) G_i \left(\frac{y}{x+y}, t \right) - \frac{1}{1-\gamma}, & t \in [t_{2i-1}, t_{2i}); \end{cases} \quad (10)$$

and it is attained by the optimal trading policy of

$$\pi(t)^* = \begin{cases} \pi_M, & t \in [t_{2i}, t_{2i+1}); \\ \pi_i^*, & t = t_{2i-1}, \end{cases}$$

where

$$G_i(\pi, t) = E \{ [1 + \pi (R(t_{2i} - t) - 1)]^{1-\gamma} \}, \quad (11)$$

$$R(u) = \exp [(\mu_n - r - \sigma_n^2/2)u + \sigma_n \mathcal{B}(u)], \quad (12)$$

$$\pi_i^* = \arg \max_{\pi \in [0,1]} \frac{G_i(\pi, t_{2i-1})}{1-\gamma}, \quad G_i^* = G_i(\pi_i^*, t_{2i-1}), \quad (13)$$

and

$$\eta(t) = r(T-t) + \frac{(\mu_d - r)^2}{2\gamma\sigma_d^2} \sum_{i=0}^N (t_{2i+1} - t_{2i} \vee t)^+ \quad (14)$$

PROOF: see Appendix A.

Theorem 1 suggests that when market is open, the investor invests the same fraction of wealth in stock as in the case without market closure. Then the investor adjusts his position at market close to take into account the effect of market closure and different return dynamics during night. In addition, since the investor cannot trade overnight, the stock position just before market open can be suboptimal and therefore another discrete adjustment is also likely at market open. These adjustments at market close and market open suggests that the trading volumes at these times are higher than in the rest of the trading hours, predicting a U-shaped trading volume pattern across trading hours. Since the support of stock price is from 0 to ∞ , the investor can never buy on margin or shortsell at market close, otherwise solvency cannot be guaranteed because of market closure. Thus when leverage is optimal during night, the effect of market closure on the optimal trading strategy is greater. Note that the optimal trading strategy during day is independent of parameter values during night. We show later that this is no longer true in the presence of transaction costs.

III. The transaction cost case

In the case where $\alpha + \theta > 0$, the problem is considerably more complicated. In this case, the investor's problem at time t becomes

$$V(x, y, t) \equiv \sup_{(D, I) \in \mathcal{A}(x, y)} E_t [u(W_T) | x_t = x, y_t = y]. \quad (15)$$

Under regularity conditions on the value function, for $i = 0, 1, 2, \dots, N$, we have the following Hamilton-Jacobi-Bellman (HJB) equations for day time

$$\max(V_t + \mathcal{L}V, (1 - \alpha)V_x - V_y, -(1 + \theta)V_x + V_y) = 0, \forall t \in [t_{2i}, t_{2i+1}), \quad (16)$$

and for night time

$$V_t + \mathcal{L}V = 0, \forall t \in (t_{2i-1}, t_{2i}), \quad (17)$$

and at market close before T

$$V(x, y, t_{2i+1}) = \max_{\Delta \in \mathcal{C}(x, y)} V(x - (1 + \theta)\Delta^+ + (1 - \alpha)\Delta^-, y + \Delta, t_{2i+1}^+), \quad (18)$$

with the terminal condition

$$V(x, y, T) = \frac{(x + (1 - \alpha)y^+ - (1 + \theta)y^-)^{1-\gamma} - 1}{1 - \gamma}, \quad (19)$$

where

$$\mathcal{L}V = \frac{1}{2}\sigma(t)^2 y^2 V_{yy} + \mu(t)y V_y + rx V_x,$$

and

$$\mathcal{C}(x, y) = \{\Delta \in \mathbb{R} : x - (1 + \theta)\Delta^+ + (1 - \alpha)\Delta^- \geq 0, y + \Delta \geq 0\}. \quad (20)$$

As we show later, (16) implies that the solvency region for the stock

$$\mathcal{S} = \left\{ (x, y) : x + (1 - \alpha)y^+ - (1 + \theta)y^- > 0 \right\}$$

at each point during a day splits into a “Buy” region (BR), a “No-transaction” region (NTR), and a “Sell” region (SR), as in Davis and Norman (1990).

The following verification theorem shows the existence and the uniqueness of the optimal trading strategy. It also ensures the smoothness of the value function except for a set of measure zero.

Theorem 2 (i) *The HJB equation (16)–(19) admits a unique viscosity solution, and the value function is the viscosity solution.*

(ii) *The value function is $C^{2,2,1}$ in $(x, y) \in \mathcal{S} \setminus (\{y = 0\} \cup \{x = 0\})$, $t \in (t_{2i}, t_{2i+1})$ and in $x > 0$, $y > 0$, $t \in (t_{2i-1}, t_{2i})$, for $i = 0, 1, \dots, N$.*

The homogeneity of the utility function u and the fact that $\mathcal{A}(\beta x, \beta y) = \beta \mathcal{A}(x, y)$ for all $\beta > 0$ imply that $V + \frac{1}{1-\gamma}$ is concave in (x, y) and homogeneous of degree $1 - \gamma$ in (x, y) [cf. Fleming and Soner (1993), Lemma VIII.3.2]. This homogeneity implies that

$$V(x, y, t) = y^{1-\gamma} \phi\left(\frac{x}{y}, t\right) - \frac{1}{1-\gamma}, \quad (21)$$

for some function $\phi : (\alpha - 1, \infty) \times [0, T] \rightarrow \mathbb{R}$.¹³

Let

$$z = \frac{x}{y} \quad (22)$$

denote the ratio of the dollar amount invested the bond to that in the stock. The homogeneity property then implies that Buy, No-transaction, and Sell regions can be described by two functions of time $z_b^*(t)$ and $z_s^*(t)$. The Buy region BR corresponds to $z \geq z_b^*(t)$, the Sell region SR to $z \leq z_s^*(t)$, and the No-Transaction region NTR to $z_s^*(t) < z < z_b^*(t)$. A time snapshot of these regions is depicted in Figure 1. As we show

¹³Since the risk premium is positive, short sale is never optimal and thus $y > 0$.

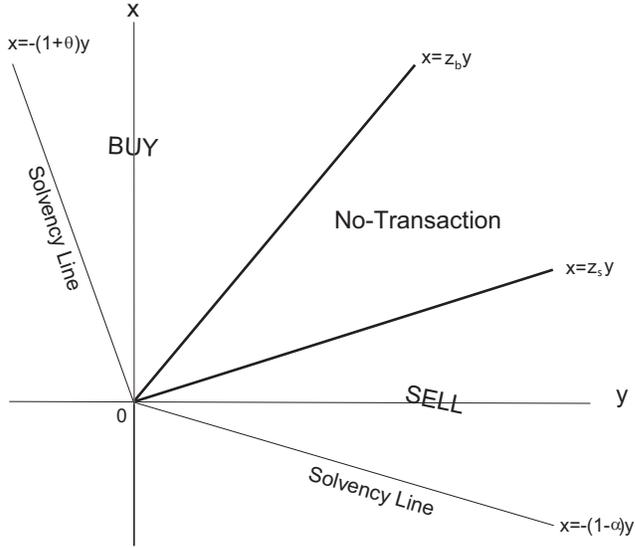


Figure 1: The Solvency Region

later, the optimal trading strategy in the stock is to transact a minimum amount to keep the ratio z_t in the optimal no-transaction region. Therefore the determination of the optimal trading strategy in the stock reduces to the determination of the optimal no-transaction region. In contrast to the no-transaction cost case, the optimal fraction of the wealth invested in the stock changes stochastically, since z_t varies stochastically due to no transaction in NTR.

The following proposition provides connection conditions at t_{2i+1} implied by (18).

Proposition 1 *There exist $z_s^*(t_{2i+1}) \in [0, \infty)$ and $z_b^*(t_{2i+1}) \in (0, \infty]$ such that*

$V(x, y, t_{2i+1})$ is given as follows:

$$\begin{cases} V(x, y, t_{2i+1}) = V(x, y, t_{2i+1}^+) & z_s^*(t_{2i+1}) < x/y < z_b^*(t_{2i+1}) \\ -(1-\alpha)V_x(x, y, t_{2i+1}) + V_y(x, y, t_{2i+1}) = 0 & x/y \leq z_s^*(t_{2i+1}) \\ (1+\theta)V_x(x, y, t_{2i+1}) - V_y(x, y, t_{2i+1}) = 0 & x/y \geq z_b^*(t_{2i+1}). \end{cases} \quad (23)$$

By transformation (21), equations (16), (17) and (19) reduce to

$$\begin{cases} \max \{ \phi_t + \mathcal{L}_1 \phi, (z + 1 - \alpha) \phi_z - (1 - \gamma) \phi, \\ \quad - (z + 1 + \theta) \phi_z + (1 - \gamma) \phi \} = 0, & t \in [t_{2i}, t_{2i+1}) \\ \phi_t + \mathcal{L}_1 \phi = 0, & t \in (t_{2i-1}, t_{2i}) \\ \phi(z, T) = \frac{1}{1-\gamma} (z + 1 - \alpha)^{1-\gamma}, \end{cases} \quad (24)$$

where

$$\mathcal{L}_1 \phi = \frac{1}{2} \sigma(t)^2 z^2 \phi_{zz} + \beta_2(t) z \phi_z + \beta_1(t) \phi,$$

with $\beta_1(t) = (1-\gamma) (\mu(t) - \frac{1}{2} \gamma \sigma(t)^2)$ and $\beta_2(t) = -(\mu(t) - r - \gamma \sigma(t)^2)$. The solvency region in trading periods becomes $(-(1-\alpha), \infty) \times [0, T] \equiv \mathcal{S}_z$ in the space for the ratio z and the connection conditions (23) at t_{2i+1} become

$$\begin{cases} \phi(z, t_{2i+1}) = \phi(z, t_{2i+1}^+), & z_s^*(t_{2i+1}) < z < z_b^*(t_{2i+1}) \\ -(z + 1 - \alpha) \phi_z(z, t_{2i+1}) + (1 - \gamma) \phi(z, t_{2i+1}) = 0, & z \leq z_s^*(t_{2i+1}) \\ (z + 1 + \theta) \phi_z(z, t_{2i+1}) - (1 - \gamma) \phi(z, t_{2i+1}) = 0, & z \geq z_b^*(t_{2i+1}). \end{cases} \quad (25)$$

The nonlinearity of this HJB equation and the time-varying nature of the free boundaries make it difficult to solve directly. Instead, we transform the above problem into a double obstacle problem, which is much easier to analyze.¹⁴ All the analytical results in this paper are obtained through this approach.

Let

$$z_M = \frac{\gamma \sigma_d^2}{\mu_d - r} - 1$$

be the daytime Merton line (i.e., the optimal ratio in the absence of transaction costs).

We then have the following comparative statics.

Proposition 2 *For any $t \in [t_{2i}, t_{2i+1})$, we have*

$$(i) \ z_b^*(t) \geq (1 + \theta) z_M; \quad (ii) \ z_s^*(t) \leq (1 - \alpha) z_M.$$

¹⁴See Dai and Yi (2009) and the references therein for description of this class of problems and solution methodology.

Proposition 2 implies that if it is suboptimal to borrow or short sell in the absence of transaction costs, then it is also suboptimal to borrow or short sell in the presence of transaction costs and in addition, the no transaction region always brackets the Merton line.

Because the market closure time is deterministic and investor can adjust his trading strategy accordingly, one might conjecture that the optimal buy and sell boundaries are always continuous in time from open to close (inclusive). The following proposition shows that this conjecture is incorrect.

Proposition 3 *The sell and buy boundaries have the following properties at t_{2i+1} :*

$$z_s^*(t_{2i+1}^-) = \min \{z_s^*(t_{2i+1}), (1 - \alpha) z_M\}; \quad (26)$$

$$z_b^*(t_{2i+1}^-) = \max \{z_b^*(t_{2i+1}), (1 + \theta) z_M\}. \quad (27)$$

As shown above, when market closes, an investor should adjust his portfolio to be within the interval $[z_s^*(t_{2i+1}), z_b^*(t_{2i+1})]$. Proposition 3 suggests that an investor may optimally wait until the market closing time to discretely adjust his portfolio. For example, in the case $z_s^*(t_{2i+1}^-) = (1 - \alpha)z_M < z_s^*(t_{2i+1})$, if the investor's position is on the sell boundary $z_s^*(t_{2i+1}^-)$ right before market closes, he will perform a discrete sale to adjust his portfolio to $z_s^*(t_{2i+1})$. Similarly, an investor may make a discrete purchase to adjust his portfolio to $z_b^*(t_{2i+1})$. This is consistent with the empirical evidence that trading volume increases at market close. By providing bounds on the boundaries, Proposition 2 and 3 also facilitate numerical computation of the boundaries.

IV. Analysis

In this section we provide some numerical analysis on the impact of market closure and time-varying return dynamics on optimal trading strategy, the liquidity premia, the

loss from market closure, and the loss from adopting the “optimal” strategy implied by the standard assumption of continuously open market and constant return dynamics.

A. Liquidity premia

Most of the existing literature find that transaction costs have a second order effect on risk premium. For example, the seminal work of Constantinides (1986) shows that for a 1% proportional transaction cost rate, an investor only needs about 0.1% compensation in risk premium (i.e., the liquidity premium is only about 0.1%). The main intuition behind this result is that investor does not need to trade much given that the investment opportunity set (such as expected return and volatility) as assumed in Constantinides (1986) is constant. Indeed, Jang et. al. (2007) shows that when there are two regimes with different volatilities, then the transaction cost can have a higher effect on liquidity premium. However, due to the infrequency of regime switching and the small difference in volatilities across regimes, the effect is still small relative to empirical findings. For example, the liquidity premium to transaction cost ratio (LPTC) only increased from 0.1 to about 0.5 in most cases in Jang et. al. (2007). In this subsection, we show that incorporating market closure and the significant difference of volatilities across day and night can make transaction cost have a first order effect on liquidity premium that is comparable to empirical findings. In other words, the liquidity premium to transaction cost ratio can be well above 1.

Let Market A be the actual market with positive transaction costs, different volatility across day and night, and market closure at night. Let Market M be exactly the same as Market A except that there is no transaction cost and no market closure in Market M (the Merton case). Let $V_M(x, y, 0; \mu)$ and $V_A(x, y, 0; \mu, \alpha)$ be the time 0 value functions in these two markets respectively given the expected returns

$\mu_d = \mu_n = \mu$. Following Constantinides (1986), we solve

$$V_M(z_M, 1, 0; \mu - \delta) = V_A(z_M, 1, 0; \mu, \alpha)$$

for δ which measures how much an investor is willing to give up in risk premium to avoid transaction cost, when he starts at the day time Merton line z_M . The liquidity premium δ is affected by the time varying volatility and the inability to trade overnight in Market A. To separate the two effects, we also compute the liquidity premium when the investor can trade with the same transaction costs day and night. Specifically, let Market B be exactly the same as Market A except that the investor can trade overnight subject to the same daytime transaction costs. We solve

$$V_M(z_M, 0, 0; \mu - \tilde{\delta}) = V_B(z_M, 0, 0; \mu, \alpha)$$

for the liquidity premium $\tilde{\delta}$.

In general, the effect of transaction cost on liquidity premium comes from two sources. One is the direct transaction cost payment incurred by trading. The other is the adoption of suboptimal trading strategy.¹⁵ To understand which one is the main driving force behind the large increase in the liquidity premium, we also compute the liquidity premium caused by the suboptimal trading strategy alone. Specifically, let (I, D) be the optimal purchase and sale strategy in the presence of transaction costs and $V_M^{(I,D)}(x, y, 0; \mu)$ be the value function from following the strategy (I, D) in Market M (without transaction costs). We solve

$$V_M(1, 0, 0; \mu - \delta^0) = V_M^{(I,D)}(1, 0, 0; \mu)$$

for the liquidity premium δ^0 that is due to the adoption of suboptimal trading strategy.

¹⁵Although the trading strategy is optimal when the investor is subject to transaction costs, it is suboptimal in the absence of transaction costs and thus would yield smaller expected utility if there were no transaction costs.

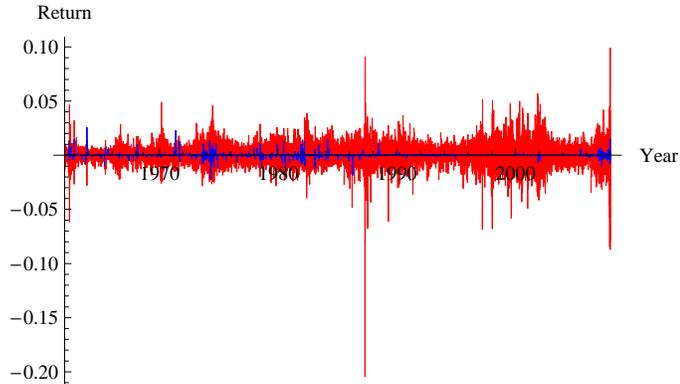


Figure 2: S&P 500 index returns

This figure plots the realized returns for S&P 500 index from January 1962 to October 2008, where the red line represents the simple return from market open to market close (“daytime” return) and the blue line represents the return from market close to next market open (“overnight” return).

For simplicity, we assume from now on that every day market opens for $\Delta_d = 6.5$ hours (from 9:30am to 4pm) and closes for $\Delta_n = 24 - 6.5 = 17.5$ hours. Let the average volatility be σ and the ratio of the day volatility to night volatility be $k \equiv \sigma_d / \sigma_n$.

Then we have

$$\begin{cases} \sigma_d = k\sigma \cdot \sqrt{\frac{\Delta t_d + \Delta t_n}{k^2 \Delta t_d + \Delta t_n}} \\ \sigma_n = \sigma \cdot \sqrt{\frac{\Delta t_d + \Delta t_n}{k^2 \Delta t_d + \Delta t_n}} \end{cases} \quad (28)$$

The existing literature on intraday price dynamics finds that the average per-hour ratio of day-time to overnight volatility is around 4.0 and that the expected returns are not significantly different across day and night (e.g., Stoll and Whaley (1990), Lockwood and Linn (1990), Tsiakas (2008)). Figure 2 plots the realized returns for S&P 500 index from January 1962 to October 2008, where the red line represents the simple return from market open to market close (“daytime” return) and the blue line represents the return from market close to next market open (“overnight” return). This figure illustrates the much higher volatility during trading periods than that

during nontrading periods, as shown in the literature.¹⁶ For subsequent numerical analysis, we choose a lower value of $k = \sigma_d/\sigma_n = 3$ as the default value, which biases against us in finding significant effects of market closure. To make the closest possible comparison with Constantinides (1986), we set other default parameter values at $\mu_d = \mu_n = \mu = 0.15$, $r = 0.10$, $\sigma = 0.20$, $\alpha = 1\%$, $\theta = 1\%$, $\gamma = 2$, and $T = 10$.¹⁷

In Table 1 we compare the LPTC ratios and the optimal no-transaction boundaries in this model with those reported by Constantinides (1986). This table shows that the LPTC ratios are much higher in this model.¹⁸ In fact, for a transaction cost rate of $< 1\%$ each way (e.g., for trading stock index such as S&P 500), transaction costs can have a first order effect. For example, at $\alpha = \theta = 0.5\%$, the LPTC ratio is as high as 3.68, more than 20 times higher than what is found by Constantinides (1986). This magnitude of LPTC ratio is consistent with empirical findings such as those by Amihud and Mendelson (1986) who find a LPTC ratio of 2.4. The second panel in Table 1 shows the results when the investor can trade overnight with the same transaction cost rate as in daytime. It suggests that the effect of the inability to trade overnight on liquidity premium is negligible. The greatest impact comes from the large difference in volatilities across day and night. Therefore market closure *per se* is not important for our results, what is important is the large volatility variation that is caused by market closure. The higher liquidity premium (compared to Constantinides

¹⁶Returns in this figure are not adjusted for duration difference between trading periods and nontrading periods. Such adjustment would make the volatility difference even more dramatic.

¹⁷Although both μ and r may be high relative to realizations in recent years, what matters for our analysis is the risk premium. $T = 10$ is chosen to match the average horizon in the equivalent interpretation of Constantinides (1986) (see footnote 5).

¹⁸As the transaction costs increase, the difference between the two model decreases. This is because the investor optimally trades less often when transaction costs increase. Indeed, in the extreme case with $\alpha = 1$, in both models the investor never invests in stock and thus in both models the liquidity premia are equal to the risk premium of the stock (i.e., $\delta = \mu - r = 0.05$), which implies that there are no difference across these two models in terms of LPTC.

Table 1: Optimal Policy and Liquidity Premia against Transaction Cost Rates

$\alpha = \theta =:$	0.005	0.01	0.02	0.03	0.04	0.05	0.10	0.15
<i>This Model with Market Closure</i>								
$z_b^*(0)$	3.590	3.608	3.644	3.680	3.718	3.753	3.932	4.189
$z_s^*(0)$	0.462	0.430	0.402	0.390	0.383	0.379	0.359	0.340
$z_b^*(t_1^-)$	3.567	3.585	3.621	3.656	3.692	3.727	3.905	4.089
$z_s^*(t_1^-)$	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000
$z_b^*(t_1)$	0.759	0.813	0.909	1.009	1.120	1.242	2.132	4.061
$z_s^*(t_1)$	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000
δ/α	3.68	1.90	1.01	0.71	0.56	0.47	0.28	0.21
δ/δ_C	23.01	13.56	8.05	5.73	4.54	3.82	2.14	1.44
$\delta^0/\delta \times 100$	97.69	94.93	90.23	86.44	83.40	80.98	76.17	81.59
<i>This Model without Market Closure</i>								
$\tilde{\delta}/\alpha$	3.67	1.90	1.00	0.71	0.56	0.46	0.28	0.21
$(\delta - \tilde{\delta}) \times 10^5$	3.7	3.7	3.9	4.0	4.1	4.4	5.2	5.7
<i>Constantinides (1986)</i>								
$z_{b,C}^*$	0.690	0.726	0.783	0.832	0.877	0.920	1.122	1.326
$z_{s,C}^*$	0.566	0.561	0.555	0.550	0.546	0.542	0.525	0.509
δ_C/α	0.16	0.14	0.13	0.12	0.12	0.12	0.13	0.14
$\delta_C^0/\delta_C \times 100$	9.50	13.79	20.44	24.38	27.49	30.08	35.79	36.10

z_b^* and z_s^* are the buying and selling boundaries. t_1^- is just before first closing and t_1 is at first closing. δ , $\tilde{\delta}$, and δ_C are the time 0 liquidity premia, δ^0 measures the loss in risk premium from using the corresponding no trading region in the absence of transaction costs, all starting from the daytime Merton line. Other parameters: $\gamma = 2$, $T = 10$, $\mu_d = \mu_n = 0.15$, $r = 0.10$, $\sigma = 0.20$, $\Delta_d = 6.5$ hours, $\Delta_n = 17.5$ hours, and $k = 3$

(1986)) found by Jang et. al (2007) also comes from volatility difference across the bear regime and the bull regime. However, since the frequency of regime switching is low and the empirically found volatility difference across the two regimes is small, the typical LPTC ratio of around 0.5 found by Jang et. al (2007) is still insufficient to match empirical evidence.

One typical explanation for a higher liquidity premium when investment opportunity set changes is the increase in trading frequency and transaction cost payment

(e.g., Jang et. al (2007)). To help understand whether higher transaction cost payment is also the main driving force behind the high LPTC ratio in our model, we also report the liquidity premium δ^0 due to the suboptimal trading strategy alone. In contrast to conventional wisdom, Table 1 shows that only a small percentage of the liquidity premium is from direct transaction cost payment. The vast majority of the liquidity premium comes from the suboptimal stock position. This finding suggests that with the large volatility difference, the investor chooses a wide no transaction region to reduce transaction cost payment at the cost of keeping significantly suboptimal average positions. Indeed, as Table 1 shows the no-transaction region in this model is much wider than that in Constantinides (1986). For example, if $\alpha = \theta = 0.01$, the time 0 NTR in this model is (0.430, 3.608) which is significantly wider than (0.561, 0.726) that is optimal in Constantinides (1986).

However, wider no transaction regions do not necessarily imply the trading frequency in this model is lower than that in Constantinides (1986), because frequent market closure may increase rebalancing needs and thus trading frequency. To compare the trading frequency and transaction cost payment across these two models, we conduct Monte Carlo simulations on these two models and report related results in Table 2.

Table 2 shows that the trading frequency in Constantinides (1986) is much *higher* (almost 30 times) than that in this model. This confirms the intuition that to avoid large transaction cost payment, the investor chooses a trading strategy to significantly reduce trading frequency. On the other hand, Table 2 also shows that even though the trading frequency is much lower, the transaction costs paid in this model is still higher than that in Constantinides (1986). For example, with 1% transaction cost rate, the present value of transaction costs paid is 0.91% of the initial wealth while

Table 2: Simulation Results

$\alpha = \theta =$	This model		<i>Constantinides</i> (1986)	
	0.005	0.01	0.005	0.01
Daily \$ trading volume $\times 10^4$	3.775	3.161	1.339	1.162
Sell/Buy (\$ ratio)	2.8293	3.3010	41.9627	35.8950
Sell/Buy (share ratio)	0.9127	0.9122	15.3734	12.1396
Buying frequency (# of purchases p.a.)	7.55	5.85	17.72	19.21
Selling frequency (# of sales p.a.)	15.93	13.80	596.71	540.47
Trading (Buying+Selling) frequency	23.48	19.65	614.42	559.68
PVTC (\$)	0.52	0.91	0.23	0.41
Expected time from buy to sell (years)	2.8077	4.4521	1.8375	2.6203

PVTC is the discounted transaction costs paid as a percentage of the initial wealth. Other parameters: $\gamma = 2$, $T = 10$, $\mu_d = \mu_n = 0.15$, $r = 0.10$, $\sigma = 0.20$, $\Delta_d = 6.5$ hours, $\Delta_n = 17.5$ hours, and $k = 3$.

it is only 0.41% in Constantinides (1986). This is mainly because trading in this model can involve large discrete trading at market close and market open, while in Constantinides (1986), only infinitesimal trading at the boundaries can happen. In other words the average per-trade trading size is much larger in this model, which is also corroborated by the trading volume numbers reported in Table 2. Table 2 also suggests that the investor sells more often than buys. This is simply because stock price grows on average.

In Figure 3, we plot the LPTC ratios against the day-night volatility ratio k for three different investment horizons of $T = 5, 10, 15$, and 50 years. This figure shows that LPTC is sensitive to and increasing in the difference between daytime and overnight volatility. For example, at $k = 2$, the LPTC ratio is about 0.99 and it increases to 1.90 when k increases to 3. It is worth noting that at $k = 1$, the LPTC ratio is close to that of Constantinides (1986). This suggests that the effect of the

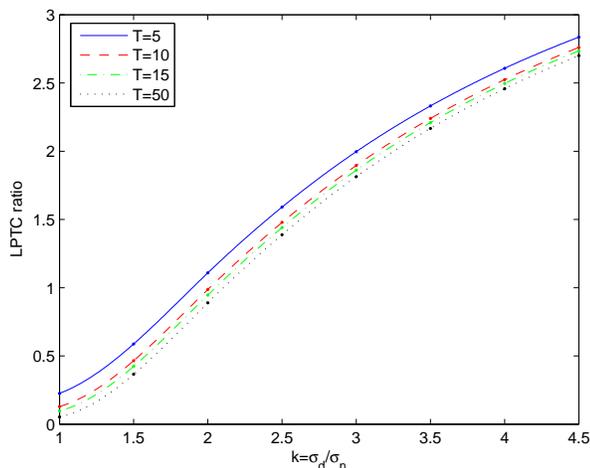


Figure 3: LPTC ratios against day-night volatility ratio k . Parameter default values: $\mu_d = \mu_n = 0.15$, $r = 0.10$, $\sigma = 0.20$, $\Delta_d = 6.5$ hours, $\Delta_n = 17.5$ hours, and $\alpha = \theta = 0.01$.

presence of intertemporal consumption on liquidity premium is small, which is verified by the case with intertemporal consumption presented in Appendix C. In addition, Figure 3 suggests that LPTC ratio decreases at a decreasing speed as the investment horizon increases. Intuitively, with a shorter horizon, an investor needs to adopt a more suboptimal strategy (if there were transaction costs) and liquidate sooner. Moreover, any further increase of horizon almost has no effect on the magnitude of the liquidity premium. This suggests that the higher liquidity premium is not caused by the choice of horizon.

Table 3 records optimal no-transaction boundaries and liquidity premia for different risk aversion coefficients γ . This table shows that the LPTC ratio is more than 10 times higher than that in Constantinides (1986) and that transaction cost clearly has a first order effect for reasonable risk aversion levels. In addition, LPTC ratio increases with risk aversion. Intuitively, as risk aversion increases, an investor invests less in the stock and therefore he is willing to give up more risk premium in exchange

for 0 transaction cost.

Table 3: Optimal Policy and Liquidity Premia against Risk Aversion Coefficients

	γ				
	2	3	4	5	6
$z_b^*(0)$	3.608	5.922	8.248	10.546	12.864
$z_s^*(0)$	0.430	1.067	1.712	2.360	3.009
$z_b^*(t_1^-)$	3.585	5.889	8.180	10.480	12.788
$z_s^*(t_1^-)$	0.000	0.000	0.038	0.295	0.552
$z_b^*(t_1)$	0.813	1.846	2.878	3.910	4.942
$z_s^*(t_1)$	0.000	0.000	0.011	0.261	0.511
δ/α	1.899	1.915	1.925	1.932	1.937
δ/δ_C	13.564	11.969	11.322	10.731	10.192
<i>Constantinides (1986)</i>					
$z_b^*(0)$	0.726	1.736	2.747	3.759	4.785
$z_s^*(0)$	0.561	1.304	2.045	2.778	3.521
δ_C/α	0.14	0.16	0.17	0.18	0.19

z_b^* and z_s^* are the buying and selling boundaries. t_1^- is just before first closing and t_1 is at first closing. δ and δ_C are the time 0 liquidity premiums starting from the daytime Merton line. Other parameters: $T = 10$, $\mu_d = \mu_n = 0.15$, $r = 0.10$, $\sigma = 0.20$, $\Delta_d = 6.5$ hours, $\Delta_n = 17.5$ hours, $k = 3$, and $\alpha = \theta = 0.01$

B. The loss from ignoring volatility variation

There exists an extensive literature on the intraday volatility and expected return dynamics. One of the most robust results is that the stock return volatility is much higher when market is open than when it is closed, while the expected returns are not significantly different across the two periods. However, most of the standard literature (e.g., Merton (1987)) assumes that market is continuously open with a constant volatility. In this subsection we show that using the “optimal” strategy derived under this assumption implies large utility loss for the investor.

Consider Market C where market closes at night and the stock has a constant

volatility of σ across day and night. Let π_t^C be the optimal trading strategy in Market C. Suppose the actual market is Market A, where market closes at night and the stock has a daytime volatility of σ_d and an overnight volatility of σ_n . We examine what would be the cost to the investor from following π_t^C in Market A.

Let $V_C(x, y, 0)$ and $V_A(x, y, 0)$ be the time 0 value functions in Market A for following the wrong strategy π_t^C and the correct strategy, respectively.¹⁹ Then we solve

$$V_A(1 - \Delta, 0, 0) = V_C(1, 0, 0)$$

for Δ which measures the percentage of initial wealth an investor is willing to give up in order to use the correct strategy.

In Figure 4 we plot the wealth loss from following the “wrong” strategy against k in the absence of transaction costs for three different levels of risk aversion: $\gamma = 2, 3, 5$. This figure shows that following the optimal strategy proposed by the standard models is costly. For example, at $k = 3$, for an investor with a risk aversion coefficient of 2, the loss is as high as 12.29% of his initial wealth. Figure 4 also shows that the wealth loss increases as the day-night volatility difference increases, which is the natural implication of the assumption of constant volatility in Market C. Interestingly, while the certainty equivalent wealth loss for a more risk averse investor is lower when the day-night volatility ratio is low, it may be higher if the ratio is high. For example, at $k = 3$, the wealth loss for an investor with a risk aversion coefficient of 3, the loss is 12.38% of his initial wealth. Intuitively, an investor overinvests (underinvests) during market open if and only if the day-night volatility ratio $k > 1$ ($k < 1$). A more risk averse investor overinvests less during market open and underinvests more during

¹⁹The explicit expressions in the no-transaction-cost case are given in (B-3) and (B-2) in Appendix B.

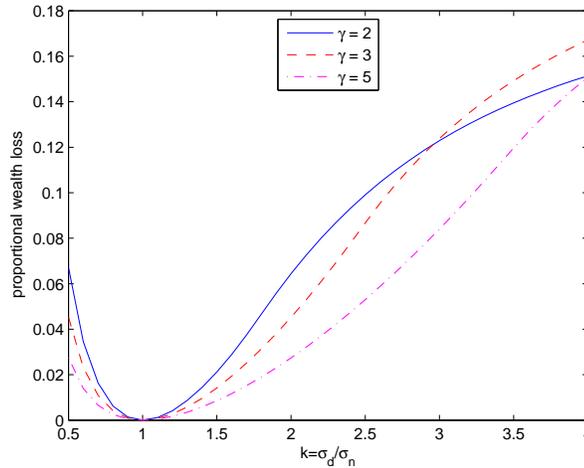


Figure 4: Wealth loss from following standard strategy against day-night volatility ratio k .

Parameter default values: $T = 10$, $\mu_d = \mu_n = 0.15$, $r = 0.10$, $\sigma = 0.20$, $\Delta_d = 6.5$ hours, $\Delta_n = 17.5$ hours, $\alpha = 0$, and $\theta = 0$.

market close. If the day-night volatility ratio is high, the more severe underinvestment during night dominates the reduction of overinvestment during day and thus a more risk averse investor incurs a greater loss.

C. Intraday trading volume

It is well known that the daily trading volume is U shaped, i.e., the trading volumes at market open and market close are much higher than the rest of a day (e.g., Chan, Christie and Schultz (1995)). Our model predicts such a trading pattern.

Figure 5 displays the fraction of total buying and selling volume that occurred within a given time interval against time. It shows that an investor trades much more at the open and the close than during other trading times. This is because investors cannot trade overnight and thus it is likely optimal to adjust his portfolio before market closes. Since there is no trading overnight, the position may be out of

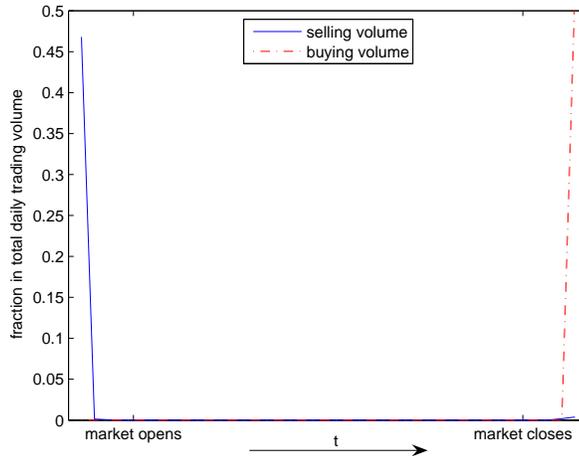


Figure 5: The distribution of the fraction of total trading volume across time. Parameter default values: $T = 10$, $\gamma = 2$, $\mu_d = \mu_n = 0.15$, $r = 0.10$, $\sigma = 0.20$, $\Delta_d = 6.5$ hours, $\Delta_n = 17.5$ hours, $\alpha = \theta = 0.01$.

the no-transaction region by the next market open, therefore they also trade more at market open. When the overnight volatility is small, it is optimal to hold more stock overnight, so investors typically buy more at market close and sell more at market open.

V. Concluding remarks

Standard portfolio models ignore periodic market closures that are commonly observed in almost all financial markets. In this article, we show that this microstructure feature is important for portfolio selection and asset pricing. First, we show that the well-established return dynamics across trading and nontrading periods implies a first order effect of transaction costs on liquidity premium, which is much greater than that found in the existing literature and matches well empirical evidence. Second, we show that adopting strategies prescribed by standard models that assume

a continuously open market with constant return dynamics can result in significant utility loss. Furthermore, our model predicts that trading volumes at market close and market open are much larger than the rest of trading times, also consistent with empirical evidence.

APPENDIX A

A.1 Proof of Theorem 1

To begin with, we point out

$$\sum_{k=0}^N (t_{2k+1} - t_{2k} \vee t)^+ = \begin{cases} \sum_{k=i+1}^N (t_{2k+1} - t_{2k}) + t_{2i+1} - t, & \text{if } t \in [t_{2i}, t_{2i+1}), \\ \sum_{k=i}^N (t_{2k+1} - t_{2k}), & \text{if } t \in [t_{2i-1}, t_{2i}), \end{cases} \quad (\text{A-1})$$

which means the cumulative time in day.

When $t \in [t_{2N}, t_{2N+1})$, the theorem is the well-known Merton's result, where we follow the Merton's strategy π_M .

When $t \in (t_{2N-1}, t_{2N})$, no trading is allowed, then

$$\begin{aligned} J(x, y, t) &= E_t^{x,y} [J(x_{2N}, y_{2N}, t_{2N})] \\ &= \frac{1}{1-\gamma} \left\{ E_t^{x,y} [(x_{t_{2N}} + y_{t_{2N}})^{1-\gamma}] e^{(1-\gamma)\eta(t_{2N})} - 1 \right\}. \end{aligned} \quad (\text{A-2})$$

It is easy to verify that

$$E_t^{x,y} [(x_{t_{2N}} + y_{t_{2N}})^{1-\gamma}] = (x + y)^{1-\gamma} e^{(1-\gamma)r(t_{2N}-t)} G_N \left(\frac{y}{x+y}, t \right).$$

Substituting into (A-2), we then get

$$J(x, y, t) = \frac{1}{1-\gamma} \left\{ (x + y)^{1-\gamma} e^{(1-\gamma)\eta(t)} G_N \left(\frac{y}{x+y}, t \right) - 1 \right\}, \quad (\text{A-3})$$

where we have used $\eta(t_{2N}) + r(t_{2N} - t) = \eta(t)$ due to (14) and (A-1).

When $t = t_{2N-1}$ at which trading is allowed, we need to determine the optimal strategy $\pi \in [0, 1]$. Due to (A-3), we get

$$\begin{aligned} J(x, y, t_{2N-1}) &= \sup_{\pi \in [0,1]} \frac{1}{1-\gamma} \left\{ (x + y)^{1-\gamma} e^{(1-\gamma)\eta(t_{2N-1})} G_N(\pi, t_{2N-1}) - 1 \right\} \\ &= \frac{1}{1-\gamma} \left\{ (x + y)^{1-\gamma} e^{(1-\gamma)\eta(t_{2N-1})} G_N^* - 1 \right\}, \end{aligned}$$

where we have chosen the optimal strategy

$$\pi(t_{2N-1})^* = \pi_N^*.$$

In terms of induction method, it is easy to see that the value function always takes the form of

$$\frac{1}{1-\gamma} \{(x+y)^{1-\gamma} A(t) - 1\} \text{ in the day time } t \in [t_{2i}, t_{2i+1}],$$

where $A(t)$ only depends on t . This allows us to use the Merton's strategy in the day time and to repeat the above derivation during $[t_{2i-1}, t_{2i+1})$ for any i . The desired result then follows.

A.2 Proof of Theorem 2

Part i) can be proved using a similar argument as in Shreve and Soner (1994). To show part ii), we can follow Dai and Yi (2009) to reduce the HJB equation to a double obstacle problem in the day time (t_{2i}, t_{2i+1}) . Then we can obtain $C^{2,2,1}$ smoothness of the value function for $t \in (t_{2i}, t_{2i+1})$. The smoothness of the value function in the night time is apparent.

A.3 Proof of Proposition 1

By definition, the value function V is concave in x and y . We then deduce

$$\begin{aligned} E_b &\triangleq \left\{ (x, y) : (1 + \theta)V_x - V_y|_{t=t_{2i+1}^+} > 0, x > 0, y > 0 \right\} \\ E_s &\triangleq \left\{ (x, y) : -(1 - \alpha)V_x + V_y|_{t=t_{2i+1}^+} > 0, x > 0, y > 0 \right\} \end{aligned}$$

must be connected. Here we confine to $x > 0$ and $y > 0$, in order to ensure solvency.

Due to the homogeneity of the value function, we can define $z_b^*(t_{2i+1})$ and $z_s^*(t_{2i+1})$

as

$$\begin{aligned} z_b^*(t_{2i+1}) &\triangleq \sup \left\{ \frac{x}{y} : (x, y) \in E_b \right\}, \\ z_s^*(t_{2i+1}) &\triangleq \inf \left\{ \frac{x}{y} : (x, y) \in E_s \right\}. \end{aligned}$$

Note that for $\Delta > 0$,

$$\begin{aligned} \frac{d}{d\Delta} V(x - (1 + \theta)\Delta, y + \Delta, t_{2i+1}^+) &= -(1 + \theta)V_x + V_y, \\ \frac{d}{d\Delta} V(x + (1 - \alpha)\Delta, y - \Delta, t_{2i+1}^+) &= (1 - \alpha)V_x - V_y. \end{aligned}$$

Combining with (18), we get the desired result.

A.4 Proof of Proposition 2

Applying the transformation

$$w(z, t) = \frac{1}{\gamma} \log(\gamma\phi),$$

we get

$$\begin{cases} \min \left\{ -w_t - \mathcal{L}_2 w, \frac{1}{z+1-\alpha} - w_z, w_z - \frac{1}{z+1+\theta} \right\} = 0, & t \in [t_{2i+1}, t_{2i+2}) \\ -w_t - \mathcal{L}_2 w = 0, & t \in (t_{2i}, t_{2i+1}) \\ w(z, T) = \log(z + 1 - \alpha) \end{cases} \quad (\text{A-4})$$

with the connection condition

$$\begin{cases} w(z, t_{2i+1}) = w(z, t_{2i}^+), & z_s^*(t_{2i+1}) < z < z_b^*(t_{2i+1}) \\ w_x(z, t_{2i+1}) = \frac{1}{z+1-\alpha}, & z \leq z_s^*(t_{2i+1}) \\ w_x(z, t_{2i+1}) = \frac{1}{z+1+\theta}, & z \geq z_b^*(t_{2i+1}). \end{cases} \quad (\text{A-5})$$

Denote $v = w_z$. Note that

$$\begin{aligned} &\frac{\partial}{\partial z} (\mathcal{L}_2 w) \triangleq \mathcal{L}v \\ &= \frac{1}{2} \sigma^2(t) z^2 v_{zz} - (\mu(t) - r - (1 + \gamma) \sigma^2(t)) z v_z \\ &\quad - (\mu(t) - r - \gamma \sigma^2(t)) v + (1 - \gamma) \sigma^2(t) (z^2 v v_z + z v^2) \end{aligned}$$

Following Dai and Yi (2009), we are able to show that v satisfies the following parabolic double obstacle problem:

$$\begin{cases} \max \left\{ \min \left\{ -v_t - \mathcal{L}v, v - \frac{1}{z+1+\theta} \right\}, \frac{1}{z+1-\alpha} - v \right\} = 0, & t \in [t_{2i+1}, t_{2i+2}) \\ -v_t - \mathcal{L}v = 0, & t \in (t_{2i}, t_{2i+1}) \\ v(z, T) = \frac{1}{z+1-\alpha} \end{cases} \quad (\text{A-6})$$

subject to the connection condition:

$$\begin{cases} v(z, t_{2i+1}) = v(z, t_{2i}^+), & z_s^*(t_{2i+1}) < z < z_b^*(t_{2i+1}) \\ v(z, t_{2i+1}) = \frac{1}{z+1-\alpha}, & z \leq z_s^*(t_{2i+1}) \\ v(z, t_{2i+1}) = \frac{1}{z+1+\theta}, & z \geq z_b^*(t_{2i+1}). \end{cases} \quad (\text{A-7})$$

We then infer that for any $t \in (t_{2i+1}, t_{2i+2})$,

$$\begin{aligned} (\mathbf{SR})_t &\triangleq \left\{ z : v(z, t) = \frac{1}{z+1-\alpha} \right\} = \{z \leq z_s^*(t)\}, \\ (\mathbf{BR})_t &\triangleq \left\{ z : v(z, t) = \frac{1}{z+1+\theta} \right\} = \{z \geq z_b^*(t)\}. \end{aligned}$$

Thanks to (A-6), we have

$$\left(-\frac{\partial}{\partial t} - \mathcal{L} \right) \left(\frac{1}{z+1-\alpha} \right) \leq 0 \text{ for } z \in (\mathbf{SR})_t \text{ (i.e. } z \leq z_s^*(t)), \quad (\text{A-8})$$

$$\left(-\frac{\partial}{\partial t} - \mathcal{L} \right) \left(\frac{1}{z+1+\theta} \right) \geq 0 \text{ for } z \in (\mathbf{BR})_t \text{ (i.e. } z \geq z_b^*(t)). \quad (\text{A-9})$$

Note that

$$\begin{aligned} \left(-\frac{\partial}{\partial t} - \mathcal{L} \right) \left(\frac{1}{z+1-\alpha} \right) &= -\mathcal{L} \left(\frac{1}{z+1-\alpha} \right) \\ &= \frac{(1-\alpha)(\mu_d - r)}{(z+1-\alpha)^3} \left[z + (1-\alpha) \frac{\mu_d - r - \gamma\sigma_d^2}{\mu_d - r} \right] \\ &= \frac{(1-\alpha)(\mu_d - r)}{(z+1-\alpha)^3} [z - (1-\alpha)z_M] \end{aligned} \quad (\text{A-10})$$

and similarly

$$\left(-\frac{\partial}{\partial t} - \mathcal{L} \right) \left(\frac{1}{z+1+\theta} \right) = \frac{(1+\theta)(\mu_d - r)}{(z+1+\theta)^3} [z - (1+\theta)z_M]. \quad (\text{A-11})$$

Combination of (A-8)-(A-11) yields the desired results.

A.5 Proof of Proposition 3

We only prove (26) as an example. First, let us show

$$z_s^*(t_{2i+1}^-) \leq z_s^*(t_{2i+1}).$$

Suppose not, i.e. $z_s^*(t_{2i+1}^-) > z_s^*(t_{2i+1})$. Let $w(z, t)$ be the solution to the problem (A-4). Since $(z_s^*(t_{2i+1}^-), t_{2i+1})$ is in the no-transaction region, $w(z, t)$ is continuous at $(z_s^*(t_{2i+1}^-), t_{2i+1})$, namely, $w(z_s^*(t_{2i+1}^-), t_{2i+1}^-) = w(z_s^*(t_{2i+1}^-), t_{2i+1})$, then for $z \in (z_s^*(t_{2i+1}), z_s^*(t_{2i+1}^-))$

$$\begin{aligned} w(z, t_{2i+1}^-) &= w(z_s^*(t_{2i+1}^-), t_{2i+1}^-) - \int_z^{z_s^*(t_{2i+1}^-)} \frac{1}{\xi + 1 - \alpha} d\xi \\ &< w(z_s^*(t_{2i+1}^-), t_{2i+1}) - \int_z^{z_s^*(t_{2i+1}^-)} w_z(\xi, t_{2i+1}) d\xi \\ &= w(z, t_{2i+1}), \end{aligned}$$

which contradicts the connection condition (A-5).

Clearly $z_s^*(t_{2i+1}^-) \leq (1 - \alpha) x_M$. So we deduce that $z_s^*(t_{2i+1}^-) \leq \min \{z_s^*(t_{2i+1}), (1 - \alpha) z_M\}$. If $z_s^*(t_{2i}^-) < \min \{z_s^*(t_{2i+1}), (1 - \alpha) z_M\}$, then for $z \in (z_s^*(t_{2i+1}^-), \min \{z_s^*(t_{2i+1}), (1 - \alpha) z_M\})$, we have $v(z, t_{2i+1}) = \frac{1}{z+1-\alpha}$ and

$$-v_t - \mathcal{L}v|_{(z, t_{2i+1})} = 0.$$

It follows that

$$\begin{aligned} v_t|_{(z, t_{2i+1})} &= -\mathcal{L} \left(\frac{1}{z + 1 - \alpha} \right) \\ &= \frac{(1 - \alpha)(\mu_d - r)}{(z + 1 - \alpha)^3} [z - (1 - \alpha) z_M] < 0, \end{aligned}$$

which conflicts with the fact $v_t|_{(z, t_{2i+1})} \geq 0$. The proof is complete.

APPENDIX B

In this Appendix, we present some no-transaction-cost case value functions used in the numerical analysis part.

- **Value function in Market A** (continuous trading with (σ_d, σ_n))

$$V_A(x, y, 0) = \frac{1}{1-\gamma} (x+y)^{1-\gamma} \exp [N(1-\gamma)(\eta_d^* \Delta_d + \eta_n^* \Delta_n)] - \frac{1}{1-\gamma} \quad (\text{B-1})$$

where

$$\eta_i^* = r + \frac{\mu - r}{2\gamma\sigma_i^2}, \quad i = d, n,$$

$$N = 250 \times T : \text{number of days in } [0, T];$$

- **Value function in Market B** (day-night trading with (σ_d, σ_n))

$$V_B(x, y, 0) = \frac{1}{1-\gamma} (x+y)^{1-\gamma} \exp \left[N(1-\gamma) \left(\eta_d^* \Delta_d + r \Delta_n + \frac{1}{1-\gamma} \log a^* \right) \right] - \frac{1}{1-\gamma} \quad (\text{B-2})$$

where

$$a^* = E \left\{ \left[1 + \pi^* \left(e^{\left(\mu - r - \frac{\sigma_n^2}{2} \right) \Delta_n + \sigma_n \sqrt{\Delta_n} B_1} - 1 \right) \right]^{1-\gamma} \right\},$$

and

$$\pi^* = \arg \max_{\pi \in [0, 1]} E \left\{ \left[1 + \pi \left(e^{\left(\mu - r - \frac{\sigma_n^2}{2} \right) \Delta_n + \sigma_n \sqrt{\Delta_n} B_1} - 1 \right) \right]^{1-\gamma} \right\};$$

- **Value function in Market B** (day-night trading with (σ_d, σ_n)) **using the optimal strategy of Market C** (continuous trading with (σ))

$$V_C(x, y, 0) = \frac{1}{1-\gamma} (x+y)^{1-\gamma} \exp \left[N(1-\gamma) \left(\eta_d \Delta_d + r \Delta_n + \frac{1}{1-\gamma} \log a \right) \right] - \frac{1}{1-\gamma} \quad (\text{B-3})$$

where

$$\eta_d = r + \frac{(\mu - r)^2}{\gamma\sigma^2} \left[1 - \frac{\sigma_d^2}{2\sigma^2} \right],$$
$$a = E \left\{ \left[1 + \min \left(\frac{\mu - r}{\gamma\sigma^2}, 1 \right) \cdot \left(e^{\left(\mu - r - \frac{\sigma_d^2}{2} \right) \Delta_n + \sigma_n \sqrt{\Delta_n} B_1} - 1 \right) \right]^{1-\gamma} \right\}.$$

APPENDIX C

C.1 Extension to the case with intertemporal consumption

We consider an investor who maximizes his constant relative risk averse (CRRA) utility from discounted terminal liquidation wealth and intertemporal consumption.

When $\alpha + \theta > 0$, the budget constraints become

$$dx_t = (rx_t - c_t) dt - (1 + \theta)dI_t + (1 - \alpha)dD_t, \quad (\text{C-1})$$

$$dy_t = \mu(t)y_t dt + \sigma(t)y_t d\mathcal{B}_t + dI_t - dD_t, \quad (\text{C-2})$$

where $c \geq 0$ is the consumption rate, the cumulative stock sales and purchases processes D and I are adapted, nondecreasing, and right continuous with $dI_t = 0$ and $dD_t = 0$ during night and $D(0) = I(0) = 0$.

The investor's problem is then

$$\sup_{(D,I,c) \in \mathcal{A}(x_0,y_0)} E \left[\int_0^T e^{-\rho s} u(c_s) ds + e^{-\rho T} u(W_T) \right], \quad (\text{C-3})$$

where $\rho > 0$ is the time discount rate.

To set up the model, we define the value function at time t by

$$V(x, y, t) \equiv \sup_{(D,I,c) \in \mathcal{A}(x,y)} E_t \left[\int_t^T e^{-\rho(s-t)} u(c_s) ds + e^{-\rho(T-t)} u(W_T) \middle| x_t = x, y_t = y \right]. \quad (\text{C-4})$$

Under regularity conditions on the value function, for $i = 0, 1, 2, \dots, N$, we obtain the same set of conditions as in (16)-(19), except that

$$\mathcal{L}V = \frac{1}{2}\sigma(t)^2 y^2 V_{yy} + \mu(t)y V_y + rx V_x - \rho V + \frac{\gamma}{1-\gamma} (V_x)^{-\frac{1-\gamma}{\gamma}}, \quad (\text{C-5})$$

By transformation (21), equations (16), (17) and (19) with the new operator $\mathcal{L}V$ defined in (C-5) reduce to (24) except that

$$\mathcal{L}_1 \phi = \frac{1}{2}\sigma(t)^2 z^2 \phi_{zz} + \beta_2(t)z \phi_z + \beta_1(t)\phi - \rho\phi + \frac{\gamma}{1-\gamma} (\phi_z)^{-\frac{1-\gamma}{\gamma}},$$

with the same connection conditions (25) at market close.

C.2 Numerical procedure

The combination of (24) and (25) provides the exact model for optimal investment and consumption with market closure. To implement the numerical procedure, we use an alternative approximation of (24) and (25), which adjusts the model during nighttime by allowing transaction but with huge transaction costs.

Thus the model for implementation of numerical procedure is

$$\left\{ \begin{array}{l} \max \{ \phi_t + \mathcal{L}_1 \phi, (z + 1 - \alpha) \phi_z - (1 - \gamma) \phi, \\ \quad -(z + 1 + \theta) \phi_z + (1 - \gamma) \phi \} = 0, \quad t \in [t_{2i}, t_{2i+1}) \\ \max \{ \phi_t + \mathcal{L}_1 \phi, (z + 1 - \alpha^N) \phi_z - (1 - \gamma) \phi, \\ \quad -(z + 1 + \theta^N) \phi_z + (1 - \gamma) \phi \} = 0, \quad t \in (t_{2i-1}, t_{2i}) \\ \phi(z, t_{2i+1}) = \phi(z, t_{2i+1}^+), \\ \phi(z, T) = \frac{1}{1-\gamma} (z + 1 - \alpha)^{1-\gamma}, \end{array} \right. \quad (\text{C-6})$$

where $\alpha^N \in [0, 1)$ and $\theta^N \in [0, \infty)$ are the nighttime proportional transaction costs. In the numerical procedure, we take $\alpha^N \rightarrow 1^-$ and $\theta^N \gg 1$, which makes the trading boundaries occur very close to the borders of solvency region. In another word, the selling boundary $z_s^*(t) \approx 0$ and the buying boundary $z_b^*(t) \approx \infty$ for $t \in (t_{2i-1}, t_{2i})$. In this way, trading will hardly happen during nighttime such that (C-6) is equivalent to (24) and (25) in the limit sense.

Table 4 shows that the presence of intertemporal consumption does not change our main results. The LPTC ratios are close to the case without intertemporal consumption.

Table 4: LPTC vs α , θ when consumption is involved

	$\alpha = \theta =:$							
	0.005	0.01	0.02	0.03	0.04	0.05	0.10	0.15
	$T = 10$							
$\delta, k = 1$	0.0011	0.0023	0.0048	0.0074	0.0102	0.0132	0.0364	<i>N.A.</i>
$\delta/\alpha, k = 1$	0.23	0.23	0.24	0.25	0.25	0.26	0.36	<i>N.A.</i>
$\delta, k = 3$	0.0183	0.0193	0.0215	0.0236	0.0255	0.0273	0.0372	<i>N.A.</i>
$\delta/\alpha, k = 3$	3.65	1.93	1.08	0.79	0.64	0.55	0.37	<i>N.A.</i>
	$T = 20$							
$\delta, k = 1$	0.0007	0.0015	0.0030	0.0046	0.0063	0.0080	0.0183	0.0348
$\delta/\alpha, k = 1$	0.15	0.15	0.15	0.15	0.16	0.16	0.18	0.23
$\delta, k = 3$	0.0177	0.0183	0.0195	0.0208	0.0223	0.0237	0.0291	0.0358
$\delta/\alpha, k = 3$	3.54	1.83	0.97	0.69	0.56	0.47	0.29	0.24

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