# (Im)Possible Frontiers: A Comment

**Moshe Levy and Richard Roll***
April 14, 2011

**Abstract**

The existence of mean-variance efficient positive portfolios – portfolios with no negative weights – is a key requirement for equilibrium in the Capital Asset Pricing Model (CAPM). Brennan and Lo (2010) define an "impossible frontier" as a frontier on which all portfolios have at least one negative weight. They prove that for randomly drawn covariance matrices the probability of obtaining an impossible frontier approaches 1 as the number of assets grows. Impossible frontiers are also found when the empirical sample parameters are employed, regardless of the specifics of the sampling method. These results seem like a deadly blow to the CAPM. Here, we show that while sample (or randomly drawn) parameter sets almost surely lead to impossible frontiers, slight variations to the parameters, well within their estimation error bounds, lead to frontiers with positive portfolio segments. Parameter sets leading to possible frontiers are somewhat like rational numbers on the real line: they occupy a zero-measure of parameter space, but there is always one close by. In reaching a mean/variance equilibrium, asset prices should change slightly from any randomly chosen set to a nearby set that produces a positive segment of the efficient frontier and is thus consistent with the CAPM.

Keywords: Mean-Variance analysis, CAPM, portfolio optimization, short selling, reverse optimization.

*Levy is from the Jerusalem School of Business Administration at the Hebrew University, and Roll is from the Anderson School of Management at UCLA.

## 1. Introduction

One of the central implications of the Capital Asset Pricing Model (CAPM) is that the market portfolio is mean-variance efficient. As the market portfolio is by definition a positive portfolio (i.e. all the portfolio weights are positive), the CAPM equilibrium requires the existence of a positive mean-variance efficient portfolio. The existence of such a portfolio, or lack thereof, has therefore attracted a great deal of research interest (among others, see Roll (1977), Rudd (1977), Roll and Ross (1977), Green (1986), Green and Hollifield (1992), Best and Grauer (1985, 1992), Jagannathan and Ma (2003), and Levy and Roll (2010)).

Green (1986) derives analytical conditions for the existence of positive efficient portfolios, and argues that these conditions conform to economic intuition about the tradeoff between risk and return. However, empirical studies have almost invariably failed to find a positive portfolio on the efficient frontier constructed from the sample parameters (Ross (1980), Gibbons (1982), Jobson and Korkie (1982), Levy (1983), Shanken (1985), Kandel and Stambaugh (1987), Gibbons, Ross, and Shanken (1989), Zhou (1991), and MacKinlay and Richardson (1991)). Furthermore, various standard shrinkage corrections to the sample parameters do not help in this regard.

In a recent paper, Brennan and Lo (2010, henceforth B&L) make a strong argument against the existence of positive efficient portfolios when the number of assets is large. They define an "impossible frontier" as a frontier on which all portfolios have at least one negative weight, and they ingeniously prove that if the covariance matrix is randomly draw from some parameter space the probability of obtaining an impossible frontier converges to 1 as the number of assets grows. They also show that empirically estimated parameter sets invariably lead to impossible frontiers. The B&L paper thus seems like a definitive final word on the subject, and a fatal criticism of the CAPM.

In this comment we show that while the probability of an impossible frontier converges to 1 with the number of assets, a slight adjustment to the return parameters, well within their estimation error bounds, suffices to yield a segment of positive portfolios on the efficient frontier. Thus, even though parameter sets leading to possible frontiers occupy a zero-measure of parameter space, as B&L prove, given any random parameter set, there is usually a parameter set nearby that yields a possible frontier. In this sense, parameter sets leading to possible frontiers resemble rational numbers: the probability of randomly sampling one from the real number line is zero, but for any point on the real number line there is always a rational number nearby.

Moreover, a possible frontier is likely in equilibrium. A mean/variance optimizing representative investor would attempt initially to hold one of the portfolios on B&L's impossible frontier, which necessarily contains short positions in some assets. But in shorting those assets, their prices will be driven down. Similarly, allocating more than 100% of wealth to the non-shorted assets will cause their prices to rise. This implies that expected returns (and possibly covariances) will change from the original B&L random values. In equilibrium, the representative investor must hold a positive position in all existing assets. This cannot happen until prices move enough to produce a possible frontier; i.e., a standard CAPM frontier with a segment of totally positive portfolios. We show here that prices do not have to move very much to achieve that equilibrium.

## 2. Methods

The main idea in our approach is to start with a given sample parameter set (or a parameter set randomly drawn from some parameter space) and to find the parameter set which is on the one hand as close as possible to the sample set, and on the other hand ensures that the efficient frontier includes positive portfolios. After this "adjusted parameter set" is found, we check whether it is statistically consistent with the sample set.

In general, one may consider adjustments to the expected returns, the standard deviations, and the correlations. In order to simplify the analysis we restrict ourselves here to adjustments only to the expected returns and standard deviations, and leave the correlations unchanged. Obviously, allowing for adjustments to the correlations as well would only improve the results, as it would allow many more degrees of freedom in the optimization.

Given a set of sample parameters $\left(\mu^{sam},\sigma^{sam}\right)$ and any other parameter set $\left(\mu,\sigma\right)$ we define the distance between these two parameter sets as:

$$D\left((\mu,\sigma),(\mu,\sigma)^{sam}\right)\equiv\sqrt{\alpha\frac{1}{N}\sum_{i=1}^{N}\left(\frac{\mu_i-\mu_i^{sam}}{\sigma_i^{sam}}\right)^2+(1-\alpha)\frac{1}{N}\sum_{i=1}^{N}\left(\frac{\sigma_i-\sigma_i^{sam}}{\sigma_i^{sam}}\right)^2}\ , \quad (1)$$

where N is the number of assets, and $0\leq\alpha\leq1$ is a parameter determining the relative weight assigned to deviations of the means relative to deviations of the standard deviations. The rationale for dividing the deviations in (1) by $\sigma_i^{sam}$ is that this distance measure "punishes" deviations in the parameters of assets with low sample standard deviations more heavily than similar deviations in assets with higher standard deviations, because the estimation error is lower for the former. Of course, one could think of other distance measures. The ultimate test of whether a set of parameters $\left(\mu,\sigma\right)$ can be considered as "reasonably close" to the sample parameters is whether one can statistically reject the adjusted parameter set given the sample parameter set, and the statistical tests employed are independent of the distance measure (1).

The optimization problem we solve is:

Minimize $D\left((\mu,\sigma)\ ,(\mu,\sigma)^{sam}\right)$

Subject to:

$$\begin{bmatrix}\sigma_1 & 0 & \cdots & 0\\ \vdots & \sigma_2 & & \\ & & \ddots & \vdots\\ & & & 0\\ 0 & \cdots & 0 & \sigma_N\end{bmatrix}\begin{bmatrix} & & \\ & \rho^{sam} & \\ & & \end{bmatrix}\begin{bmatrix}\sigma_1 & 0 & \cdots & 0\\ \vdots & \sigma_2 & & \\ & & \ddots & \vdots\\ & & & 0\\ 0 & \cdots & 0 & \sigma_N\end{bmatrix}\begin{bmatrix}\omega_1\\ \omega_2\\ \vdots\\ \\ \omega_N\end{bmatrix}=q\cdot\begin{bmatrix}\mu_1-r_z\\ \mu_2-r_z\\ \vdots\\ \\ \mu_N-r_z\end{bmatrix} \quad (2)$$

where $\rho^{sam}$ is the sample correlation matrix (which remains unadjusted), $r_z$ is the zero-beta rate, q>0 is a constant of proportionality, and the vector $\omega$ is a vector of portfolio weights. Condition (2) implies that the portfolio given by $\omega$ is mean-variance efficient (see, for example, Merton (1972), and Roll (1977)). Any pre-specified *positive* vector $\omega$ ($\omega_i > 0$ for all i) that is employed in (2) thus ensures the existence of a positive portfolio on the efficient frontier. Levy and Roll (2010) take $\omega$ as the vector of weighted market capitalizations to examine the mean-variance efficiency of a given market proxy. Here we employ two alternatives for positive $\omega$ (value weighted and equal weighted), and examine the segment of possible efficient portfolios implied by each,[1] as well as the magnitude of the required parameter adjustments. There are 2N+2 variables in the optimization: N $\mu$'s, N $\sigma$'s, q and $r_z$. We are looking for the set of parameter vectors $\left(\mu^*,\sigma^*\right)$ that satisfy the mean-variance efficiency condition (2) and are as closest as possible to the sample parameters.[2]

[1] Best and Grauer (1992) have shown that if positive portfolios exist on the efficient frontier, they form a continuous segment of the frontier.

[2] This optimization problem is similar in spirit to Sharpe's [2007] and Levy's [2007] "reverse optimization" problem. To the best of our knowledge, this approach was first used in a very innovative paper by Best and Grauer [1985].

## 3. Data and Results

Following B&L, we take all stocks listed on the S&P 500 in December of 1995 for which monthly return data were available for the period from January 1980 through December 2005 (312 monthly returns).

Of these stocks we randomly draw 100 stocks and analyze the efficient set. Below we give a detailed description of one typical 100-stock set, and we later provide statistics for a large number of other randomly drawn stock sets.

As explained above, any vector of positive portfolio weights, $\omega$, that is employed in (2) ensures the existence of at least one positive portfolio on the efficient frontier. However, different $\omega$'s imply different positive portfolio segments, and also different parameter adjustments. We show results for two cases: the case were $\omega$ reflects the relative market caps of the stocks (as of December 1995), i.e. a value weighted portfolio, and the case were $\omega_i = 1/N$ for all stocks, i.e. an equal-weighted portfolio. Table 1 shows the parameters and adjusted parameters for the value-weighted case (due to space considerations the parameters for only the first 20 stocks are shown; the entire table can be found in the supplementary materials section). As can be seen in the table, all the adjusted parameters are well within the estimation error bounds of the sample parameters. For the set of all 100 stocks, the minimal t-value is -0.94 and the maximal value is 1.08. Given 312 observations, the 95% confidence interval for the ratio $\left(\sigma_i^*\right)^2 / \left(\sigma_i^{sam}\right)^2$ is (0.86-1.18).[3] The minimal ratio we find is 0.91 and the maximal value is 1.06.

While these are univariate statistics, the picture does not change when multivariate tests are employed. We employ a bootstrap test to estimate the multivariate sampling error involved when the true parameter set is $(\mu^*, \sigma^*)$ and one obtains a sample estimate $\left(\mu^{BS}, \sigma^{BS}\right)$ from 312 bootstrap observations. We compare the distance between $\left(\mu^{BS}, \sigma^{BS}\right)$ and $(\mu^*, \sigma^*)$, i.e. the sampling error for the entire set, with the distance between $(\mu^*, \sigma^*)$ and $\left(\mu^{sam}, \sigma^{sam}\right)$, i.e. the adjustment required to obtain a positive portfolio segment (see the appendix for the details of the bootstrap analysis). We

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[3] The ratio $\frac{(n-1)s^2}{\sigma^2}$ is distributed according to the $\chi^2_{n-1}$ distribution, where $\sigma^2$ is the population variance, $s^2$ is the sample variance (or $\left(\sigma^{sam}\right)^2$ in the notation used in this paper), and *n* is the number of observations. We have 312 monthly return observations, hence *n*=312. As we are looking for the 95% confidence interval for $s^2/\sigma^2$, we need to find the critical values $c_1$ and $c_2$ for which $P\left(\chi^2_{311} > c_1\right) = 0.025$, and $P\left(\chi^2_{311} < c_2\right) = 0.025$. For large *n*, $\sqrt{2\chi^2_n} - \sqrt{2n-1}$ can be approximated by the standard normal distribution. Thus, the critical values $c_1$ and $c_2$ satisfy $\sqrt{2c_1} - \sqrt{2 \cdot 311 - 1} = 1.96$ and $\sqrt{2c_2} - \sqrt{2 \cdot 311 - 1} = -1.96$, which yield: $c_1 = 361.3$ and $c_2 = 263.6$. Thus, the 95% confidence interval for $s^2/\sigma^2$ is given by $263.6 < 311 \cdot s^2/\sigma^2 < 361.3.6$ or: $0.85 < s^2/\sigma^2 < 1.16$. Alternatively, this range can be also stated as $0.86 < \sigma^2/s^2 < 1.18$.

employ 10,000 random draws of 312 returns. We find that the sampling error is larger than the adjustment required for *all* 10,000 draws. The average sampling error distance across the 10,000 draws is 0.0766, and the standard deviation is 0.0112. In comparison, the adjustment distance is only 0.0447.

Figure 1 shows the efficient frontier based on the adjusted parameters, and the segment of efficient portfolios that have all positive weights. These results show that even a small adjustment to the sample parameters yields an efficient frontier with a rather substantial segment of positive portfolios.

(Please insert Table 1 and Figure 1 about here)

Table 2 and Figure 2 show the corresponding results for the case of the equal weighted portfolio (i.e. $\omega_i = 1/N$ for all stocks in eq.(2)). The results are similar, and again all of the 200 parameters are within their estimation error bounds. The multivariate bootstrap analysis confirms this: in all 10,000 bootstrap draws the sampling error is larger than the adjustment required. Figure 2 reveals that in this case the segment of positive efficient portfolios is even larger than in the value weighted case. This is not very surprising, as the optimal portfolio weights on the efficient frontier are continuous functions of the zero-beta rate employed, and the equal-weighted case represents the case where all weights are "as far as possible" from zero.

(Please insert Table 2 and Figure 2 about here)

The results above are for one particular random sample of 100 stocks. However, they are very typical. When we repeat the analysis for different draws of 100 stocks we find very similar results. For 100 draws of 100 stocks we have a total of 20,000 parameters (100 sets $\times$ (100 $\mu$ 's +100 $\sigma$ 's)). Of these, we find that the adjusted parameter is outside of the 95% confidence interval of its sample counterpart in only 0.85% of the cases. This again indicates that the adjustments required are typically much smaller than the estimation error.

One may suspect that the existence of a near-by parameter set leading to a positive portfolio segment depends crucially on the number of assets. After all, B&L's results are for the limit of $N \rightarrow \infty$. In order to examine this we repeat the analysis above for the case of 50 stocks and for the case of 150 stocks. In each case we draw 100 random sets of stocks. In the case of 50 stocks, we find that 2.76% of the parameters are outside of the 95% confidence interval of their sample counterparts. In the case of 150 stocks the percentage goes down to 0.39%. Thus, if anything, it seems that finding a near-by parameter set that ensures a positive portfolio segment on the efficient frontier becomes *easier* when the number of assets increases.

## 4. Conclusion

Brennan and Lo (2010, B&L) provide a beautiful and powerful proof showing that the probability of drawing parameters that lead to the existence of even a single positive portfolio on the efficient frontier approaches zero as the number of assets increases. They show that this is also the case empirically, regardless of the sampling specifics. These are strong results, which seems to imply that the CAPM equilibrium cannot possibly hold when there are many assets.

However, this is only one part of the story. The other part is that even though parameter sets leading to possible frontiers occupy a zero-measure of the parameter space, there is always one nearby. We demonstrate this with the same empirical data employed by B&L. Like them, we find that the sample parameters lead to an impossible frontier. But we show that a slight modification of the parameters, well within their estimation error bounds, leads to a segment of positive portfolios on the frontier. Moreover, this segment can be quite large. Thus, the sample parameters are perfectly consistent with a *possible* frontier.

It may be instructive to think of the situation from an equilibrium perspective. Imagine first that a covariance matrix and vector of expected returns are randomly selected a la B&L. As they show, for these parameters there generally will be no positive efficient portfolio, but, as shown here, there will be a positive efficient portfolio for points “nearby” in parameter space. With the B&L random parameters, the mean/variance optimizing representative agent will want to short some assets and would not want to hold the aggregate portfolio. Yet the aggregate portfolio *must* be held because assets exist in positive net supply.

In attempting to short, prices of the assets being shorted will fall. Similarly, allocating more than 100% of wealth to the non-shorted assets will cause their prices to rise. This implies that expected returns (and possibly covariances) will change from the original B&L random values. Clearly, there can be no mean/variance economic equilibrium until prices move to the point that the representative agent will be satisfied holding positive positions in all existing assets; i.e., the equilibrium will be the standard CAPM frontier with at least some totally positive portfolios. We show that prices would usually not have to move all that much to achieve this equilibrium.

In summary, while B&L’s results are insightful and perfectly correct, they do not at all imply that the sample parameters are inconsistent with positive efficient portfolios. In fact, the opposite is true. So don’t bury the CAPM just yet.

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## Appendix – Details of the Bootstrap Procedure

To carry out the bootstrap, we first adjust the empirical TxN return matrix (T monthly returns for N stocks) to create a "true" return matrix with parameters $\mu^*$ and $\sigma^*$. Then, we resample randomly from this return matrix and calculate the parameters ($\mu^{BS}, \sigma^{BS}$) obtained in each random draw of T periods. For each draw, a "distance" is calculated between ($\mu^{BS}, \sigma^{BS}$) and ($\mu^*, \sigma^*$) and compared with the distance between ($\mu^{sam}, \sigma^{sam}$) and ($\mu^*, \sigma^*$). If the bootstrap distance exceeds the original sample distance in a large fraction of cases, one can conclude that the sample and adjusted parameters are reasonably close.

Below are the step by step details:

1. The sample returns, $r_{i,t}$, are adjusted to create returns with the desired parameters, ($\mu^*, \sigma^*$), by the simple linear transformation $r^*_{i,t} = a_i + b_i r_{i,t}$, with $b_i = \sigma^* / \sigma^{sam}$ and $a_i = \mu^* - b_i \mu^{sam}$. (Obviously, the correlations are unaltered.) The adjusted returns are arranged in a matrix with T columns and N rows.
2. From this (T X N) matrix, T columns are drawn randomly with replacement, thus maintaining the underlying cross-sectional dependence, and ($\mu^{BS}, \sigma^{BS}$) are computed for this (re-)sample. The returns are assumed to be independent over time.
3. The "distance" between the sample parameters ($\mu^{BS}, \sigma^{BS}$) and the true parameters ($\mu^*, \sigma^*$) is computed as the simple Euclidean distance:

   $$d \equiv \sqrt{\sum_{i=1}^{N} \left(\mu_i^{BS} - \mu_i^*\right)^2 + \sum_{i=1}^{N} \left(\sigma_i^{BS} - \sigma_i^*\right)^2}.$$

   One could employ various other more sophisticated distance measures (e.g. the distance D in eq. (1)). The results described in the text are very strong, and they are robust to the distance measure employed. Obviously, we employ the same measure d for the distance between ($\mu^{sam}, \sigma^{sam}$) and ($\mu^*, \sigma^*$) and between ($\mu^{BS}, \sigma^{BS}$) and ($\mu^*, \sigma^*$).

4. This distance is compared with the corresponding distance between the parameters ($\mu^{sam}, \sigma^{sam}$) and ($\mu^*, \sigma^*$).

**Table 1**

**The Sample Parameters and Closest Adjusted Parameters Ensuring that the Value-Weighted Portfolio is Mean-Variance Efficient**

The sample parameters are shown in columns (2) and (4). The adjusted parameters that are closest to the sample parameters and ensure that the value-weighted portfolio is efficient are given in columns (3) and (5). For the sake of brevity, the table reports only the first 20 of the 100 stocks (the complete table is available at the supplementary materials section). The t-values for the expected returns are given in column (6), which shows that none of these values are significant at the 95% level (this is true for all 100 stocks). Column (7) reports the ratio between the optimized variances $\left(\sigma^*\right)^2$ and the sample variances. The 95% confidence interval for this ratio is [0.86-1.18] (see footnote 3). All of the ratios in the table, as well as the ratios for all other 80 stocks not shown here, fall well within this interval. These results are obtained with a value of $\alpha = 0.97$ in the minimized distance measure D (see eq.(1)). Higher values of $\alpha$ reduce the variation in the expected returns (at the expense of increasing the deviations in the standard deviations).

| (1) Stock # (*i*) | (2) $\mu_i^{sam}$ | (3) $\mu_i^*$ | (4) $\sigma_i^{sam}$ | (5) $\sigma_i^*$ | (6) t-value for $\mu_i^*$ | (7) $\left(\sigma_i^*\right)^2 / \left(\sigma_i^{sam}\right)^2$ (the 95% confidence interval for this value is [0.86-1.18] ) |
|---|---|---|---|---|---|---|
| 1 | 0.0117 | 0.0145 | 0.0862 | 0.0806 | 0.3541 | 0.9349 |
| 2 | 0.0150 | 0.0140 | 0.0693 | 0.0703 | -0.1466 | 1.0155 |
| 3 | 0.0120 | 0.0141 | 0.0811 | 0.0788 | 0.2795 | 0.9716 |
| 4 | 0.0134 | 0.0141 | 0.0971 | 0.0954 | 0.0768 | 0.9835 |
| 5 | 0.0184 | 0.0152 | 0.1088 | 0.1108 | -0.3190 | 1.0185 |
| 6 | 0.0078 | 0.0153 | 0.1487 | 0.1393 | 0.5438 | 0.9367 |
| 7 | 0.0151 | 0.0135 | 0.0650 | 0.0666 | -0.2707 | 1.0244 |
| 8 | 0.0137 | 0.0128 | 0.0520 | 0.0527 | -0.1865 | 1.0126 |
| 9 | 0.0120 | 0.0128 | 0.0497 | 0.0494 | 0.1754 | 0.9937 |
| 10 | 0.0115 | 0.0141 | 0.0685 | 0.0653 | 0.4086 | 0.9538 |
| 11 | 0.0072 | 0.0136 | 0.0758 | 0.0706 | 0.9262 | 0.9322 |
| 12 | 0.0143 | 0.0142 | 0.0713 | 0.0706 | -0.0144 | 0.9911 |
| 13 | 0.0142 | 0.0133 | 0.0502 | 0.0507 | -0.1903 | 1.0098 |
| 14 | 0.0153 | 0.0140 | 0.0821 | 0.0829 | -0.1777 | 1.0093 |
| 15 | 0.0158 | 0.0139 | 0.0641 | 0.0657 | -0.3127 | 1.0249 |
| 16 | 0.0132 | 0.0151 | 0.0977 | 0.0950 | 0.2158 | 0.9723 |
| 17 | 0.0146 | 0.0150 | 0.0929 | 0.0917 | 0.0445 | 0.9869 |
| 18 | 0.0106 | 0.0137 | 0.0805 | 0.0780 | 0.4176 | 0.9685 |
| 19 | 0.0011 | 0.0150 | 0.1396 | 0.1270 | 1.0851 | 0.9100 |
| 20 | 0.0102 | 0.0147 | 0.1099 | 0.1050 | 0.4430 | 0.9557 |

**Table 2**

**The Sample Parameters and Closest Adjusted Parameters Ensuring that the Equal-Weighted Portfolio is Mean-Variance Efficient**

The sample parameters are shown in columns (2) and (4). The adjusted parameters that are closest to the sample parameters and ensure that the equal-weighted portfolio is efficient are given in columns (3) and (5). For the sake of brevity, the table reports only the first 20 of the 100 stocks (the complete table is available at the supplementary materials section). The t-values for the expected returns are given in column (6), which shows that none of these values are significant at the 95% level (this is true for all 100 stocks). Column (7) reports the ratio between the optimized variances $\left(\sigma^*\right)^2$ and the sample variances. The 95% confidence interval for this ratio is [0.86-1.18] (see footnote 3). All of the ratios in the table, as well as the ratios for all other 80 stocks not shown here, fall well within this interval.

| (1) Stock # (*i*) | (2) $\mu_i^{sam}$ | (3) $\mu_i^*$ | (4) $\sigma_i^{sam}$ | (5) $\sigma_i^*$ | (6) t-value for $\mu_i^*$ | (7) $\left(\sigma_i^*\right)^2 / \left(\sigma_i^{sam}\right)^2$ (the 95% confidence interval for this value is [0.86-1.18] ) |
|---|---|---|---|---|---|---|
| 1 | 0.0117 | 0.0143 | 0.0862 | 0.0836 | 0.3249 | 0.9697 |
| 2 | 0.0150 | 0.0138 | 0.0693 | 0.0702 | -0.1761 | 1.0138 |
| 3 | 0.0120 | 0.0140 | 0.0811 | 0.0794 | 0.2621 | 0.9797 |
| 4 | 0.0134 | 0.0139 | 0.0971 | 0.0965 | 0.0545 | 0.9938 |
| 5 | 0.0184 | 0.0149 | 0.1088 | 0.1114 | -0.3465 | 1.0243 |
| 6 | 0.0078 | 0.0148 | 0.1487 | 0.1433 | 0.5116 | 0.9634 |
| 7 | 0.0151 | 0.0135 | 0.0650 | 0.0663 | -0.2725 | 1.0197 |
| 8 | 0.0137 | 0.0128 | 0.0520 | 0.0525 | -0.1806 | 1.0090 |
| 9 | 0.0120 | 0.0128 | 0.0497 | 0.0493 | 0.1836 | 0.9920 |
| 10 | 0.0115 | 0.0140 | 0.0685 | 0.0659 | 0.3992 | 0.9618 |
| 11 | 0.0072 | 0.0135 | 0.0758 | 0.0716 | 0.9073 | 0.9447 |
| 12 | 0.0143 | 0.0142 | 0.0713 | 0.0711 | -0.0259 | 0.9981 |
| 13 | 0.0142 | 0.0133 | 0.0502 | 0.0508 | -0.1968 | 1.0123 |
| 14 | 0.0153 | 0.0139 | 0.0821 | 0.0830 | -0.1843 | 1.0108 |
| 15 | 0.0158 | 0.0139 | 0.0641 | 0.0657 | -0.3139 | 1.0255 |
| 16 | 0.0132 | 0.0150 | 0.0977 | 0.0955 | 0.2063 | 0.9772 |
| 17 | 0.0146 | 0.0149 | 0.0929 | 0.0921 | 0.0326 | 0.9914 |
| 18 | 0.0106 | 0.0136 | 0.0805 | 0.0784 | 0.4019 | 0.9734 |
| 19 | 0.0011 | 0.0149 | 0.1396 | 0.1285 | 1.0732 | 0.9207 |
| 20 | 0.0102 | 0.0146 | 0.1099 | 0.1054 | 0.4373 | 0.9597 |

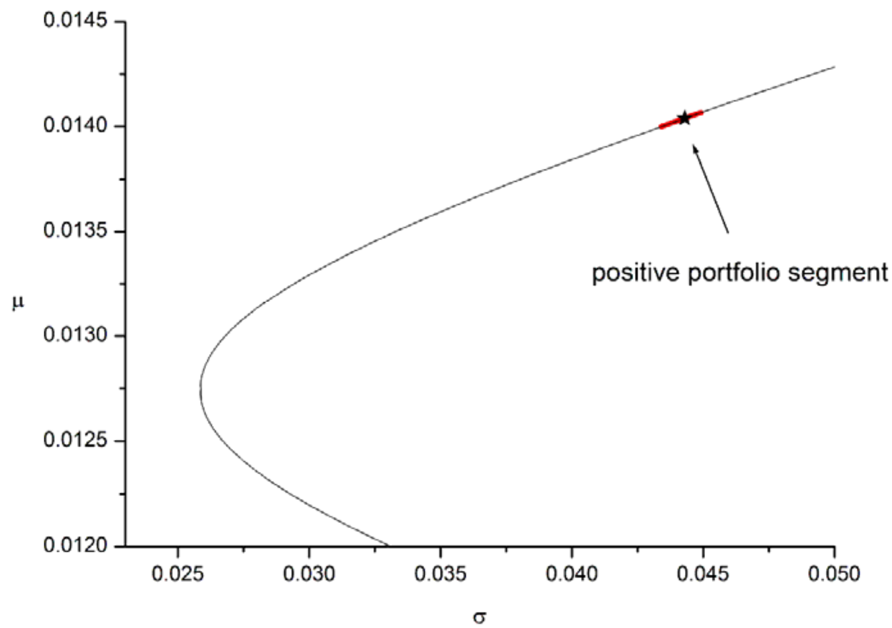


**Figure 1**

The efficient frontier derived from the adjusted parameters that ensure that the value-weighted portfolio is mean-variance efficient (the adjusted parameters in Table 1). The bold segment is the segment of all-positive efficient portfolios. The star marks the value-weighted portfolio.

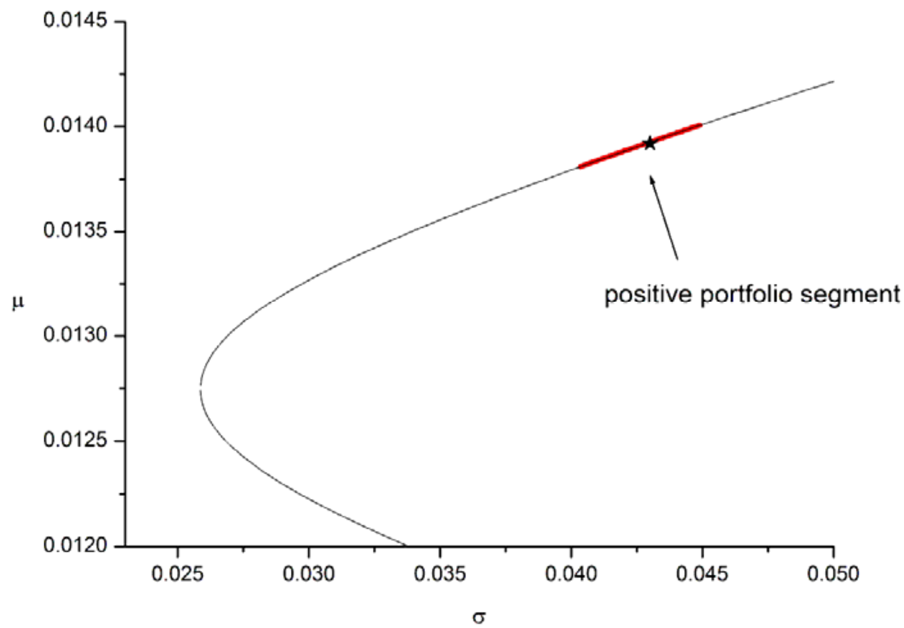


**Figure 2**

The efficient frontier derived from the adjusted parameters that ensure that the equal-weighted portfolio is mean-variance efficient (the adjusted parameters in Table 2). The bold segment is the segment of all-positive efficient portfolios. The star marks the equal-weighted portfolio.