

Decentralized Exchange¹

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Abstract Assets are increasingly traded at new types of exchanges that operate along with traditional, centralized trading venues. We develop an equilibrium model of decentralized markets that accommodates any coexisting exchanges including graphs and more general, common network market structures represented by hypergraphs. We identify gains to trade in decentralized markets which have no centralized market counterparts. An intermediated market can give rise to higher welfare in Pareto sense than a centralized market with the same traders and assets. This can happen only in a market with a core-periphery structure. Changes in market structure that reduce liquidity may increase utility of every agent.

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1 Introduction

In the classical theory, markets are centralized. All units are exchanged through a single market clearing at terms of trade that apply to all agents equally. In today's markets, many types of assets are traded increasingly, or exclusively *over the counter* (OTC). Trade away from centralized exchanges is common not only for assets and goods with units as heterogeneous as real estate, but also for homogeneous assets. Indeed, most bonds (government, municipal, and corporate) are traded over the counter, as are foreign exchange, loans, and more recently stocks.⁴ Over the past decade and a half, a new type of markets have also

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⁴ In the U.S. equity market, the NYSE executes less than a quarter of the volume in its listed stocks; the remaining volume is created in over 10 public exchanges, more than 30 liquidity pools, and over 200 broker-dealers. Over the past few years, trading in private exchanges (liquidity pools) has grown by more than 50% in the U.S., and more than doubled in Europe (Schapiro (2010)). Similarly, while prior to 2007, equity markets in Europe were characterized by dominant exchanges in each domestic market, a regulatory reform of MiFID in 2007 created more than 200 new trading venues in which equities, bonds, and even derivatives are traded. As of 2012, these alternative venues accounted for at least 30% of total equity turnover. In the

emerged online that offer different types of market clearing – direct matching with an intermediary, trading in a dealer network, or an electronic centralized exchange – to different types of traders, institutional and retail.⁵ This paper examines the potential for market decentralization to improve efficiency. What are the economic mechanisms in decentralized markets that do not have centralized market counterparts?

The growing literature on decentralized trading has emphasized frictions due to decentralization of trade – associated with search, information, or counterparty risk. Since ‘decentralization’ is introduced as a friction, it takes away from welfare. We consider trading without exogenous frictions with an arbitrary number of strategic traders and divisible assets. The sole assumption of the centralized market model which we relax is that a single market clearing determines all agents’ allocations. Namely, the *market* consists of *exchanges*, each defined by the subset of agents who trade there and the subset of assets traded. Thus, a market is *decentralized* if there are multiple market clearing mechanisms (exchanges) for a given set of traders and assets. A market corresponds to a hypergraph⁶ and accommodates essentially arbitrary market structures with coexisting exchanges, including centralized markets, networked markets on graphs, and empirically common market structures which cannot be studied as graphs without loss.⁷ Description of preferences and assets is the standard setting with CARA utilities and Gaussian payoffs. Gains to trade come from risk sharing: endowments and preferences (risk aversion) are heterogeneous and are agents’ *private information*.⁸ We assume that each exchange operates as the (uniform-price) double auction. Thus, the model is a decentralized market counterpart of the double auction models with divisible assets in the tradition of Kyle (1989), Vives (2011), and the CAPM. This permits a direct comparison of the predictions for centralized and decentralized markets.

MAIN RESULTS. Why might one expect a decentralized market to create gains to trade?

U.S. bond market, the outstanding debt nearly tripled over the last two decades, reaching approximately \$3.7 trillion in 2012 (*SIFMA, U.S. Bond Market Statistics, June 2012*). Duffie (2012) provides an overview.

⁵ E.g., BondDesk, BrokerTec, eSpeed, MarketAxess, and TradeWeb.

⁶ A hypergraph generalizes a graph by allowing an edge to connect any number of nodes, not just two (see Berge (1973)). Agents can participate in many different types of trading venues for possibly non-disjoint subsets of traders (such as a centralized exchange, a dealer network, liquidity pools), some of which may involve multilateral clearing. Nodes may reside on many graphs (e.g., traders may participate in different types of trading venues), and edges may originate and point to different graphs (e.g., there can be intermediaries between different types of trading venues).

⁷ E.g., consider any trading network whose participants also have access to private exchanges (e.g., dealer-intermediated platforms) or a public, centralized exchange. This simple and common market structure cannot be studied as a graph without loss of generality. A data set that only accounts for pairwise interactions when groups may interact as well will, in general, introduce biases in inference and may lead to artefacts in the data in the study of topological properties such as connectivity. Hypergraph representations of markets that are complex networks allow the study of interactions among groups of traders and aspects other than the traders, such as assets.

⁸ With divisible goods, equilibrium is invariant to the distribution of private uncertainty. By the Myerson-Satterthwaite theorem, with two-sided private information, there is no mechanism that can attain an efficient allocation.

In markets with strategic traders, the Pareto efficiency of allocation of the First Welfare Theorem does not apply. As we show, in general, traders in decentralized markets have strictly positive price impact in the exchanges in which they participate, even if the total number of traders is large. In centralized markets, equilibrium price impacts of all traders are always *proportional* to the primitive asset covariance; hence is the strategic demand reduction in response to price impact. With decentralized trading, this is not the case. In fact, we show that equilibrium price impacts are always *concave* in the to the primitive asset covariance. This affects the extent to which gains to trade are exhausted. In particular, by contrast to a centralized market, agents who trade assets with more risky payoffs do *not* necessarily reduce their orders more and changes in market structure can increase agents’ utility from *both* idiosyncratic and diversified (aggregate) risk. We identify a weaker than proportionality structural property of equilibrium to be the key to why decentralized and centralized markets equilibria differ – commutativity – whose absence in decentralized markets in general implies that a reduction in liquidity for all traders may make allocation of some types of risk more efficient, improving overall efficiency.⁹

We develop three main sets of results. First, we establish a general complementarity result for price impacts in inter-connected (via traders and assets) markets: Any trader’s price impact is lower than it would be if exchanges traded in isolation. In addition, any change in the market structure that lowers price impact ‘locally’ in an exchange lowers price impact in other exchanges, even if only indirectly connected. In any decentralized market, creating a new exchange (weakly) lowers the price impacts in all exchanges, whereas breaking up an exchange increases price impact – a trader’s price impact in any exchange is *decreasing in the inclusion* of traders and assets and, thus the lowest in a centralized market. Thus, creating private exchanges separately from public ones improves the liquidity in all exchanges.¹⁰

⁹ Matrices A and B commute if $AB = BA$. Diagonalizable matrices A and B commute if, and only if, they are simultaneously diagonalizable (e.g., Horn and Johnson (2013), Theorem 1.3.12). We show that in centralized markets, equilibrium price impacts always commute (across agents and with the covariance matrix). Intuitively, commutativity captures a certain symmetry of centralized market equilibria that naturally does not hold in decentralized markets. The riskiness of assets in any given exchange is the *residual* riskiness, as determined by the traders’ participation in other exchanges. As a result, in equilibrium, agents trade as if the riskiness of assets differed from the primitive payoff covariance and were heterogeneous across agents, even if information is complete and all primitives of the market, including the distribution of assets’ payoffs, are commonly known. In particular, payoff complementarity or substitutability of the same assets traded in different exchanges is endogenous. We show that the commutativity of equilibrium is nongeneric in decentralized markets.

¹⁰ “The competition induced by the proliferation of alternative trading venues such as dark pools has been a mostly positive development, as bid-offer spreads and trading costs have fallen” (*Financial Times*, Jan 6, 2013, Rhodri Preece, Director of Capital Markets Policy at CFA Institute). Competition among exchanges has substantially decreased liquidity costs in many markets. For example, in the U.S. stock market, after electronic trading platforms decreased the liquidity costs of trading NASDAQ stocks, the SEC adopted Regulation National Market System (NMS) in 2005, which cleared regulatory impediments to electronic trading, further increasing competition among exchanges. Similar decreases in trading costs occurred in

However, no general link exists between equilibrium price impact and utility in decentralized markets. While a new exchange always lowers price impact, it may improve or reduce total welfare. We show that equilibrium utility in a decentralized market can be higher in *Pareto* sense, compared to the centralized market with the same traders and assets, if cash transfers are allowed. The main underlying mechanism is an increase in illiquidity due to decentralization of trade, which gives rise to a non-linear change in the equilibrium compensation for risk, driving the market risk premium and the allocation closer to the efficient benchmark. Because of this liquidity-based pecuniary externality, simply splitting traders into groups to create multiple disjoint exchanges may be Pareto improving. Introducing intermediation between these exchanges may increase welfare even more.

We develop several implications of the non-trivial in decentralized markets relation between market structure and welfare: Breaking up an exchange may increase utility of every trader, while it increases all traders' price impacts; restricting participation may increase welfare;¹¹ moving an asset from OTC to centralized clearing may lower welfare; intermediaries may increase welfare; financial innovation may reduce welfare. The key to these results are (i) the concavity of equilibrium price impacts in the primitive covariance matrix and (ii) the non-commutativity of equilibrium demand slopes across agents. Intuitively, removing an agent from an exchange corresponds to a decrease in the markets' risk bearing capacity. The concavity implies that this decrease is tempered in a decentralized market structure and, hence, is less adverse than if trading were centralized. In turn, the non-commutativity of the slopes of demand schedules in decentralized market implies that equilibrium allocation may actually be closer to the frictionless efficient outcome than the allocation in the more liquid centralized market with the same traders and assets. Indeed, for some initial allocations of risks, non-commuting demand schedules effectively "rotate" initial holdings into different directions: the liquidity-efficient (agent specific, determined by the equilibrium price impact matrix) and the risk-efficient (determined by the common variance-covariance matrix) portfolios. In equilibrium, this may lead to a better alignment of risk exposures and liquidity needs across traders. The link between the market structure and utility contribution of the type of risk taken (aggregate vs idiosyncratic) points to profitability of distinct types of intermediation, such as brokers (who do not trade on their own account) or dealers (who do), specialist intermediaries (who trade a particular asset) or non-specialists.

Finally, while our analysis takes the market structure as given, the model predicts a

Canada, Europe, and Asia, where different regulatory environments allowed electronic exchanges to develop earlier than those in the United States (Knight Capital Group (2010); Angel, Harris and Spatt (2011); and O'Hara and Ye (2011)).

¹¹ In practice, markets are decentralized in our sense simply because different participants trade different assets – by choice or regulation. Even large financial institutions typically participate in only a few trading venues and trade a small subset of existing securities; e.g., pension funds cannot trade many types of derivatives and most hedge funds have a clear specialization in trading a limited number of securities.

general topological property of *equilibrium* that captures how the interconnectedness among exchanges – via traders or assets – affects prices, allocations and price impacts in a decentralized market. We show that, in a decentralized market with any set of exchanges, equilibrium is *as if* the market structure were a *forest* (in graph-theoretic terms, a disjoint union of trees).¹² In general, the same assets may trade at different prices in different exchanges. We characterize the type of interconnectedness among exchanges for an asset to trade at the same price in different exchanges – as a necessary and sufficient condition on the market structure. We also characterize the type of interconnectedness with which decentralized markets behave essentially as centralized markets. We show that its absence gives rise to a ‘local monopoly power,’ or various types of intermediation.

RELATED LITERATURE. Our model is part of the growing literature on decentralized markets. Most modern models are based on graphs, random or fixed. One approach assumes that trade occurs through random matching in large markets among a continuum of non-strategic traders.¹³ Empirically, while some markets are best described by random meetings, in others (e.g., dealer networks or interbank systems), relationships are not random,¹⁴ and dealing with strategic behavior and price impact often serves as a primary motivation to create an OTC exchange.¹⁵ We consider markets with any number of traders, all of whom are strategic, and are thus closer to the strand of literature that views agents as interacting on a fixed network.¹⁶ Both types of models typically consider markets in which all transactions are bilateral (graphs),¹⁷ while we study networked market structures with coexisting

¹² Forests encompass markets with various forms of intermediation, including brokers, dealers, or specialists who trade different assets as well as non-specialist intermediaries, and the empirically common core-periphery and hub-and-spoke architectures (e.g., the U.S. Federal Funds Market, the interbank markets, the U.S. municipal bonds market; see Bech and Atalay (2010); Craig and Peter (2010); Afonso, Kovner, and Schoar (2012)).

¹³ E.g., Gale (1986a,b); Duffie, Garleanu and Pedersen (2005, 2007); Weill (2008); Vayanos and Weill (2008); Duffie, Malamud and Manso (2009, 2013); Lagos and Rocheteau (2009); Lagos, Rocheteau and Weill (2011); Afonso and Lagos (2012).

¹⁴ It is well documented that dealers or brokers trade via an established network structure and trading relationships exist between banks. An average bank trades with a small number of counterparties and most banks form stable relationships with at least one lending counterparty; e.g., the U.S. Federal Funds market (Bech and Atalay (2010); Afonso, Kovner, and Schoar (2012)), interbank markets (Craig and Peter (2010); Cocco, Gomes and Martins (2009)), and the U.S. municipal bonds market (Li and Schürhoff (2012)).

¹⁵ E.g., Biais, Bisière, and Spatt (2010), Knight Capital Group (2010), Angel, Harris and Spatt (2011). Biais and Green (2007) attribute the historical shift in the U.S. municipal and corporate bond markets toward OTC trading to the increased importance of institutional investors.

¹⁶ E.g., Kranton and Minehart (2001); Gale and Kariv (2007); Blume, Easley, Kleinberg and Tardos (2009); Manea (2011); Nava (2011); Condorelli and Galeotti (2012); Elliott (2012); Fainmesser (2012); Babus and Kondor (2013); Bramoullé, Kranton, and D’Amours (2013); Rahi and Zigrand (2013); Glode and Opp (2013).

¹⁷ Corominas-Bosch (2004) and Elliott (2011) allow for multilateral bargaining with search. Some models (see Duffie, Garleanu and Pedersen (2005); Lagos and Rocheteau (2009); and Lagos, Rocheteau and Weill (2011)) assume that trade can only happen through special intermediaries (dealers) who provide liquidity. Rahi and Zigrand (2013) study trade of price-taking investors intermediated by arbitrageurs.

exchanges that are hypergraphs.

In addition, both types of models typically derive the terms of trade from bargaining (e.g., take-it-or-leave-it offers) or fixed prices.¹⁸ In our model, prices are determined by the standard divisible-good double auction protocol in which agents submit demand or supply schedules (or limit and market orders) in the exchanges where they participate. Auctions are increasingly used in OTC markets. Traditionally, trading over the counter involved bilateral market clearing with a dealer over the phone. During the past two decades, large volumes of transactions that occur outside open exchanges have shifted to new electronic auctions, which have been introduced for large institutional traders, dealers, and retail investors. The ongoing transition in the market clearing itself is sometimes called “call to electronic.”¹⁹ As we show, in a market for divisible assets, traders have nonnegligible price impact – the behavior of prices and allocations is different than in trading environments with efficient surplus sharing.

TECHNIQUES. The methods we introduce will be useful to other researchers studying games on networks. To the best of our knowledge, this paper is the first analysis of the class of games on hypergraphs. Relative to the literature on games on networks, the games we study allow applications with strategies that have multiple dimensions, applications in which static equilibria exhibit dependence on the entire network not only direct neighbors,²⁰ and have equilibrium strategies that are nonlinear in the actions of others. The techniques we introduce extend to other contexts described as complex networks with nonlinear effects, where the methods useful for linear equilibrium strategies are not applicable and summary statistics, such as the smallest eigenvalue (e.g., Bramoullé, Kranton, and D’Amours (2013); the *Handbook* chapter by Jackson and Zenou (2012)) are not sufficient to capture the properties of the network. We identify the appropriate counterpart – the commutativity – and show its key role in the new in decentralized markets economic implications.

Second, we characterize the comparative statics of equilibrium and welfare with respect to preferences, assets, and market structure (the hypergraph). For comparative statics in centralized markets, lattice theoretic arguments can be invoked. As we show, decentralized market games do not have the lattice structure, because of the non-comutativity of equilib-

¹⁸ Like this paper, Babus and Kondor (2013) use a double auction. Traditional network optimization focuses on a network populated by nonstrategic (“obedient”) users; more recently, pricing (e.g., Kranton and Minehart (2001); Blume, Easley, Kleinberg, and Tardos (2009); Nava (2011); Jackson and Zenou (2012); and Babus and Kondor (2013)), or bargaining on networks, has been examined.

¹⁹ E.g., Hendershott and Madhavan (2012) examine the changes in the U.S. corporate bonds market. Innovations in trading technology allow traders to engage in multilateral trading as opposed to sequentially contacting dealers.

²⁰ This feature, due to the divisibility of the asset, is common in the network models with continuous actions (e.g., survey by Jackson and Zenou (2012)). In models of decentralized market for indivisible goods, equilibrium prices and allocations depend on the network through *dynamic* trading.

rium.²¹ Using the concavity and monotonicity of the best responses, we develop monotone comparative statics of equilibrium and welfare for decentralized markets with arbitrary market structures, multiple assets, and any number of strategic agents.

STRUCTURE OF THE PAPER. Section 2 presents the model of decentralized markets. Section 2.5 characterizes equilibrium. Section 3 introduces an equivalent characterization of equilibrium. Section 4 studies equilibrium liquidity, allocations and prices. Section 5 derives the comparative statics of equilibrium liquidity. Section 6 characterizes risk sharing and welfare in decentralized markets. Section 7 concludes. All proofs appear in the Appendix.

2 A Decentralized Market Model

2.1 Notation

All vectors are column vectors. A^T denotes the transpose of matrix A . The same symbol is used for a set and the number of its elements. \oplus is the direct sum. For any vector $q \in \mathbb{R}^N$, let $q_{N(i)} \in \mathbb{R}^{N(i)}$ denote its restriction to the vector's elements in $\mathbb{R}^{N(i)}$. We denote by $A_{N(i)}$ the sub-matrix of A with rows and columns in $N(i)$. For symmetric matrices A and B , we write $A \geq B$ if $A - B$ is positive semidefinite, and $A > B$ if $A - B$ is positive definite.

2.2 Decentralized Market Setting

MARKET: TRADERS, ASSETS AND EXCHANGES. Consider a market with I traders who trade K risky assets in N exchanges, each with a separate clearing price. We index agents by i , assets by k , and exchanges by n . An *exchange* $n \in N$ is identified by the subset of agents $I(n) \subseteq I$ who trade there and the subset of assets traded $K(n) \subseteq K$. The set of exchanges $\{(I(n), K(n))\}_n$, which we take as a primitive, represents the *market structure* and together with the set of agents I and assets K corresponds to a hypergraph.²² We assume that at

²¹ This involves comparing (endogeneous) positive semidefinite matrices defined on different assets and agents, as determined by the market structure (the hypergraph). Absent commutativity, the order is incomplete (and, hence, the set of the symmetric matrices to-be-compared is not a lattice) and agents' payoffs are not supermodular in general even when matrices can be ranked. While the standard arguments that derive from Tarski's fixed point theorem do not apply, monotone comparative statics can be characterized with our results: the uniqueness of the maximal and minimal equilibria (applying the logic analogous to Milgrom and Roberts (1990)); the positive semidefinite property of equilibrium price impacts (which gives the notion of monotonicity in the comparative statics); the monotonicity and convexity of the (matrix) harmonic mean which naturally arises in aggregation. For the comparative statics in centralized markets, lattice theoretic arguments can be invoked

²² A hypergraph is defined as a pair (X, E) , where X is a set of elements called *nodes* and E is a set of nonempty subsets of X called (*hyper-*)*edges*. Thus, E is a nonempty subset of the power set of X . In our model, $X = (I, K)$ and an edge $(I(n), K(n))$ represents exchange n with $I(n)$ agent classes and $K(n)$ assets. The hypergraph generates a graph on the set of N exchanges.

least three agents participate in every exchange, $I(n) > 2$ for all n .²³

The K risky assets have jointly normally distributed payoffs $R \sim \mathcal{N}(d, \Sigma)$ with positive definite covariance Σ ; a riskless asset with a zero interest rate (a numéraire) is also available. In the analysis, we treat assets traded at different exchanges as different assets, regardless of whether the same asset is traded. That is, we treat a market with K assets traded in N exchanges $\{(I(n), K(n))\}_n$ as a market with $\sum_n K(n)$ (replicas of) assets and $(\sum_n K(n)) \times (\sum_n K(n))$ positive semidefinite covariance matrix \mathcal{V} , induced by covariance Σ and the set of exchanges. The *hypergraph on the sets of agents and assets* (I, K) generates a *graph on the set of N exchanges* $\{(I(n), K(n))\}_n$. Thus, \mathcal{V} describes interconnectedness among the exchanges via traders and assets.

Example 1 (i) The centralized market: All I agents trade K assets in one exchange; $N = \{(I, K)\}$, and $\mathcal{V} = \Sigma$.

(ii) Liquidity pools: In addition to the centralized exchange as in (i), there are L exchanges (liquidity pools) in which only subsets of agents can trade. There are L (possibly non-disjoint) subsets $I(1), \dots, I(L) \subset I$ of agents, each trading in exchange $l \in L$ organized as a centralized exchange among agents $i \in I(l)$; $N = \{(I, K)\} \cup \{(I(l), K(l))\}_l$ with $K + \sum_l K(l)$ assets.

Let $N(i) \subseteq N$ denote the subset of exchanges in which trader i participates. Each trader i , endowed with wealth w_i and a vector $q_i^0 \in \mathbb{R}^{N(i)}$ of risky assets of dimension $\sum_{n \in N(i)} K(n)$, maximizes the expected CARA utility

$$E[-e^{-\alpha_i(w_i - q_i^T p_{N(i)} + (q_i^0 + q_i)^T R_{N(i)})}],$$

where α_i is agent i 's absolute risk aversion, $q_i \in \mathbb{R}^{N(i)}$ denotes the vector of risky assets trades of i and $p_{N(i)}$ is the vector of prices. Initial holdings are *private information* with a nondegenerate distribution. Equivalently, trader i maximizes the quasilinear-quadratic utility function

$$U(q_i) = (q_i^0)^T d_{N(i)} + q_i^T (d_{N(i)} - p_{N(i)}) - \frac{\alpha_i}{2} (q_i^0 + q_i)^T \mathcal{V}_{N(i)} (q_i^0 + q_i), \quad (1)$$

where $d_{N(i)}$ is the sub-vector of the expected payoffs d and $\mathcal{V}_{N(i)}$ is the sub-matrix of the covariance matrix \mathcal{V} that correspond to the assets traded by agent i in exchanges $N(i)$. We use $q_i \in \mathbb{R}^N$ and $q_i^0 \in \mathbb{R}^N$ to also denote the vectors $q_i \in \mathbb{R}^{N(i)}$ and $q_i^0 \in \mathbb{R}^{N(i)}$ ‘completed’ by zeros in their $\mathbb{R}^{N \setminus N(i)}$ coordinates. With a “limited participation” interpretation, given

²³ As is well known in centralized markets, with two traders, equilibrium with trade does not exist (e.g., Kyle (1989)).

the agents I and assets K , the market structure $\{(I(n), K(n))\}_n$ is equivalently described by traders' *participation* $\{N(i)\}_i$.

DECENTRALIZED DOUBLE AUCTION. Each exchange n is organized as a centralized market for traders $i \in I(n)$ and operates as the standard uniform-price double auction (e.g., Kyle (1989); Vives (2011); CAPM). Trader i submits a (net) demand schedule $q_i(p_{N(i)}) : \mathbb{R}^{N(i)} \rightarrow \mathbb{R}^{N(i)}$, which specifies demanded quantities of assets in the exchanges in which he participates; the demand is strictly downward-sloping in each exchange.²⁴ The aggregate net demand in exchange n determines the clearing price p_n^* in this exchange, $\sum_{i \in I(n)} q_{i,n}(p_n^*, p_{N(i) \setminus \{n\}}^*) = 0$, and the allocations; trader i receives $q_i \equiv q_i(p_{N(i)}^*)$ and pays $p_{N(i)}^* \cdot q_i$. All traders are strategic; in particular, there are no noise traders.

2.3 Decentralized Market Equilibrium

As is standard in strategic centralized markets models for divisible goods or assets, we study the equilibrium in linear bid schedules (hereafter, *equilibrium*); that is, the linear equilibrium that is robust to adding noise in trade (robust Nash Equilibrium) which, with independent private values about endowments, coincides with the linear Bayesian Nash Equilibrium.²⁵ Theorem 2.1 characterizes equilibrium through two conditions, which correspond to an individual optimization problem and aggregation in decentralized markets.

TRADER OPTIMIZATION. In equilibrium, the (net) demand schedule of trader i equalizes his marginal utility with the marginal payment, for each price,

$$d_{N(i)} - \alpha_i \mathcal{V}_{N(i)}(q_i^0 + q_i) = p_{N(i)} + \Lambda_i q_i, \quad (2)$$

where Λ_i is the $N(i) \times N(i)$ Jacobian matrix of the residual inverse supply of trader i , which is defined through market clearing by the schedules submitted by the traders in exchanges $N(i)$, $\{q_j(p_{N(j)}) : \mathbb{R}^{N(j)} \rightarrow \mathbb{R}^{N(j)}\}_{j \neq i}$. Λ_i measures the *price impact* of trader i in the exchanges in which he participates (i.e., a decentralized market counterpart of 'Kyle's lambda'). Entry (k, l) represents the price change of asset l that results from a marginal increase in demanded

²⁴ This rules out trivial equilibria with no trade.

²⁵ Equilibrium is *linear* if schedules have the functional form of $q_i(\cdot) = \alpha_0 + \alpha_{i,q} q_i^0 + \alpha_{i,p} p$. Trader strategies are not restricted to linear schedules; rather, we analyze equilibria in which it is optimal for a trader to submit a linear demand, given that others do. The approach to analyze the symmetric linear equilibrium is common in the centralized market models (e.g., Kyle (1989), Vayanos (1999), Vives (2011)). Our analysis does not assume equilibrium symmetry.

All results in this paper hold for complete information and incomplete information with independent value endowments q_i^0 (random marginal utility intercepts, $\tilde{d} = d - \alpha \Sigma \tilde{q}_i^0$). Equilibrium schedules are optimal even if the traders learn the types (endowments) of all other agents; that is, equilibrium is *ex post* Bayesian Nash. The key to the *ex post* property is that by permitting pointwise optimization – for each price – the equilibrium demand schedules are optimal for any distribution of independent private information and are independent of agents' expectations about others' endowments or marginal utility intercepts.

quantity of asset k , $\frac{\partial(p_1, \dots, p_{N(i)})}{\partial(q_1, \dots, q_{N(i)})} = \left(\frac{\partial p_l}{\partial q_k} \right)_{k,l}$. It follows from (2) that *if* trader i knew his price impact Λ_i , which is endogenous, he could determine his equilibrium demand by equalizing his marginal utility and marginal payment pointwise: Let $q_i(\cdot, \Lambda_i) : \mathbb{R}^{N(i)} \rightarrow \mathbb{R}^{N(i)}$ be the schedule defined by the pointwise optimization (2) for all prices $p_{N(i)}$, given his assumed price impact Λ_i ²⁶

$$q_i(p_{N(i)}, \Lambda_i) = (\alpha_i \mathcal{V}_{N(i)} + \Lambda_i)^{-1} (d - p_{N(i)} - \alpha_i \mathcal{V}_{N(i)} q_i^0). \quad (3)$$

MARKET CLEARING. Equilibrium price impacts $\{\Lambda_i\}_i$ can now be determined by market clearing, which cannot be written exchange by exchange. Price impacts live in different assets, are of different dimensionality and, in general, are not independent across exchanges. To apply market clearing to all assets in all exchanges, we use the procedure of *lifting*, which restores the common dimensionality. For a given subset $N(i) \subset N$, decompose $\mathbb{R}^N = \mathbb{R}^{N(i)} \oplus \mathbb{R}^{N \setminus N(i)}$ as a direct sum of two subspaces corresponding to coordinates that agent i trades and those that he does not trade, where \mathbb{R}^N is the space of asset holdings of dimension $\sum_{n \in N} K(n)$. Any symmetric matrix A can be decomposed into a block form

$$A = \begin{pmatrix} A_{11} & A_{12} \\ A_{12}^T & A_{22} \end{pmatrix}$$

where $A_{11} = A_{N(i)}$ lives in subspace $\mathbb{R}^{N(i)}$, $A_{22} = A_{N \setminus N(i)}$ lives in the complementary subspace $\mathbb{R}^{N \setminus N(i)}$, and A_{12} is a rectangular block. For any matrix A_{11} living in $\mathbb{R}^{N(i)}$, with an abuse of notation, let \bar{A}_{11} denote the *lifted* matrix, which lives in \mathbb{R}^N and let \bar{A}_{11}^{-1} denote its corresponding inverse,

$$\bar{A}_{11} \equiv \begin{pmatrix} A_{11} & 0 \\ 0 & 0 \end{pmatrix}, \quad \bar{A}_{11}^{-1} \equiv \begin{pmatrix} A_{11}^{-1} & 0 \\ 0 & 0 \end{pmatrix}.$$

EQUILIBRIUM. Treating assets traded at different exchanges as distinct assets and dealing with aggregation through lifting allows us to characterize (linear) equilibria in the general model of decentralized markets by two conditions on schedules and price impacts: (i) each trader submits a schedule that equalizes his marginal utility and the marginal payment given price impact (i.e., submits $q_i(\cdot, \Lambda_i)$) and (ii) the price impact Λ_i in $q_i(\cdot, \Lambda_i)$ is correct (i.e., it equals the slope of the residual supply resulting from the aggregation of other traders' schedules, projected on the assets relevant for trader i).

Theorem 2.1 (Equilibrium Characterization) *A profile $\{q_i(p_{N(i)})\}_i$ is a robust Nash*

²⁶ Section 3 shows that demand (3) is well defined, even with (replicas of) the same assets traded in different exchanges.

Equilibrium in a decentralized market with participation $N(i)\}_i$ if, and only if,

(i) each agent i submits $q_i(p_{N(i)}, \Lambda_i) = (\alpha_i \mathcal{V}_{N(i)} + \Lambda_i)^{-1}(d_{N(i)} - p_{N(i)} - \alpha_i \mathcal{V}_{N(i)} q_i^0)$, and

(ii)

$$\Lambda_i = \left(\left(\sum_{j \neq i} (\alpha_j \bar{\mathcal{V}}_{N(j)} + \bar{\Lambda}_j)^{-1} \right) \right)_{N(i)}^{-1}, \quad i \in I. \quad (4)$$

Although the matrices $\mathcal{V}_{N(j)}$ and Λ_j are for “local” exchanges, when lifted, the same aggregation condition as in a centralized market applies. Theorem 2.1, thus, allows a direct comparison of equilibrium predictions in decentralized and centralized markets.^{27,28}

Let us remark that if a trader knows his own utility, then Theorem 2.1, if he knows his own price impact in the exchanges he participates, the opacity of decentralized markets is without loss of generality for trader optimization and equilibrium in a decentralized market – in trading environments with independent private values. In particular, a trader’s strategy $q_i(\cdot, \Lambda_i)$ would not be altered by knowledge of the market structure $\{N(i)\}_i$, the terms of trade in exchanges $N \setminus N(i)$, or even the submitted schedules or preferences of those traders. Through the fixed-point condition (4), price impact Λ_i is the sufficient statistic for trader i for the optimality of his schedule in exchanges $N(i)$, given the schedules of all traders $j \neq i$ in *all* exchanges $n \in N$ (i.e., $\Lambda_i \in \mathbb{R}^{N(i) \times N(i)}$ is sufficient for $\{q_j(\cdot) : \mathbb{R}^{N(j)} \rightarrow \mathbb{R}^{N(j)}\}_{j \neq i}$).

2.4 Centralized vs Decentralized Markets

Throughout, a centralized market (Example 1 (i)) serves as a benchmark: All traders participate in a single exchange, $N(i) = N = \{(I, K)\}$ and $\mathcal{V}_{N(i)} = \Sigma = \mathcal{V}$, for all $i \in I$.

Proposition 2.1 (Centralized Market Equilibrium) *In a centralized market, equilibrium exists and is unique. Price impact Λ_i of agent i is proportional to the covariance matrix \mathcal{V} ,*

$$\Lambda_i = \beta_i \Sigma \quad (5)$$

²⁷ For centralized markets, the linear equilibrium coincides with Kyle (1989, without nonstrategic traders and assuming independent values), Klemperer and Meyer (1989, for quadratic utilities), Vayanos (1999), Vives (2011), Weretka (2011) who provides a nonstrategic characterization of the equilibrium (i.e., in terms of levels rather than demand schedules) for centralized market general equilibrium settings and Rostek and Weretka (2011) who show the relationship in Theorem 2.1 for centralized market games. The linear-in-trade price impact function is the prevalent assumption in the financial industry (the “quadratic cost model;” see, e.g., Almgren (2009)).

²⁸ The harmonic mean is standard and results from aggregation of individual demands; for instance, in a centralized market with I strategic traders, the price impact of each trader i is $\Lambda_i = \left(\sum_{j \neq i} (\alpha_j \mathcal{V} + \Lambda_j)^{-1} \right)^{-1}$.

with a coefficient β_i that depends on risk aversion α_i , $\beta_i = (2 - \alpha_i b + \sqrt{(\alpha_i b)^2 + 4})/2b$, and $b \in \mathbb{R}_+$ is the unique solution to $\sum_i (2 + \alpha_i b + \sqrt{(\alpha_i b)^2 + 4})^{-1} = 1/2$. Letting $\gamma_i \equiv \frac{\alpha_i}{\alpha_i + \beta_i}$, agent i 's equilibrium risky holdings are

$$q_i = \gamma_i \alpha_i^{-1} q^{Av} + (1 - \gamma_i) q_i^0 \quad (6)$$

where the average portfolio q^{Av} is given by $q^{Av} \equiv b^{-1} \sum_i \frac{\alpha_i}{\alpha_i + \beta_i} q_i^0$.

If $\alpha_i = \alpha$ for all i , then $\gamma = (I-2)/(I-1)$, $\Lambda_i = (1/(I-2))\alpha\mathcal{V}$, and $q_i = \gamma q^{Av} + (1-\gamma)q_i^0$.²⁹

Due to positive price impact, idiosyncratic risk in allocations is not fully diversified. Agents' risky asset holdings comprise a combination of the (per capita) market portfolio q^{Av} and initial endowment q_i^0 . More risk averse agents face more elastic residual supply and submit more competitive schedules; if $\alpha_i > \alpha_j$, then $\beta_i < \beta_j$ and $\alpha_i + \beta_i > \alpha_j + \beta_j$.

With strategic agents, one expects the positive price impact and inefficiency to be present in a decentralized market. Let us motivate the new mechanisms to be explored in our analysis of decentralized trade settings with the following example. We start by noting that, with linear-quadratic preferences, the sum of indirect utilities (1) computed at equilibrium allocation and prices is a natural welfare criterion. Indeed, since utilities are linear in cash (numeraire), we immediately get that there exist deterministic monetary transfers that make one market Pareto dominate the other if and only if the sum of indirect utilities is higher in the corresponding market.

Example 2 (Decentralized Market Welfare Can Be Higher) *Fix an arbitrary centralized market with I classes, class i having M_i traders with risk aversion α_i , and suppose that $M_1 > 2$ and $M_2 > 2$. Then, there exist thresholds $\varepsilon_1 < \varepsilon_2 < \varepsilon_3$ such that:*

- *The total welfare in the market with two exchanges $\{(M_1, \alpha_1)\} \cup \{(M_i, \alpha_i)\}_{i>1}$ is strictly higher than that in the centralized market $\{(M_i, \alpha_i)\}_{i \geq 1}$. Furthermore, there exists initial endowments for which traders $\{(M_i, \alpha_i)\}_{i>1}$ can make a monetary transfer to class $\{(M_1, \alpha_1)\}$ so that all traders are better off in the splitted market than the centralized market.*
- *Making an agent of class 2 an intermediary improves total welfare even further. This intermediary is worse off than his "class-mates" who do not intermediate, and should therefore be compensated for intermediation.*

²⁹ This is the equilibrium in Rostek and Weretka (2011). To the best of our knowledge, the uniqueness of the linear equilibrium for many assets (divisible goods) in Proposition 2.1 and the analytic characterization of equilibrium for asymmetric risk aversion $\{\alpha_i\}$, are both new.

The result of Example 2 is striking. Indeed, in a competitive market, removing a class of agents from the market cannot be Pareto-improving for the remaining agents because the competitive allocation is in the core. It is particularly surprising that the conclusions of Example 2 holds even with an arbitrary large number of traders.³⁰ That is, even for markets that are arbitrarily close to being competitive, decentralization may lead to welfare gains. The intuition behind this surprising result is based on Proposition 2.1: As we can see, the weights γ_i in the market portfolio depend non-linearly on the distribution of risk aversions in the economy. By removing an agent from the market, we distort this distribution and may shift the allocation closer to the efficient (competitive) benchmark.³¹

Let us give a preview of why might equilibria and welfare predictions in decentralized markets be different. In a centralized market, equilibrium price impacts are proportional to Σ (see (5)). This is not the case in generic decentralized markets. Why? We show agents' incentives to trade an asset are determined by the *residual* riskiness of that asset, defined by the traders' participation in other exchanges. Consequently, in equilibrium, agents trade as if the riskiness of assets differed from the primitive payoff covariance Σ and were heterogeneous across agents (Section 5.1). The extent to which a trader reduces his demand in response to price impact is not determined by the primitive asset covariance alone $(\alpha_i^{-1} - (\alpha_i + \beta_i)^{-1})\Sigma^{-1}$, but depends on the assets riskiness and preferences in *all* exchanges $((\alpha_i \mathcal{V}_{N(i)})^{-1} - (\alpha_i \mathcal{V}_{N(i)} + \Lambda_i)^{-1})$. Thus, decentralized trading alters the surplus exhausted in equilibrium by strategic traders. In fact, as we show, with the endogenous effective riskiness, equilibrium price impacts do not commute— across agents and with the covariance matrix.³² Whether changes in the market structure that lower price impact increase equilibrium utility depends on whether agents' endowments are composed of idiosyncratic or diversifiable risk. In particular, equilibrium utility from idiosyncratic or aggregate risk is generally *not* monotone in price impact decentralized markets (Section 6).

The next section gives a game theoretic interpretation of the commutativity of price impacts. Proposition 2.2 shows that commutativity is nongeneric in decentralized markets.

³⁰ With trader subsets interpreted as coalitions, Example 2 implies that the centralized market is not in the core among the market structures defined by any hypergraph and surplus sharing via the uniform-price mechanism. For any risk aversion α_1, α_2 satisfying $\alpha_2 < \varepsilon_1 < \varepsilon_2 < \alpha_1 < \varepsilon_3$, the centralized market structure would be blocked by coalition $\{(M_i, \alpha_i)\}_{i>1}$ which, moreover, can compensate class $\{(M_1, \alpha_1)\}$ so that the market with two exchanges Pareto improves over a centralized one.

³¹We will come back to this point in Example 5.

Recall that inefficient allocation in our model arises due to asymmetric information about each trader's endowment shocks. In light of Example 2, we could interpret monetary transfers as intermediation fees: before the endowment shocks are realized, agents agree on the decentralized market structure that maximizes total welfare. All agents in the market then pay agent 2 a fee so that he agrees to intermediate and also make a transfer to the agents of class 1. Such a market structure Pareto dominates the centralized market.

³² Recall that matrices A and B commute if $AB = BA$. Diagonalizable matrices A and B commute if, and only if, they are simultaneously diagonalizable.

Proposition 2.2 (Commutativity is Nongeneric) *In any decentralized market, for traders i and j , let $N(i) \neq N(j)$ and suppose that the cardinality of $N(i) \cap N(j)$ is strictly greater than 1. Then, for generic covariance Σ and generic risk aversions $\{\alpha_i\}_i$, price impacts $(\Lambda_i)_{N(i) \cap N(j)}$ and $(\Lambda_j)_{N(i) \cap N(j)}$ do not commute.*

2.5 Equilibrium: General Properties

Theorem 2.2 establishes the existence and uniqueness properties of the decentralized markets equilibrium $\{(q_i(\cdot, \Lambda_i), \Lambda_i)\}_i$. Let us first show that, unlike the centralized market game, the decentralized market game is not supermodular in general. The double auction – centralized and decentralized – can be equivalently seen as a game in which agents choose their demand slope (which determines the demand reduction, relative to the competitive schedule), $S_i \equiv (\alpha_i \mathcal{V}_{N(i)} + \Lambda_i)^{-1}$. For centralized markets, the double auction game with strategies $\{S_i\}_i$ is supermodular: This follows from the proportionality of matrices $\{\{\Lambda_i\}_i, \Sigma\}$ (Proposition 2.1) which implies that the set of price impact tuples $\{\Lambda_i\}_i$ is a lattice, and hence so is the set of demand slope tuples $\{S_i\}_i$. Absent commutativity, the set of demand slope tuples (i.e., the set \mathcal{S}^I of tuples of positive semidefinite matrices) is not a lattice in general. By Proposition 2.2, the decentralized-market game is not supermodular.³³ Consequently, Tarski’s fixed point theorem cannot be applied to prove the existence of equilibrium and comparative statics. Nevertheless, one can show that the equilibrium demand slope of each agent is monotone increasing in demand slopes of all other agents in the market (Lemma A.3 in Appendix A) and that the symmetric positive definite order emerges as the relevant order for monotonicity (Theorem 2.2) and demand slopes (Lemma A.4). Standard arguments then imply that equilibrium exists and is locally unique. Theorem 2.2 summarizes these equilibrium properties.

Theorem 2.2 (Equilibrium Existence and Local Uniqueness) *Equilibrium exists and it is locally unique, generically risk aversions and the elements of the covariance matrix. Equilibrium price impacts are positive semidefinite. If the covariance matrix $\mathcal{V}_{N(i)}$ is invertible for any i , the price impacts are positive definite.*

³³ It is generally not possible to define the greatest lower bound and the least upper bound for a bounded set of positive semidefinite matrices. For example, consider $A = \begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix}$, $B = \begin{pmatrix} 1 & 1 \\ 1 & 2 \end{pmatrix}$. Note that both $A \not\geq B$ and $A \not\leq B$ hold, because the positive semidefinite order is incomplete. By definition, matrix $C = \begin{pmatrix} a & c \\ c & b \end{pmatrix}$ is the least upper bound of A and B if $C \geq A$, $C \geq B$, and any other matrix C' satisfying $C' \geq A$, $C' \geq B$ also satisfies $C \geq C'$. However, $C \geq A$, $C \geq B$ is equivalent to $a > 1, b > 2, (a-1)(b-2) \geq \max\{c^2, (c-1)^2\}$. Clearly, one can decrease a and increase b without violating these inequalities, which implies that C cannot be the least upper bound.

Induced by the covariance matrix Σ and participation $\{N(i)\}_i$, matrix $\mathcal{V}_{N(i)}$ can be singular only if agent i can trade the same asset at different exchanges.³⁴ While the lattice structure required for Tarski’s fixed point theorem is absent, we show in the next result that the maximal and minimal fixed points exist.

Proposition 2.3 *For any decentralized market, the set of equilibrium price impact tuples has unique maximal and minimal elements that are given by $\{\Lambda_{i,\max}\}_i$ and $\{\Lambda_{i,\min}\}_i$, respectively.*

By Proposition 2.3, the set of equilibria $\{(q_i(\cdot, \Lambda_i), \Lambda_i)\}_i$ has a unique maximal and a unique minimal element.³⁵ In Appendix B, we give sufficient conditions for the minimal and maximal equilibria to coincide.³⁶ Given that the strategy space is not a lattice, existence of the maximal and the minimal equilibria is not obvious. To tackle the existence, we introduce a technique based on iterative procedures.

3 Equivalence Loops and Regularization

In this section, we show that, in any market described by a complex network (a hypergraph), equilibrium is outcome equivalent to – and, hence, can be analyzed as – a simpler market (Theorem 3.1), using the conditions on the market structure we define next. This result reveals a general topological property of *equilibria* that captures how interconnectedness

³⁴ If matrix $\mathcal{V}_{N(i)}$ is degenerate, then the inverse demand slope $(\alpha_i \mathcal{V}_{N(i)} + \Lambda_i)$ is also degenerate if the rows of Λ_i corresponding to the replica assets are identical. This is the only type of degeneracy that may occur (Corollary B.1 in the Appendix). In Section 3, we show that, in this case, agents are indifferent about which exchange to trade in and (replicas of) assets then trade at the same prices. Equilibrium, defined as $\{\Lambda_i, q_i(\cdot, \Lambda_i)\}_i$, is locally unique.

³⁵ In all of the examples in this paper, the linear equilibrium is globally unique. With multiple assets (divisible goods), the uniqueness of a *linear* equilibrium in the set of all (potentially nonlinear) equilibria remains an open problem even for centralized markets with quadratic utilities. The linear equilibrium in our decentralized market model is globally unique under the conditions we provide in the Appendix. The comparative-statics results we develop involve monotone comparative statics rather than the Implicit Function Theorem – in particular, monotonicity conditions are separate from second-order conditions.

³⁶ It is interesting to compare the best response map G to games on networks with continuous strategies (e.g., Bramoullé, Kranton, and D’Amours (2013); the survey by Jackson and Zenou (2013)) and linear social interactions (e.g., Blume et al. (2012)). There, the best response maps are typically linear in the actions of others and conditions on the invertibility of a linear map (a matrix) verify uniqueness. In our model, the best response map is nonlinear in strategies $\{(\alpha_i \bar{\mathcal{V}}_{N(i)} + \bar{\Lambda}_i)^{-1}\}_i$ (i.e., it is a harmonic mean of matrices) and, unless the lifted demand slopes commute, does not reduce to a linear map (an arithmetic mean) and equilibrium uniqueness involves invertibility of a nonlinear best response map on tuples of positive definite matrices. This differs from the model of Bramoullé, Kranton, and D’Amours (2013), where a constraint $x \geq 0$ on the actions introduces a non-linearity and equilibrium multiplicity: The best response is $\max\{0, f(x)\}$, where f is linear; if the constraint does not bind, the linear equilibrium is unique, when it binds, the nonlinearity induces the multiplicity, depending on whose constraint is binding.

Note that the conditions derived from Gershgorin disc theorem, such as diagonal dominance, often used by researchers cannot be used in our model, as the theorem applies to matrices that are maps from $\mathbb{R}^N \rightarrow \mathbb{R}^N$, whereas we deal with maps of matrix tuples into matrix tuples. The isomorphism of \mathcal{S}^I with a cone in $\times_{i=1}^I \mathbb{R}^{N(i) \times N(i)}$ is not useful here, as it would ignore the positive definite structure.

among exchanges affects equilibrium in a decentralized market (Corollaries 3.1 and 3.2) and allows us to completely characterize price behavior in decentralized markets (Theorem 3.2).

Definition 3.1 *Given two exchanges n and n' and asset k , an equivalence path connecting these exchanges with respect to asset k is a sequence of exchanges $\{n_l\}_l$ and a sequence of agents $\{a_l\}_{l=1}^{L-1}$ such that $n_1 = n$, $n_L = n'$, and for all l , $a_l \in I(n_l) \cap I(n_{l+1})$ and $k \in K(n_l)$. Two equivalence paths are disjoint if the corresponding sets of agents are disjoint. Two disjoint equivalence paths form an equivalence loop.*

Let us describe a *regularization* procedure, which removes the equivalence loops from the hypergraph. Fix a decentralized market with exchanges N . For any pair of exchanges n and n' and any asset k , if there is an equivalence loop with respect to asset k between exchanges n and n' , remove k from all the exchanges in the equivalence loop $\{n_l\}_l$ between $n_1 = n$ and $n_L = n'$, and create a new exchange $(k, \cup_l I(n_l))$ in which only asset k is traded; the initial endowment of asset k for any trader $i \in \cup_l I(n_l)$ in the new exchange is the sum of i 's initial endowments of k in $\{n_l\}_l$. Iterating this procedure, we arrive at a decentralized market with no equivalence loops. Denote by N^* the set of exchanges in the so constructed *regularized* market associated with the market with exchanges N .

Theorem 3.1 (Regularization: Equilibrium Equivalence) *For any market with exchanges N , equilibrium prices and allocations in the associated regularized market with exchanges N^* coincide with those in N . Namely, for any pair of exchanges n_1, n_2 and any agent i , the element $(\Lambda_i)_{n_1, n_2}$ of agent i 's price impact coincides with $(\Lambda_i^*)_{n_1^*, n_2^*}$ where n_1^* and n_2^* are the exchanges in the regularized hypergraph to which n_1 and n_2 belong, respectively. Furthermore, equilibrium price impacts in the market with exchanges N^* are nonsingular.*

Hence, for any decentralized market, there is an equivalent and unique representation without singularities in the equilibrium price impact which may arise when identical assets are traded in different exchanges. Theorem 3.1 explicitly characterizes the set of pairs of exchanges for which the singularities occur.

EQUILIBRIUM AND INTERCONNECTEDNESS AMONG EXCHANGES. In this paper, we aim to explore ways in which, given the traders and assets, incentives and welfare differ with decentralized and centralized trading. A condition on trader participation alone implies that, with sufficient participation, this is not the case.

Corollary 3.1 *Equilibrium schedules and price impacts $\{(\Lambda_i, q_i(\cdot, \Lambda_i))\}_i$ in a market with assets K and agents I coincide with those in a centralized market with the same assets and agents if, and only if, (i) for any agent i and any asset k , we have $k \in K(N(i))$; and (ii) for any asset k and any two exchanges n, n' with $k \in K(n) \cap K(n')$, there exists an equivalence loop with respect to k connecting n and n' .*

Loops thus capture the type of interconnectedness across exchanges with which decentralized markets behave essentially as centralized markets. Absence of an equivalence loop gives rise to a ‘local monopoly power,’ or intermediation. For such more general decentralized markets, Theorem 3.1 makes a topological prediction about how equilibrium is related across exchanges – equilibrium prices, allocations and price impacts are *as if* the graph on the exchanges (equivalently, the hypergraph on traders and assets) had a tree-like market structure. Precisely, denote by N_k^* the exchanges in the regularized market in which asset k is traded. By construction, the graph of N_k^* does not contain loops and is, thus, a forest.³⁷

Definition 3.2 *Agent i is a monopolistic bridge with respect to asset k if removing i from the market strictly increases the number of connected components in the graph with exchanges N_k^* .*

Corollary 3.2 (Equilibrium Hypergraph) *A market is regularized if, and only if, the hypergraph is such that if $I(n) \cap I(n') \neq \emptyset$, then $K(n) \cap K(n') = \emptyset$ or $I(n) \cap I(n')$ is a monopolistic bridge, for any $n' \neq n$.*

To the extent that a trader participation in exchanges is a proxy for whether the assets in these exchanges can hedge his endowment risks, taken together, Corollaries 3.1 and 3.2 suggest the following interpretation: If assets K are *standardized* in that they can hedge risks of sufficiently many market participants³⁸ and provided there is no ‘local monopoly power’ (in the sense of Definition 3.2), decentralized markets behave essentially as centralized markets.³⁹ Nevertheless, many products traded over the counter are notoriously hard to standardize. Types of financial products traded OTC are typically either standardized but relatively illiquid assets or bespoke (personalized) products, which do not have the level of standardization required for trading on organized exchanges.⁴⁰ Our results suggest that

³⁷ A *connected component* is a subgraph in which any two vertices (here, exchanges in N^*) are connected by paths, and which is connected to no additional vertices in the supergraph. A *forest* is an (undirected) graph, all of whose connected components are trees; its graph consists of a disjoint union of trees. Equivalently, a forest is a cycle-free graph.

³⁸ While we do not study standardization in this paper, observe that the model allows for arbitrary K assets described by $\mathcal{N}(d, \Sigma)$ – standardized or bespoke.

³⁹ In a study of the U. S. equity market, O’Hara and Ye (2011) conclude that the creation of new exchanges has increased competition and resulted in a single virtual market with multiple points of entry. Similar evidence is now available for other countries (see Introduction).

⁴⁰ Market participants looking to hedge specific risks may not find a standardized product that would effectively match their exposure and instead may prefer to use a bespoke product. For many, there may be no secondary market pricing sources. Demand for bespoke products comes from a variety of market participants, including end-users of the non-financial sector (e.g., airlines), end-users of the financial sector (e.g., banks and insurance companies), and institutional investors (e.g., mutual funds, pension funds, university endowments, and sovereign wealth funds). Furthermore, a variety of financial instruments that may not have an official market are traded by multilateral trading facilities (MTFs), which have fewer restrictions on the admittance of financial instruments for trading. Standardization of assets is considered as the central challenge in the ongoing changes in the regulation of OTC markets (Financial Stability Board (2010); Duffie (2013)).

differences in the standardization of assets endogenously give rise to different types of decentralized markets which, for homogeneous assets, are forests with respect to traders and, for multiple assets or bespoke products – with respect to assets and traders.⁴¹

PRICES IN DECENTRALIZED MARKETS. With decentralized trading, the value of the same asset can be different across exchanges. Building on Theorem 3.1, Theorem 3.2 gives the condition on the market structure for the same asset to trade at the same prices in different exchanges.⁴²

Theorem 3.2 (Price Equalization) *In equilibrium, generically in the initial asset holdings, asset k is traded at the same price in exchanges n and n' if, and only if, there exists an equivalence loop connecting these two exchanges with respect to asset k .*

For the intuition, suppose that there is no equivalence loop connecting n and n' . Consider a single-agent node (a “network monopolist”) connecting exchanges n and n' . If removed from the market, exchanges n and n' become isolated. Let $N = N_1 \cup N_2$ be the partition of exchanges that is obtained if we remove agent i from the market. Then, in the decomposition $\mathbb{R}^N = \mathbb{R}^{N_1} \oplus \mathbb{R}^{N_2}$, price impact of agent i is block diagonal. Therefore, since no other agent trades simultaneously in exchanges from N_1 and N_2 , Equation (4) further implies that price impacts of all other agents are also block diagonal – no agent is able to equalize prices in these exchanges. By Theorem 3.1, unless there is another path connecting two exchanges, a single-agent node is also the only instance of an absence of an equivalence loop and, hence, a sufficient condition for price discrimination in decentralized markets. More subtly, the value

⁴¹ A core-periphery and hub-and-spoke architectures are robustly documented for many markets for homogeneous assets. 1,000 banks participate in the U.S. Federal Funds market of overnight unsecured loans; however, each bank provides loans on average to only 3.3 other banks, and most banks have few counterparties while few have many (Bech and Atalay (2010); Afonso, Kovner, and Schoar (2012)). Craig and Peter (2010) document a core-periphery structure in the German banking system and Li and Schürhoff (2012) in the U.S. municipal bonds market. See also Cocco, Gomes and Martins (2009) for the Portuguese interbank market. Moreover, in these markets, gross notional outstanding is highly asymmetric between smaller and larger institutions (e.g., Atkeson, Eisfeldt, and Weill (2013); Schachar (2013)). This is consistent with a (nontrivial) forest topology being associated with ‘local’ monopoly power and asymmetric distribution of endowment risks in the market (see Section 6).

In turn, bespoke financial products are typically traded by large financial institutions that design complex products for other institutional investors – a form of intermediation (Financial Stability Board (2010)). OTC credit default swaps markets are segmented; most small banks do not participate at all, and large banks serve as intermediaries between medium-size banks (e.g., Bech and Atalay (2010); Atkeson, Eisfeldt, and Weill (2013)).

⁴² Institutional investors choose to participate in liquidity (dark) pools seeking privacy in execution of trades – to avoid exposure of their orders and front running (Knight Capital Group (2010); Angel, Harris, and Spatt (2011)). Consistent with our results, liquidity pools (e.g., internal crossing networks in large banks), which trade publicly listed stocks, typically have no internal price setting mechanism but execute orders at the best price currently available at the public exchange (“price matching”). While there is no hard requirement for broker-dealers to match prices this way and the term “best execution” has not been defined clearly by U.S. regulation, this is the general business practice; e.g., <http://www.wallstreetandtech.com/internalization-is-it-really-that-bad/60404324>.

of an asset depends on market characteristics other than the span itself. Yet, no arbitrage incentives exist – the schedules are *ex post* optimal.

While the converse echoes Corollary 3.1, let us explain just why a loop is sufficient for price equalization. Consider the market in Figure 2. In the market with four links (Panel A), prices equalize along the links on which there are two traders. The two traders *equalize the within-exchange with across-exchange price impacts* for each other. Thus, equilibrium price impacts per unit change in demand are the same as in a market in which traders from n_1 and n_2 trade in one exchange and traders from n_3 and n_4 in another (Panel B) – prices and the agents’ total trades of the asset are the same as well. In this market with two exchanges, consider traders i_1 and i_2 , who participate in both exchanges. Then, by the same logic, each equalizes prices for the other, and all other agents. Price impacts and prices in all exchanges along an equivalence loop with respect to an asset coincide and correspond to the one liquidity and one price in the associated regularized market.

4 Equilibrium Price Impact, Allocations and Prices

Do traders have non-negligible price impact when trading is decentralized? Proposition 4.1 shows that, even if the total number of traders in the market is large, traders do not act as price takers in a decentralized market. In a (non-regularized) market, take an exchange n and an asset k traded on n let $I^*(n, k)$ denote the number of agents who trade in all exchanges, including n , connected with n via an equivalence loop with respect to asset k .

Proposition 4.1 (Equilibrium Noncompetitiveness) *Consider a sequence of markets, indexed by $l \geq 1$, with a fixed market structure $\{(I(n), K(n))\}_n$ and changing characteristics $\{\alpha_{i,l}, I^*(n, k)_l\}_l$. Then, in the limit as $l \rightarrow \infty$, price impact vanishes in exchange n for asset k if, and only if, either $I^*(n, k) \rightarrow \infty$ or $\alpha_{i,l} \rightarrow 0$ for at least two agents in the exchanges that are in the equivalence loop with n for asset k .*

When some agents in exchange n are almost risk neutral or the number of agents in the loop with exchange n for asset k becomes large, exchange n is essentially perfectly liquid: Equilibrium schedules in n correspond to the marginal utilities. Having access to a perfectly liquid exchange n , agents will, in general, have strictly positive price impact in the other exchanges $N \setminus \{n\}$ in which they participate, such as liquidity pools or a dealer network, unless (by Theorem 3.1) there is an equivalence loop linking these exchanges.

In the analysis to follow, we exploit the equilibrium structure of traders’ price impact to identify economic mechanisms that are present in decentralized but not centralized markets. The noncommutativity (Proposition 2.2) is one such property. Another is the interdependence among price impacts, anticipated by Theorem 2.1: In general, the equilibrium price

impact of trader i in exchanges $N(i)$ depends not only on the price impacts of other traders in exchanges $N(i)$, but also on price impacts of traders in all other exchanges $N \setminus N(i)$, even if the sets of agents are disjoint from those participating in exchanges from $N(i)$. The cross-exchange interconnectedness – via traders and assets – carries over to equilibrium prices and allocations. Theorem 4.1 characterizes equilibrium trades and prices in decentralized markets. Let $\gamma_i \equiv (\alpha_i \mathcal{V}_{N(i)} + \Lambda_i)^{-1} \alpha_i \mathcal{V}_{N(i)}$.

Theorem 4.1 (Equilibrium Prices and Trades) *Let*

$$\mathbf{Q} \equiv \left(\sum_j \alpha_j^{-1} \gamma_j \right)^{-1} \sum_j \gamma_j q_j^0$$

be the ‘market portfolio.’ Then, in equilibrium, the trade of agent i is

$$q_i = \gamma_i ((\alpha_i \mathcal{V}_{N(i)})^{-1} (\mathcal{V} \mathbf{Q})_{N(i)} - q_i^0)$$

and the vector of market clearing prices is given by $p = d - \mathcal{V} \mathbf{Q}$.

Portfolio $\{(\alpha_i \mathcal{V}_{N(i)})^{-1} (\mathcal{V} \mathbf{Q})_{N(i)}\}_i$ is the decentralized market counterpart of the (unique) efficient portfolio in a centralized competitive market. Given the strictly positive equilibrium price impact (Proposition 4.1), trader i demands (or sells) less, relative to the competitive schedule in exchanges $N(i)$ and his equilibrium allocation is $q_i + q_i^0 = \gamma_i (\alpha_i \mathcal{V}_{N(i)})^{-1} (\mathcal{V} \mathbf{Q})_{N(i)} + (\text{Id} - \gamma_i) q_i^0$. Mapping to the centralized market equilibrium in Proposition 2.1, with decentralized trading, the fraction γ_i of gains to trade $(\alpha_i \mathcal{V}_{N(i)})^{-1} (\mathcal{V} \mathbf{Q})_{N(i)} - q_i^0$ that is exhausted in equilibrium depends, through price impact, on the market structure, all preferences and assets, as does the efficient allocation $\{(\alpha_i \mathcal{V}_{N(i)})^{-1} (\mathcal{V} \mathbf{Q})_{N(i)}\}_i$.⁴³

5 Comparative Statics of Price Impact

In Sections 5 and 6, we examine the comparative statics of equilibrium liquidity and welfare. Two questions are of central interest: How does a trader’s price impact change when a market becomes more decentralized, in the sense that there are more market clearing mechanisms? And what is the impact of market decentralization on welfare? Consider two decentralized markets characterized by $\{\{\alpha_i\}_i, \Sigma, \{(I(n), K(n))\}_n\}$ and $\{\{\alpha'_i\}_i, \Sigma', \{(I'(n), K'(n))\}_n\}$ and let $\{\Lambda_i\}_{i \in I}$ and $\{\Lambda'_i\}_{i \in I'}$ be the corresponding equilibrium price impacts. We can rank price

⁴³ The linearity of the equilibrium prices leads us to interpret Theorem 4.1 in CAPM terms. A decentralized market CAPM holds: In each exchange, expectations of asset payoffs lie on a security-market line defined by an agent-specific portfolio. With many assets, agents diversify risk through different (and multiple) funds, which depend on the agents’ participation in exchanges (Appendix B).

impacts using the positive semidefinite order:⁴⁴ We say that price impact $\{\Lambda'_i\}_{i \in I'}$ is larger than $\{\Lambda_i\}_{i \in I}$, and write

$$(\Lambda_i)_{N(i) \cap N'(i)} \leq (\Lambda'_i)_{N(i) \cap N'(i)}, \quad (7)$$

if (7) holds for all i for the minimal and maximal equilibria of Proposition 2.3. Recall that if $A \geq B$ then the diagonal elements satisfy $A_{ii} \geq B_{ii}$ for all i (however, no implication for the ordering of the off-diagonal elements follows); $A_{N(i)} \geq B_{N(i)}$ for any i ; and $A^{-1} \leq B^{-1}$.

We consider three comparative statics for price impact. Proposition 5.1 characterizes how equilibrium price impact depends on the (primitive) characteristics of traders and assets, given the traders' participation in the exchanges $\{N(i)\}_i$. We then ask how price impact changes with market structure $\{N(i)\}_i$ (Theorem 5.1). We also compare price impact across traders (Proposition 5.2).

Proposition 5.1 (Price Impact and Traders, Assets) *Equilibrium price impact tuple $\{\Lambda_i\}_i$ is*

- (i) *increasing and concave in risk aversion α_j , for any j ;*
- (ii) *increasing and concave in the covariance matrix Σ ;*
- (iii) *decreasing in the number of agents $I(n)$ in an exchange, for any n .*

Relative to the observation that equilibrium price impacts are interdependent across exchanges (Theorem 2.1), Proposition 5.1 demonstrates a general *complementarity* of price impacts in a decentralized market: A trader's price impact in exchanges $N(i)$ depends *positively* (in the sense of positive semidefinite order) on the market characteristics in both exchanges $N(i)$ as well as $N \setminus N(i)$.⁴⁵ The decentralized market effect is strict, even for disjoint groups of agents in $N(i)$ and $N \setminus N(i)$, so long as the agents are indirectly connected through a sequence of counterparties and assets in $N(i)$ and $N \setminus N(i)$ are not independent. Notably, the *concavity* result implies that lower risk (Σ) or risk aversion $\{\alpha_i\}_i$ play a greater role in the determination of a trader's price impact than high-risk assets and high risk aversion traders. Being connected to such low risk assets or low risk aversion traders – whether

⁴⁴ We can apply lattice-theoretic arguments based on Tarski's fixed point theorem for centralized but not decentralized markets, given the non-commutativity. General monotone comparative statics can still be characterized with our results: the existence of the maximal and minimal equilibria (Proposition 2.3); and the positive semidefinite property of equilibrium price impacts (Theorem 2.2); and the monotonicity and concavity properties of the matrix harmonic mean (used in our main results, Proposition 5.1, Theorem 5.1, Proposition H.2).

⁴⁵ For the intuition, an agent's price impact from increasing his trade in exchanges $N(i)$ represents the price concessions required for other agents in exchanges $N(i)$ to absorb the trade. With decreasing marginal utility ($\alpha_j > 0$), more risk averse counterparties j in exchanges $N(i)$ demand larger price concessions to compensate for the trade's impact on their own marginal utilities (cf. the first-order condition (2)), thus making the residual supply less elastic and, hence, price impact larger for all other agents in $N(i)$. In addition, when trading is decentralized, the fewer and more risk averse agent i 's counterparties' trading partners in exchanges $N \setminus N(i)$, the larger the counterparties' price impacts in the exchanges $N(j)$ and the larger the price concessions they require in exchanges $N(i)$.

directly in own exchanges $N(i)$ or indirectly in the interconnected through counterparties' exchanges $N \setminus N(i)$ – lowers the price impact of i 's trades in $N(i)$. Example 9 (Appendix B) illustrates the interdependence in price impact between exchanges.

We now turn to changes in market structure in a market with traders I and assets K . Suppose a new exchange is created for a subset of agents which operates along with the existing exchanges. This corresponds to an increase in the subset of exchanges in which trader i from the subset participates, $N'(i) \supseteq N(i)$. Since, unlike in the experiments of Proposition 5.1, the dimensions of price impacts change, we compare the price impacts for exchanges $N(i)$ only (i.e., in the sub-hypergraphs induced by $N(i) \cap N'(i)$ in definition (7)).

Theorem 5.1 (Price Impact and Participation) *An increase in participation of any trader i lowers equilibrium price impact of all agents in all exchanges.*

Hence, creating a new exchange always (weakly) lowers price impact (improves liquidity) in all existing exchanges. Suppose instead that the number of exchanges in a market increases by breaking up an existing exchange. Theorem 5.1 implies that breaking up exchanges always (weakly) increases price impact. Furthermore, this holds for any asset structure in the new market.

Corollary 5.1 *Suppose exchange $(I(n), K(n))$ is split into two exchanges $(I_1(n), K_1(n))$ and $(I_2(n), K_2(n))$ with arbitrary (not necessarily disjoint) subsets of assets $K_1(n) \cup K_2(n) = K(n)$ and agents $I_1(n) \cup I_2(n) = I(n)$. Equilibrium price impact increases in all exchanges.*

In general, price impact is monotone in both the set inclusion of traders and assets and is, thus, minimal in a centralized market. It follows from Theorem 5.1 that the lowest price impact that agents I who trade assets K can achieve is when all agents participate in all potential exchanges – a market structure equivalent to a centralized market. By Corollary 3.1, exchanges in which disjoint subsets of all traders participate may attain liquidity as high as the centralized market with a single exchange $n = (K, I)$. By Theorem 3.1, a change in the market structure has impact on equilibrium liquidity so long as it affects the hypergraph of the associated regularized market, with correlated assets.⁴⁶ The liquidity effects of market decentralization (i.e., increasing the number of exchanges) characterized by Theorem 5.1 and Corollary 5.1 hold for any number and risk aversions of traders, and the covariance matrix of assets in the exchanges before and after the change.

⁴⁶ A change in participation affects liquidity if, and only if, it changes, for some trader i , the subsets of agents I in some exchanges $N(i)$ or the subsets of assets K he trades. Respectively, these changes in participation correspond to a change in cross-exchange price impact, for a fixed set of exchanges, and the creation of a new exchange with some assets that are not part of an equivalence loop (the dimensions of price impacts changes). With uncorrelated assets, price impacts are diagonal, and there is no relationship between liquidity for different assets in indirectly connected exchanges. Hence, strict liquidity improvement may fail for nongeneric covariance matrices.

Relative to centralized markets results (Proposition 2.1), Proposition 5.1 and Theorem 5.1 report two complementarity results on how the interconnectedness among exchanges $\{N(i)\}_i$ affects a trader's equilibrium price impact in exchange $N(i)$. Respectively, changes in market characteristics or market structure that lower any trader's price impact in an exchange lower price impacts of all traders in all directly and indirectly connected exchanges. Finally, how does an agent's participation in the market affect his price impact, relative to other agents in the exchanges $N(i)$ in which he trades? Suppose that $N(i) \supset N(j)$; for instance, trader i is better connected with the market or more centrally located than j . A concavity property of the harmonic mean implies the following relationship among the price impacts in the exchanges in which both traders i and j participate, $(\Lambda_i)_{N(j)}$ and Λ_j .

Proposition 5.2 (Relative Price Impact) *Suppose that $N(i) \supset N(j)$ and that trader j participates in a single exchange for one asset. Then, if $\alpha_i \leq \alpha_j$, the equilibrium price impact of trader i in exchanges $N(j)$ is larger than that of trader j ,*

$$(\Lambda_i)_{N(j)} \geq \Lambda_j. \tag{8}$$

That is, more connected agents face a less elastic residual supply – their price impact is higher.

In particular, if agents are equally risk averse, $\alpha_i = \alpha_j$, price impact in exchanges $N(j)$ is higher for the more connected agents i (e.g., dealers or intermediaries) than their less connected counterparties. (In contrast, in a centralized market with the assets from exchanges $N(j)$ and agents i and j , the agents would have the same price impact.) Intuitively, since an agent's price impact in an given exchange is determined by the diversification possibilities of other agents in that exchange, trader i 's ability to diversify risk in exchanges $N(i) \setminus N(j)$ lowers his risk to be diversified in exchanges $N(j)$ relative to trader j 's risk and, thus, the price impact of trader j in exchanges $N(j)$. Nevertheless, with many assets, one cannot extrapolate Proposition 5.2 to conclude that $(\Lambda_i)_{N(j)} \geq \Lambda_j$. The non-commutativity is the key.⁴⁷ In Section 6, we show that this non-monotonicity has important welfare implications.

5.1 Complementarity and Substitutability of Asset Payoffs

In centralized markets, equilibrium price impact is determined solely by the primitive covariance matrix Σ (cf. Proposition 2.1). We show that, with decentralized trading, agents' participation in different exchanges alters the riskiness of assets relative to Σ , and payoff substitutability or complementarity is no longer determined by Σ . Furthermore, both the

⁴⁷ If $(\Lambda_i)_{N(j)}$ and Λ_j commute, they can be diagonalized in the same basis so that one can rank the diagonal matrices. In Appendix B we present the results for multiple assets, showing that one can still compare price impacts through an eigenvalue order instead of the positive semidefinite order.

riskiness and the covariance of assets traded in decentralized markets are typically heterogeneous across traders.

We illustrate the endogeneity in the riskiness of asset payoffs in a family of markets which include a perfectly liquid exchange and are otherwise arbitrary. The family allows us to isolate the endogenous cross-asset effects in a fully explicit fashion. The effects discussed carry over to general decentralized markets.

Example 3 Markets with a liquid exchange: I agents all trade assets $\{1, \dots, K_1\}$ in exchange n . In addition, there are exchanges $\{n_l\}_l$, $l \in L$, $n_l \neq n$ in which assets $\{K_1 + 1, \dots, K\}$ are traded, and each agent i can trade in a subset of these exchanges together with a class I_j , $I_j \cap I = \emptyset$. Assume $\alpha_j/|I_j|$ is sufficiently small and $|I_j| \geq 2$. Let, for any i ,

$$\mathcal{V}_{N(i)} = \begin{pmatrix} \mathcal{V}_i^{11} & \mathcal{V}_i^{12} \\ \mathcal{V}_i^{21} & \mathcal{V}_i^{22} \end{pmatrix}$$

be the block decomposition of \mathcal{V} in $\mathbb{R}^{N(i)} = \mathbb{R}^{N(i) \setminus N(j)} \oplus \mathbb{R}^{N(i) \cap N(j)}$. Define, for any $i \neq j$, $\mathcal{V}_{i \setminus j} \equiv \mathcal{V}_i^{11} - \mathcal{V}_i^{12}(\mathcal{V}_i^{22})^{-1}\mathcal{V}_i^{21} \in \mathbb{R}^{(N(i) \setminus N(j)) \times (N(i) \setminus N(j))}$.

Proposition 5.3 (Price Impact and *Residual* Riskiness) Let I_j be a set of agents with risk aversion α_j and the same set of exchanges $N(j)$ in which they participate. Assume $|I_j| \geq 2$. Then, in the limit as $\alpha_j/|I_j| \rightarrow 0$, equilibrium price impacts in exchanges $N(j)$ vanish, $\Lambda_j \rightarrow 0$, whereas equilibrium price impacts in exchanges $N(i) \setminus N(j)$, $\Lambda_{i \setminus j} \equiv \Lambda_{i, N(i) \setminus N(j)}$, solve the system

$$\Lambda_{i \setminus j} = \left(\left(\sum_{k \notin I_j} (\alpha_k \bar{\mathcal{V}}_{k \setminus j} + \bar{\Lambda}_{k \setminus j})^{-1} \right)^{-1} \right)_{N(i) \setminus N(j)}, \quad i \in I.$$

Furthermore, the demand slope of agent i in exchanges $N(i)$ coincides with $(\alpha_i \bar{\mathcal{V}}_{i \setminus j} + \bar{\Lambda}_{i \setminus j})^{-1}$ and $(\mathcal{V}\mathbf{Q})_{N(j)} = 0$.

Proposition 5.3 can be understood through the classic conditioning arguments: Consider agent i who chooses portfolio $q = \begin{pmatrix} y \\ x \end{pmatrix} \in \mathbb{R}^{N(i) \cap N(j)} \oplus \mathbb{R}^{N(i) \setminus N(j)}$. The variance of this portfolio is given by $\text{Var}(R_{N(i)}^T q) = q^T \mathcal{V} q$, and the minimal risk that the agent can achieve by trading in exchanges $N(j)$ is given by

$$\min_{y \in \mathbb{R}^{N(i) \cap N(j)}} \text{Var} \left(R_{N(i)}^T \begin{pmatrix} y \\ x \end{pmatrix} \right) = x^T \mathcal{S}(\mathcal{V}_i, N(i) \setminus N(j)) x. \quad (9)$$

Thus, $\mathcal{V}_{i \setminus j}$ is the agent's covariance matrix for the residual risks in $N(i) \setminus N(j)$, which cannot be hedged in the liquid exchange $N(j)$. The greater the extent to which the market

participation of i overlaps with $N(j)$, the less the residual risk that i must bear. Thus, $\mathcal{V}_{i \setminus j}$ is the conditional variance of assets traded by i , given participation $\{N(i)\}_i$. Observe that matrices $\mathcal{V}_{i \setminus j}$ do *not* correspond to sub-matrices of the covariance matrix \mathcal{V} ; the conditional covariances $\mathcal{V}_{i \setminus j}$ are linked to \mathcal{V} in a nonlinear way. Decentralized trading changes the riskiness of assets for different agents, even if it is common knowledge that asset payoffs are distributed $N(d, \Sigma)$. Example 4 illustrates.

Example 4 (Endogenous Payoff Riskiness) *Consider a market from Example 3.*

- (1) *Class I_j with a sufficiently small $\alpha_j/|I_j|$ trades J assets κ_i , $i = 1, \dots, J$, in I_1 exchanges $N(j) = \{\kappa_1, \dots, \kappa_J\}$; that is, we identify exchange κ_i with asset κ_i traded in this exchange. Some of the assets κ_i can be identical. In addition, I_1 agents $i \notin I_j$ can all participate in a common exchange n for $K(n)$ assets with the covariance matrix $V_{\{n\}} = (V_{k_1, k_2})_{k_1, k_2=1}^{K(n)}$, and agent i can also trade asset κ_i in exchange κ_i ; that is, $N(i) = \{n \cup \kappa_i\}$, $i = 1, \dots, I_1$. Let $V_{\kappa_i, k} = \text{Cov}(R_{\kappa_i}, R_k) = V_{\kappa_i, k}$, $k = 1, \dots, K(n)$, $i = 1, \dots, I_1$. Then, $N(i) \setminus N(j) = \{n\}$, and Proposition 5.3 implies that, in element-by-element notation, $V_{i \setminus j} \in R^{K(n) \times K(n)}$ of agent i is given by*

$$\mathcal{V}_{i \setminus j} = \mathcal{V}_{\{n\}} - \mathcal{V}_{\kappa_i, \kappa_i}^{-1} (\mathcal{V}_{\kappa_i, k_1} \mathcal{V}_{\kappa_i, k_2})_{k_1, k_2=1}^{|N(j)|}, \quad (10)$$

where $\mathcal{V}_{\kappa_i, \kappa_i}$ is the variance of asset κ_i . Thus, effective riskiness for the assets in exchange n differs across agents; in particular, the implied covariance between assets k_1 and k_2 for agent i is $\mathcal{V}_{k_1, k_2} - \mathcal{V}_{\kappa_i, \kappa_i}^{-1} \mathcal{V}_{\kappa_i, k_1} \mathcal{V}_{\kappa_i, k_2}$.

- (2) *Suppose that the covariances $\mathcal{V}_{\kappa_i, k}$ and the variances $\mathcal{V}_{\kappa_i, \kappa_i}$, $i = 1, \dots, I_1$, $k = 1, \dots, K(n)$ are multiplied by a common factor β , $|\beta| > 1$. Then, the implied covariance matrices decrease and a slight modification of Proposition 5.1 implies that equilibrium price impacts of all classes in exchange n decrease. That is, the more the assets in $K(n)$ covary with those in the liquid exchanges, the more risk can be hedged; this improves liquidity in exchange n . In particular, when the set $K(n)$ contains a single asset, the price impacts of all agents in n are monotone decreasing in the absolute value of any covariance $|\mathcal{V}_{\kappa_i, k}|$.*

As Example 4 shows, with decentralized trading, within-exchange price impact $(\Lambda)_{nn}$ and cross-exchange price impact $(\Lambda)_{nm}$, $n \neq m$, are affected by asset covariances traded in different exchanges. Note that these results contrast sharply with centralized markets, where the equilibrium price impact matrix of every agent is proportional to matrix Σ ; thus, introducing new assets (i.e., market completion) does not influence liquidity in trading the existing assets.

6 Welfare in Decentralized Markets

In centralized markets, a decrease in the price impact of an agent's trades also increases his equilibrium utility and, hence, welfare. In this section, we show that the link between price impact and equilibrium utility does not hold in decentralized markets. Consequently, changes in market structure that lower liquidity may increase an agent's utility and total welfare. Let us define

$$\Gamma_i(\Lambda_i) \equiv \gamma_i^T \left(\frac{1}{2} \alpha_i \mathcal{V}_{N(i)} + \Lambda_i \right) \gamma_i.$$

Let also

$$\mathcal{Q}_i \equiv (\alpha_i \mathcal{V}_{N(i)})^{-1} (\mathcal{V} \mathbf{Q})_{N(i)}. \quad (11)$$

The following is true.

Proposition 6.1 (Equilibrium Utility) *The equilibrium utility of trader i with initial holdings q_i^0 can be decomposed as*

$$U_i(\Lambda_i; q_i^0) = \underbrace{(q_i^0)^T d_{N(i)} - \frac{1}{2} (q_i^0)^T \alpha_i \mathcal{V}_{N(i)} q_i^0}_{\text{Utility in Autarky}} + \underbrace{(Q_i - q_i^0)^T \Gamma_i(\Lambda_i) (Q_i - q_i^0)}_{\text{Equilibrium Surplus from Trade}}. \quad (12)$$

Proposition 6.1 shows that, compared to centralized markets, decentralized trading changes the size of the surplus exhausted *in equilibrium* by strategic traders, in two ways. First, it changes the surplus matrix $\Gamma_i(\Lambda_i)$. Second, the gains to trade $(Q_i - q_i^0)$ available to strategic traders change with the aggregate risk $\{Q_i\}_i$, determined by the market structure. In particular, $\{Q_i\}_i$ plays the role of the efficient portfolio for the decentralized market. By contrast, in centralized markets, the aggregate risk is determined solely by endowments, risk aversion and the number of traders $\{\{q_i^0, \alpha_i\}_i, I\}$.

The relation between price impact and welfare in a decentralized market depends on how the aggregate (diversified) and idiosyncratic risk in the equilibrium allocation contributes to an agent's utility. In a competitive market, the surplus matrix is given by $\Gamma_i(0) = 0.5 \alpha_i \mathcal{V}_{N(i)}$. Define

$$\Delta_i(\Lambda_i) \equiv \Gamma_i(0) - \Gamma_i(\Lambda_i)$$

to be the “surplus matrix gap.” Using the identity $\Delta_i(\Lambda_i) = \frac{1}{2}\Lambda_i\gamma_i^T(\alpha_i\mathcal{V}_{N(i)})^{-1}\gamma_i\Lambda_i$, equilibrium utility can be written as

$$U_i(\Lambda_i; q_i^0) = (q_i^0)^T d_{N(i)} \quad \underbrace{+(\mathcal{Q}_i)^T \Gamma_i(\Lambda_i) \mathcal{Q}_i}_{\text{Nondiversifiable Risk Utility}} \quad (13)$$

$$\quad \underbrace{-(q_i^0)^T \Delta_i(\Lambda_i) q_i^0}_{\text{Idiosyncratic Risk Disutility}} \quad (14)$$

$$\quad \underbrace{-2(q_i^0)^T \Gamma_i(\Lambda_i) \mathcal{Q}_i}_{\text{Risk Covariance Disutility}}. \quad (15)$$

Since matrices $\Gamma_i(\Lambda_i)$ and $\Delta_i(\Lambda_i)$ are positive definite, it is immediate that the nondiversifiable risk utility (13) (the idiosyncratic risk disutility (14)) is positive (negative) and increasing (decreasing) in the aggregate (per capita) risk $\mathbf{Q}_{N(i)}$ and the idiosyncratic risk in the initial portfolio q_i^0 respectively.

As we show in the Appendix, in any decentralized market, $\Gamma_i(\Lambda_i) \leq \Gamma_i^{CM}$. Hence, in markets in which market structure does not affect aggregate risk $\{\mathcal{Q}_i\}_i$, the centralized market is the unique market structure that is Pareto optimal and that maximizes total welfare. As we show in Example 2, through its impact on non-diversifiable risk, welfare in a decentralized market may dominate *in the Pareto sense a centralized market with the same traders and assets if monetary transfers are allowed. In fact, Example 3 shows that this effect holds in the case of a single asset, in which case endogenous riskiness in the sense of Example 4 is irrelevant. In this section, we explore ways in which welfare implications in decentralized markets differ from centralized markets for the case of multiple traded assets. In this case, concavity of price impact (Proposition 5.1) and endogenous riskiness (Example 4) combined with non-commutativity (Proposition 2.2) lead to the following surprising welfare implications:*

- Risk sharing: In centralized markets, in equilibrium, utility compensation always rewards exposure to diversified (aggregate) risk and penalizes exposure to idiosyncratic risk. When trading is decentralized, *both* an agent’s compensation for aggregate risk exposure may increase and his utility loss from idiosyncratic risk exposure may decrease when his price impact increases (Proposition 6.3). In fact, increasing participation for all agents (in the sense of Theorem 5.1), while increasing price impacts of all agents, *may increase welfare in the Pareto sense.*
- Financial innovation: In a centralized market (with quasilinear utilities), adding a non-redundant asset weakly increases utilities for all traders. In a decentralized market, all traders can be strictly worse off (Proposition 6.3).

Furthermore, implications for market design follow: Breaking up an exchange may increase utility of every trader, while it increases all traders' price impacts (Proposition 6.3); intermediaries may increase or decrease welfare; restricting participation may increase welfare (Proposition 6.3).

PRICE IMPACT AND WELFARE. Suppose that, for some i , Λ_i and $\mathcal{V}_{N(i)}$ are proportional; $\Lambda_i = \lambda \mathcal{V}_{N(i)}$. Then the centralized market predictions follow: Larger price impact unambiguously lowers equilibrium utility; the utility compensation for bearing aggregate risk and the negative of the loss from residual idiosyncratic risk exposure both decrease.⁴⁸ Proposition 6.1 establishes that the proportionality is also necessary for each of the monotone relations.⁴⁹

Proposition 6.2 *Fix the market portfolio \mathbf{Q} . Then, an increase in price impact always decreases the utility of an agent from class i if, and only if, $(\mathcal{Q}_i - q_i^0)$ is an eigenvector of γ_i .*

Two matrices are proportional only if every vector is an eigenvector for both of them. Proposition 6.2 therefore yields the following result.

Corollary 6.1 *Fix the market portfolio \mathbf{Q} . In any decentralized market with equilibrium price impact $\{\Lambda_i\}_i$, a change in market structure $\{N(i)\}_i$ that increases in price impact decreases equilibrium utility from diversified risk and increases equilibrium utility loss from idiosyncratic risk if, and only if, Λ_i is proportional to $\mathcal{V}_{N(i)}$.*

In a decentralized market, Λ_i and \mathcal{V} are linked non-linearly. In fact, by Proposition 5.1 price impact is concave in risk. From the first order condition (2), a trader who buys y

⁴⁸ When Λ_i and $\mathcal{V}_{N(i)}$ are scalar (i.e., one-dimensional), by direct calculation, $\Gamma_i(\Lambda_i)$ and $-\Delta_i(\Lambda_i)$ are monotone decreasing in Λ_i . For proportional price impact (i.e., if $\Lambda_i = \lambda \mathcal{V}_{N(i)}$),

$$\Gamma(\Lambda_i) = \frac{0.5\alpha_i + \lambda}{(\alpha_i + \lambda)^2} \mathcal{V}_{N(i)}^{-1} \text{ and } \Delta_i(\Lambda_i) = \frac{1}{2} \frac{\lambda^2 \alpha_i}{(\alpha_i + \lambda)^2} \mathcal{V}_{N(i)},$$

and, hence, the monotonicity in λ follows.

⁴⁹ In equilibrium, a change in the market structure affects equilibrium utility through price impact Λ_i directly and indirectly via \mathbf{Q} . While the latter effect depends on the distribution of endowments in the economy, we can characterize the former explicitly. Proposition 6.2 concerns the utility effects of the changes in the market structure when the impact of \mathbf{Q} is sufficiently small.

For an example of a class of distributions for which the risk premium vector \mathbf{Q} is independent of the equilibrium price impacts, take an arbitrary vector $Q' \in \mathbb{R}^N$ and partition the agents into classes according to their risk aversion and participation, so that class i contains M_i agents with risk aversion α_i , all participating in the same exchanges $N(i)$. Suppose that the total initial holding of class i is $Q_i^0 = M_i (\alpha_i \mathcal{V}_{N(i)})^{-1} Q'_{N(i)}$, for all i . Then, $\mathbf{Q} = \mathcal{V}^{-1} Q'$, there is no inter-class trade, and only agents within the same class trade with one other – by direct calculation using Theorem 4.1. Thus, the risk premium vector \mathbf{Q} is independent of the equilibrium price impacts. In particular, if participation of some agents decreases from $N(i)$ to $N'(i) \subset N(i)$, $\alpha_i \mathcal{V}_{N'(i)} Q_i^0 = M_i Q'_{N'(i)}$ continues to hold and, hence, $\mathbf{Q} = \mathcal{V}^{-1} Q'$ does too. Note that, in the absence of aggregate risk, if traders had zero price impact, all risk could be diversified within each class and the classes would have no incentive to trade with one another. With positive equilibrium price impact, changes in participation affect the diversification opportunities within and across classes.

aims to minimize diversification inefficiency $\frac{1}{2}\alpha_i y^T \mathcal{V}_{N(j)} y$ and price impact inefficiency $y^T \Lambda_i y$. Since matrices Λ_i and $\mathcal{V}_{N(i)}$ do not commute in general, the portfolio that minimizes price impact inefficiency is *not* the same as the one that minimizes diversification inefficiency. Accordingly, $\Gamma_i(\Lambda_i)$ can be decomposed into the contribution of diversification and liquidity parts

$$\Gamma_i(\Lambda_i) = \underbrace{\frac{1}{2}\gamma_i^T \alpha_i \mathcal{V}_{N(i)} \gamma_i}_{\text{Diversification}} + \underbrace{\gamma_i^T \Lambda_i \gamma_i}_{\text{Price Impact}}$$

As we show above, in a centralized market an increase in liquidity implies that Γ_i scales proportionally by a constant $\lambda > 1$. In this case, components (13) and (14) of an agent's utility increase by

$$(\lambda - 1)(\mathcal{Q}_i)^T \Gamma_i(\Lambda_i) \mathcal{Q}_i \text{ and } (\lambda - 1)(q_i^0)^T \Gamma_i(\Lambda_i) q_i^0$$

respectively and the welfare implications of higher liquidity is unambiguous. By contrast in a decentralized market, price impact (and hence the surplus matrix Γ_i) change non-linearly along different subspaces. As a consequence, both components (13) and (14) may increase or decrease independently of each other in response to a change in the structure of the decentralized market. Proposition 6.3 below shows that this *non-linear response across different sub-spaces* may lead to unexpected welfare effects.

Proposition 6.3 *For generic markets from Example 3, there exist initial endowments such that*

- *one can increase participation (in the sense of Theorem 5.1) in such a way that components (13) and (14) increase or decrease independently of each other, simultaneously for all agents in the economy;*
- *the utility of every agent increases if some exchanges are split or some agents are excluded from participating in some exchanges;*
- *financial innovation (introducing a new, unspanned security) may reduce welfare in the Pareto sense;*
- *financial innovation in the form of an OTC market may Pareto dominate an innovation via a commonly accessible centralized exchange.*

NON-DIVERSIFIABLE RISK AND WELFARE. When the market portfolio \mathbf{Q} is not independent of price impact, any change in the market structure also changes these portfolios. As a result, this may lead to *nonlinear* changes in the allocation of risk across agents. Example 5 illustrates the adverse welfare effects of increased participation, financial innovation and a positive endowment shocks in a decentralized market.

Example 5 (Non-Diversifiable Risk and Welfare) Consider markets from Example 3 with only one asset traded in exchange n and normalize $M_i = 1$, for simplicity. Let $v_i \equiv \mathcal{V}_{i \setminus j} = \mathcal{V}^{11} - \mathcal{V}_i^{12}(\mathcal{V}_i^{22})^{-1}(\mathcal{V}_i^{12})^T$ be the effective riskiness of asset 1 for trader i . Then, Propositions 2.1 and 5.3 imply that price impact $\lambda_i \equiv \Lambda_{i \setminus j}$ satisfies

$$\Lambda_i = (2 - \alpha_i v_i b + \sqrt{(\alpha_i v_i b)^2 + 4}) / (2b)$$

and b is the unique positive solution to $\sum_i (2 + \alpha_i v_i b + \sqrt{(\alpha_i v_i b)^2 + 4})^{-1} = 1/2$. Furthermore, equilibrium surplus from trade of agent i is given by

$$\frac{0.5\alpha_i v_i + \Lambda_i}{(\alpha_i v_i + \Lambda_i)^2} (\alpha_i^{-1} \mathbf{Q} - q_i^0)^2$$

with $\mathbf{Q} \equiv b^{-1} \sum_i \gamma_i q_i^0$, where $\gamma_i = (\alpha_i v_i) / (\alpha_i v_i + \Lambda_i)$ is the order reduction of agent i .

Suppose that either (i) participation of some agents increases, or (ii) a new, liquid exchange for a nonredundant asset is introduced. Then, for all i , v_i decreases and, hence, Λ_i does too; however, $\{v_i, \Lambda_i\}_i$ are otherwise unrestricted. It follows that, for generic changes in participation and generic new assets introduced, vector $\{\gamma_i\}_i$ of equilibrium order reduction changes non-proportionally. Suppose that $\{\gamma_i\}_i$ changes to $\{\hat{\gamma}_i\}_i$. One can pick $\{q_i^0\}_i$ that is orthogonal to $\{\gamma_i\}_i$ but not orthogonal to $\{\hat{\gamma}_i\}_i$.⁵⁰ Then, a change in market structure gives rise to aggregate risk (i.e., q^{Av} becomes non-zero), which may reduce total welfare. That is, (i) as participation of some traders $\{N(i)\}_i$ increases or (ii) a new exchange is created, while price impacts of all traders decrease, the endogenous weights reweigh the traders' endowments $\{q_i^0\}_i$ differently across traders $\mathbf{Q}_{N(i)}$. And, (iii) an increase in a trader's endowment with fixed participation does not change price impact, but may decrease gains to trade of some traders.

In a centralized \mathbf{Q} is always a *linear* combination of the initial asset holdings with *scalar* coefficients. The decentralized market coefficients with which initial asset holdings enter \mathbf{Q} are (non-commuting) matrices. Depending on the endogenous conditional covariances, the coordinates of \mathbf{Q} may get amplified, thus improving the agents' compensation for the risks taken.

By Example 5, if the traders' liquidity needs measured by endowment size are heterogeneous and the number of traders is not large, moving an asset traded OTC to a single centralized exchange can create systemic risk and lower welfare.⁵¹ The market portfolio \mathbf{Q} is

⁵⁰A similar phenomenon may occur in a centralized market if we remove some traders. See Example 2 above.

⁵¹ Indeed, OTC trading is considered superior for more customized products, but also for products that are standardized but less frequently traded, such as many single-name credit default swaps (e.g., Duffie (2012, 2013)). For the pros and cons of introducing centralized clearing houses for standardized assets traded OTC, see the discussion between the Director of Research at Bloomberg (Litan (2013)) and Duffie (2013).

not monotone in price impact and its equilibrium response to changes in the market structure may overturn the effects due to a change in Λ_i .

WHO PROFITS FROM INTERMEDIATION? We have shown various ways in which changes in market structure may increase equilibrium utility, even if price impact increases. Example 2 shows that a decentralized market structure may Pareto-dominate a centralized market with the same set of agents and assets in monetary transfers are allowed. In light of Proposition 6.2, a decentralized market may permit a better alignment of traders' gains to trade with trading opportunities in exchanges $\{N(i)\}_i$ in a way that is not feasible with centralized trading.⁵² Introducing a strategic intermediary (a monopolistic bridge in the sense of Definition 3.2) may substantially change risk allocation in the economy. Intuitively, one might expect that the intermediary will use his monopoly power and extract large rents at the cost of other (periphery-based) market participants. Quite surprisingly, we find that this is not the case in general. In fact, the double auction mechanism does not permit the intermediary to exploit his strategic position: in agreement with Proposition 5.2, in equilibrium, traders provide less liquidity to the intermediary. Below, we construct an example of a decentralized market with the following properties:

- the market is a star: there is a monopolistic bridge that intermediates trades between two classes of agents (clients), who cannot trade with each other directly;
- intermediary has zero endowment (pure intermediation);
- the decentralized market utilities for clients are strictly higher than those in the centralized market with same traders and assets;
- the intermediary's utility in the decentralized market is strictly lower than that in the centralized market with same traders and assets.

Example 6 *Consider the market from Example 9 with two assets (e.g., complete market with two states), two classes of agents of size M_1 and M_2 with risk aversions α_1, α_2 and a third class with a single agent (an intermediary; $M_3 = 1$) whose risk aversion is normalized to one. Suppose first that there are two exchanges, A in which class-1 agents trade asset 1 with the intermediary and B in which class-2 agents trade asset 2 with the intermediary*

⁵² The proportionality constant in the eigenvector condition represents a 'single degree of freedom' in exhausting gains to trade $(\alpha_i \mathcal{V}_{N(i)})^{-1}(\mathcal{V}\mathbf{Q})_{N(i)} - q_i^0$ in a centralized market. In a decentralized market, Γ_i does not commute with \mathcal{V} , generically, and agents can benefit from different effective riskiness of the assets. Additionally, the impact of the initial holdings on the risk premium vector \mathbf{Q} differs from that in a centralized market, where $\mathcal{V}^{-1}\mathbf{Q}$ is always a *linear* combination of the initial asset holdings. The decentralized market coefficients with which initial asset holdings enter \mathbf{Q} are (non-commuting) matrices. Depending on the endogenous effective covariances, the coordinates of the risk premium \mathbf{Q} may get amplified, thus improving the agents' compensation for the risks taken.

($N_1 = \{A\}$ and $N_2 = \{B\}$). Thus, the two classes cannot trade directly. The initial holdings of asset j of the agents in class i are q_i^j , $j = 1, 2$.

If α_1 is sufficiently large, α_2 is sufficiently small, and the covariance between the two assets is positive and sufficiently small. Then, agents of classes 1 and 2 are strictly better off in the decentralized market, whereas the intermediary is strictly better off in the centralized market.

7 Conclusions

This paper takes the exchanges and the assets available to trade in decentralized markets as exogenous. Our analysis suggests that the study of endogenous formation of exchanges in decentralized markets, with respect to welfare or other objectives, should not be separated from security design. Furthermore, the endogeneity and heterogeneity of residual riskiness of assets traded in decentralized markets imply that efficiency and profit opportunities from security design as well as specialization in trading certain assets exist that are not available in centralized markets. Endogenizing exchange creation and asset structure in decentralized markets is studied in Malamud and Rostek (2013).

References

- Afonso, G., A. Kovner, and A. Schoar (2012): “Trading Partners in the Interbank Lending Market,” working paper.
- Afonso, G. and R. Lagos (2012): “Trade Dynamics in the Market for Federal Funds,” working paper.
- Almgren, R. (2009): “Execution Costs,” in: Encyclopedia of Quantitative Finance, R. Cont (ed.), Wiley.
- Anderson, W. N. (Jr.) (1971): “Shorted Operators,” SIAM Journal of Applied Mathematics 20, 3, 520-525.
- Anderson, W. N. (Jr.) and R. J. Duffin (1969): “Series and Parallel Addition of Matrices,” Journal of Mathematical Analysis and Applications 26, 576-594.
- Anderson, W. N. (Jr.) and G. E. Trapp (1975): “Shorted Operators II,” SIAM Journal of Applied Mathematics, 28, 1, 60-71.
- Angel, J., L. Harris, and C. Spatt (2011): “Equity Trading in the 21st Century,” Quarterly Journal of Finance, 1, 1-53.
- Atkeson, A. G., A. L. Eisfeldt, and P.-O. Weill (2013): “The Market for OTC Derivatives,” working paper.

- Babus, A. and P. Kondor (2013): “Trading and Information Diffusion in Over-The-Counter Markets”, working paper.
- Bech, M., and E. Atalay (2010): “The Topology of the Federal Funds Market,” *Physica A: Statistical Mechanics and its Applications*, 389, 22, 5223-5246.
- Berge, C. (1973): *Graphs and Hypergraphs*, North-Holland Mathematical Library.
- Biais, B. and R. Green (2007): “The Microstructure of the Bond Market in the 20th Century,” IDEI Working Paper, 482.
- Biais, B. C. Bisière, and C. Spatt (2010): “Imperfect Competition in Financial Markets: An Empirical Study of Island and Nasdaq,” *Management Science*, 56, 2237-2250.
- Blume L., D. Easley, J. Kleinberg and E. Tardos (2009): “Trading Networks with Price Setting Agents,” *Games and Economic Behavior*, 67, 1, 36-50.
- Blume, L., S. Durlauf, W. Brock, and Y. Ioannides (2012): “Identification of Social Interactions,” forthcoming in *The Handbook of Social Economics*, J. Benhabib, A. Bisin, and M. Jackson, eds.
- Bramoullé, Y., R. Kranton, and M. D’Amours (2013): “Strategic Interaction and Networks,” forthcoming in *American Economic Review*.
- Cocco, J. F., F. J. Gomes, and N. C. Martins (2009): “Lending Relationships in the Interbank Market”, *Journal of Financial Intermediation*, 18, 24-48.
- Condorelli, D. and A. Galeotti (2012): “Bilateral Trading in Networks,” working paper.
- Corominas-Bosch, M. (2004): “Bargaining in a Network of Buyers and Sellers,” *Journal of Economic Theory* 115, 1, 35-77.
- Craig, B., and G. Peter (2010): “Interbank Tiering and Money Center Banks,” BIS Working Paper, 322.
- Donoghue, W. F. Jr. (1974): *Monotone Matrix Functions and Analytic Continuation*. Springer-Verlag, New York, Heidelberg, Berlin.
- Duffie, D. (2012): *Dark Markets*, Princeton Lectures in Finance, Y.Ait-Sahalia (ed.), Princeton University Press.
- Duffie, D. (2013): “Futurization of Swaps. Stanford. Duffie Offers Another Viewpoint on This Emerging Trend,” Bloomberg Government, BGOV Analysis: Counterpoint, January 28, 2013.
- Duffie, D., N. Garleanu and L. H. Pedersen (2005): “Over-the-Counter Markets,” *Econometrica*, 73, 1815-1847.
- Duffie, D., N. Garleanu and L. H. Pedersen (2007): “Valuation in Over-the-Counter Markets,” *Review of Financial Studies*, 20, 5, 1865-1900.
- Duffie, D., S. Malamud and G. Manso (2009): “Information Percolation with Equilibrium Search Dynamics,” *Econometrica*, 77, 1513-1574.

- Duffie, D., S. Malamud and G. Manso (2013): “Information Percolation in Segmented Markets,” working paper.
- Elliott, M. (2011): “Search with Multilateral Bargaining,” working paper.
- Elliott, M. (2012): “Inefficiencies in Networked Markets,” working paper.
- Fainmesser, I. (2012): “Intermediation and Exclusive Representation in Financial Networks,” working paper.
- Financial Stability Board (2010): “Implementing OTC Derivatives Market Reforms”.
- Gale, D. (1986a): “Bargaining and Competition Part I: Characterization,” *Econometrica*, 54, 785-806.
- Gale, D. (1986b): “Bargaining and Competition Part II: Existence,” *Econometrica*, 54, 807-818.
- Gale, D. and S. Kariv (2007): “Financial Networks,” *American Economic Review*, 97, 2, 99-103.
- Glode, V., and C. Opp (2013): “Adverse Selection and Intermediation Chains,” working paper.
- Hendershott, T. and A. Madhavan (2012): “Click or Call? Auction Versus Search in the Over-the-Counter Market,” working paper.
- Horn, R. A. and C. R. Johnson (2013): *Matrix Analysis*, 2nd edition, Cambridge University Press.
- Jackson, M. and Y. Zenou (2012): “Games on Networks,” forthcoming in *Handbook of Game Theory Vol. 4*, P. Young and S. Zamir (eds.), Elsevier Science.
- Klemperer, P. D. and M. A. Meyer (1989): “Supply Function Equilibria in Oligopoly under Uncertainty,” *Econometrica*, 57, 6, 1243-1277.
- Knight Capital Group (2010): *Current Perspectives on Modern Equity Markets: A Collection of Essays by Financial Industry Experts*: Knight Capital Group, Inc.
- Kranton, R. E. and D. F. Minehart (2001): “A Theory of Buyer-Seller Networks,” *American Economic Review*, 91, 3, 485-508.
- Kyle, A. S. (1989): “Informed Speculation and Imperfect Competition,” *Review of Economic Studies*, 56, 517-556.
- Lagos, R. and G. Rocheteau (2009): “Liquidity in Asset Markets with Search Frictions,” *Econometrica*, 77, 2, 403-426.
- Lagos, R., G. Rocheteau and P.-O. Weill (2011): “Crises and Liquidity in Over-the-Counter Markets,” *Journal of Economic Theory*, 146, 6, 2169-2205.
- Li, D. and N. Schürhoff (2012), “Dealer Networks,” working paper.
- Litan, R. (2013): “Futurization of Swaps. A Clever Innovation Raises Novel Policy Issues for Regulators,” Bloomberg Government, BGOV Analysis, January 14, 2013.
- Malamud, S. and M. Rostek (2013): “Decentralized Market Design,” working paper.
- Manea, M. (2011): “Bargaining on Stationary Networks,” *American Economics Review*, 101, 5, 2042-80.

- Milgrom, P. and J. Roberts (1990): “Rationalizability, Learning and Equilibrium in Games With Strategic Complementarities,” *Econometrica*, 58, 6, 1255-1278.
- Nava, F. (2011): “Efficiency in Decentralized Oligopolistic Markets,” working paper.
- O’Hara, M. and M. Ye (2011): “Is Market Fragmentation Harming Market Quality?,” *Journal of Financial Economics*, 100, 3, 459-474.
- Preece, R. (2013): “The Pros and Cons of Dark Pools of Liquidity,” *Financial Times*, www.ft.com, January 6, 2013.
- Rahi, R. and J.-P. Zigrand (2013): “Market Quality and Contagion in Fragmented Markets,” working paper.
- Rostek, M. and M. Weretka (2011): “Dynamic Thin Markets,” working paper.
- Schachar, O. (2013): “Exposing The Exposed: Intermediation Capacity in the Credit Default Swap Market,” Federal Reserve Bank of New York, working paper.
- Schapiro, L. M. (2010): Strengthening Our Equity Market Structure, U.S. SEC, New York, <http://www.sec.gov/news/speech/2010/spch090710mls.htm>.
- Vayanos, D. (1999): “Strategic Trading and Welfare in a Dynamic Market,” *Review of Economic Studies*, 66, 219-254.
- Vayanos, D. and P.-O. Weill (2008): “A Search-based Theory of the On-the-run Phenomenon,” *Journal of Finance*, 63, 1351-1389.
- Vives, X. (2011): “Strategic Supply Function Competition with Private Information”, *Econometrica* 79, 6, 1919-1966.
- Weill, P.-O. (2008): “Liquidity Premia in Dynamic Bargaining Markets,” *Journal of Economic Theory*, 140, 66-96.
- Weretka, M. (2011): “Endogenous Market Power,” *Journal of Economic Theory*, 146, 6, 2281-2306.

A Appendix I

A. Equilibrium Existence and Uniqueness

A profile of demand functions $\{q_i(\cdot)\}_i$ is a *robust Nash Equilibrium* if, for each trader i , $q_i(p_{N(i)})$ is a best response given $\{q_j(p_{N(j)})\}_{j \neq i}$ for an arbitrary additive noise.

Proof of Theorem 2.1. Consider market $\mathbb{M} = \{\{\alpha_i, M_i, N(i)\}_i, (K, d, \mathcal{V})\}$ and let N be the total number of exchanges in \mathbb{M} . Let $q_i(p_{N(i)}, \Lambda_i)$ be trader i ’s optimal demand given his assumed price impact Λ_i , defined in Equation (2), given the quasilinear utility function (1): for all prices $p_{N(i)}$, i equalizes marginal utility and marginal payment,

$$d_{N(i)} - (\alpha_i \mathcal{V}_{N(i)} + \Lambda_i) q_i(p_{N(i)}, \Lambda_i) = p_{N(i)} + \alpha_i \mathcal{V}_{N(i)} q_i^0. \quad (16)$$

For the “if” part, fix demand schedules $\{q_i(p_{N(i)}, \hat{\Lambda}_i)\}_i$ submitted by all traders given their assumed price impact $\hat{\Lambda}_i$. Assume that the Jacobian of each trader’s residual supply, defined by $\{q_j(p_{N(j)}, \hat{\Lambda}_j)\}_{j \neq i}$, is $\hat{\Lambda}_i = -(\sum_{j \neq i} D_p q_j(\cdot, \hat{\Lambda}_j))_{N(i)}^{-1}$, for each i . The price \hat{p}_n that clears exchange n is determined by $\sum_{i \in I(n)} q_i(p_n, p_{N(i) \setminus \{n\}}, \hat{\Lambda}_i) = 0$. Since for each i and n , demand $q_i(\cdot, \hat{\Lambda}_i)$ satisfies condition (16) for all prices p_n , it does so for the exchange-clearing price \hat{p}_n . By the global concavity of the maximization problem of each trader, demand functions $\{q_i(p_{N(i)}, \hat{\Lambda}_i)\}_i$ are best responses at \hat{p}_n . With (nondegenerate support) uncertainty about the traders’ initial holdings, trader i ’s residual supply with slope $\hat{\Lambda}_i$ has a stochastic intercept. Since, for each i and n , condition (16) holds for all prices p_n , it holds for each realization of the residual supply. Hence, given the global concavity, $\{q_i(p_{N(i)}, \hat{\Lambda}_i)\}_i$ is a Nash equilibrium with an arbitrary additive noise and, thus, a robust Nash Equilibrium.

For the “only if” part, suppose that traders submit demand functions $\{q_i(p_{N(i)}, \hat{\Lambda}_i)\}_i$ such that price impact $\hat{\Lambda}_i \neq -(\sum_{j \neq i} D_{p_{N(j)}} q_j(\cdot, \hat{\Lambda}_j))^{-1}$ for some i . Then, schedule $q_i(p_{N(i)}, \hat{\Lambda}_i)$ is not a best response to $\{q_j(p_{N(j)}, \hat{\Lambda}_j)\}_{j \neq i}$ at the exchange-clearing price \hat{p}_n , defined by $\sum_{i \in I(n)} q_i(\hat{p}_n, p_{N(i) \setminus \{n\}}, \hat{\Lambda}_i) = 0$. With an additive perturbation, by the linearity of demands, for each trader i , for almost all prices p_n , $d_{N(i)} - (\alpha_i \mathcal{V}_{N(i)} + \Lambda_i) q_i(p_{N(i)}, \Lambda_i) \neq p_{N(i)} + \alpha_i \mathcal{V}_{N(i)} q_i^0$, where Λ_i is the Jacobian of the residual supply of trader i . The prices for which the equality is violated have measure one and the schedule that equalizes marginal utility with marginal payment for all prices p_n (i.e., satisfies condition (16)) gives a strictly higher utility for measure one of noise realizations (and, hence, a strictly higher expected utility). It follows that $q_i(p_{N(i)}, \hat{\Lambda}_i)$ is not a robust best response and noise exists for which $\{q_i(p_{N(i)}, \hat{\Lambda}_i)\}_i$ is not a robust Nash Equilibrium. ■

We will need the following auxiliary Lemma.⁵³

Lemma A.1 *For a positive definite matrix A , $A^{-1} \geq \bar{A}_{N(i)}^{-1}$.*

Let $D = \text{diag}(z)$, $z \in \mathbb{R}^N$, be a diagonal matrix. Multiplication of (4) by $D_{N(i)}$ from the left and from the right gives the following scale invariance property of price impacts.

Lemma A.2 *Let $D = \text{diag}(z)$, $z \in \mathbb{R}^N$, be a diagonal matrix and $\mathcal{V}' = DVD$. Then, the map $\{\Lambda_i\}_i \rightarrow \{D_{N(i)} \Lambda_i D_{N(i)}\}$ defines a one-to-one correspondence between equilibria in markets defined by \mathcal{V} and \mathcal{V}' , respectively.*

Proof of Proposition 2.1. The fact that this is indeed an equilibrium follows by direct calculation. For the uniqueness, diagonalize \mathcal{V} by multiplying Equations (4) from left and right by $\mathcal{V}^{-1/2}$ and denote $\tilde{\Lambda}_i = \mathcal{V}^{-1/2} \Lambda_i \mathcal{V}^{-1/2}$,

$$\tilde{\Lambda}_i = \left(\left(\sum_{j \neq i} (\alpha_j \text{Id} + \tilde{\Lambda}_j)^{-1} \right)^{-1} \right)_{N(i)}, \quad i \in I. \quad (17)$$

⁵³This is a folklore result in linear algebra. See Online Appendix for a Proof.

One expects that any solution to this equation is of the form $\tilde{\Lambda}_i = \beta_i \text{Id}$ for some $\beta_i > 0$, $i \in I$, and consequently $\Lambda_i = \beta_i \mathcal{V}$. Lemma C.3 in the Online appendix shows that this is indeed the case. ■

Let \mathcal{S}^I be the set of I -tuples $\{\Lambda_i\}_i$ of positive semidefinite matrices with $\Lambda_i \in \mathbb{R}^{N(i) \times N(i)}$. On this set, we introduce a partial order: $\{\Lambda_i\}_i \leq \{\Lambda'_i\}_i$ for a pair of tuples $\{\Lambda_i\}_i, \{\Lambda'_i\}_i$ if $\Lambda_i \leq \Lambda'_i$ for all $i \in I$. Recalling that the negative X_i of the slope of i 's demand and i 's price impact are linked through $X_i \equiv (\alpha_i \mathcal{V}_{N(i)} + \Lambda_i)^{-1}$, we can rewrite the fixed point condition (4) as a fixed point condition for demand slopes, as follows.⁵⁴ Define map $G = \{G_i\}_i : \mathcal{S}^I \rightarrow \mathcal{S}^I$ via

$$G_i(\{X_i\}_i) = \left(\left(\sum_{j \neq i} \bar{X}_j \right)^{-1} \right)_{N(i)} + \alpha_i \mathcal{V}_{N(i)} \Big)^{-1}, \quad i \in I. \quad (18)$$

Essentially, the decentralized market model can be seen as a game in which agents choose their demand slopes. Let us denote by $G^n(\{\Lambda_i\}_i)$ the n^{th} iteration of the best response map. Standard properties of the positive semidefinite order imply that map G is monotone increasing in $\{X_i\}_i$.

Lemma A.3 *Map G is monotone increasing on \mathcal{S}^I .*

Let $F = \{F_i\}_i : \mathcal{S}^I \rightarrow \mathcal{S}^I$ be the map defined by the right-hand side of (4). By construction, the maps F and G are simple transformation of each other. However, analytically, it is more convenient to work directly with the map F and all the proofs in the sequel use this map. Passing from F to G is then straightforward. The following result then follows directly from Theorem 2.1.

Lemma A.4 *A tuple of linear demand schedules with slopes $\{X_i\}_i$ is an equilibrium if, and only if, it is a fixed point of the best response map. That is, $\{X_i\}_i = G(\{X_i\}_i)$.*

Existence of equilibrium follows directly from the following result.⁵⁵

Proposition A.1 (Monotone Convergence) *Pick an arbitrary starting tuple $\{X_i^0\}_i$ such that $\{X_i^0\}_i \leq G(\{X_i^0\}_i)$ ($\{X_i^0\}_i \geq G(\{X_i^0\}_i)$). Then, iterations $G^n(\{X_i^0\}_i)$ are monotone increasing (decreasing) in n and converge to an equilibrium tuple.*

Proof of Proposition 2.2. By standard arguments based on the Sard Theorem and the analytic implicit function theorem, it suffices to show that, locally, we can always find a small perturbation of the covariance matrix such that price impacts do not commute. This follows by direct calculation, using the expressions in Lemma C.1. ■

⁵⁴ Note that we are assuming that equilibrium is symmetric within a class: all agents from the same class i submit identical schedules. Lemma D.2 in the Appendix implies that this is indeed the case.

⁵⁵By Proposition A.1, equilibrium exists if we can find a tuple $\{X_i^0\}_i$ satisfying the inequality conditions. We explicitly construct such a tuple. See Online Appendix for a proof

B Equivalence Loops: Proof of Theorems 3.1 and 3.2

Proof of Theorems 3.1 and 3.2. For notational convenience, in this proof, we relabel exchanges in which multiple assets are traded by creating a separate exchange for each asset without changing participation. That is, in the new notation, a single asset is traded in every exchange.

The proof of the “if” part (liquidity in two exchanges coincides *if* there is an equivalence loop connecting these two exchanges) consists of several steps. Our goal is to show that, for any price impact Λ , the rows corresponding to n and n' coincide. This is done in Lemmas B.1 and B.2. The proof of Lemma B.1 is established through a sequence of auxiliary results. The goal is to show that price impacts are singular. To this end, we use Lemma A.1 to conclude that price impacts in a market are lower than those in a smaller market in which only trading along the loop is possible. To proceed further, we need to get hold of equilibria in this simpler “single loop” market. Since we cannot exclude existence of asymmetric equilibria, we use the concavity of map F to reduce the problem to studying symmetric equilibria. The latter can be characterized explicitly as solutions to a simple algebraic system.

Lemma B.1 *Suppose that an asset k is traded along an equivalence loop $n = n_1, n_2, \dots, n_L = n'$. Then, for any $i \in I(n_l) \cap I(n_{l+1})$, price impact $(\Lambda_i)_{n_1 \cup n_2}$ is proportional to $\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}$.*

Proof. For simplicity, we normalize variance \mathcal{V}_{kk} to 1. As in the proof of Theorem 2.2, we add an ε to the diagonal of \mathcal{V} and then study the limit as $\varepsilon \downarrow 0$. Let $a_l, l \in L$, be the traders who define an equivalence loop and let $\Lambda_l^c \equiv (\Lambda_l)_{n_l \cup n_{l+1}} \in \mathbb{R}^{2 \times 2}$ their price impacts in the corresponding exchanges. By assumption, there are at least three agents trading in every exchange. Denote by A_l the projection of their demand slope on the exchange n_l . Since all $\{\Lambda_i\}_i$ are uniformly bounded from above, the demand slopes A_l are uniformly bounded away from zero by a constant $A > 0$, independent of ε . Furthermore, let also $\alpha = \max_l \alpha_l$ be the maximal risk aversion for the agents in the sequence that defines the equivalence loop. Denote by $\hat{\Lambda}_l^c$ the corresponding price impacts, lifted only to the collection of exchanges in the equivalence loop. Then, by Lemma A.1,

$$\Lambda_l^c \leq \left(\left(\sum_{k \neq l} (\hat{\Lambda}_k^c + \alpha(\mathbf{1} + \varepsilon \text{Id}))^{-1} + A \text{Id} \right)^{-1} \right)_{n_l \cup n_{l+1}}. \quad (19)$$

Denote by \hat{F} the map on the right-hand side of this inequality. Then, we can rewrite (19) as $\{\Lambda_l^c\}_l \leq \hat{F}(\{\Lambda_l^c\}_l)$. Let $Z \equiv \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$. Then, it follows from Theorem 5 in Anderson (1971) that $\{Z\Lambda_l^c Z\}_l \leq \hat{F}(\{Z\Lambda_l^c Z\}_l)$. Let $\Lambda_l^a \equiv 0.5(\Lambda_l^c + Z\Lambda_l^c Z)$, $l = 1, \dots, L$. Then, Theorem 24 in Anderson and Duffin (1969) implies that $\{\Lambda_l^a\}_l \leq \hat{F}(\{\Lambda_l^a\}_l)$. Note that, for each matrix Λ_l^a , the two diagonal elements are identical. Note further that inequalities (19) are symmetric with respect to Λ_l .⁵⁶

⁵⁶ In fact, the equivalence loop is a circle, and so a rotation of the loop moves the classes without affecting their price impacts.

Therefore, adding up the inequalities $\Lambda_l^a \leq \hat{F}_l(\{\Lambda_k^a\}_k)$ and using the concavity of map F (based on Theorem 24 in Anderson and Duffin (1969)), we get that $\{\Lambda_l^{av}\}_l \leq \hat{F}(\{\Lambda_l^{av}\}_l)$, where $\Lambda_l^{av} = \frac{1}{L} \sum_{l=1}^L \Lambda_l^a$ is independent of l .⁵⁷

Proposition A.1 implies that iterations $\hat{F}^n(\{\Lambda_l^{av}\}_l)$, $n \geq 1$, converge to a fixed point $\{\Lambda_l^*\}_l$ satisfying $\{\Lambda_l^{av}\}_l \leq \{\Lambda_l^*\}_l$. Furthermore, by symmetry, $\Lambda_l^* = Z\Lambda_l^*Z$ for all l and Λ_l^* is independent of l . That is, these matrices are all identical and have identical diagonal elements. Let

$$(\hat{\Lambda}_l^c + \alpha(\mathbf{1} + \varepsilon\text{Id}))^{-1} = \begin{pmatrix} x_1 & x_2 \\ x_2 & x_1 \end{pmatrix}, \quad l = 1, \dots, L.$$

A direct (but tedious) calculation, based on Lemma D.2, implies that

$$x_1 - x_2 = \varepsilon^{-1}g(\varepsilon/w), \quad (20)$$

where $w = y_1 - y_2$ and

$$Y = \begin{pmatrix} y_1 & y_2 \\ y_2 & y_1 \end{pmatrix} = \left((\hat{B} + A\text{Id})^{-1} \right)_{n_1 \cup n_2},$$

where $\hat{B} = \sum_{l=1}^L (\hat{\Lambda}_l^c + \alpha(\mathbf{1} + \varepsilon\text{Id}))^{-1}$. We now show that, in the limit as $\varepsilon \rightarrow 0$, we have $w \rightarrow 0$. Suppose the contrary. Then, by the definition of the function $g(a)$, we have $\varepsilon^{-1}g(\varepsilon/w) \rightarrow w^{-1}$ as $\varepsilon \rightarrow 0$, and hence, $x_1 - x_2 \rightarrow (y_1 - y_2)^{-1}$. A direct calculation implies that

$$y_1 = \frac{1}{N} \sum_{n=0}^{N-1} \frac{1}{2x_1 + A + 2x_2 \cos(2\pi n/N)} \quad (21)$$

$$y_2 = \frac{1}{N} \sum_{n=0}^{N-1} \frac{\cos(2\pi n/N)}{2x_1 + A + 2x_2 \cos(2\pi n/N)}. \quad (22)$$

and a simple monotonicity arguments implies that $y_1 - y_2 < \frac{1}{x_1 - x_2}$, which is a contradiction. Hence we conclude that $w \rightarrow 0$ when $\varepsilon \rightarrow 0$. But this cannot happen if x_1, x_2 stay bounded in this limit. Thus, x_1 must blow up, that is, the matrix $\hat{\Lambda}_l^c + \alpha\mathbf{1}$ becomes singular when $\varepsilon \rightarrow 0$. But this can only happen if $\hat{\Lambda}_l^c$ is proportional to $\mathbf{1}$. Indeed, $\mathbf{1}$ is a projection onto $\begin{pmatrix} 1 \\ 1 \end{pmatrix}$. If $\hat{\Lambda}_l^c$ is not proportional to $\mathbf{1}$, it has an eigenvector not proportional to $\begin{pmatrix} 1 \\ 1 \end{pmatrix}$, implying that the image of $\hat{\Lambda}_l^c + \alpha\mathbf{1}$ spans \mathbb{R}^2 , and hence the invertibility. We conclude that $\hat{\Lambda}_l^c = \kappa\mathbf{1}$ for some $\kappa > 0$. The inequality $\Lambda_l^m \leq \hat{\Lambda}_l^c$ implies that Λ_l^m cannot have any eigenvectors except for $\begin{pmatrix} 1 \\ 1 \end{pmatrix}$ and hence $\Lambda_l^m = \kappa_1\mathbf{1}$ for some $\kappa_1 > 0$. The inequality

$$\Lambda_l^c \leq \frac{1}{2L} \Lambda_l^a \leq \frac{1}{2L} \sum_l \Lambda_l^a = \Lambda_l^m = \kappa_1\mathbf{1}$$

implies by the same argument that Λ_l^c is proportional to $\mathbf{1}$. ■

⁵⁷ I.e., tuple $\{\Lambda_l^{av}\}$ consists of identical elements, all equal to the average of Λ_l^a .

Lemma B.2 *Suppose that $\Lambda \in \mathbb{R}^{N \times N}$ is a symmetric, positive semidefinite matrix. If $\Lambda_{11} = \Lambda_{22} = \Lambda_{12} = \lambda$ then the first two rows of Λ coincide.*

Proof. Indeed, for any $i > 2$, the sub-matrix

$$\begin{pmatrix} \Lambda_{11} & \Lambda_{12} & \Lambda_{1i} \\ \Lambda_{12} & \Lambda_{22} & \Lambda_{2i} \\ \Lambda_{1i} & \Lambda_{2i} & \Lambda_{ii} \end{pmatrix}$$

must be positive semidefinite. This implies that

$$\begin{pmatrix} \lambda & \lambda \\ \lambda & \lambda \end{pmatrix} - \begin{pmatrix} \Lambda_{1i} \\ \Lambda_{2i} \end{pmatrix} \Lambda_{ii}^{-1} (\Lambda_{1i}, \Lambda_{2i})$$

is positive semidefinite, which is only possible if $\Lambda_{1i} = \Lambda_{2i}$. ■

It follows that the rows of Λ_l corresponding to exchanges n_l and n_{l+1} coincide. Therefore, for any vector in the image of $\Lambda_l + \alpha_l \mathcal{V}_{N(l)}$, the corresponding coordinates are equal; that is, prices in the two exchanges coincide and the equivalence result of Theorem 3.2 also follows.

Suppose now that there is no equivalence loop connecting exchanges n and n' . Treating the set of exchanges in which asset k is traded as a graph, where two vertices (exchanges) are connected if there is an agent who trades in both exchanges, consider its associated regularized graph obtained using the procedure described in the main text. On this tree, there is a unique path connecting n and n' . Since, by assumption, n and n' are not connected by a loop, an agent i exists on this path who, if removed from n and n' , the exchanges become isolated. Let $N = N_1 \cup N_2$ be the disjoint partition of exchanges that is obtained if we remove agent i from the market. By the equilibrium Equation (4), in the decomposition $\mathbb{R}^N = \mathbb{R}^{N_1} \oplus \mathbb{R}^{N_2}$, price impact of agent i is block diagonal. Furthermore, since no other agent can trade simultaneously in exchanges from both N_1 and N_2 , by (4), price impacts of all other agents are also block diagonal. Thus, no agent is able to equalize prices in these exchanges, and the claim follows. ■

Corollary B.1 *The only singularity in $\Lambda_i + \alpha_i \mathcal{V}_{N(i)}$ that may occur in equilibrium is when some rows and columns of this matrix are identical.*

Proof. The claim follows directly from the proof of Theorem 3.1 above: After the regularization procedure (i.e., the procedure of removing identical rows and columns from $\Lambda_i + \alpha_i \mathcal{V}_{N(i)}$), the resulting matrices are nonsingular. ■

II. Online Appendix

C Proofs omitted from the main text

Proof of Lemma A.1. Let $B = A^{-1}$, $x \in \mathbb{R}^{N(i)}$ and $y \in Y^{N \setminus N(i)}$. Then,

$$\min_{y \in \mathbb{R}^{N \setminus N(i)}} \langle B(x, y), (x, y) \rangle = \langle (B_{11} - B_{12}B_{22}^{-1}B_{21})x, x \rangle. \quad (23)$$

By the Frobenius formula (Lemma D.4), $B_{11} - B_{12}B_{22}^{-1}B_{21} = ((B^{-1})_{11})^{-1} = A_{11}^{-1}$. Therefore,

$$\langle A^{-1}(x, y), (x, y) \rangle = \langle B(x, y), (x, y) \rangle \geq \langle A_{11}^{-1}x, x \rangle = \langle \bar{A}_{11}^{-1}(x, y), (x, y) \rangle,$$

for any $(x, y) \in \mathbb{R}^N$, and the claim follows since $A_{11} = A_{N(i)}$. ■

Proof of Lemmas A.4 and A.3. By Theorem 2.1, $\{\Lambda_i\}_i$ is an equilibrium if and only if $\{\Lambda_i\}_i = F(\{\Lambda_i\}_i)$. By direct calculation, defining $X_i = (\Lambda_i + \alpha_i \mathcal{V}_{N(i)})^{-1}$, we get that $\{\Lambda_i\}_i = F(\{\Lambda_i\}_i)$ if and only if $\{X_i\}_i = G(\{X_i\}_i)$. The fact that both F and G are monotone increasing follows because the map $Y \rightarrow Y^{-1}$ is monotone decreasing, whereas $Y \rightarrow Y_{N(i)}$ is monotone increasing in the positive semi-definite order. ■

Proof of Proposition A.1. Pick an arbitrary starting tuple $\{X_i^0\}_i$ such that $\{X_i^0\}_i \leq G(\{X_i^0\}_i)$. By direct calculation, the corresponding price impacts $\Lambda_i^0 = (X_i^0)^{-1} - \alpha_i \mathcal{V}_{N(i)}$ satisfy $\{\Lambda_i^0\}_i \geq F(\{\Lambda_i^0\}_i)$. Since map F is continuous and monotonic with respect to the defined partial order, recursively applying F to the inequality $\{\Lambda_i^0\}_i \geq F(\{\Lambda_i^0\}_i)$, we get that $F^n(\{\Lambda_i^0\}_i)$ is monotone decreasing and, therefore, converges to a fixed point of F . For the second case, we get a price impact tuple satisfying $\{\Lambda_i^0\}_i \leq F(\{\Lambda_i^0\}_i)$, so that the sequence $F^n(\{\Lambda_i^0\}_i)$ is monotone increasing. Therefore, to prove convergence to a fixed point, we need to show that it is bounded from above. To this end, pick $\alpha > 0$ sufficiently large so that $\{\tilde{\Lambda}_i\}_i$ defined by

$$\tilde{\Lambda}_i = \alpha \text{diag}((M(n) - 2)^{-1})_{n \in N(i)} \quad (24)$$

satisfies $\{\Lambda_i^0\}_i \leq \{\tilde{\Lambda}_i\}_i$, where $M(n)$ is the total number of agents in exchange n . An analogous argument implies that

$$F(\{\tilde{\Lambda}_i\}_i) \leq \{\tilde{\Lambda}_i\}_i.$$

Let $\Omega = \{\{\Lambda_j\}_j \in \mathcal{S}^M : \Lambda_j \leq \tilde{\Lambda}_j, \forall j\}$. Then, for any $\{\Lambda_j\}_j \in \Omega$,

$$F(\{\Lambda_j\}_j) \leq F(\{\tilde{\Lambda}_j\}_j) \leq \{\tilde{\Lambda}_j\}_j \quad (25)$$

and, hence, F maps Ω into itself. Therefore, the sequence $F^n(\{\Lambda_i^0\}_i)$ is monotone increasing, bounded from above by $\{\tilde{\Lambda}_i\}_i$, and hence converges to a fixed point of F . ■

Proof of Theorem 2.2. Existence of equilibria for the case when $\mathcal{V}_{N(i)}$ is nonsingular for any i follows from Proposition A.1. For the general case, let $\mathcal{V}^\varepsilon \equiv \mathcal{V} + \varepsilon \text{Id}$ and let F^ε be the corresponding

map. By Proposition A.1, for any $\varepsilon > 0$, there exists an equilibrium $\{\Lambda_i^\varepsilon\}_i$ corresponding to \mathcal{V}^ε . Pick a sequence $\varepsilon_k = 1/k$ and an equilibrium $\{\Lambda_i^{\varepsilon_k}\}_i$. Since F^ε is monotone increasing in ε , we have

$$\{\Lambda_i^{\varepsilon_k}\}_i = F^{\varepsilon_k}(\{\Lambda_i^{\varepsilon_k}\}_i) \geq F^{\varepsilon_{k+1}}(\{\Lambda_i^{\varepsilon_k}\}_i)$$

and, hence, by Proposition A.1, there exists an equilibrium $\{\Lambda_i^{\varepsilon_{k+1}}\}_i \leq \{\Lambda_i^{\varepsilon_k}\}_i$. Thus, we can construct a monotone decreasing sequence $\{\Lambda_i^{\varepsilon_k}\}_i$ which converges to an equilibrium corresponding to $\varepsilon = 0$.

To prove generic determinacy, let, for each i , $X_i \equiv \Lambda_i + \alpha_i \mathcal{V}_{N(i)}$, and define a map $\Phi : \mathcal{S}^I \rightarrow \mathcal{S}^I$ via

$$\Phi_i(\{X_j\}_j) = X_i - \left(\left(\sum_{j \neq i} X_j^{-1} \right)^{-1} \right)_{N(i)}.$$

The equilibrium equation can be written as $\Phi(\{X_j\}_j) = \{\alpha_j \mathcal{V}_{N(j)}\}_j$. Let Ψ^I be the image of the map $\mathcal{V} \rightarrow \{\alpha_i \mathcal{V}_{N(i)}\}_i$ defined on the set of positive semidefinite matrices. Let Θ^I be the subset of \mathcal{S}^I such that, for any $\{X_j\}_j \in \Theta^I$, we have that $\Phi(\{X_j\}_j) = \{\alpha_j \mathcal{V}_{N(j)}\}_j$ for some positive definite matrix \mathcal{V} . Θ^I is an algebraic variety and, therefore, can be represented as a finite union of irreducible algebraic sets that are smooth manifolds. The same is true for Ψ^I . By Sard's theorem, almost every $\{\alpha_i \mathcal{V}_{N(i)}\}_i \in \Psi^I$ has a regular pre-image under Φ , that is, equilibria are determinate for generic covariance matrices. The finiteness of the set of equilibria follows by the standard compactness arguments and the fact that, by Proposition 2.3, all equilibria belong to a compact set. ■

Equilibrium uniqueness is equivalent to the uniqueness of the fixed point of map F . It therefore suffices to show that F is a contraction on a suitably defined normed space. We can identify the strategy of an agent in the game (i.e., his demand schedule) with its slope $(\alpha_i \mathcal{V}_{N(i)} + \Lambda_i)^{-1}$, and find conditions on the demand slopes.

Lemma C.1 *For any i , suppose that $0 \leq \{B_i\}_i \leq \{A_i\}_i$ are such that any equilibrium tuple $\{\Lambda_i\}_i$ satisfies $\{B_i\}_i \leq \{(\alpha_i \bar{\mathcal{V}}_{N(i)} + \bar{\Lambda}_i)^{-1}\}_i \leq \{A_i\}_i$. Suppose that, for any $\{X_i\}_i$, $\{B_i\}_i \leq \{X_i\}_i \leq \{A_i\}_i$,*

$$(M_j - 1)X_j^2 + \sum_{i \neq j} M_i X_i^2 < \left((M_j - 1)X_j + \sum_{i \neq j} M_i X_i \right)^2, \quad j = 1, \dots, I. \quad (26)$$

Then, map F is a contraction on the set $\{B_i\}_i \leq \{X_i\}_i \leq \{A_i\}_i$ and, hence, there exists a unique equilibrium.

Proof of Lemma C.1. Let us calculate the derivative of map F . That is, consider an infinitesimal change $\{\Lambda_i\}_i \rightarrow \{\Lambda_i + \varepsilon Y_i\}_i$. Then, a direct calculation based on the identity

$$(U + \varepsilon V)^{-1} \approx U^{-1} - \varepsilon U^{-1} V U^{-1}$$

implies that the Frechet derivate of F , $\frac{\partial F_j}{\partial \{\Lambda_i\}_i}(\{Y_i\}_i)$, is given by

$$\left(\left(\sum_{i \neq j} X_i \right)^{-1} \left(\sum_{i \neq j} X_i Y_i X_i \right) \left(\sum_{i \neq j} X_i \right)^{-1} \right)_{N(j)}.$$

Introduce a norm of the set of I -tuples of positive semidefinite matrices via $\|\{Y_i\}_i\| = \max_i \|Y_i\|_{N(i)}$, where $\|\cdot\|_{N(i)}$ is the standard norm on matrices in $\mathbb{R}^{N(i)}$ defined by

$$\|Y\| = \max_{x \in \mathbb{R}^N, x \neq 0} \frac{\|Yx\|}{\|x\|}.$$

For simplicity, in the sequel we omit the index $N(i)$ for the norms. For a symmetric matrix, $\|Y\| = \max |\text{eig}(Y)|$ and, therefore, $Y_i \in [-\|Y_i\| \text{Id}_{N(i)}, \|Y_i\| \text{Id}_{N(i)}]$. Suppose now that condition (26) holds. Then,

$$\left\| \left(\sum_{i \neq j} X_i \right)^{-1} \left(\sum_{i \neq j} X_i^2 \right) \left(\sum_{i \neq j} X_i \right)^{-1} \right\| \leq 1,$$

and hence

$$\begin{aligned} \frac{\partial F_j}{\partial \{\Lambda_i\}_i}(\{Y_i\}_i) &= \left(\left(\sum_{i \neq j} X_i \right)^{-1} \left(\sum_{i \neq j} X_i \|\{Y_i\}_i\| \text{Id}_{N(i)} X_i \right) \right)_{N(j)} \\ &= \left(\sum_{i \neq j} X_i \right)^{-1} < \|\{Y_i\}_i\| \text{Id}_{N(j)}. \end{aligned}$$

The same argument implies

$$\frac{\partial F_j}{\partial \{\Lambda_i\}_i}(\{Y_i\}_i) > -\|\{Y_i\}_i\| \text{Id}_{N(j)}.$$

That is,

$$\left\| \frac{\partial F_j}{\partial \{\Lambda\}_i}(\{Y_i\}_i) \right\| < \|\{Y_i\}_i\|.$$

That is, map F is a contraction on this set and, consequently, cannot have more that one fixed point. ■

Note that, when X_i are positive numbers (or commuting matrices, in which case they can be simultaneously diagonalized), a direct calculation implies that condition (26) holds. However, absent commutativity, this is generally not true. The usefulness of Lemma C.1 depends on a good choice of the bounds $\{B_i\}_i$ and $\{A_i\}_i$. The next result provides a simple and easily verifiable condition that guarantees the applicability of Lemma C.1, based on the choice $\{B_i\}_i = \{(\bar{\Lambda}_{i,\max}^0 + \alpha_i \bar{\mathcal{V}}_{N(i)})^{-1}\}_i$ and $\{A_i\}_i = \{(\bar{\Lambda}_{i,\min}^0 + \alpha_i \bar{\mathcal{V}}_{N(i)})^{-1}\}_i$.⁵⁸

⁵⁸ As an example, consider the case when all pairs of assets are equally correlated with correlation ρ , all

Corollary C.1 *Suppose that*

$$\min_n \frac{M(n) - 2}{\lambda^*(n)} \geq \max_n \frac{M(n) - 2}{(M(n) - 1)\lambda^*(n)}. \quad (27)$$

Then, equilibrium is unique.

Roughly, the left-hand side of (27) measures how competitive an exchange is, whereas the right-hand side reflects the dispersion of payoff riskiness across exchanges. If this dispersion is high, there is a lot of ‘room’ for non-commutativity and uniqueness can only be guaranteed when strategic effects are small; that is, when $M(n)$ is sufficiently large.

To proceed further, we first establish auxiliary results.

Lemma C.2 *If there is only one asset (and only one exchange), $K = 1$, then equilibrium is unique.*

Proof. The proof follows directly from Lemma C.1. Indeed, in this case, the conditions of Lemma C.1 hold and, therefore, map F is a contraction and has a unique fixed point. ■

Lemma C.3 *Let $\{A_i\}_i \in \mathcal{S}^I$ be a tuple of diagonal matrices. Consider the map $F_A : \mathcal{S}^I \rightarrow \mathcal{S}^I$ defined via*

$$F_i(\{\Lambda_j\}_j) = \left(\left(\sum_{j \neq i} (\bar{A}_j + \alpha_j \bar{\Lambda}_j)^{-1} \right)^{-1} \right)_{N(i)}.$$

Then, this map has a unique fixed point in the class of diagonal matrices.

Proof. The proof follows directly from Lemma C.1 because, on the set of diagonal matrices, F is a contraction. ■

Lemma C.4 *Let $\{A_i\}_i \in \mathcal{S}^I$ be a tuple of diagonal matrices. Then, map F_A from Lemma C.3 has a unique fixed point.*

Proof. Let $\{\Lambda_i^A\}_i$ be an arbitrary fixed point of F_A , and let $\{\Lambda_i^*\}_i$ be the diagonal fixed point, which is unique by Lemma C.3. Pick $\beta_1 \in \mathbb{R}_+$ so that β_1 satisfies $\beta_1 \text{Id}_{N(i)} \leq \Lambda_i^A$ for all i and $\beta_1 \leq \min_i \min(\text{eig}(A_i))$. Similarly, pick $\beta_2 \in \mathbb{R}_+$ so that β_2 satisfies $\beta_2 \text{Id}_{N(i)} \geq \Lambda_i^A$ for all i and $\beta_2 \geq \max_i \max(\text{eig}(A_i))$. Define $\{B_{1k}\}_i \equiv \{\beta_k \text{Id}_{N(i)}\}_i$, $k = 1, 2$, and let F_{B_k} , $k = 1, 2$ be the corresponding maps. Then, define $\{\Lambda_i^{B_k}\}_i = \beta_k \{\text{diag}((M(n) - 2)^{-1})_{N(i)}\}_i$. We have,

$$\{\Lambda_i^{B_k}\}_i = F^{B_k}(\{\Lambda_i^{B_k}\}_i).$$

agents have same risk aversion α , and $\max_i |N(i)| \leq \hat{N}$ Then, $\max(\text{eig}(C(\mathcal{V}_{N(i)}))) \leq \max\{1 + \rho(\hat{N} - 1), 1 - \rho\}$ and $\min(\text{eig}(C(\mathcal{V}_{N(i)}))) \geq \min\{1 + \rho(\hat{N} - 1), 1 - \rho\}$ and (27) thus imposes upper and lower bounds on the correlation ρ . For example, in the symmetric case when $M(n) = \hat{M}$ is independent of n and $\rho > 0$, we obtain the simple condition $\rho < \frac{\hat{M} - 2}{\hat{M} + \hat{N} - 2}$. For Corollary C.1, one could also pick $\{B_i\}_i = \{(\bar{\Lambda}_{i,\max}^k + \alpha_i \bar{\mathcal{V}}_{N(i)})^{-1}\}_i$ and $\{A_i\}_i = \{(\bar{\Lambda}_{i,\min}^k + \alpha_i \bar{\mathcal{V}}_{N(i)})^{-1}\}_i$, for any $k \geq 1$.

Therefore, iterating the inequality

$$\{\Lambda_i^{B_1}\}_i = F^{B_1}(\{\Lambda_i^{B_1}\}_i) \leq F_A(\{\Lambda_i^{B_1}\}_i),$$

we obtain that $F_A^n(\{\Lambda_i^{B_1}\}_i)$ converges to a diagonal fixed point of F_A , and hence, by Lemma C.3, converges to $\{\Lambda_i^*\}_i$. A similar argument implies that $F_A^n(\{\Lambda_i^{B_2}\}_i)$ also converges to $\{\Lambda_i^*\}_i$.

Now, by the definition of β_k , $k = 1, 2$, we also have

$$F^{B_1}(\{\Lambda_i\}_i) \leq F_A(\{\Lambda_i\}_i) \leq F^{B_2}(\{\Lambda_i\}_i),$$

for any $\{\Lambda_i\}_i \in \mathcal{S}^I$. Therefore, by the monotonicity of map F_A ,

$$F_A^n(\{\Lambda_i^{B_1}\}_i) \leq F_A^n(\{\Lambda_i^A\}_i) \leq F_A^n(\{\Lambda_i^{B_2}\}_i).$$

Taking $n \rightarrow \infty$ and using the fact that $F_A^n(\{\Lambda_i^A\}_i) = \{\Lambda_i^A\}_i$, we get $\{\Lambda_i^*\}_i \leq \{\Lambda_i^A\}_i \leq \{\Lambda_i^*\}_i$, and the claim follows. ■

Let $C(\mathcal{V}) = \text{diag}(\{\mathcal{V}_{nn}^{-1/2}\})\mathcal{V}\text{diag}(\{\mathcal{V}_{nn}^{-1/2}\})$ be the correlation matrix of the assets. For any exchange n and agent i , define $\alpha_{i,*} \equiv \alpha_i \min(\text{eig}(C(\mathcal{V})_{N(i)}))$ and $\alpha_i^* \equiv \alpha_i \max(\text{eig}(C(\mathcal{V})_{N(i)}))$; $\alpha_{i,*}$ and α_i^* can be interpreted as the bounds on the effective riskiness (see Section 5.1). For any exchange n , define two constants $\lambda^*(n) \equiv \min_{i \in I(n)} \alpha_{i,*}$ and $\bar{\lambda}^*(n) \equiv \max_{i \in I(n)} \alpha_i^*$ and let $M(n) = \sum_{i \in I(n)} M_i$ be the number of agents trading in n . Let further $\{\Lambda_{i,\min}^0\}_i = \left\{ \text{diag}\left(\frac{\lambda^*(n)\mathcal{V}_{nn}}{M(n)-2}\right)_{N(i)} \right\}_i$ and $\{\Lambda_{i,\max}^0\}_i = \left\{ \text{diag}\left(\frac{\bar{\lambda}^*(n)\mathcal{V}_{nn}}{M(n)-2}\right)_{N(i)} \right\}_i$ and $\{X_{i,\min}^0\}_i = \{(\alpha_i \mathcal{V}_{N(i)} + \Lambda_{i,\max}^0)^{-1}\}_i$, $\{X_{i,\max}^0\}_i = \{(\alpha_i \mathcal{V}_{N(i)} + \Lambda_{i,\min}^0)^{-1}\}_i$. A direct calculation implies that

$$\{X_{i,\min}^0\}_i \leq G(\{X_{i,\min}^0\}_i) \text{ and } \{X_{i,\max}^0\}_i \geq G(\{X_{i,\max}^0\}_i).$$

and, similarly,

$$\{\Lambda_{i,\min}^0\}_i \leq F(\{\Lambda_{i,\min}^0\}_i) \text{ and } \{\Lambda_{i,\max}^0\}_i \geq F(\{\Lambda_{i,\max}^0\}_i).$$

We now construct the minimal and maximal equilibria by the explicit iterative procedure described in Proposition A.1. To this end, define recursively two sequences $\{\Lambda_{i,\min}^k\}_i \in \mathcal{S}^I$ and $\{\Lambda_{i,\max}^k\}_i \in \mathcal{S}^I$, $k \geq 1$ via $\{\Lambda_{i,\min}^k\}_i \equiv F(\{\Lambda_{i,\min}^{k-1}\}_i)$ and $\{\Lambda_{i,\max}^k\}_i \equiv F(\{\Lambda_{i,\max}^{k-1}\}_i)$. By Proposition A.1, the sequence $\{\Lambda_{i,\min}^k\}_i$, $k \geq 0$, is monotone increasing, whereas $\{\Lambda_{i,\max}^k\}_i$, $k \geq 0$, is monotone decreasing and they converge to equilibria (the fixed points of map F) that we denote by $\{\Lambda_{i,\min}\}_i$ and $\{\Lambda_{i,\max}\}_i$, respectively. The corresponding demand slopes are determined via $\{X_{i,\min}\}_i = \{(\alpha_i \mathcal{V}_{N(i)} + \Lambda_{i,\max})^{-1}\}_i$, $\{X_{i,\max}\}_i = \{(\alpha_i \mathcal{V}_{N(i)} + \Lambda_{i,\min})^{-1}\}_i$.

Proof of Proposition 2.3. Pick an arbitrary equilibrium $\{\Lambda_i\}_i$. Then, for all $i \in I$,

$$\Lambda_i = \left(\left(\sum_{j \neq i} (\alpha_j \mathcal{V} + \bar{\Lambda}_j)^{-1} \right)^{-1} \right)_{N(i)} \leq \left(\left(\sum_{j \neq i} (\alpha_j^* \text{Id}_{N(j)} + \bar{\Lambda}_j)^{-1} \right)^{-1} \right)_{N(i)}.$$

Let F_A be the map corresponding to the right-hand side of (28). Then, iterating F_A and using Proposition A.1, we get that F_A has a fixed point $\{\Lambda_i^*\}_i$ satisfying $\{\Lambda_i\}_i \leq \{\Lambda_i^*\}_i$. By Lemma C.4, this is the unique diagonal fixed point. Then, Λ_i^* is diagonal, and, for any exchange n , the scalar price impacts $\{(\Lambda_i^*)_{nn}\}_i$ coincide with price impacts in a centralized exchange for a single asset with variance 1 and risk aversions α_i^* . The same iteration argument as above implies that these price impacts are monotone increasing in α_i^* and therefore satisfy

$$\Lambda_i^* \leq \frac{\lambda^*(n)}{M(n) - 2} \text{Id}_{N(i)}, \quad i \in I.$$

Therefore, by the monotonicity of map F ,

$$\{\Lambda_i\}_i = F^n(\{\Lambda_i\}_i) \leq F^n(\{\Lambda_{i,\max}^0\}_i) \rightarrow \{\Lambda_{i,\max}\}_i.$$

Similarly,

$$\Lambda_i = \left(\left(\sum_{j \neq i} (\alpha_j \mathcal{V} + \bar{\Lambda}_j)^{-1} \right)^{-1} \right)_{N(i)} \leq \left(\left(\sum_{j \neq i} (\alpha_{j,*} \text{Id}_{N(j)} + \bar{\Lambda}_j)^{-1} \right)^{-1} \right)_{N(i)}.$$

and the same argument as above implies that

$$\Lambda_i^* \geq \frac{\lambda^*(n)}{M(n) - 2} \text{Id}_{N(i)}, \quad i \in I,$$

and the same iterative procedure implies that

$$\{\Lambda_i\}_i = F^n(\{\Lambda_i\}_i) \geq F^n(\{\Lambda_{i,\min}^0\}_i) \rightarrow \{\Lambda_{i,\min}\}_i.$$

This completes the proof. ■

Everywhere in the sequel, we use the following convenient notation:

Notation. For any $x, y \in \mathbb{R}^N$, we write $y^T x = \langle x, y \rangle$.

Given a triplet X, A, B of symmetric matrices of same dimension, we use the notation $X \in [B, A]$ when $B \leq X \leq A$. We need the following auxiliary result.

Lemma C.5 *If $X \in [B, A]$ and $Xq = z$ then*

$$\langle q, z \rangle \geq \max\{\langle Bq, q \rangle, \langle A^{-1}z, z \rangle\}.$$

Proof. Since $X \geq B$, we have $\langle Bq, q \rangle \leq \langle Xq, q \rangle = \langle q, z \rangle$ and the first claim follows. To prove the second claim, pick an $\varepsilon > 0$. Then, $X \leq A$ implies $(X + \varepsilon \text{Id})^{-1} \geq (A + \varepsilon \text{Id})^{-1}$ and, therefore,

$$XA^{-1}X \leq X(X + \varepsilon \text{Id})^{-1}X.$$

Since $x^2(x + \varepsilon)^{-1}x \leq x$ of any $x \geq 0$, the functional calculus implies that $X(X + \varepsilon \text{Id})^{-1}X \leq X$. Taking the limit as $\varepsilon \downarrow 0$, we get $XA^{-1}X \leq X$. Therefore,

$$\langle A^{-1}z, z \rangle = \langle A^{-1}Xq, Xq \rangle = \langle XA^{-1}Xq, q \rangle \leq \langle Xq, q \rangle = \langle z, q \rangle,$$

and the proof is complete. ■

Define $M_i = \mathbf{1}_{i=j}$, $i = 1, \dots, I$.

Lemma C.6 Consider the function

$$\Psi_j(z_1, \dots, z_I) \equiv \sum_i \|z_i\|^2 - \left\| \sum_{i \neq j} z_i \right\|^2$$

and let $\mu(q) = \max\{\Psi_j(z_1, \dots, z_I) : z_i \in \mathbb{R}^{N^{(i)}}, \langle q, z_i \rangle \geq \max\{\langle \bar{B}_i q, q \rangle, \langle A_i^{-1} z_i, z_i \rangle\}, i \in I\}$. If

$$\max_{q \in \mathbb{R}^N} \mu(q) < 0,$$

then the conditions of Lemma C.1 are satisfied.

Proof. The claim follows directly from Lemma C.5 if we define $\bar{X}_i q = z_i$. ■

Lemma C.7 Let $a_i = \|A_i\|$ and $a = \max_{i \in I} a_i$. Suppose that

$$a \text{Id} \leq \sum_i \bar{B}_i.$$

Then, the hypothesis of Lemma C.6 is satisfied.

Proof. Pick a tuple $z_i \in \mathbb{R}^{N^{(i)}}, i \in I$, satisfying $\langle q, z_i \rangle \geq \max\{\langle \bar{B}_i q, q \rangle, \langle A_i^{-1} z_i, z_i \rangle\}, i \in I$. Then,

$$a_i^{-1} \|z_i\|^2 \leq \langle A_i^{-1} z_i, z_i \rangle \leq \langle q, z_i \rangle, i \in I.$$

Normalize q so that $\|q\| = 1$. Then, we can decompose $z_i = \langle q, z_i \rangle q + z_i^\perp$ with $z_i^\perp \in \mathbb{R}^N, \langle z_i^\perp, q \rangle = 0$. Let $\beta_i \equiv \langle q, z_i \rangle$. Then,

$$\left\| \sum_{i \neq j} z_i \right\|^2 = \left(\sum_i \beta_i \right)^2 + \left\| \sum_i z_i^\perp \right\|^2 \geq \left(\sum_i \beta_i \right)^2$$

and, therefore,

$$\begin{aligned} \Psi_j(z_1, \dots, z_I) &\equiv \sum_i \|z_i\|^2 - \left\| \sum_{i \neq j} z_i \right\|^2 \leq \sum_i a_i \beta_i - \left\| \sum_{i \neq j} z_i \right\|^2 \leq \sum_i a_i \beta_i - \left(\sum_i \beta_i \right)^2 \\ &\leq \left(\sum_i \beta_i \right) \left(a - \sum_i \beta_i \right). \end{aligned}$$

and the claim follows because, by assumption,

$$\sum_i \beta_i \geq \sum_i \langle q, z_i \rangle \geq \sum_i \langle q, \bar{B}_i q \rangle \geq a.$$

■

Proof of Corollary C.1. For simplicity, we work directly with the correlation matrix and assume that $\mathcal{V} = C(\mathcal{V})$. By Proposition 2.3, any equilibrium $\{\Lambda_i\}_i$ satisfies

$$\Lambda_{i,\min}^0 \leq \Lambda_{i,\min} \leq \Lambda_i \leq \Lambda_{i,\max} \leq \Lambda_{i,\max}^0, \quad i \in I.$$

Therefore,

$$\text{diag} \left(\frac{M(n) - 2}{(M(n) - 1)\lambda_*(n)} \right)_{N(i)} \leq (\Lambda_i + \alpha_i \mathcal{V}_{N(i)})^{-1} \leq \text{diag} \left(\frac{M(n) - 2}{(M(n) - 1)\lambda_*(n)} \right)_{N(i)}, \quad i \in I,$$

and the claim follows from Lemma C.7. ■

D Price Impact Characterization

We now use Lemma D.4 and the following result.

Lemma D.1 (Functional Calculus for Symmetric Matrices) *For any continuous function $f(x)$ and any symmetric matrix A , we can define $f(A)$ as follows. By the eigen-decomposition theorem, there exists an orthogonal matrix U and a diagonal matrix D such that $A = U^T D U$ where $D = \text{diag}(d_i)$ where d_i are the eigenvalues of A . Then,*

$$f(A) = U^T \text{diag}(f(d_i)) U.$$

In general, the matrix U is not unique. The uniqueness holds only if eigenvalues of A are all distinct. However, even if U is not unique, $f(A)$ is uniquely determined, and so it is well-defined. The following lemma explicitly links price impact Λ_i with the aggregate liquidity measure B . Let $f_1(a) = (2 - a + \sqrt{a^2 + 4})/2$ and $f(a) = f_1(a)/a$.

Lemma D.2 *Let $Y_i = (B^{-1})_{N(i)}$. Then,*

$$\Lambda = Y_i^{1/2} f_1(Y_i^{-1/2} \alpha_i \mathcal{V}_{N(i)} Y_i^{-1/2}) Y_i^{1/2}.$$

If $\mathcal{V}_{N(i)}$ is invertible then

$$\Lambda_i = \alpha_i \mathcal{V}_{N(i)}^{1/2} f(\alpha_i \mathcal{V}_{N(i)}^{1/2} Y_i^{-1} \mathcal{V}_{N(i)}^{1/2}) \mathcal{V}_{N(i)}^{1/2}. \quad (28)$$

Lemma is a direct consequence of the following auxiliary result.

Lemma D.3 *Let Y, Z be nonnegative definite, with Y positive definite. The unique positive definite symmetric matrix Λ solving*

$$\Lambda = (Y^{-1} - (Z + \Lambda)^{-1})^{-1}$$

is given by

$$\Lambda = Y^{1/2} f_1(Y^{-1/2} Z Y^{-1/2}) Y^{1/2},$$

where $f_1(a) = (2 - a + \sqrt{a^2 + 4})/2$. If Z is invertible, then we can also write

$$\Lambda = Z^{1/2} f(Z^{1/2} Y^{-1} Z^{1/2}) Z^{1/2}$$

with $f(a) = f_1(a)/a$. Furthermore,

$$(Z + \Lambda)^{-1} = Z^{-1/2} g(Z^{1/2} Y^{-1} Z^{1/2}) Z^{-1/2}$$

with $g(a) = (f(a) + 1)^{-1} = 2a/(2 + a + \sqrt{a^2 + 4})$.

Proof. Multiplying by $(Y^{-1} - (Z + \Lambda)^{-1})$ from the right gives

$$\Lambda(Y^{-1} - (Z + \Lambda)^{-1}) = \text{Id}.$$

Multiplying by Λ from the left gives

$$Y^{-1} = \Lambda^{-1} + (Z + \Lambda)^{-1}. \tag{29}$$

Multiplying from the left and right by $Y^{1/2}$ (we do this to preserve symmetry), we have

$$\text{Id} = L^{-1} + (Y^{-1/2} Z Y^{-1/2} + L)^{-1},$$

where we defined $L = (Y^{-1/2} \Lambda Y^{-1/2})$. Let $A = Y^{-1/2} Z Y^{-1/2}$. Let us first show that A and L commute. Indeed, multiplying $(A + L)$ from the left and the right, gives

$$(A + L)L^{-1} + \text{Id} = (A + L) = L^{-1}(A + L) + \text{Id}.$$

Subtracting Id from both sides and multiplying by L from the left and the right gives

$$LA + L^2 = L(A + L) = (A + L)L = AL + L^2$$

and the claim follows.

Thus, A and L commute and, therefore, there exists an orthonormal basis such that both A and L are diagonal in this basis. For an orthogonal matrix U , both UAU^T and ULU^T are diagonal, and

$$\text{Id} = U\text{Id}U^T = UL^{-1}U^T + U(A + L)^{-1}U^T = (ULU^T)^{-1} + (UAU^T + ULU^T)^{-1}.$$

Since all matrices on both sides are diagonal, each diagonal element has the same form with the unique positive solution $f(a)$ of

$$1 = \frac{1}{x} + \frac{1}{a+x}.$$

Therefore, we obtain

$$L = U^T f(UAU^T)U = f(A) = f(Y^{-1/2}ZY^{-1/2}).$$

Similarly, assume that Z is invertible (also, symmetric and positive definite). Then, there exists a positive-definite invertible matrix $Z^{1/2}$. Multiplying (29) by $Z^{1/2}$ from the left and right, we get

$$K = B^{-1} + (\text{Id} + B)^{-1},$$

where $K = Z^{1/2}Y^{-1}Z^{1/2}$ and $B = Z^{-1/2}\Lambda Z^{-1/2}$. Multiplying $(\text{Id} + B)$ from the left and right,

$$K + BK = (\text{Id} + B)K = B^{-1} + 2\text{Id} = K(\text{Id} + B) = K + KB,$$

which implies that K and B commute. By the argument analogous to the above, with the unique positive solution $f_1(a)$ to

$$a = \frac{1}{x} + \frac{1}{1+x},$$

we get that $B = f_1(K)$. ■

Proof of Propositions 4.1 and 5.3. Suppose that $M_i \geq 2$. Then, since, by assumption, there are at least three agents participating in each exchange, there exists an $\varepsilon > 0$ such that

$$\begin{aligned} \Lambda_i &= \left(\left(\bar{\Lambda}_i \right)^{-1} + \sum_{j \neq i} (\alpha_j \bar{\mathcal{V}}_{N(j)} + \bar{\Lambda}_j)^{-1} \right)^{-1}_{N(i)} \\ &\leq \left((\varepsilon \text{Id} + (M_i - 1)(\alpha_i \|\mathcal{V}\| \text{Id} + \bar{\Lambda}_i)^{-1})^{-1} \right)_{N(i)} = (\varepsilon \text{Id} + (M_i - 1)(\alpha_i \|\mathcal{V}\| \text{Id} + \Lambda_i)^{-1})^{-1}. \end{aligned}$$

Let $\ell \geq 0$ be the largest eigenvalue of Λ_i . Then, we get that

$$\ell \leq (\varepsilon \text{Id} + (M_i - 1)(\alpha_j \|\mathcal{V}\| \text{Id} + \ell)^{-1})^{-1}.$$

By direct calculation, this inequality implies that $\ell \rightarrow 0$ when $\alpha_j \rightarrow 0$ or $M_j \rightarrow \infty$.

Now, pick any trader $i \neq j$. Then,

$$(\Lambda_i)_{N(j) \cap N(i)} \leq ((M_j(\alpha_j \|\mathcal{V}\| + \bar{\Lambda}_j)^{-1} + \varepsilon \text{Id})^{-1})_{N(j) \cap N(i)} = ((M_j(\alpha_j \|\mathcal{V}\| + \Lambda_j)^{-1} + \varepsilon \text{Id})^{-1})_{N(j) \cap N(i)}.$$

Since $(\alpha_j \|\mathcal{V}\| + \Lambda_j)^{-1}$ converges to ∞ , we get the required.

Finally, the last claim follows because

$$\lim_{\alpha_j \rightarrow 0} \Lambda_i = (\bar{\Lambda}_{i \setminus j})_{N(i)}$$

and hence, by the Frobenius formula (Lemma D.4)

$$((\mathcal{V}_{N(i)} + \Lambda_i)^{-1})_{N(j) \cap N(i)} \rightarrow (\Lambda_{i \setminus j} + \mathfrak{S}(\mathcal{V}_{N(i)}, N(i) \setminus N(j)))^{-1}.$$

To prove the result about the limit allocation, we need to study the asymptotic behaviour in greater detail. This is done in the following proposition.

Proposition D.1 *Let $M_j > 2$. Then, for sufficiently small $\alpha = \alpha_j$, there exists an equilibrium price impact tuple $\{\Lambda_i(\alpha)\}_i$ satisfying $\Lambda_j(\alpha) \approx \frac{\alpha}{M_j-2} \mathcal{V}_{N(j)}$ and, for all $i \neq j$, to the first order in α ,*

$$\Lambda_i(\alpha) \approx \begin{pmatrix} \alpha \mathcal{V}_{N(j)} \frac{M_j-1}{(M_j-2)M_j} & -\alpha \frac{M_j-1}{(M_j-2)M_j} \mathcal{V}_{N(j)} W_{12}(i) W_{22}(i)^{-1} \\ -\alpha \frac{M_j-1}{(M_j-2)M_j} W_{22}(i)^{-1} W_{12}(i)^T \mathcal{V}_{N(j)} & \Lambda_{i \setminus j} + \alpha \Lambda_{i \setminus j}^{(1)} \end{pmatrix}_{N(i)},$$

where

$$W(i) = \begin{pmatrix} W_{11}(i) & W_{12}(i) \\ W_{12}^T(i) & W_{22}(i) \end{pmatrix} \equiv \sum_{k \neq i, j} (\alpha_k \bar{\mathcal{V}}_{N(k)} + \bar{\Lambda}_k(0))^{-1}.$$

The first order equilibrium response $\{\Lambda_{i \setminus j}^{(1)}\}_{j \neq i}$ is the unique solution to the system

$$\Lambda_{i \setminus j}^{(1)} = \begin{pmatrix} W_{22}(i)^{-1} \left(\sum_{k \neq i, j} Z_k \Lambda_{k \setminus j}^{(1)} Z_k + W_{12}(i)^T \mathcal{V}_{11} \frac{M_j-1}{(M_j-2)M_j} W_{12}(i) \right) W_{22}(i)^{-1} \end{pmatrix}_{N(i) \setminus N(i)}.$$

where $Z_i \equiv (\alpha_i \bar{\mathcal{V}}_{N(i)} + \bar{\Lambda}_i(0))^{-1}$, $i \neq j$.

Proof. The fixed point equation is

$$\Lambda_i(\alpha) = (((\alpha \bar{\mathcal{V}}_{N(j)} + \bar{\Lambda}_j)^{-1} + W(i, \alpha))^{-1})_{N(i)}$$

and the claim follows by direct calculation from the Frobenius formula (Lemma D.4). ■

Furthermore,

$$B^{-1} \approx \begin{pmatrix} \alpha_j \mathcal{V}_{N(j)} \frac{M_j-1}{(M_j-2)M_j} & -\alpha_j \frac{M_j-1}{(M_j-2)M_j} \mathcal{V}_{N(j)} W_{12} W_{22}^{-1} \\ -\alpha_j \frac{M_j-1}{(M_j-2)M_j} W_{22}^{-1} W_{12}^T \mathcal{V}_{N(j)} & W_{2 \setminus 1} + \alpha_j W_{2 \setminus 1}^{(1)} \end{pmatrix}$$

where

$$W = \begin{pmatrix} W_{11} & W_{12} \\ W_{12}^T & W_{22} \end{pmatrix} \equiv \sum_{k \neq j} M_k (\alpha_k \bar{\mathcal{V}}_{N(k)} + \bar{\Lambda}_k(0))^{-1}.$$

Thus, the trade of an agent of class j is approximately given by

$$(\alpha_j \mathcal{V}_{N(j)} + \Lambda_j)^{-1} \mathbf{Q}_{N(j)} - \frac{M_j-2}{M_j-1} q_j^0 = \alpha_j^{-1} \frac{M_j-2}{M_j-1} \mathcal{V}_{N(j)}^{-1} \mathbf{Q}_{N(j)} - \frac{M_j-2}{M_j-1} q_j^0.$$

And we have

$$\begin{aligned}
& \alpha_j^{-1} \frac{M_j - 2}{M_j - 1} \mathcal{V}_{N(j)}^{-1} \mathbf{Q}_{N(j)} \\
& \approx \alpha_j^{-1} \frac{M_j - 2}{M_j - 1} \mathcal{V}_{N(j)}^{-1} \alpha_j \mathcal{V}_{N(j)} \frac{M_j - 1}{(M_j - 2)M_j} X_{N(j)}^{(0)} - \alpha_j^{-1} \frac{M_j - 2}{M_j - 1} \mathcal{V}_{N(j)}^{-1} \alpha_j \frac{M_j - 1}{(M_j - 2)M_j} \mathcal{V}_{N(j)} W_{12} W_{22}^{-1} X_{N \setminus N(j)}^{(0)} \\
& = M_j^{-1} \left(X_{N(j)}^{(0)} - W_{12} W_{22}^{-1} X_{N \setminus N(j)}^{(0)} \right).
\end{aligned}$$

where

$$X^{(0)} = \sum_{j \neq i} (\alpha_j \bar{\mathcal{V}}_{N(j)} + \bar{\Lambda}_j(0))^{-1} \alpha_j \bar{\mathcal{V}}_{N(j)} Q_j^0 + \frac{M_i - 2}{M_i - 1} Q_i^0.$$

In contrast, agents $i \neq j$ trade

$$(\alpha_i \mathcal{V}_{N(i)} + \bar{\Lambda}_i(0))^{-1} (\mathbf{Q}_{N(i) \setminus N(j)} - \alpha_i \mathcal{V}_{N(i)} q_i^0),$$

because $\mathbf{Q}_{N(j)} = 0$. ■

Proof of Theorem 4.1. The market clearing takes the form

$$\sum_{j=1}^I (\alpha_j \bar{\mathcal{V}}_{N(j)} + \bar{\Lambda}_j(B))^{-1} (d - p - \alpha_j \bar{\mathcal{V}}_{N(j)} q_j^0) = 0$$

and the required expressions follow by direct calculation. ■

We need the following auxiliary lemma that can be verified by direct calculation.

Lemma D.4 (Frobenius Formula) *We have*

$$\begin{pmatrix} A & B \\ B^T & D \end{pmatrix}^{-1} = \begin{pmatrix} (A - BD^{-1}B^T)^{-1} & -A^{-1}B(D - B^T A^{-1}B)^{-1} \\ -(D - B^T A^{-1}B)^{-1} B^T A^{-1} & (D - B^T A^{-1}B)^{-1} \end{pmatrix}.$$

E Comparative Statics

Proof of Proposition 5.1. Fix a parameter α and let us rewrite equilibrium equation as $\{\Lambda_j\}_j = F(\{\Lambda_j\}_j, \alpha)$. By definition, both for $\alpha = M_i$ and $\alpha = \alpha_j$, map F is monotone increasing in α in the sense of the positive semidefinite partial order.

Fix $\alpha_1 \leq \alpha_2$ and let $\{\Lambda_j(\alpha_1)\}_j$ be an equilibrium. Then,

$$\{\Lambda_j(\alpha_1)\}_j = F(\{\Lambda_j(\alpha_1)\}_j, \alpha_1) \leq F(\{\bar{\Lambda}_j(\alpha_1)\}_j, \alpha_2).$$

By Proposition A.1, there exists an equilibrium $\{\Lambda_j(\alpha_2)\}_j$ satisfying $\{\Lambda_j(\alpha_2)\}_j \geq \{\Lambda_j(\alpha_1)\}_j$.

The last claim follows by a similar argument. Namely, let $\hat{F}(\{\Lambda_j\}_{j=1}^I, \Lambda_{I+1})$ be the equilibrium map corresponding to the market with $I + 1$ traders and let $\{\Lambda_j^*\}_{j=1}^{I+1}$ be an equilibrium in that market. Define the map

$$\hat{F}_1(\{\Lambda_j\}_{j=1}^I) \equiv \hat{F}(\{\Lambda_j\}_{j=1}^I, \Lambda_{I+1}^*).$$

Then, using the same monotonicity arguments as above, we get that

$$\hat{F}_1(\{\Lambda_j\}_{j=1}^I) \leq F(\{\Lambda_j\}_{j=1}^I)$$

for any tuple $\{\Lambda_j\}_{j=1}^I$. By construction, $\{\Lambda_j^*\}_j$ is a fixed point of \hat{F}_1 and therefore

$$\{\Lambda_j^*\}_{j=1}^I = \hat{F}_1(\{\Lambda_j^*\}_{j=1}^I) \leq F(\{\Lambda_j^*\}_{j=1}^I),$$

and Proposition A.1 implies that there exists an equilibrium $\{\Lambda_j\}_{j=1}^I$ with I traders satisfying $\{\Lambda_j\}_{j=1}^I \geq \{\Lambda_j^*\}_{j=1}^I$.

It remains to prove the concavity results. We only prove concavity in the covariance matrix. The other Pick two covariance matrices \mathcal{V}^m , $m = 1, 2$, and let $\{\Lambda_j^m\}_j$ be two corresponding equilibria. Let $\mathcal{V}^3 = 0.5(\mathcal{V}^1 + \mathcal{V}^2)$ and denote by F^m , $m = 1, 2, 3$ the equilibrium maps corresponding to these covariance matrices. A direct application of Theorem 24 in Anderson and Duffin (1969) implies that, for any $\{\Lambda_j^1\}_j, \{\Lambda_j^2\}_j \in \mathcal{S}^I$,

$$0.5(F^1(\{\Lambda_j^1\}_j) + F^1(\{\Lambda_j^2\}_j)) \leq F^3(\{0.5(\Lambda_j^1 + \Lambda_j^2)\}_j).$$

Consequently, for any two fixed points satisfying $\{\Lambda_j^m\}_j = F^1(\{\Lambda_j^m\}_j)$, Proposition A.1 implies that F^3 has a fixed point $\{\Lambda_j^3\}_j$ satisfying $\{\Lambda_j^3\}_j \geq 0.5(\Lambda_j^1 + \Lambda_j^2)_j$, and the required concavity follows. ■

Proof of Theorem 5.1. Consider two markets which differ only in participation $\{N(i)\}_i$ and $\{N'(i)\}_i$, $N'(i) \geq N(i)$ for all $i \in I$. Pick an equilibrium $\{\Lambda_i\}_i$ corresponding to $\{N'(i)\}_i$. By Lemma A.1,

$$\begin{aligned} (\Lambda_i)_{N(i)} &= \left(\left(\sum_{j \neq i} (\alpha_j \bar{\mathcal{V}}_{N'(j)} + \bar{\Lambda}_j)^{-1} \right)^{-1} \right)_{N(i)} \\ &\leq \left(\left(\sum_{j \neq i} (\alpha_j \bar{\mathcal{V}}_{N(j)} + (\bar{\Lambda}_j)_{N(j)})^{-1} \right)^{-1} \right)_{N(i)}. \end{aligned}$$

For all $i \in I$. Therefore, by Proposition A.1, there exists an equilibrium $\{\hat{\Lambda}_i\}_i$ corresponding to participation $\{N(i)\}_i$ and satisfying $\{\hat{\Lambda}_i\}_i \geq \{(\Lambda_i)_{N(i)}\}_i$ and the claim follows. ■

Proof of Corollary 5.1. Let \mathbb{M}_{-n} be market \mathbb{M} with all exchanges but $n = (I(n), K(n))$.

Consider exchanges $(I_1(n), K_1(n))$ and $(I_2(n), K_2(n))$ in market \mathbb{M}' . Let all traders $i \in I(n)$ trade in each exchange for assets $K_1(n)$ and $K_2(n)$. Then, by Theorem 5.1, price impact in the resulting market $\tilde{\mathbb{M}} = (\mathbb{M}_{-n}, (K_1(n), I(n)), (K_2(n), I(n)))$ is lower. Thus, price impact in \mathbb{M}' is higher than that in \mathbb{M} .

Instead, in market \mathbb{M}' , let the participation of traders $I_1(n)$ in the new exchanges increase from $K_1(n)$ to $K(n)$ and traders $I_2(n)$ from $K_2(n)$ to $K(n)$ such that each group of agents trades assets $K(n)$ in a separate exchange. Then, by Theorem 5.1, price impact in the resulting market

$\tilde{\mathbb{M}}' = (\mathbb{M}_{-n}, (K(n), I_1(n)), (K(n), I_2(n)))$ is lower. Define $n^l = (K(n), I_l(n))$, $l = 1, 2$. Next, let the participation of traders $I_1(n)$ increase to include n^2 and the participation of traders $I_2(n)$ increases to include n^1 . Denote by $\tilde{\mathbb{M}}'' = (\mathbb{M}_{-n}, (I(n), K(n)), (I(n), K(n)))$ the corresponding decentralized market. Then, by Theorem 5.1, price impact in $\tilde{\mathbb{M}}''$ is lower than that in \mathbb{M}' . Since market $\tilde{\mathbb{M}}''$ is equivalent to one with the centralized exchange $n = (I(n), K(n))$, price impact in $\tilde{\mathbb{M}}''$ and \mathbb{M} coincides.

Thus, price impact in $(\mathbb{M}_{-n}, (I_1(n), K_1(n)), (I_2(n), K_2(n))) = \mathbb{M}'$ is higher than in $(\mathbb{M}_{-n}, (K(n), I_1(n)), (K(n), I_2(n)))$, which is higher than in $(\mathbb{M}_{-n}, (K_1(n), I(n)), (K_2(n), I(n)))$, which equals that in $(\mathbb{M}_{-n}, (I(n), K(n))) = \mathbb{M}$. ■

F Welfare

Proof of Proposition 6.1 and Corollary 6.2. We need a couple of lemmas. The first one is a direct consequence of the Frobenius formula (Lemma D.4).

Lemma F.1 *Let $H \subset \mathbb{R}^N$ be a subspace, B a symmetric positive definite matrix on H and A a positive definite matrix on \mathbb{R}^N . Then, $A \geq \bar{B}$ if, and only if,*

$$(A^{-1})_H \leq B^{-1}.$$

Proof. We have

$$A - \bar{B} = \begin{pmatrix} A_{11} & A_{12} \\ A_{12}^T & A_{22} - B \end{pmatrix}$$

and hence, by (23), $A - \bar{B} \geq 0$ if, and only if, $A_{22} - A_{12}^T A_{11}^{-1} A_{12} - B \geq 0$. By Lemma D.4, this is equivalent to $(A^{-1})_{22} \leq B^{-1}$. ■

Lemma F.2 *There exists a matrix $B \leq A$ such that $Bq = z$ if, and only if, $\langle A^{-1}z, z \rangle \leq \langle q, z \rangle$.*

Proof. We normalize z so that $\|z\| = 1$. Suppose first that $B \leq A$ satisfies $Bq = z$. Then, $\langle q, z \rangle = \langle B^{-1}z, z \rangle \geq \langle A^{-1}z, z \rangle$. Now, suppose that $\langle A^{-1}z, z \rangle \leq \langle q, z \rangle$ and define $B = (\langle q, z \rangle)^{-1} \langle \cdot, z \rangle z$. Let H be the span of the vector z . By Lemma F.1, it suffices to check that $(A^{-1})_H \leq B^{-1}$. But $(A^{-1})_H = \langle A^{-1}z, z \rangle$ and B^{-1} acts as $(\langle q, z \rangle)$ on this subspace. The claim follows. ■

Now, we have

$$\langle \Gamma_i(\Lambda)q, q \rangle = \langle X_1q, q \rangle - 0.5\alpha_i \langle \mathcal{V}_{N(i)}X_1q, X_1q \rangle,$$

where $X_1 = (\alpha_i \mathcal{V}_{N(i)} + \Lambda)^{-1}$ and $q = \mathbf{Q}_{N(i)} - \alpha_i \mathcal{V}_{N(i)} q_i^0$. Denote $X_2 = (\alpha_i \mathcal{V}_{N(i)} + \hat{\Lambda})^{-1}$. Suppose first that we do not change the participation of trader i . Then, $\hat{\Lambda} \geq \Lambda$ if, and only if, $X_2 \leq X_1$ and the problem becomes

$$\max_{X_2 \leq X_1} \{ \langle X_2q, q \rangle - 0.5\alpha_i \langle \mathcal{V}_{N(i)}X_2q, X_2q \rangle \}.$$

If the participation of trader i increases from $N(i)$ to $N'(i) \supset N(i)$, it follows from Theorem 5.1 that $\hat{\Lambda}_i > (\Lambda_i)_{N(i)}$ and, therefore, by Lemma A.1,

$$X_1 = (\alpha_i \mathcal{V}_{N'(i)} + \Lambda_i)^{-1} \geq (\alpha_i \bar{\mathcal{V}}_{N(i)} + (\bar{\Lambda}_i)_{N(i)})^{-1} \geq (\alpha_i \bar{\mathcal{V}}_{N(i)} + \bar{\Lambda}_i)^{-1} = X_2,$$

and so the same argument applies.

Denote $z = X_2 q$. Then, by Lemma F.2, the problem is equivalent to

$$\max_{z: \langle X_1^{-1} z, z \rangle \leq \langle q, z \rangle} \{ \langle z, q \rangle - 0.5 \alpha_i \langle \mathcal{V}_{N(i)} z, z \rangle \}.$$

This is a concave maximization problem over a convex domain and, hence, the global maximum is achieved when $z = (\alpha_i \mathcal{V}_{N(i)})^{-1} q$. By direct calculation, $\langle X_1^{-1} z, z \rangle > \langle q, z \rangle$ holds and, hence, the constraint is binding at the optimum. Therefore, the first-order condition is

$$q - \alpha_i \mathcal{V}_{N(i)} z - \lambda (2X_1^{-1} z - q) = 0 \Rightarrow z = (1 + \lambda)(\alpha_i \mathcal{V}_{N(i)} + 2\lambda X_1^{-1})^{-1} q,$$

where the Lagrange multiplier λ is determined from the binding constraint

$$0 = (1 + \lambda)^2 \langle X_1^{-1} (\alpha_i \mathcal{V}_{N(i)} + 2\lambda X_1^{-1})^{-1} q, (\alpha_i \mathcal{V}_{N(i)} + 2\lambda X_1^{-1})^{-1} q \rangle - (1 + \lambda) \langle (\alpha_i \mathcal{V}_{N(i)} + 2\lambda X_1^{-1})^{-1} q, q \rangle.$$

Denote the right-hand side of this equation by $\varphi(\lambda)$. Then,

$$\varphi(0) = \langle X_1^{-1} (\alpha_i \mathcal{V}_{N(i)})^{-1} q, (\alpha_i \mathcal{V}_{N(i)})^{-1} q \rangle - \langle (\alpha_i \mathcal{V}_{N(i)})^{-1} q, q \rangle > 0,$$

because $X_1 < (\alpha_i \mathcal{V}_{N(i)})^{-1}$ implies $(\alpha_i \mathcal{V}_{N(i)})^{-1} X_1^{-1} (\alpha_i \mathcal{V}_{N(i)})^{-1} > (\alpha_i \mathcal{V}_{N(i)})^{-1}$. Furthermore, $\varphi(\infty) = (1/4) \langle X_1 q, q \rangle - (1/2) \langle X_1 q, q \rangle < 0$. The maximum is achieved at $z = X_1 q$ if, and only if,

$$(1 + \lambda)(\alpha_i \mathcal{V}_{N(i)} + 2\lambda X_1^{-1})^{-1} q = X_1 q$$

or, equivalently, $(\alpha_i \mathcal{V}_{N(i)} + 2\lambda X_1^{-1}) X_1 q = (1 + \lambda) q$. This happens if, and only if, q is an eigenvector of $\alpha_i \mathcal{V}_{N(i)} X_1 = (\text{Id} + \Lambda(\alpha_i \mathcal{V}_{N(i)})^{-1})^{-1}$ and the assertion follows because the latter matrix has the same eigenvectors as $\Lambda \mathcal{V}_{N(i)}^{-1}$. ■

G Proof of Proposition 6.3

We will construct the required examples in the class of markets with liquid exchanges (Example 3, Section 5.1) and a single illiquid exchange in which all agents participate, so that $n = N(i) \setminus N(j)$ is the same for all agents. By Proposition 5.3, the problem reduces to studying the price impact $\tilde{\Lambda}_i = \Lambda_{i \setminus j}$ of class i in the (illiquid) exchange n . Let $\Pi_{K(n)}$ be the orthogonal projection onto the subspace of assets traded in exchange n and let $\tilde{\mathbf{Q}} \equiv (\mathcal{V} \mathbf{Q})_{K(n)}$ and $\tilde{q}_i^0 \equiv (q_i^0)_{K(n)}$. With $\tilde{\mathcal{V}}_i = \mathcal{V}_{i \setminus j}$ defined as in Proposition 5.3, and $\tilde{\Gamma}_i \equiv (\alpha_i \tilde{\mathcal{V}}_i)^{-1} \tilde{\Gamma}_i (\tilde{\mathcal{V}}_i, \tilde{\Lambda}_i) (\alpha_i \tilde{\mathcal{V}}_i)^{-1}$, $\tilde{\Delta}_i \equiv (\alpha_i \tilde{\mathcal{V}}_i)^{-1} \tilde{\Delta}_i (\tilde{\mathcal{V}}_i, \tilde{\Lambda}_i) (\alpha_i \tilde{\mathcal{V}}_i)^{-1}$, and $\tilde{\Lambda}_i^{-1} + (\tilde{\Lambda}_i + \alpha_i \tilde{\mathcal{V}}_i)^{-1} = \tilde{B}$ for all i , which implies a global upper bound on the price impact of all

agents,

$$\tilde{\Lambda}_i < 2\tilde{B}^{-1}. \quad (30)$$

Lemma G.1 *In the markets from Example 3, $\tilde{\Gamma}_j = 0.5(\tilde{B}\tilde{\Lambda}_j\tilde{B} - \tilde{B})$ and $2\tilde{\Delta}_j = 3\tilde{\Lambda}_j\tilde{B}\tilde{\Lambda}_j - \tilde{\Lambda}_j\tilde{B}\tilde{\Lambda}_j\tilde{B}\tilde{\Lambda}_j$.*

Proof of Lemma G.1. Let, for brevity $\Gamma = \Gamma_{j \setminus i}$, $\mathcal{V} = \alpha_j \tilde{\mathcal{V}}_j$, $\Lambda = \tilde{\Lambda}_j$, $B = \tilde{B}$. Then,

$$\begin{aligned} & (\alpha_j \mathcal{V}_{N(j)})^{-1} \Gamma (\alpha_j \mathcal{V}_{N(j)})^{-1} \\ &= 0.5(\mathcal{V} + \Lambda)^{-1} + 0.5(\mathcal{V} + \Lambda)^{-1} \Lambda (\mathcal{V} + \Lambda)^{-1} = 0.5(B - \Lambda^{-1}) + 0.5(B - \Lambda^{-1}) \Lambda (B - \Lambda^{-1}) \end{aligned}$$

and the claim follows. For Δ , we have

$$\begin{aligned} & (\alpha_j \mathcal{V}_{N(j)})^{-1} \Delta (\alpha_j \mathcal{V}_{N(j)})^{-1} = \Lambda (\Lambda + \mathcal{V})^{-1} \mathcal{V} (\Lambda + \mathcal{V})^{-1} \Lambda \\ &= \Lambda (\Lambda + \mathcal{V})^{-1} (\mathcal{V} + \Lambda - \Lambda) (\Lambda + \mathcal{V})^{-1} \Lambda \\ &= \Lambda (\Lambda + \mathcal{V})^{-1} \Lambda - \Lambda (\Lambda + \mathcal{V})^{-1} \Lambda (\Lambda + \mathcal{V})^{-1} \Lambda = \Lambda (B - \Lambda^{-1}) \Lambda - \Lambda (B - \Lambda^{-1}) \Lambda (B - \Lambda^{-1}) \Lambda \end{aligned}$$

and the claim follows by direct calculation. ■

Proposition G.1 (Commutativity, Connectedness and Price impact) *If $\tilde{B}^{1/2} \tilde{\mathcal{V}}_{j_1} \tilde{B}^{1/2}$ and $\tilde{B}^{1/2} \tilde{\mathcal{V}}_{j_2} \tilde{B}^{1/2}$ commute and $\tilde{\mathcal{V}}_{j_1} \leq \tilde{\mathcal{V}}_{j_2}$, then $\tilde{\Lambda}_{j_1} \geq \tilde{\Lambda}_{j_2}$. However, for any \tilde{B} and $\tilde{\mathcal{V}}_{j_1}$ that do not commute and satisfy $\tilde{B} > 2\tilde{\mathcal{V}}_{j_1}^{-1}$, there exists $\mathcal{V}_{j_2} \geq \mathcal{V}_{j_1}$ such that $\tilde{\Lambda}_{j_1} \not\geq \tilde{\Lambda}_{j_2}$.*

Proof of Proposition G.1. For simplicity, we normalize all risk aversions to 1. Let $j_1 = 1$, $j_2 = 2$. We first show that, for any $\tilde{\mathcal{V}}_1$, $\tilde{\mathcal{V}}_2$ and \tilde{B} there exists a market in which they are realized. To prove this, consider a market with three classes and let us show that we can pick $\tilde{\mathcal{V}}_3$ accordingly. First, equation $\tilde{\Lambda}_i^{-1} + (\tilde{\mathcal{V}}_i + \tilde{\Lambda}_i)^{-1} = \tilde{B}$ implies (by Lemma D.2) that

$$\tilde{\Lambda}_i = \tilde{\mathcal{V}}_i^{1/2} f(\tilde{\mathcal{V}}_i^{1/2} \tilde{B} \tilde{\mathcal{V}}_i^{1/2}) \tilde{\mathcal{V}}_i^{1/2}$$

and

$$(\tilde{\Lambda}_i + \tilde{\mathcal{V}}_i)^{-1} = \tilde{\mathcal{V}}_i^{-1/2} g(\tilde{\mathcal{V}}_i^{1/2} \tilde{B} \tilde{\mathcal{V}}_i^{1/2}) \tilde{\mathcal{V}}_i^{-1/2}.$$

Denote

$$A \equiv (\tilde{\mathcal{V}}_1 + \tilde{\Lambda}_1)^{-1} + (\tilde{\mathcal{V}}_2 + \tilde{\Lambda}_2)^{-1}.$$

Then, $\tilde{\Lambda}_3$ satisfies

$$\tilde{\Lambda}_3 = (A + (M_3 - 1)(\tilde{\mathcal{V}}_3 + \tilde{\Lambda}_3)^{-1})^{-1} = (\tilde{B} - (\tilde{\mathcal{V}}_3 + \tilde{\Lambda}_3)^{-1})^{-1}$$

and therefore, to complete the proof, it suffices to show that there exist positive definite matrices $\tilde{\Lambda}_3$, $\tilde{\mathcal{V}}_3$ satisfying

$$\tilde{\Lambda}_3^{-1} - (M_3 - 1)(\tilde{\mathcal{V}}_3 + \tilde{\Lambda}_3)^{-1} = A, \quad \tilde{\Lambda}_3^{-1} + (\tilde{\mathcal{V}}_3 + \tilde{\Lambda}_3)^{-1} = \tilde{B}.$$

Solving this system, we get

$$(\tilde{\mathcal{V}}_3 + \tilde{\Lambda}_3)^{-1} = M_3^{-1}(\tilde{B} - A), \quad \tilde{\Lambda}_3^{-1} = M_3^{-1}(A + (M_3 - 1)\tilde{B}^{-1}).$$

Since $A + (M_3 - 1)\tilde{B}^{-1} > \tilde{B}^{-1} - A$ whenever $M_3 > 1$, the matrix

$$\tilde{\mathcal{V}}_3 = (M_3^{-1}(\tilde{B} - A))^{-1} - (M_3^{-1}(A + (M_3 - 1)\tilde{B}^{-1}))^{-1}$$

is positive definite. Finally, $\tilde{B} > 2\tilde{\mathcal{V}}_1^{-1} > \tilde{\mathcal{V}}_1^{-1} + \tilde{\mathcal{V}}_2^{-1} > A$, completing the proof of existence.

Lemma G.2 *None of the functions f , f_1 , g is matrix monotone.*

Proof. By the Löwner Theorem (Donoghue (1974)), it suffices to show that none of these functions can be analytically continued to the whole upper half-plane. This follows directly from the fact that $2i$ is a branching point for all these functions. ■

By Lemma, $f_1 = \frac{2-a+\sqrt{a^2+4}}{2}$ is not matrix monotone on any interval and, consequently, for any positive definite matrix X_1 of sufficiently high dimension, there exists a matrix $X_2 \geq X_1$ such that $f_1(X_2) \not\leq f_1(X_1)$. Let $X_1 = \tilde{B}^{-1/2}\tilde{\mathcal{V}}_1\tilde{B}^{-1/2}$. Then, define $\tilde{\mathcal{V}}_2 \equiv \tilde{B}^{1/2}X_2\tilde{B}^{1/2}$.

Now, by Lemma D.2,

$$\Lambda_1 = \tilde{B}^{1/2}f_1(X_1)\tilde{B}^{1/2} \not\leq \tilde{B}^{1/2}f_1(X_2)\tilde{B}^{1/2} = \Lambda_2,$$

and the proof is complete. ■

Proof of Proposition 6.3. The proof is based on the following auxiliary “anything goes” result.

Lemma G.3 *Within Example 3, suppose that equilibrium price impact of trader 1 is $\tilde{\Lambda}_1$. Then, any $\hat{\Lambda}_1 \leq \Lambda_1$ can be attained as an equilibrium price impact by adding an additional trader to the exchange.*

Proof. The proof follows directly from the arguments in the proof of Proposition G.1. Namely, suppose that there are N traders such that

$$\tilde{\Lambda}_i^{-1} + (\alpha_i\tilde{\mathcal{V}}_i + \tilde{\Lambda}_i)^{-1} = \tilde{B}^{-1}, \quad i = 1, \dots, N.$$

Pick an arbitrary $\hat{\Lambda}_1 \leq \Lambda_1$ and define \hat{B} via

$$\hat{\Lambda}_1^{-1} + (\alpha_1\tilde{\mathcal{V}}_1 + \hat{\Lambda}_1)^{-1} = \hat{B}^{-1}.$$

Then, define $\hat{\Lambda}_i$, $i \geq 2$ via

$$\hat{\Lambda}_i^{-1} + (\alpha_i\tilde{\mathcal{V}}_i + \hat{\Lambda}_i)^{-1} = \hat{B}^{-1}, \quad i = 2, \dots, N.$$

The argument in the proof of Proposition G.1 implies that these price impacts can be sustained in equilibrium if we add an $N + 1$ -st class with some perceived covariance matrix $\tilde{\mathcal{V}}_{N+1}$, and the proof is complete. ■

Lemma G.3 implies that, by adding/removing traders to/from the illiquid exchange, arbitrary changes in the price impact can be achieved. The required claim follows now from Proposition 6.1. ■

Lemma G.4 *Denote*

$$\tilde{\Gamma}_i \equiv (\alpha_i \tilde{\mathcal{V}}_j + \tilde{\Lambda}_j)^{-1} \left(\frac{1}{2} \alpha_i \tilde{\mathcal{V}}_j + \tilde{\Lambda}_j \right) (\alpha_i \tilde{\mathcal{V}}_j + \tilde{\Lambda}_j)^{-1}$$

and

$$\tilde{\Delta}_j \equiv \frac{1}{2} \tilde{\Lambda} (\alpha_i \tilde{\mathcal{V}}_j + \tilde{\Lambda}_j)^{-1} \alpha_i \tilde{\mathcal{V}}_j (\alpha_i \tilde{\mathcal{V}}_j + \tilde{\Lambda}_j)^{-1} \tilde{\Lambda}_j.$$

Then, the utility U_j of agent from lass j with initial holdings q_k^0 with $(q_k^0)_{N \setminus K(n)} = 0$ (i.e., no initial holdings in exchanges $N \setminus K(n)$) is given by

$$U_j(\Lambda_j; q_k^0) = \langle \tilde{\Gamma}_j \tilde{\mathbf{Q}}, \tilde{\mathbf{Q}} \rangle - 2 \langle \alpha_j \tilde{\mathcal{V}}_j \tilde{\Gamma}_j \tilde{\mathbf{Q}}, \tilde{q}_k^0 \rangle - \langle \tilde{\Delta}_j \tilde{q}_k^0, \tilde{q}_k^0 \rangle. \quad (31)$$

Proof. Let $\mathcal{V}_j \equiv \mathcal{V}_{N(j)}$. Then,

$$\Gamma_j = 0.5(\alpha_i \mathcal{V}_j + \Lambda_j)^{-1} + 0.5(\alpha_j \mathcal{V}_j + \Lambda_j)^{-1} \Lambda_j (\alpha_j \mathcal{V}_j + \Lambda_j)^{-1},$$

and hence,

$$\langle \Gamma_i \mathbf{Q}_{N(i)}, \mathbf{Q}_{N(i)} \rangle = \langle \tilde{\Gamma}_j \tilde{\mathbf{Q}}, \tilde{\mathbf{Q}} \rangle,$$

because $\mathbf{Q}_{\kappa(n)} = 0$ and price impact in exchanges not from $\kappa(n)$ also vanishes.

Furthermore, a direct calculation implies that

$$\alpha_i \mathcal{V}_i \Gamma_i = 0.5(\text{Id} - \Lambda_i (\alpha \mathcal{V}_i + \Lambda_i)^{-1}) + 0.5(\text{Id} - \Lambda_i (\alpha \mathcal{V}_i + \Lambda_i)^{-1}) \Lambda_i (\alpha \mathcal{V}_i + \Lambda_i)^{-1},$$

and hence,

$$\langle \alpha_i \mathcal{V}_i \Gamma_i \mathbf{Q}_{N(i)}, q_k^0 \rangle = \langle \alpha_j \tilde{\mathcal{V}}_j \tilde{\Gamma}_j \tilde{\mathbf{Q}}, \tilde{q}_k^0 \rangle.$$

■

UTILITY AND ENDOWMENT RISK. Example 5 suggests that markets in which specialists intermediate trading of particular assets can be associated with higher welfare than markets in which these assets are all traded in a single exchange. More generally, our analysis sheds light on the instances in which different forms of intermediation improve efficiency; for instance, when dealers or brokers (who as opposed to dealers need not trade on their own account) or specialists (in trading particular assets) would be efficient. While adding a trader with a zero net endowment always increases liquidity, it may decrease utility. The model points to a relation between distribution of endowment risk and the market structure (e.g., core versus periphery traders) and, thus, types of

intermediation. Incentives of a potential intermediary to set up a monopolistic link (cf. Corollary 3.2) depend on his and others' endowments and participation.⁵⁹

While price impact is independent of endowments, through non-commutativity, decentralized trading changes the way aggregate and idiosyncratic endowment risks affect agents' utility. In particular, both an agent's compensation for aggregate risk exposure may increase and his utility loss from idiosyncratic risk exposure may decrease if his price impact increases – Propositions 6.3 and G.2, Example 7 illustrate.

Proposition G.2 (Non-commutativity and Idiosyncratic Risk) *For generic markets from Example 3, the loss due to idiosyncratic risk exposure can be smaller for the traders with larger price impact.*

In Proposition 5.2, we showed that agents who are better connected with the market, in the sense of having more links or being more centrally located, may have larger price impact. Example 7 shows that the agents who face larger price impact may also have greater equilibrium utility.

Example 7 *In the markets from Example 3, consider exchange n with at least two assets traded, and agents j_1 and j_2 with identical risk aversions, non-zero initial endowments $\tilde{q}_{j_1}^0 = \tilde{q}_{j_2}^0 > 0$ and no aggregate risk ($\tilde{\mathbf{Q}} = 0$), so that the agents only receive negative utility from the residual idiosyncratic risk exposure. Then, there exists participation such that $N(j_1) \supset N(j_2)$ and agent j_1 has a larger price impact than j_2 , and yet agent j_1 attains a higher equilibrium utility.*

Proof of Proposition G.2. If all matrices commute, we can diagonalize them in a common basis, and the required inequality is equivalent to verifying monotonicity of the scalar function $\delta(\lambda) = 3\lambda^2b - \lambda^3b^2$ for $\lambda \in (0, 2b^{-1})$. This follows by direct calculation.

Without commutativity, consider a small one dimensional perturbation $\tilde{\Lambda}_{j_2} = \tilde{\Lambda}_{j_1} + \alpha P$ with $P = \langle \cdot, y \rangle y$ for some $y \in \mathbb{R}^{\kappa(n)}$ and a small $\varepsilon > 0$. Then,

$$\begin{aligned} \langle (\Delta_{j_1} - \Delta_{j_2})q, q \rangle &\approx \varepsilon \langle (3\Lambda BP + 3PBA - PB\Lambda B\Lambda - \Lambda BPBA - \Lambda B\Lambda BP)q, q \rangle \\ &= 3\langle q, y \rangle \langle B\Lambda y, q \rangle + 3\langle q, y \rangle \langle B\Lambda q, y \rangle - \langle B\Lambda B\Lambda q, y \rangle \langle q, y \rangle - \langle B\Lambda q, y \rangle \langle \Lambda B y, q \rangle \\ &\quad - \langle q, y \rangle \langle \Lambda B\Lambda B y, q \rangle. \end{aligned}$$

This is monotone decreasing in α for any $q \perp y$ unless y is an eigenvector of $B\Lambda$, which precisely guarantees that P and $B\Lambda$ commute. ■

⁵⁹ Although the term “dealer” carries some legal distinctions in certain markets, the main difference between a dealer and other market participants is that, by convention, an OTC dealer is usually expected to make “two-way markets”; a dealer usually has less difficulty in adjusting inventories. Central dealers hold larger and more volatile (risky) inventories on average and keep bonds longer than peripheral dealers (Li and Schürhoff (2012)). In the market for CDSs, Atkeson, Eisfeld and Weil (2013) document a relation between order imbalances and trader position in the network; Shachar (2012) finds that order imbalances of end-users in the CDSs market are associated with significant price impact, which depends on their direction relative to the direction of dealers' inventory.

Proof of Example 7. For simplicity we consider the case when there are only two classes, 1 and 2, that initially have access to the same liquid exchange and normalize their risk aversion to 1. Then,

$$\Lambda_1 = \Lambda_2 = \frac{1}{M_1 + M_2 - 2} \tilde{\mathcal{V}} \equiv \Lambda(0).$$

Now, suppose that agents of class 1 get access to an additional exchange for an asset that has covariance $\varepsilon \{y_k\} = \varepsilon y \in \mathbb{R}^{K(n)}$ with assets $k \in K(n)$ and whose variance is normalized to 1. Then,

$$\tilde{V}_1 = \tilde{V} - \varepsilon \langle \cdot, y \rangle y \equiv \tilde{V} - \varepsilon P.$$

Substituting conjectured Taylor expansions $\Lambda_i = \Lambda(0) + \varepsilon \Lambda_i^{(1)}$, we have

$$\Lambda_1 = ((M_1 - 1)(\tilde{V}_1 + \Lambda_1)^{-1} + M_2(\tilde{V} + \Lambda_2)^{-1})^{-1}, \quad \Lambda_2 = (M_1(\tilde{V}_1 + \Lambda_1)^{-1} + (M_2 - 1)(\tilde{V} + \Lambda_2)^{-1})^{-1}.$$

We get the system

$$\begin{aligned} \Lambda_1^{(1)} &\approx \Lambda(0)((M_1 - 1)(\Lambda(0) + \tilde{V})^{-1}(\Lambda_1^{(1)} - P)(\Lambda(0) + \tilde{V})^{-1} + M_2(\Lambda(0) + \tilde{V})^{-1}\Lambda_2^{(1)}(\Lambda(0) + \tilde{V})^{-1})\Lambda(0) \\ \Lambda_2^{(1)} &\approx \Lambda(0)(M_1(\Lambda(0) + \tilde{V})^{-1}(\Lambda_1^{(1)} - P)(\Lambda(0) + \tilde{V})^{-1} + (M_2 - 1)(\Lambda(0) + \tilde{V})^{-1}\Lambda_2^{(1)}(\Lambda(0) + \tilde{V})^{-1})\Lambda(0). \end{aligned}$$

Substituting the expression for $\Lambda(0)$, we get that both $\Lambda_i^{(1)} = -\zeta_i P$ for some $0 < \zeta_1 < \zeta_2$ that only depend on M_1, M_2 . Consequently, Λ_1 and Λ_2 differ from each other by a one-dimensional projection and the arguments in (G) apply. ■

Proof of Proposition G.1. For simplicity, we normalize all risk aversions to 1. Let $j_1 = 1, j_2 = 2$. We first show that, for any $\tilde{\mathcal{V}}_1, \tilde{\mathcal{V}}_2$ and \tilde{B} there exists a market in which they are realized. To prove this, consider a market with three classes and let us show that we can pick $\tilde{\mathcal{V}}_3$ accordingly. First, equation $\tilde{\Lambda}_i^{-1} + (\tilde{\mathcal{V}}_i + \tilde{\Lambda}_i)^{-1} = \tilde{B}$ implies (by Lemma D.2) that

$$\tilde{\Lambda}_i = \tilde{\mathcal{V}}_i^{1/2} f(\tilde{\mathcal{V}}_i^{1/2} \tilde{B} \tilde{\mathcal{V}}_i^{1/2}) \tilde{\mathcal{V}}_i^{1/2}$$

and

$$(\tilde{\Lambda}_i + \tilde{\mathcal{V}}_i)^{-1} = \tilde{\mathcal{V}}_i^{-1/2} g(\tilde{\mathcal{V}}_i^{1/2} \tilde{B} \tilde{\mathcal{V}}_i^{1/2}) \tilde{\mathcal{V}}_i^{-1/2}.$$

Denote

$$A \equiv (\tilde{\mathcal{V}}_1 + \tilde{\Lambda}_1)^{-1} + (\tilde{\mathcal{V}}_2 + \tilde{\Lambda}_2)^{-1}.$$

Then, $\tilde{\Lambda}_3$ satisfies

$$\tilde{\Lambda}_3 = (A + (M_3 - 1)(\tilde{\mathcal{V}}_3 + \tilde{\Lambda}_3)^{-1})^{-1} = (\tilde{B} - (\tilde{\mathcal{V}}_3 + \tilde{\Lambda}_3)^{-1})^{-1}$$

and therefore, to complete the proof, it suffices to show that there exist positive definite matrices $\tilde{\Lambda}_3, \tilde{\mathcal{V}}_3$ satisfying

$$\tilde{\Lambda}_3^{-1} - (M_3 - 1)(\tilde{\mathcal{V}}_3 + \tilde{\Lambda}_3)^{-1} = A, \quad \tilde{\Lambda}_3^{-1} + (\tilde{\mathcal{V}}_3 + \tilde{\Lambda}_3)^{-1} = \tilde{B}.$$

Solving this system, we get

$$(\tilde{\mathcal{V}}_3 + \tilde{\Lambda}_3)^{-1} = M_3^{-1}(\tilde{B} - A), \quad \tilde{\Lambda}_3^{-1} = M_3^{-1}(A + (M_3 - 1)\tilde{B}^{-1}).$$

Since $A + (M_3 - 1)\tilde{B}^{-1} > \tilde{B}^{-1} - A$ whenever $M_3 > 1$, the matrix

$$\tilde{\mathcal{V}}_3 = (M_3^{-1}(\tilde{B} - A))^{-1} - (M_3^{-1}(A + (M_3 - 1)\tilde{B}^{-1}))^{-1}$$

is positive definite. Finally, $\tilde{B} > 2\tilde{\mathcal{V}}_1^{-1} > \tilde{\mathcal{V}}_1^{-1} + \tilde{\mathcal{V}}_2^{-1} > A$, completing the proof of existence.

Lemma G.5 *None of the functions f , f_1 , g is matrix monotone.*

Proof. By the Löwner Theorem (Donoghue (1974)), it suffices to show that none of these functions can be analytically continued to the whole upper half-plane. This follows directly from the fact that $2i$ is a branching point for all these functions. ■

By Lemma, $f_1 = \frac{2-a+\sqrt{a^2+4}}{2}$ is not matrix monotone on any interval and, consequently, for any positive definite matrix X_1 of sufficiently high dimension, there exists a matrix $X_2 \geq X_1$ such that $f_1(X_2) \not\leq f_1(X_1)$. Let $X_1 = \tilde{B}^{-1/2}\tilde{\mathcal{V}}_1\tilde{B}^{-1/2}$. Then, define $\tilde{\mathcal{V}}_2 \equiv \tilde{B}^{1/2}X_2\tilde{B}^{1/2}$.

Now, by Lemma D.2,

$$\Lambda_1 = \tilde{B}^{1/2}f_1(X_1)\tilde{B}^{1/2} \not\leq \tilde{B}^{1/2}f_1(X_2)\tilde{B}^{1/2} = \Lambda_2,$$

and the proof is complete. ■

Example 8 builds on Proposition G.1.

Example 8 *Within Example 3, consider exchange n with at least two assets traded and agents j_1 and j_2 with identical risk aversions and zero initial endowments $\tilde{q}_{j_1}^0 = \tilde{q}_{j_2}^0 = 0$. Then, there exists participation such that agent j_1 is better connected than agent j_2 , that is, $N(j_1) \supset N(j_2)$, and agent j_1 attains lower equilibrium utility.*

Proof of Example 8. Suppose that agent j_1 participates in a greater number of liquid exchanges than agent j_2 so that his perceived covariance matrix $\tilde{\mathcal{V}}_{j_1}$ satisfies $\tilde{\mathcal{V}}_{j_1} \leq \tilde{\mathcal{V}}_{j_2}$. Pick $\tilde{\mathcal{V}}_{j_1}$, $\tilde{\mathcal{V}}_{j_2}$ and \tilde{B} satisfying the conditions of Proposition G.1. Then, there exists a vector π such that

$$\langle \Lambda_{j_1} \pi, \pi \rangle < \langle \Lambda_{j_2} \pi, \pi \rangle.$$

Now, pick a distribution of initial holdings so that $\tilde{\mathbf{Q}} = \tilde{B}^{-1}\pi$. Denote by U_i the utility of agent $j_i, i = 1, 2$. Then, by Lemma G.1,

$$U_1 + \langle \tilde{B}\tilde{\mathbf{Q}}, \tilde{\mathbf{Q}} \rangle = \langle 0.5\tilde{B}\tilde{\Lambda}_{j_1}\tilde{B}\tilde{\mathbf{Q}}, \tilde{\mathbf{Q}} \rangle = 0.5\langle \Lambda_{j_1}\pi, \pi \rangle < 0.5\langle \Lambda_{j_2}\pi, \pi \rangle = \langle 0.5\tilde{B}\tilde{\Lambda}_{j_1}\tilde{B}\tilde{\mathbf{Q}}, \tilde{\mathbf{Q}} \rangle = U_2 + \langle \tilde{B}\tilde{\mathbf{Q}}, \tilde{\mathbf{Q}} \rangle.$$

■

H Additional Results

A. Decentralized Market CAPM. Denote by Q_n^k the coordinate of $Q \in R^{\bar{K}}$ corresponding to asset k traded in exchange n . Let $\Sigma_{\kappa(n)}$ be the covariance matrix of the assets that are traded in exchange n and let $\Gamma_n \equiv \Sigma_{\kappa(n)}^{-1} \{\mathbf{Q}_n^k\}_{k \in \kappa(n)} = \{\Gamma_n^k\}_{k \in \kappa(n)} \in \mathbb{R}^{\kappa(n)}$. By the linearity, there exists a vector $\mu_{n,k} = \{\mu_{n,k}^{i,l}\}_{i,l} \in R^{I \times K}$ such that

$$\Gamma_n^k = \langle \mu_{n,k}, Q^0 \rangle = \sum_{i,l} \mu_{n,k}^{i,l} Q_i^{0,l},$$

where $Q^0 = \{Q_i^{0,k}\}_{i,k}$ is the vector of initial holdings. Let $\mathbf{R}_n = \sum_{k \in \kappa(n)} \Gamma_n^k R_k$ denote the random payoff of the portfolio that contains Γ_n^k units of asset k , $k \in K$.

Proposition H.1 (Decentralized CAPM) *For any exchange n , the price at which asset $k \in \kappa(n)$ is traded in exchange n is equal to the expected payoff net the risk premium, given by the covariance of the asset payoff with the payoff of the exchange-specific market portfolio, \mathbf{R}_n ,*

$$p_n(k) = \delta_k - \text{Cov}(R_k, \mathbf{R}_n).$$

Proposition [H.1](#) implies that exchange-specific market portfolios depend in a nontrivial way on the market structure. Even in a public exchange, the market-capitalization-weighted portfolio may not command the highest risk premium if some agents have access to other exchanges.

B. Comparative Statics of Price Impact. Example [9](#) illustrates the interdependence in price impact Λ_i between exchanges with disjoint sets of traders.

Example 9 (Indirectly Connected Markets) *Consider partition $\{N_1, N_2\}$ of the set of exchanges N and assume that there exist nonempty sets I_1 and I_2 , such that $I_1 = \{i \in I; N(i) \in N_1\}$ and $I_2 = \{i \in I; N(i) \in N_2\}$. If we define another set of agents, $I_3 = I - (I_1 \cup I_2)$, then $\{I_1, I_2, I_3\}$ is a partition of the set of agents I . For instance, agents I_3 are dealers who trade assets in a centralized exchange with one another and two sets of clients, I_1 and I_2 , who trade a disjoint set of assets. Let M_l be the number of agents in I_l , $l = 1, 2, 3$, $M_1, M_2 \geq 2$. We assume that all agents in I_l , $l = 1, 2, 3$, have the same risk aversion α_l and trade assets in exchanges N_l , where $N_3 = N$. Equation [\(4\)](#) implies that price impact Λ_l is identical for all agents in I_l . Thus,*

$$B = \left(\sum_{l=1}^3 M_l (\alpha_l \bar{\mathcal{V}}_{P_l} + \bar{\Lambda}_l)^{-1} \right).$$

When $I_3 = \emptyset$ (i.e., the network is disconnected), then

$$B = \begin{pmatrix} M_1 (\alpha_1 \mathcal{V}_{P_1} + \Lambda_1)^{-1} & 0 \\ 0 & M_2 (\alpha_2 \mathcal{V}_{P_2} + \Lambda_2)^{-1} \end{pmatrix},$$

and, hence, $\Lambda_l = (B_{11} - (\alpha_l \mathcal{V}_{P_l} + \Lambda_l)^{-1})^{-1} = (M_1 - 1)^{-1}(\alpha_l \mathcal{V}_{P_l} + \Lambda_l)$. The price impact of $i \in I_l$ does not depend on the price impacts, risk aversion, or number of traders in $N \setminus N_l$, as expected.

Assume now $I_3 \neq \emptyset$, $M_3 \geq 1$, so that exchanges N_1 and N_2 are connected via I_3 . The market-wide liquidity is

$$B = \begin{pmatrix} M_1 \Sigma_1 + M_3 \Sigma_{3,11} & M_3 \Sigma_{3,12} \\ M_3 \Sigma_{3,12}^T & M_2 \Sigma_2 + M_3 \Sigma_{3,22} \end{pmatrix},$$

where $\Sigma_l \equiv (\alpha_l \mathcal{V}_l + \Lambda_l)^{-1}$, $l = 1, 2, 3$ and

$$\Lambda_3 \equiv \begin{pmatrix} \Lambda_{3,11} & \Lambda_{3,12} \\ \Lambda_{3,12}^T & \Lambda_{3,22} \end{pmatrix}, \quad \Sigma_3^{-1} = \begin{pmatrix} \Sigma_{3,11} & \Sigma_{3,12} \\ \Sigma_{3,12}^T & \Sigma_{3,22} \end{pmatrix}^{-1} = \begin{pmatrix} \alpha_3 \mathcal{V}_1 + \Lambda_{3,11} & \alpha_3 \mathcal{V}_{12} + \Lambda_{3,12} \\ \alpha_3 \mathcal{V}_{12}^T + \Lambda_{3,12}^T & \alpha_3 \mathcal{V}_2 + \Lambda_{3,22} \end{pmatrix}.$$

Then, in particular, the price impact of agents in I_1 is

$$\Lambda_1 = ((M_1 - 1)\Sigma_1 + M_3 \Sigma_{3,11} - (M_3 \Sigma_{3,12})(M_2 \Sigma_2 + M_3 \Sigma_{3,22})^{-1}(M_3 \Sigma_{3,12})^T)^{-1}.$$

Assume that Λ_l is a positive definite matrix for all $l = 1, 2, 3$. We consider three comparative statics for Λ_1 and one for Λ_3 .

(1) Price impact in exchanges N_1 depends positively on the price impact in the indirectly connected exchanges N_2 : Consider a matrix $\tilde{\Lambda}_2 \geq \Lambda_2$, such that $(\tilde{\Lambda}_2 - \Lambda_2)$ is a positive semidefinite matrix. By the properties of positive definite matrices, we have $\tilde{\Sigma}_2^{-1} \equiv \alpha_2 \mathcal{V}_2 + \tilde{\Lambda}_2 \geq \Sigma_2^{-1}$ and, thus, $\tilde{\Sigma}_2 \leq \Sigma_2$ and

$$\tilde{\Lambda}_1 \equiv ((M_1 - 1)\Sigma_1 + M_3 \Sigma_{3,11} - (M_3 \Sigma_{3,12})(M_2 \tilde{\Sigma}_2 + M_3 \Sigma_{3,22})^{-1}(M_3 \Sigma_{3,12})^T)^{-1} \geq \Lambda_1.$$

(2) The more risk averse the traders in exchanges N_2 , the greater the price impacts in exchanges N_1 : For any $\hat{\alpha}_2 \geq \alpha_2 > 0$, $\hat{\Sigma}_2^{-1} \equiv \hat{\alpha}_2 \mathcal{V}_2 + \Lambda_2 \geq \Sigma_2^{-1}$ and $\hat{\Sigma}_2 \leq \Sigma_2$. By the same argument, $\hat{\Lambda}_1 \geq \Lambda_1$.

(3) The price impacts of traders in exchanges N_1 increase when the number of traders decreases in N_1 or in the indirectly connected exchanges N_2 : Consider an increase in the number of traders M_3 in the linking classes on the price impact,

$$\Lambda_1 \equiv (M_1 - 1)\Sigma_1 + M_3(\Sigma_{3,11} - \Sigma_{3,12}(\frac{M_2}{M_3}\Sigma_2 + \Sigma_{3,22})^{-1}\Sigma_{3,12}^T).$$

When M_3 increases, $(\Sigma_{3,11} - \Sigma_{3,12}(\frac{M_2}{M_3}\Sigma_2 + \Sigma_{3,22})^{-1}\Sigma_{3,12}^T)$ decreases. However, by Proposition 5.2, if $\alpha_3 < \min\{\alpha_1, \alpha_2\}$, the more connected agents of class I_3 have larger price impact. This also illustrates how the equilibrium effects of market segmentation (i.e., traders in exchanges N_1 and N_2 are disjoint) differ from those of the absence of lack of payoff correlation (between the assets traded in N_1 and N_2).

(4) Cross-exchange effects: The dealers' price impacts in exchanges N_1 and N_2 are not independent,

$$\Lambda_3 = \left((M_3 - 1) \begin{pmatrix} \Sigma_{3,11} & \Sigma_{3,12} \\ \Sigma_{3,12}^T & \Sigma_{3,22} \end{pmatrix} + \begin{pmatrix} M_1 \Sigma_1 & 0 \\ 0 & M_2 \Sigma_2 \end{pmatrix} \right)^{-1},$$

as long as the assets traded in N_1 and N_2 are not independent. Depending on the substitutability or complementarity of assets N_1 and N_2 , the dealer's liquidity is higher or lower, relative to the case of independent assets in N_1 and N_2 ($\Lambda_{3,12}^T \sim \alpha_3 \mathcal{V}_{12}^T$).

C. Relative Price Impact: Multiple Assets. We have

$$\Lambda_i = \left(\underbrace{((B^{-1})_{N(i)})^{-1} - (\alpha_i \mathcal{V}_{N(i)} + \Lambda_i)^{-1}}_{\text{Residual liquidity of } i\text{'s in exchanges } N(i)} \right)^{-1}. \quad (32)$$

where

$$B \equiv \sum_j (\alpha_j \bar{\mathcal{V}}_{N(j)} + \bar{\Lambda}_j)^{-1}, \quad (33)$$

We can restate it in the following form.

Lemma H.1 (Equilibrium Price Impact) *In equilibrium, a trader's (gross) liquidity in exchanges $N(i)$, $((B^{-1})_{N(i)})^{-1}$, is the Schur complement of $B_{-i,-i}$ in B , $((B^{-1})_{N(i)})^{-1} = \mathcal{S}(B, N(i)) \equiv B_{i,i} - B_{i,-i} B_{-i,-i}^{-1} B_{-i,-i}^T$.*

Hence, the equilibrium price impacts of different market participants are linked through the aggregate liquidity measure B . Namely, let

$$\Phi(\Lambda_i, \alpha_i \mathcal{V}_{N(i)}) \equiv (\Lambda_i^{-1} + (\alpha_i \mathcal{V}_{N(i)} + \Lambda_i)^{-1})^{-1} \quad (34)$$

be the harmonic mean of two matrices Λ_i and $\Lambda_i + \alpha_i \mathcal{V}_{N(i)}$. By Equation (32), $\Phi(\Lambda_i, \alpha_i \mathcal{V}_{N(i)}) = (B^{-1})_{N(i)}$, for any trader i . In particular, the price impacts of two traders i and j who are connected (i.e., $N(i) \cap N(j) \neq \emptyset$) are related as follows

$$(\Phi(\Lambda_i, \alpha_i \mathcal{V}_{N(i)}))_{N(i) \cap N(j)} = (\Phi(\Lambda_j, \alpha_j \mathcal{V}_{N(j)}))_{N(i) \cap N(j)} = (B^{-1})_{N(i) \cap N(j)}. \quad (35)$$

Suppose that $N(i) \supset N(j)$; for instance, trader i is better connected than j . A concavity property of the harmonic mean (34) implies the following relationship among the price impacts in the exchanges in which both traders i and j participate, $(\Lambda_i)_{N(j)}$ and Λ_j .

Proposition H.2 *Suppose that trader i has greater market participation than trader j , $N(i) \supset N(j)$. Then,*

$$\Phi((\Lambda_i)_{N(j)}, \alpha_i \mathcal{V}_{N(j)}) \geq \Phi(\Lambda_j, \alpha_j \mathcal{V}_{N(j)}). \quad (36)$$

Proof of Proposition H.2. By (32),

$$(\Phi(\Lambda_i, \alpha_i \mathcal{V}_{N(i)}))_{N(j)} = \Phi(\Lambda_j, \alpha_j \mathcal{V}_{N(j)}).$$

By Theorem 5 in Anderson (1971),

$$(\Phi(\Lambda_i, \alpha_i \mathcal{V}_{N(i)}))_{N(j)} \leq \Phi((\Lambda_i)_{N(j)}, \alpha_i \mathcal{V}_{N(j)}),$$

and the claim follows. ■

Function $\Phi(\Lambda, \alpha\mathcal{V})$ is monotone increasing in Λ and, therefore, for the case of scalar Λ_j , inequality (36) immediately yields Lemma 5.2.

Let us explain why, unless price impact matrices commute, Proposition 5.2 does not carry over to multiple assets by using (36) to conclude that $(\Lambda_i)_{N(j)} \geq \Lambda_j$. Let $A_1 \equiv \Phi((\Lambda_i)_{N(j)}, \alpha_i \mathcal{V}_{N(j)})$ and $A_2 \equiv \Phi(\Lambda_j, \alpha_j \mathcal{V}_{N(j)})$. Then (using Lemma D.2),

$$(\Lambda_i)_{N(j)} = \alpha_i \mathcal{V}_{N(j)}^{1/2} f(\alpha_i \mathcal{V}_{N(j)}^{1/2} A_1^{-1} \mathcal{V}_{N(j)}^{1/2}) \mathcal{V}_{N(j)}^{1/2}, \quad \Lambda_j = \alpha_j \mathcal{V}_{N(j)}^{1/2} f(\alpha_j \mathcal{V}_{N(j)}^{1/2} A_2^{-1} \mathcal{V}_{N(j)}^{1/2}) \mathcal{V}_{N(j)}^{1/2},$$

where $f(a) = (1/2a)/(2 - a + \sqrt{a^2 + 4})$, is monotone decreasing in a . Inequality $A_1 \geq A_2$ (Proposition H.2) implies $X_1 \equiv \mathcal{V}_{N(j)}^{1/2} A_1^{-1} \mathcal{V}_{N(j)}^{1/2} \leq \mathcal{V}_{N(j)}^{1/2} A_2^{-1} \mathcal{V}_{N(j)}^{1/2} \equiv X_2$. However, given two non-commuting symmetric matrices X_1 and X_2 and a monotone decreasing function $f(x)$, inequality $X_1 \leq X_2$ does *not* generally imply $f(X_1) \geq f(X_2)$. A function f that satisfies $f(X_1) \geq f(X_2)$ for any $X_1 \leq X_2$ is called *matrix monotone*. In particular, to conclude that $(\Lambda_i)_{N(j)} \geq \Lambda_j$, function f in (28) must be matrix-monotone, which is not the case.⁶⁰

One can still compare price impacts through an eigenvalue order instead of the (weaker) positive semidefinite order, using that with positive semidefinite matrices, there is a min-max interpretation of eigenvalues. For the eigenvalues of a symmetric $m \times m$ matrix A ordered to be decreasing, $\text{eig}(A) = \{\mu_1(A) \geq \dots \geq \mu_m(A)\}$, we write $\text{eig}(A) \geq \text{eig}(B)$ if $\mu_i(A) \geq \mu_i(B)$ for all $i = 1, \dots, m$.⁶¹

Proposition H.3 (Relative Price Impact: Many Assets) *Suppose that trader i has greater market participation than trader j , $N(i) \supset N(j)$. Then, if $\alpha_i \leq \alpha_j$, equilibrium price impact of trader i in exchanges $N(j)$ is larger than that of trader j in the following sense:*

$$\text{eig}(\alpha_i^{-1} \mathcal{V}_{N(j)}^{-1/2} (\Lambda_i)_{N(j)} \mathcal{V}_{N(j)}^{-1/2}) \geq \text{eig}(\alpha_j^{-1} \mathcal{V}_{N(j)}^{-1/2} \Lambda_j \mathcal{V}_{N(j)}^{-1/2}).$$

If the matrices $\mathcal{V}_{N(j)}^{-1/2} (\Lambda_i)_{N(j)} \mathcal{V}_{N(j)}^{-1/2}$ and $\mathcal{V}_{N(j)}^{-1/2} \Lambda_j \mathcal{V}_{N(j)}^{-1/2}$ commute, then the stronger inequality (8) holds.

Proof of Proposition H.3. Let $W_1 = \alpha_i^{-1} \mathcal{V}_{N(j)}^{-1/2} (\Lambda_i)_{N(j)} \mathcal{V}_{N(j)}^{-1/2}$ and $W_2 = \alpha_j^{-1} \mathcal{V}_{N(j)}^{-1/2} \Lambda_j \mathcal{V}_{N(j)}^{-1/2}$. Then, $W_k = f(\alpha_i \mathcal{V}_{N(j)}^{1/2} A_k^{-1} \mathcal{V}_{N(j)}^{1/2})$, $k = 1, 2$. By Lemma H.2, eigenvalues are increasing in the positive semi-definite order and hence, by Proposition H.2

$$\text{eig}(\alpha_i \mathcal{V}_{N(j)}^{1/2} A_1^{-1} \mathcal{V}_{N(j)}^{1/2}) \leq \text{eig}(\alpha_j \mathcal{V}_{N(j)}^{1/2} A_2^{-1} \mathcal{V}_{N(j)}^{1/2}).$$

⁶⁰ In fact, f is not matrix monotone on any interval. This noteworthy property does not have any scalar analogues. This implies that, with sufficiently many assets, for any $A \geq 0$ there exists B , $B \leq A$, such that B is sufficiently close to A and the monotonicity fails (by the Löwner's Theorem). A function $f(z)$ is matrix monotone on some (even an arbitrarily small) interval if, and only if, it can be approximated by convex combinations of simple hyperbolic functions $\frac{\alpha}{z+\beta}$, $\alpha \in R_+$, $\beta \in R$. For the general theory of monotone matrix functions, see Donoghue (1974).

⁶¹ The eigenvalue functional is not a linear functional (except in dimension one), but it is a minimax expression of linear functionals. Eigenvalue order and the positive semidefinite order are equivalent for commuting matrices. We also use techniques from the theory of monotone matrix functions to prove Proposition 6.2 in the welfare section.

Therefore,

$$\text{eig}(W_1) = f(\text{eig}(\alpha_i \mathcal{V}_{N(j)}^{1/2} A_1^{-1} \mathcal{V}_{N(j)}^{1/2})) \geq f(\text{eig}(\alpha_j \mathcal{V}_{N(j)}^{1/2} A_2^{-1} \mathcal{V}_{N(j)}^{1/2})) = \text{eig}(W_2).$$

If W_1 and W_2 commute, diagonalizing them in the same basis implies that eigenvalue order and the positive semi-definite order are equivalent. ■

In particular, if agents are equally risk averse, $\alpha_i = \alpha_j$, liquidity in exchanges $N(j)$ is lower for the more connected agents i (e.g., dealers or intermediaries) than their less connected counterparties. (In contrast, in a centralized market with assets from exchanges $N(j)$ and agents i and j , the agents would have the same price impact.) This can be understood through Lemma H.1: Since the price impact of each agent in a given exchange is determined by the risk exposure of other agents in that exchange, trader i 's ability to diversify risk in exchanges $N(i) \setminus N(j)$ lowers their risk exposure in exchanges $N(j)$ relative to trader j 's exposure and, thus, the price impact of trader j in exchanges $N(j)$.

The eigenvalues of matrix $\mathcal{V}_{N(j)}^{-1/2} \Lambda_j \mathcal{V}_{N(j)}^{-1/2}$ have a clear economic interpretation based on the min-max representation of eigenvalues.

Lemma H.2 (Courant-Fisher Theorem) *Let $U \subset \mathbb{R}^{N(j)}$ be a subspace. For any symmetric matrix $A \in \mathbb{R}^{N(j) \times N(j)}$, the eigenvalues of $\mathcal{V}_{N(j)}^{-1/2} A \mathcal{V}_{N(j)}^{-1/2}$, $\mu_1 \leq \dots \leq \mu_{N(j)}$ satisfy*

$$\begin{aligned} \mu_k &= \min_{\dim U = N(j) - k + 1} \left\{ \max \left\{ y^T A y \mid y \in U, y^T \mathcal{V}_{N(j)} y \leq 1 \right\} \right\} \\ &= \max_{\dim U = k} \left\{ \min \left\{ y^T A y \mid y \in U, y^T \mathcal{V}_{N(j)} y \leq 1 \right\} \right\}. \end{aligned} \quad (37)$$

D. Conditional Riskiness. The following corollary of Proposition 5.3 demonstrates that, in a decentralized market, depending on the market structure, an introduction of an asset (e.g., through an increase in participation) may give rise to essentially arbitrary⁶² heterogeneity in conditional riskiness, and consequently, in the cross-exchange price impact.

Corollary H.1 *Consider a market with K assets with the covariance matrix Σ and I agents with risk aversion $\{\alpha_i\}_i$. Let Z_i , $i \in I$ be an arbitrary collection of positive semidefinite matrices satisfying $Z_i \leq \Sigma$ for all i and such that either $\sum_i (\Sigma - Z_i) > \Sigma$, or $Z_1 \leq Z_2 \leq \dots \leq Z_I$. Then, one can increase participation $\{N(i)\}_i$ that introduces additional assets available for trading such that effective riskiness for trader i for the K assets is given by Z_i .*

⁶² Specifically, the diagonal elements $(\mathcal{V}_{i \setminus j})_{nn}$ of the effective riskiness matrix $\mathcal{V}_{i \setminus j}$ are lower than those for \mathcal{V} , whereas the off-diagonal elements $(\mathcal{V}_{i \setminus j})_{nm}$, $n \neq m$, (i.e., effective cross-exchange covariances) can take any values satisfying $|(\mathcal{V}_{i \setminus j})_{nm}| < (\mathcal{V}_{nn} \mathcal{V}_{mm})^{1/2}$ (the Cauchy-Schwartz inequality).