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SAVINGS BONDS, RETRACTABLE BONDS AND CALLABLE BONDS

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Savings bonds, retractable bonds and callable bonds are each equivalent to a straight bond with an option. Neglecting default risk the value of these contingent claims depends upon the riskless interest rate. This paper employs the option pricing framework to value these bonds, under the assumptions that the interest rate follows a Gauss–Wiener process and that the pure expectations hypothesis holds.

I

The savings or parity bond, the retractable or extendible bond and the callable bond are, like the convertible bond, distinct types of fixed income security which represent contingent claims. Two major questions arise with respect to these securities: how they should be valued, and how the party who holds the option to cash the savings bond, to retract the retractable bond, or to call the callable bond, should exercise his option; these two issues are logically inter-related and are the subject of this paper.

The savings or parity bond is a fixed income security with a given maturity, which can be cashed at par at the discretion of the holder: it is in effect a payable-on-demand loan with a predetermined interest rate. Such securities are frequently used by national governments to tap the savings of small investors, and often specify that only a limited amount of bonds may be heid by any individual investor: examples include the U.S. Savings Bond, the Canada Savings Bond, and various similar instruments issued in the U.K. The bond may be a discount instrument, as for example, the Series E U.S. Savings Bond which have an original maturity of five years and permit redemption at an escalating series of redemption prices, designed so that the rate of return from holding the bond rises as the length of time the bond is held increases; or the bond may be a coupon instrument redeemable at par, as for example the Canada Savings Bond, and the Series H U.S. Savings Bond. The coupon

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stream may be either level or rising through time: the latter scheme has the effect of discouraging purchasers of the bond from redeeming them early. Occasionally, coupon rates on outstanding issues of Canada Savings Bonds are raised, also to discourage redemptions, at a time of rising interest rates. This change in the terms of an outstanding issue is formally equivalent to the redemption of the outstanding bonds and the issuance of new bonds, but is preferred by the issuer and the holder, since it eliminates the transactions costs that would otherwise be incurred.[1] The coupon bearing savings bond is identical in concept to the parity bonds issued from time to time by Provincial Governments and Crown Corporations in Canada: as its name implies, the parity bond is redeemable at par at any time. While thin secondary markets for parity bonds exist occasionally, savings bonds are typically non-transferable so that they cannot be sold, and no secondary market for them exists; however, this presents no obstacle to the investor in an efficient capital market since the bond can be priced and the optimal exercise strategy determined even if trading of the bond is not allowed.

The guaranteed yield and guaranteed redemption value of these instruments permit the purchaser to profit by declines in interest rates while protecting him from the capital loss associated with rises in interest rates: as a result, such bonds may be floated on a lower yield basis than straight bonds of similar risk and maturity. In fact, however, savings bond yields are sometimes set above the yields on comparable straight debt instruments, while the type of purchaser and the amount that may be purchased is restricted. Thus in the United States, 'during World War II and the early postwar period, yields to maturity of all series of savings bonds exceeded yields on long-term governments; and indeed the E-bond rate generally exceeded market yields on Aaa corporate bonds'.[2] In Canada, savings bond issues frequently yield more than regular Government of Canada bonds of equivalent maturity as shown in table 1. Although the two yield series in this table are not directly comparable since the Government of Canada bonds were not selling at par and so offered some tax advantages as well as differing in their duration from the savings bonds, it does seem that on occasion the Canada Savings Bonds have offered a remarkable bargin to those eligible to purchase them.

Savings bonds present a decision problem to the investor who must decide at what time to redeem them. It is clearly not optimal to redeem them if the yield on equivalent straight bonds is less than the coupon rate on the savings bond, and it is possibly not optimal to redeem even if the yield on straight bonds exceeds the coupon rate on the savings bond, either because the coupon rate on the savings bond rises through time or, more generally, because of the possibility that bond yields will rise even higher in the future. Therefore the investor

[1] The cost to the government of redeeming a Canada Savings Bond and issuing a new bond is of the order of 1% of the face value of the bond.

[2] Hanc (1962, p. 24).

must weigh carefully his current yield disadvantage against the possibility of a capital loss if he switches into the straight bonds. Savings bonds also present a valuation problem to the issuer who must determine the terms of the issue so that the bond is attractive to purchasers, but not so attractive as to be unnecessarily expensive to the issuer: in a perfect capital market the terms will be set so that the market value of the savings bond, if it were traded, would be equal to the par value at time of issue.

Table 1

Comparative yields of newly issued Canada Savings Bonds (CSB) and Government of Canada Bonds (GCB) of similar maturity.

Date of issue	CSB: Yield to maturity (%)	GCB: Yield to maturity (%)	Gross sales of CSB $(billion)
11.1.68	6.75	7.31	3.13
11.1.69	8.00	8.44	4.74
11.1.70	7.75	8.18	1.76
11.1.71	7.19	6.26	2.53
11.1.72	7.30	6.56	1.78
11.1.73	7.54	7.83	1.06
11.1.74	9.75	7.86	6.00
11.1.75	9.38	9.06	3.47

Although savings bonds appear to have received comparatively little attention in the literature,[3] this neglect cannot be attributed entirely to their unimportance. The total of Canada Savings Bonds outstanding in Canada is in the order of $17 billion representing 39% of the debt of the Canadian Federal Government, and in the United States the $64 billion of savings bonds outstanding in 1975 was equal to 21% of the total marketable debt of the Federal Government.

Retractable and extendible bonds, unlike savings bonds which are issued primarily by governments, have been used as a financing instrument by private corporations in Canada, and were especially popular in the late 1960's. A retractable bond is a long-term bond which, at the holder's option, may be retracted or redeemed on a specified date prior to expiration.[4] For example, the Gulf Oil of Canada $8\frac{1}{2}$'s of 1989, issued in December 1969, were prepayable in December 1974 at the holder's option, exercisable between June 1973 and June

[3]See however Burrell (1953) and Hanc (1962).

[4]Note that the distinction between a savings bond and a retractable bond is akin to the distinction between American and European options, in that the savings bond can be redeemed at par at any time before maturity, while the retractable bond can be redeemed at par only at a specified date.

1974.[5] An extendible bond is similar to a retractable bond, but is nominally a shorter-term instrument which may be extended for a longer period at the holder's option, possibly at a higher coupon rate. An example is the Bell Telephone of Canada 8's of 1977, issued in May 1969, and exchangeable at the holder's option from November 1970 to November 1975 into $8\frac{1}{4}\%$ bonds due in May 1990: in this case, the higher coupon rate available upon extension may make it advantageous to extend the bond before the last possible date. Since this paper is expository in nature, we make the simplifying assumptions that the extension/retraction decision must be made on a single specified date, that the option elected takes effect immediately, and that only a single coupon rate is specified: generalization of the methods employed here to take account of the specific features of individual bond indentures is straightforward. With these simplifying assumptions, retractable and extendible bonds are identical and each is equivalent to either a long-term bond with a put option on the bond exercisable on the extension/retraction date, or a short-term bond together with a call option on a long-term bond exercisable on the extension/retraction date: in either case, the exercise price of the option is equal to the par value of the bond.

The attraction of extendibles and retractables to corporate issuers is that they permit the issuance of a long-term bond in the case of a retractable or a medium-term bond in the case of the extendible at a lower coupon rate than would be required on an otherwise similar straight bond. On the other hand, the corporation faces the risk, not present with a straight long-term bond, that the bonds will be redeemed early if interest rates rise.

Given our assumptions about the nature of the investor's option, his optimal exercise strategy is straightforward: he should retract the bond if, on the retraction date, the value of the bond, unretracted, is below par value: otherwise, he should not exercise the retraction option. Hence, the major problem is that of valuing the retractable bond.

Callable bonds are much more familiar to students of finance and have been extensively analyzed in the literature.[6] However, none of the earlier analyses have explicitly treated the problems of valuing callable bonds and of determining an optimal call strategy within the option pricing framework which has proved such a versatile tool for the analysis of financial instruments since the seminal work of Black–Scholes (1973) and Merton (1973). For an excellent review of recent developments in option pricing see Smith (1976).

In the following section we show that, under certain assumptions about the term structure of interest rates, the value of all three instruments must obey the same partial differential equation, which is almost the same as the equation

[5]It was clearly optimal for the investor to delay his decision on retraction to the last possible date.

[6]See Elton and Gruber (1972), Jen and Wert (1966, 1967, 1968), Kalymon (1971), Pye (1966, 1967), Weingartner (1967) and others.

which must be obeyed by the value of a straight default-free discount bond. However, the boundary conditions are different for the four different types of bonds. We then show how this differential equation may be solved by numerical methods for the different types of boundary conditions, and offer some numerical examples. The paper concludes with an indication of possible extensions of this type of analysis.

II

The stochastic nature of interest rates is central to the analysis of the three instruments treated in this paper, and it is necessary therefore to start our analysis from an assumption about the term structure of interest rates and its stochastic behaviour over time. Lacking a well developed theory of the term structure under uncertainty, we make the simplest possible assumption, namely, that the pure expectations hypothesis holds, so that the instantaneous expected rate of return on default-free securities of all maturities is equal to the instantaneously risk-free rate of interest.[7] We assume further that the instantaneously risk-free rate of interest follows the stochastic process,

$$\mathrm{d}r = \mu(r)\,\mathrm{d}t + \sigma(r)\,\mathrm{d}z, \tag{1}$$

where $\mathrm{d}z$ is a Gauss–Wiener process with $\mathrm{E}[\mathrm{d}z] = 0$, and $\mathrm{E}[\mathrm{d}z^2] = \mathrm{d}t$. Without a general equilibrium model of growth under uncertainty[8] the only restrictions which may be placed on the functional forms $\mu(r)$ and $\sigma(r)$ arise from the existence of money: to avoid dominance by money the nominal interest rate must be non-negative. A sufficient condition for this is that $\sigma(0) = 0$ and $\mu(0) \geqq 0$ so that there is a natural absorbing or reflecting barrier at $r = 0$.

It is assumed that the capital market is perfect, with no transaction costs, taxes, restrictions on short sales or other institutional frictions.

While callable bonds are issued by governments as well as corporations, retractable and extendible bonds are issued only by corporations and are therefore subject to default risk. We assume in the interest of simplicity that all bonds discussed here are default free. The consequences of relaxing this assumption are discussed in section V.

Given the above assumptions, the market value of any of the three securities under consideration, which we denote generically by G, is a function only of the current value of the riskless rate, r, and time, t, or, more conveniently, time to maturity, $\tau = T - t$, where T is the maturity date, so that we may write the value of the generic bond as $G(r, \tau)$ where it is to be understood that the value of the bond is also a function of its coupon rate and the boundary conditions.

[7]Note that under uncertainty this is not the same as assuming that forward rates are equal to expected future spot rates.

[8]For an example see Merton (1975).

Similarly, we may write the value of the pure discount bond, promising \$1 at maturity, as a function of the current riskless rate and time to maturity, $B(r, \tau)$.

Using Ito's Lemma [McKean (1969)], the instantaneous return on the riskless discount bond is given by

$$dB = B_1\,dr + B_2\,d\tau + \tfrac{1}{2}B_{11}(dr)^2, \tag{2}$$

where the subscripts denote partial derivatives. Then, since from (1), $dr^2 = \sigma^2(r)\,dt$, and $d\tau = -dt$, (2) may be written

$$\frac{dB}{B} = \frac{-B_2 + \mu(r)B_1 + \frac{1}{2}\sigma^2(r)B_{11}}{B}\,dt + \frac{\sigma(r)B_1}{B}\,dz. \tag{3}$$

The pure expectations hypothesis implies that, since $E[dz] = 0$, the coefficient of dt in (3) is equal to the instantaneously risk-free rate of interest, so that

$$\tfrac{1}{2}\sigma^2(r)B_{11} + \mu(r)B_1 - rB - B_2 = 0. \tag{4}$$

Thus the pure expectations hypothesis implies that the value of a default-free discount bond satisfies the partial differential equation (4). We assume that the value of the bond tends to zero for very high interest rates at any time prior to maturity, so that

$$\lim_{r\to\infty} B(r, \tau) = 0, \qquad \tau > 0. \tag{5}$$

Since the value of the bond at maturity is one dollar for sure, we have

$$B(r, 0) = 1. \tag{6}$$

The singularity of the diffusion coefficient at $r = 0$ yields a natural boundary at the origin [see Feller (1952)]. Setting $r = 0$ in (4) and using $\sigma(0) = 0$, we have

$$\mu(0)B_1(0, \tau) - B_2(0, \tau) = 0. \tag{7}$$

Thus the value of a pure discount bond may be obtained as a function of the instantaneous riskless rate and time to maturity by solving the differential equation (4), subject to the boundary conditions [(5), (6), (7)]. The solution procedure will be described in the subsequent section. Observe that the whole term structure of interest rates for a given riskless rate, r, is given by the values of $B(r, \tau)$, $\tau > 0$, for the yield to maturity on a τ-period bond, R_τ, is defined by

$$R_\tau = -\tau^{-1} \ln B(r, \tau), \qquad \tau > 0. \tag{8}$$

Applying Ito's Lemma in a similar fashion to the value of the generic bond under consideration,[9] its instantaneous rate of price appreciation is given by

$$dG = G_1\,dr - G_2\,dt + \tfrac{1}{2}G_{11}(dr)^2, \tag{9}$$

so that, using (1) and dividing by G,

$$dG/G = \gamma\,dt + \delta\,dz, \tag{10}$$

where

$$\gamma = (-G_2 + \mu(r)G_1 + \tfrac{1}{2}\sigma^2(r)G_{11})/G \quad \text{and} \quad \delta = \sigma(r)G_1/G.$$

Similarly, using the pure expectations condition (4), (3) may be written as

$$dB/B = r\,dt + \beta\,dz, \tag{11}$$

where

$$\beta = \sigma(r)B_1/B.$$

Consider forming a zero risk, zero net investment portfolio, consisting of investments in the generic bond, the discount bond and the instantaneously riskless security. Then, the classical hedging argument [see Merton (1973)] implies that the value of the generic bond satisfies

$$\tfrac{1}{2}\sigma^2(r)G_{11} + \mu(r)G_1 - rG - G_2 + c = 0, \tag{12}$$

where c is the continuous rate of coupon payment on the generic bond.

This partial differential equation differs from that which must be satisfied by the discount bond (4) only by inclusion of the coupon rate, c. It is apparent that (12) is the requirement that the expected rate of return on the generic bond, including both coupon and price change, be equal to the instantaneous risk-free rate. Thus we have established that, if the pure expectations hypothesis holds, it must also hold for default-free parity bonds, retractable bonds and callable bonds. Eq. (12) may be regarded as the analogue of the Black–Scholes differential equation, where here the underlying stochastic variable is the level of interest rates, rather than the value of a common stock, as in their case; note however the assumptions about the stochastic structure of interest rates which were required to obtain (12).

[9]We use the term 'generic bond' to refer to savings bonds, retractable bonds and callable bonds.

While the market values of the three different types of bond we have collectively labelled 'the generic bond' satisfy (12), they differ in the nature of their contingent claim and therefore in the boundary conditions which must be satisfied by the solution to the differential equation. It is of course these different boundary conditions which give rise to the different market values of the three instruments: we consider them in turn.

(i) *The savings bond*

The bond is assumed to have a par value of unity so that at maturity, $\tau = 0$, the bond value is unity for all values of r, yielding the terminal value condition,

$$G(r, 0) = 1. \tag{13}$$

The assumed singularity of the diffusion coefficient at the origin yields the natural boundary,

$$\mu(0)G_1(0, \tau) - G_2(0, \tau) + c = 0. \tag{14}$$

Finally, since the bond can always be redeemed at a predetermined value $p(\tau)$, it will be redeemed as soon as the market value falls to $p(\tau)$, giving rise to the redemption condition,

$$G(r, \tau) \geqq p(\tau). \tag{15}$$

(ii) *The retractable bond*

The rectractable bond is assumed to have a par value of unity, so that we have the same terminal value condition (13). If the interest rate falls to zero, it will certainty not pay to retract the bond so that we have the zero interest rate condition (14). Further, we assume that for very high interest rates the value of the bond approaches zero except at final maturity and the maturity corresponding to the retraction date, τ_e,

$$\lim_{r \to \infty} G(r, \tau) = 0, \qquad \tau > 0 \neq \tau_e. \tag{16}$$

On the retraction date, it will be optimal for the investor to retract if the redemption value of unity exceeds the value of the bond if it is not retracted: using τ_e^- to denote the time to maturity of the bond at the instant after the retraction date, this latter value is given by $G(r, \tau_e^-)$, so that the value of the bond the instant before retraction $G(r, \tau_e^+)$ is

$$G(r, \tau_e^+) = \max[G(r, \tau_e^-), 1]. \tag{17}$$

Since after the retraction date the retractable bond is equivalent to an otherwise identical straight bond, $G(r, \tau_e^-)$ is equal to the value of a straight bond with maturity τ_e^- and coupon rate c.

Finally, we observe that since the investor may at his option treat the retractable bond as a straight bond with a maturity corresponding either to the retraction date or to the final maturity, the value of the retractable bond prior to the retraction date can fall below the values of neither of these straight bonds.

(iii) *The callable bond*

Again the maturity value of the bond is given by the terminal value condition (13), and since the bond will never be called if the interest rate is very high, the limiting interest rate condition (16) holds.

When the bond is non-callable the natural boundary condition (14) obtains.

Finally, the optimal call policy must be determined. While this has been treated elsewhere [Brennan and Schwartz (1975), Ingersoll (1976)] for the case of callable convertible bonds and the same considerations apply here, we restate the argument briefly for the sake of completeness. First, the Modigliani–Miller Theorem (1958) assures us that the value of a firm is independent of its financing policies under reasonable assumptions, so that by adopting a call policy which minimizes the value of the bonds, the issuing corporation will *ipso facto* be maximizing the value of the equity, which we take to be the objective of the management. The value of the bonds will be minimized if they are called at the point at which their uncalled value is equal to the call price: to call when the uncalled value is below the call price is to confer a needless gain on the bondholders, while to allow the bond value to rise above the call price is clearly incompatible with minimizing the value of the bond.[10] Hence the optimal call policy gives rise to the call policy condition,

$$G(r, \tau) \leqq 1, \qquad \tau < \tau_c, \tag{18}$$

where τ_c is the maximum maturity at which the bond may be called.

The results of this section are summarized in table 2 for the discount bond and the three different types of bond under consideration. For notational consistency we have used the expression $G(r, \tau)$ to denote the value of the discount bond also.

III

In general there will exist no analytical solution to the differential equations which must be satisfied by either the discount bond or the generic bond, so that

[10]While the above argument applies strictly to corporate bonds only, we assume that a government should also minimize the value of its callable bonds.

Table 2
Differential equation and boundary conditions satisfied by the values of discount bonds, savings bonds, retractable bonds and callable bonds.

	Discount bond	Savings bond	Retractable bond	Callable bond
Differential equation				
$\frac{1}{2}\sigma^2(r)G_{11}+\mu(r)G_1-rG-G_2+c=0$	X[a]	X	X	X
Boundary conditions				
Terminal value: $G(r,0)=1$	X	X	X	X
For $r=0$: $\mu(0)G_1(0,\tau)-G_2(0,\tau)+c=0$	X[a]	X	X	X[b]
$\lim_{r\to\infty} G(r,\tau)=0,\ r\neq 0$	X		X[c]	X
Redemption: $G(r,\tau)\geqq p(\tau)$		X		
Call policy: $G(r,\tau)\leqq 1,\ \tau\leqq\tau_c$				X
Retraction policy: $G(r,\tau_e^{+})=\max[G(r,\tau_e^{-}),1]$			X	

[a]For $c=0$.
[b]For $\tau>\tau_c$.
[c]For $\tau\neq\tau_e$.

it is necessary to employ a finite difference approximation to the equations in order to find a solution. The three sets of boundary conditions satisfied by the different types of generic bond, and the distinct problem posed by the discount bond require somewhat different solution procedures and we describe them in turn.

We deal first with the savings bond, since the solution procedures for the discount bond, the retractable bond and the callable bond are somewhat similar.

(i) *The savings bond*

Writing finite differences in place of partial derivatives, the differential equation (12) which must be satisfied by the parity bond can be approximated by[11]

$$U_i G_{i-1,j} + V_i G_{i,j} + W_i G_{i+1,j} = G_{i,j-1} + ck, \qquad i = 1, \ldots, n-1, \quad j = 1, \ldots, m, \tag{19}$$

where U_i, V_i and W_i are known, and

$$G(r, \tau) = G(r_i, \tau_j) = G(ih, jk) = G_{i,j};$$

h and k are the discrete increments in the interest rate and the time to maturity, respectively. In the example which follows, these parameters were set at 0.005% per month and one half month respectively. The number of steps in the time dimension, m, is chosen to correspond to the maturity of the savings bond under consideration ($mk = T$), and n is chosen sufficiently large that the redemption condition (15) holds as an equality, for some value of r smaller than nh, for all τ considered: larger values of r are irrelevant since the differential equation does not apply when the interest rate is so high that the bond has been redeemed.

From the natural boundary condition (14), $U_0 = 0$, so that (19) is a tri-diagonal system of n linear equations in the $(n+1)$ unknowns $G_{i,j}$ $(i = 0, 1, \ldots, n)$ for each j, which may be transformed into

$$u_i G_{i,j} + v_i G_{i+1,j} = w_i, \qquad i = 0, \ldots, n-1, \tag{20}$$

where the coefficients of (20) are known, given the values of $G_{i,j-1}$ $(i = 0, 1, \ldots, n)$. (20) is a system of (n) equations in the $(n+1)$ unknowns $G_{i,j}$ $(i = 0, \ldots, n)$.

[11] See McCracken and Dorn (1964) for a detailed explanation.

Given the terminal value condition (13), $G_{i,0} = 1$ $(i = 0, 1, \ldots, n)$, the remaining values of $G_{i,j}$ $(i = 0, \ldots, n; j = 1, \ldots, m)$ are determined recursively by the following procedure, which takes account of the redemption constraint (15), written in the finite difference notation as

$$G_{i,j} \geqq p(jk). \tag{21}$$

Set $G_{n,j} = p(jk)$ and solve for $G_{n-1,j}$ from (20). If $G_{n-1,j} \geqq p(jk)$ solve the remaining equations for $G_{i,j}$; if $G_{n-1,j} < p(jk)$, set $G_{n-1,j} = p(jk)$ and solve for $G_{n-2,j}$. Again check whether $G_{n-2,j} \gtreqless p(jk)$, and continue in this manner until a set of $G_{i,j}$ $(i = 0, \ldots, n_c)$ is obtained which satisfies the first (n_c) equations of (20): the remaining $(n-n_c)$ values of $G_{i,j}$ are equal to $p(jk)$. We have thus obtained a set of $G_{i,j}$ values which satisfy the differential equation subject to the redemption condition (21). The minimum value of i for which $G_{i,j} = p(jk)$, $i = n_c$, corresponds to the critical interest rate r_c, above which the bond, should optimally be redeemed $(r_c = hn_c)$. Thus the solution procedure, in addition to providing a value of the savings bond, $G_{i,j}$, for each time increment, for the whole range of interest rates considered, presents the optimal redemption strategy in the form of a time series of critical interest rates at which the bond should optimally be redeemed: the critical interest rate is that interest rate at which the bond if it were traded, would sell at par.

(ii) *The discount bond*

In the case of the savings bond, the redemption constraint (15) avoided the need to consider interest rates above the maximum critical interest rates, which is finite. With the discount bond, and retractable and callable bonds, however, there is no such natural limit on the range of interest rates to be considered, the limiting interest rate condition (16) applying only as $r \to \infty$. It is therefore necessary to transform the state variable so that only a bounded range need be considered. We do this by defining the state variable as $s(r)$, where

$$s(r) = 1/(1+r). \tag{22}$$

Note that $0 \leqq s(r) \leqq 1$. Applying Ito's Lemma to $s(r)$,

$$\mathrm{d}s = s_r\,\mathrm{d}r + \tfrac{1}{2}s_{rr}(\mathrm{d}r)^2, \tag{23}$$

and since $s_r = -s^2$, $s_{rr} = 2s^3$, we have, using (1) and the fact that $r = (1-s)/s$,

$$\mathrm{d}s = [-s^2\mu_*(s) + s^3\sigma_*^2(s)]\,\mathrm{d}t - s^2\sigma_*(s)\,\mathrm{d}z, \tag{24}$$

where

$$\mu_*(s) \equiv \mu(r) \quad \text{and} \quad \sigma_*(s) \equiv \sigma(r).$$

Using lower case letters to denote the bond values as a function of s and τ,

$$b(s,\tau) \equiv B(r,\tau) \quad \text{and} \quad g(s,\tau) \equiv G(r,\tau).$$

Applying Ito's Lemma to $b(s,\tau)$

$$\frac{\mathrm{d}b}{b} = \frac{[-s^2\mu_*(s)+s^3\sigma_*^2(s)]b_1 - b_2 + \frac{1}{2}s^4\sigma_*^2(s)b_{11}}{b}\,\mathrm{d}t - \frac{s^2\sigma_*(s)b_1}{b}\,\mathrm{d}z. \tag{25}$$

Arguments similar to those presented in section II imply that the coefficient of $\mathrm{d}t$ in (25) be equal to $r \equiv (1-s)/s$, so that we obtain the following partial differential equation for $b(s,\tau)$:

$$\tfrac{1}{2}s^4\sigma_*^2(s)b_{11} + [-s^2\mu_*(s)+s^3\sigma_*^2(s)]b_1 - \frac{1-s}{s}\,b - b_2 = 0, \tag{26}$$

and the boundary conditions corresponding to the limiting interest rate condition (5), the terminal value condition (6) and the zero interest rate condition (7) are

$$b(0,\tau) = 0, \tag{27}$$

$$b(s,0) = 1, \tag{28}$$

$$\mu_*(1)b_1(1,\tau) + b_2(1,\tau) = 0. \tag{29}$$

Since the boundary conditions are in the form of equalities, the differential equation (26) can be solved numerically by the methods described in McCracken and Dorn (1964). Then, using the relationship between r and s, (22), we obtain values of $B(r,\tau)$ for different values of r and τ.

(iii) *The retractable bond*

Arguments similar to those just given for the discount bond lead to a differential equation for the value of the coupon paying retractable and callable bonds identical to that derived for the discount bond (26) except for the inclusion of the coupon, c,

$$\tfrac{1}{2}s^4\sigma_*^2(s)g_{11} + [-s^2\mu_*(s)+s^3\sigma_*^2(s)]g_1 - \frac{1-s}{s}\,g - g_2 + c = 0. \tag{30}$$

We know from (17) that the value of the retractable bond on the retraction date is equal to the greater of unity and the value of a straight bond with the same coupon and a maturity equal to the longer maturity of the retractable bond. Let τ_j^* denote the maturity of the retractable bond on the retraction date. Then the value of the corresponding straight coupon bond may be computed from the solutions to the discount bond problem discussed earlier, $b(s, \tau)$. In particular, $b(s, \tau)$ is the present value of \$1 to be delivered in τ periods if $s(r) = s(r = (1-s)/s)$. Hence the value of the straight bond, $R(s)$ is approximated by

$$R(s) = ck \sum_{j=0}^{j^*} b(s, \tau_j) + b(s, \tau_j^*), \tag{31}$$

where c is the coupon rate on the straight bond.

Now, measuring the maturity of the retractable relative to the retraction date, we have as the maturity value condition,

$$g(s, 0) = \max[R(s), 1]. \tag{32}$$

Prior to the retraction date, the value of the retractable bond obeys the differential equation (30). The boundary conditions which must be satisfied correspond to $s = 0$ ($r \to \infty$) and $s = 1$ ($r = 0$) and are given from (16) and (14) by

$$g(0, \tau) = 0, \qquad \tau > 0, \tag{33}$$

$$\mu^*(1)g_1(1, \tau) + g_2(1, \tau) - c = 0, \qquad \tau > 0. \tag{34}$$

Differential equation (30) with its boundary conditions can be solved by standard numerical techniques to obtain the value of the bond as a function of s and hence also of the instantaneously riskless rate, r.

(iv) *The callable bond*

The value of the callable bond, $g(s, \tau)$, satisfies the differential equation (30). At maturity the value of the bond is given by the terminal value condition (13), so that

$$g(s, 0) = 1. \tag{35}$$

While the bond is callable it satisfies the limiting interest rate condition (16) and the call policy condition (18) which we write as

$$g(1, \tau) = 0, \quad \tau > 0, \tag{36}$$

$$g(s, \tau) \leqq 1, \quad \tau < \tau_c, \tag{37}$$

where τ_c is the time to maturity at the first call date.

Since the solution to the differential equation (30) for the callable bond must satisfy the inequality (37), as well as the boundary conditions (35) and (36), it is derived by the same procedure employed for the parity bond, which also determines the optimal call policy as a function of s (or r) and τ.

When the bond is not callable the call policy condition is replaced by the zero interest rate condition (34), and the differential equation (30) with boundary conditions (35), (36) and (34) is solved by standard numerical methods.

IV

While several alternative models of the stochastic process of the riskless rate of interest have been proposed,[12] no empirical evidence exists as to the relative merits of these models. Since empirical estimation is beyond the scope of this paper a simple stochastic process for the interest rate was assumed, to illustrate the analysis of the previous sections. In particular we assume $\mu(r) = 0$ and $\sigma(r) = r\sigma$ (with σ constant), which correspond to a driftless geometric Brownian motion (σ is taken as 0.045 per month).

To examine the implications of our assumptions about the term structure of interest rates, we valued a pure discount bond considering all maturities up to 20 years, and then used (8) to convert the bond values into yields to maturity. This yielded a term structure of interest rates for each value of the instantaneous riskless rate. Three representative term structures are shown in fig. 1. The assumptions we made about the term structure were, first, that the instantaneous riskless rate follows a random walk without trend, so that the expected future spot rate is equal to the current spot rate; and secondly, that the pure expectations hypothesis holds so that the expected instantaneous rates of return on riskless bonds of all maturities are equal. As we observed in footnote 7 this does not imply that the term structure of interest rates is flat; rather, as we see in fig. 1, the term structure is downward sloping, so that forward rates are downward biased estimates of future spot rates, the degree of bias increasing with the current level of the spot rate. Thus, if the instantaneous riskless rate is 4.81%, the yield to maturity on a 20 year bond is 4.48%, while a current spot rate of 12.12% implies a yield to maturity on a 20 year bond of only 10.38%.

[12]See for example Merton (1975), Vasicek (1976) and Cox, Ingersoll and Ross (1976).

Using the methods of sections II and III we valued:

(i) a 5-year savings bond with an 8% coupon rate;
(ii) a 5-year bond with an 8% coupon rate, callable at any time;
(iii) a 20-year bond with an 8% coupon rate, retractable to 5 years.

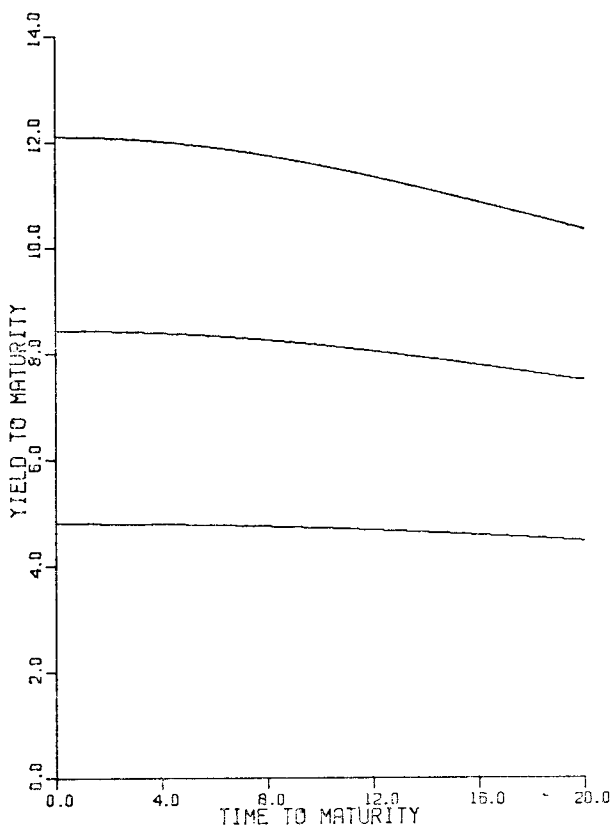

Fig. 1. Term structure of interest rates (for three alternative values of the instantaneous riskless rate) derived from the pure expectations hypothesis when the spot rate follows a geometric Brownian motion without drift.

To provide a standard of comparison we also valued a straight 5-year bond with an 8% coupon rate. This was done by noting that $B(r, \tau)$, the value of a discount bond promising \$1 in τ periods when the current riskless rate is r is the present value of \$1 in τ periods when the riskless rate is r. Applying these present value factors to the coupon stream and maturity value of a straight coupon bond we may calculate the value of the straight bond for any assumed level of the current riskless rate, r. The values of the three types of generic bond and the straight coupon bond are shown as functions of the riskless rate in fig. 2.

The retractable bond may be regarded as a straight coupon bond plus the option to purchase a 15-year coupon bond after five years. This option is most valuable when interest rates are low so that at low rates of interest the value

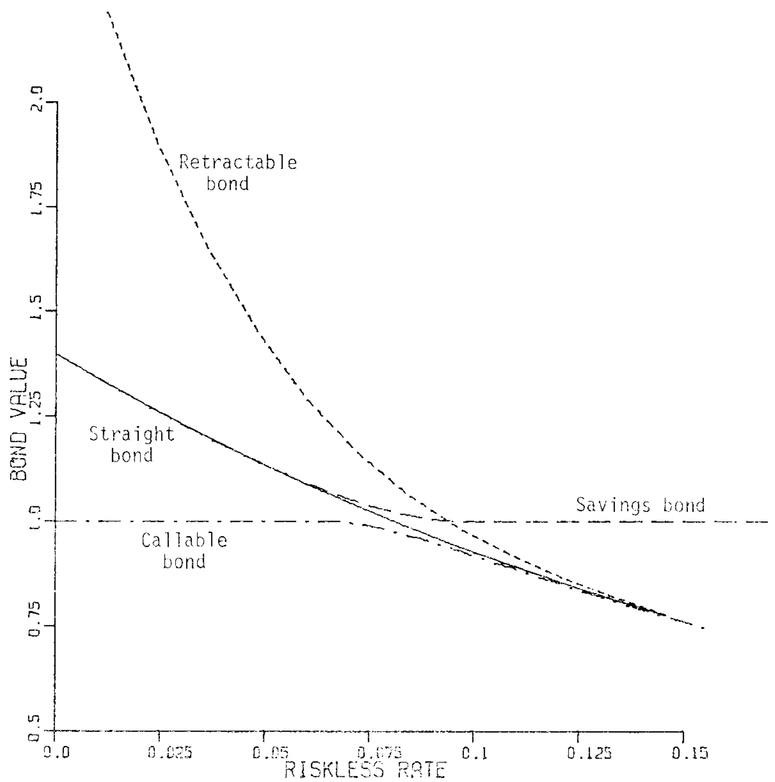

Fig. 2. Bond values as a function of the instantaneous riskless rate. Coupon rate: 8%. Time to maturity: retractable bond, 20 years (retractable after 5 years) and others, 5 years.

of the retractable bond substantially exceeds that of the straight bond. At high rates of interest it becomes less likely that the option will have any exercise value so that the value of the retractable bond approximates that of the straight bond.

The savings bond is equivalent to a straight coupon bond plus an American put option to sell the bond at par. At low interest rates the straight bond is above par so that the value of the put option is low and the savings bond value approximates that of the straight bond. At higher rates of interest it becomes optimal to exercise the put option so that the value of the savings bond becomes equal to the par value of unity.

The callable bond is equivalent to a straight bond less the value of an American call option to purchase the bond for the value of unity. Hence, unlike the retractable and savings bonds whose values are never exceeded by that of the straight bond, the value of the callable bond is never greater than that of the straight bond. At high interest rates, the value of the straight bond is low so that the value of the call option is also low, and the callable bond value is close to that of the straight bond. On the other hand, the optimal call policy ensures that the value of the callable bond never rises above the call price of unity which is therefore the upper bound for low interest rates.

In fig. 3 the values of the three types of bond we are considering are shown as functions of the value of the straight coupon bond. In the figure the value of the option feature on the bond is given by the distance of the bond value from the 45° ray.

As we explained in sections II and III the valuation procedure also determines the optimal strategy for exercising the option inherent in the three types of bond. The optimal strategy for the retractable is to retract if the value of a 15-year bond on the retraction date is less than par. For the savings bond and the callable bond the optimal strategy for exercising the option is expressed as a time series of critical interest rates at which the option should be exercised. These are shown in fig. 4 and correspond to the critical interest rates for exercising a call (the callable bond) and a put (the savings bond) on an 8% coupon 5-year bond. The discontinuities in the critical interest rate series are attributable to the discreteness of the solution procedure.

Finally, in order to compare the explicit coupon costs of financing with the three different types of instrument we constructed table 3. In this table it is assumed that all bonds are floated at par, and the table gives, for various assumed coupon rates on a straight bond sold at par, the coupon which would have to be placed on a savings bond, a retractable bond or a callable bond if that bond were to be sold at par. The table was constructed by interpolating to find the instantaneous riskless rate at which an $X\%$ straight bond would sell at par. Then for each type of generic bond we found by interpolation the coupon rate which would be required for it to sell at par, given that riskless rate. For example, if an 8% coupon straight bond could be sold at par, the required coupon rate on a savings bond would be 6.57%, on a retractable bond 6.85% and on a callable bond 9.45%.

V

In this paper we have shown, under certain assumptions about the term structure of interest rates, how three kinds of fixed income security, each of which involves a different type of contingent claim, may be valued. The interest of the models developed lies in their explicit attention to the stochastic nature of interest rates. Besides the possibilities for testing the models, further work is

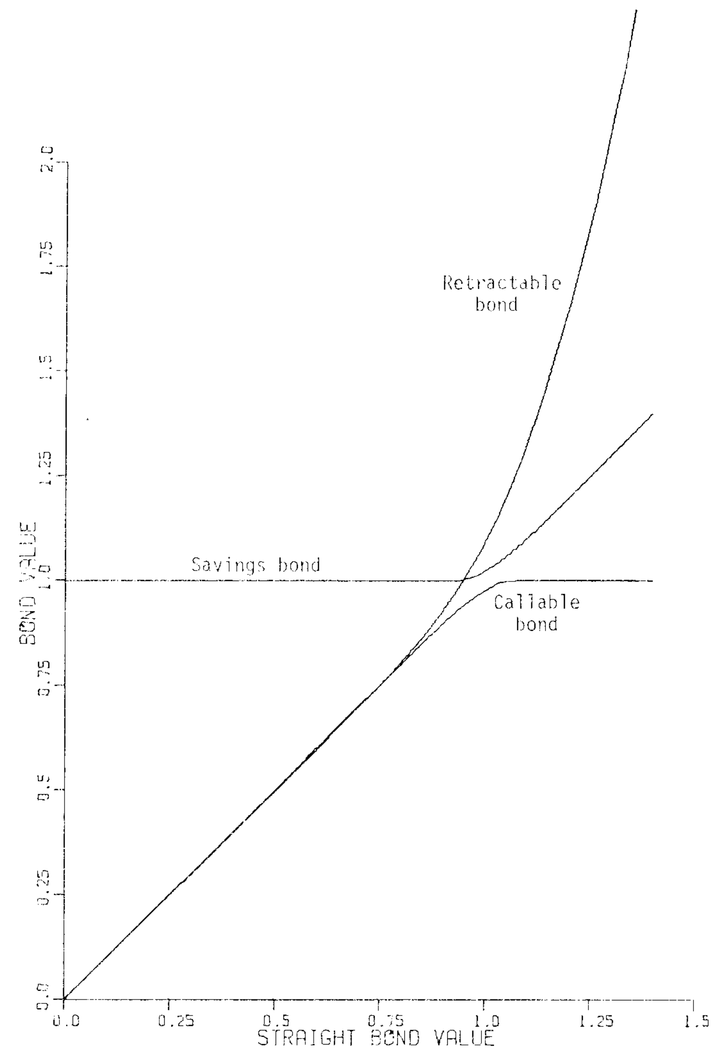

Fig. 3. Bond values at time of issue as a function of the value of a straight bond with the same coupon rate: 8%. Time to maturity: retractable bond, 20 years (retractable after 5 years) and others, 5 years.

required in two major directions. First, the model of the stochastic process for interest rates is high restrictive, and extension of the analysis to encompass more general stochastic specifications would be highly desirable.

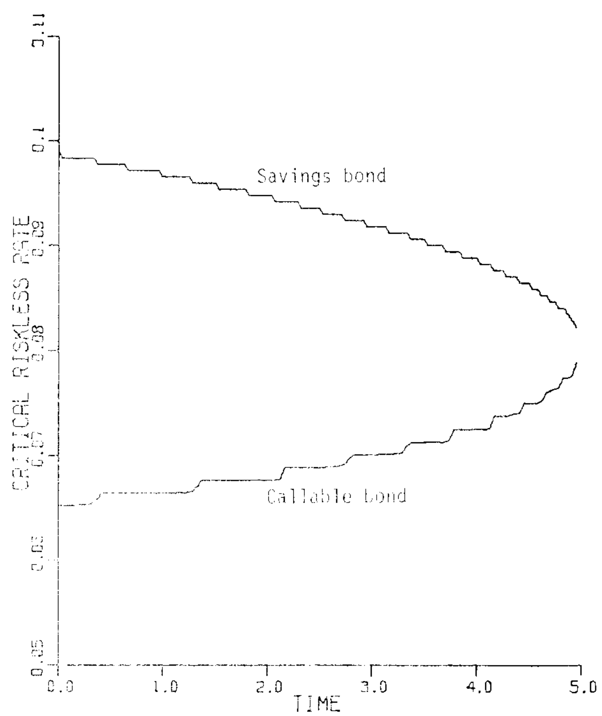

Fig. 4. Critical riskless rate for exercise of bond options as a function of time: 8% 5-year savings bond and 8% 5-year callable bond.

Table 3
Coupon rates required for bonds to sell at par as a function coupon required for the straight bond to sell at par.

Coupon on straight bond	Required coupon on savings bond	Required coupon on retractable bond	Required coupon on callable bond
2.00	1.63	1.67	2.19
3.00	2.45	2.50	3.33
4.00	3.25	3.35	4.53
5.00	4.05	4.21	5.78
6.00	4.89	5.08	6.83
7.00	5.74	5.96	8.26
8.00	6.57	6.85	9.45
9.00	7.38	7.75	10.62
10.00	8.20	8.65	12.06
11.00	9.06	9.55	13.07
12.00	9.91	10.46	14.27
13.00	10.77	11.39	15.67
14.00	11.63	12.33	16.62
15.00	12.50	13.28	18.20
16.00	13.36	14.24	19.27

Secondly, while in this paper we have considered interest rates as the sole stochastic variable, the analysis may be extended to the pricing of contingent claims whose value depends not only on the level of interest rates, but also on the value of some other security whose return follows a similar but distinct stochastic process; examples of such contingent claims include not only the convertible bond and convertible preferred share, but also warrants and other option contracts, as well as corporate bonds subject to default risk.

Thus, suppose that the value of the contingent claim may be written as $G(x, r, \tau)$, where x is the value of the underlying security which follows the stochastic process,

$$\mathrm{d}x = (\alpha x - C)\,\mathrm{d}t + \sigma x\,\mathrm{d}z, \tag{38}$$

where C is the rate of distribution to holders of the underlying security.

Then, considering the formation of a zero net investment hedge portfolio consisting of investments in the contingent claim, the underlying security, a long-term bond and the instantaneously riskless security, it may be shown that the value of the contingent claim must satisfy the partial differential equation,[13]

$$\begin{aligned}&\tfrac{1}{2}\gamma^2 x^2 G_{11} + \sigma\gamma\rho x G_{12} + \tfrac{1}{2}\sigma^2 G_{22} \\ &\qquad + (rx - C)G_1 + \mu G_2 - G_3 - rG + c = 0,\end{aligned} \tag{39}$$

where c is the continuous coupon payment on the contingent claim, and ρ is the instantaneous correlation between the return on the underlying security and the change in the riskless rate of interest. Given the boundary conditions which are determined by the specific nature of the contingent claim under consideration, this equation may be solved for the value of the contingent claim.

The equation should be contrasted with the similar equation also derived by Merton (1973, eq. (34)), assuming that interest rates are stochastic. This analysis avoids our strong assumptions about the term structure of interest rates by assuming that the value of the contingent claim is homogeneous of the first degree in the value of the underlying security and the value of a discount bond with a maturity equal to that of the contingent claim. While this assumption may be appropriate for a European type option which receives no distributions, or for an American type option which will not be exercised prior to expiration, it does not seem reasonable for a continuously exercisable contingent claim which may also receive distributions; for such securities the methods of this paper are more appropriate.

[13]See Merton (1970).

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