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A Simple Approach to Valuing Risky Fixed and Floating Rate Debt

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ABSTRACT

We develop a simple approach to valuing risky corporate debt that incorporates both default and interest rate risk. We use this approach to derive simple closed-form valuation expressions for fixed and floating rate debt. The model provides a number of interesting new insights about pricing and hedging corporate debt securities. For example, we find that the correlation between default risk and the interest rate has a significant effect on the properties of the credit spread. Using Moody's corporate bond yield data, we find that credit spreads are negatively related to interest rates and that durations of risky bonds depend on the correlation with interest rates. This empirical evidence is consistent with the implications of the valuation model.

THE TRADITIONAL BLACK-SCHOLES (1973) and Merton (1974) contingent-claims-based approach to valuing corporate debt has become an integral part of the theory of corporate finance. In this approach, interest rates are assumed to be constant, and the default risk of a bond is modeled using option pricing theory. This framework for valuing risky debt has been applied in a number of articles including Geske (1977), Ingersoll (1977a, 1977b), Merton (1977), Smith and Warner (1979), and many others.

One of the drawbacks of this approach is that default is assumed to occur only when the firm exhausts its assets. This is clearly unrealistic, since firms usually default long before the firm's assets are exhausted. In addition, Jones, Mason, and Rosenfeld (1984) and Franks and Torous (1989) show that this aspect of the model implies credit spreads much smaller than actual credit spreads. In an important article, Black and Cox (1976) relax this assumption and allow default to occur when the value of the firm's assets reaches a lower threshold. This feature makes the model consistent with

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either net-worth or cash-flow-based insolvency. By incorporating this more realistic default condition, the Black and Cox model is able to generate credit spreads more consistent with those observed in corporate debt markets.

Despite this advantage, the Black and Cox model shares some of the other limitations of the traditional Black-Scholes-Merton framework for valuing risky debt. Specifically, this framework assumes that interest rates are constant. This assumption is difficult to justify in a valuation model for risky fixed-income securities. In addition, this framework assumes that assets are allocated among corporate claimants according to strict absolute priority rules if the firm defaults. However, recent evidence by Franks and Torous (1989, 1994) Eberhart, Moore, and Roenfeldt (1990), LoPucki and Whitford (1990), Weiss (1990), Betker (1991, 1992), and others shows that strict absolute priority is rarely upheld in distressed reorganizations.

This article develops a simple new approach to valuing risky debt by extending the Black and Cox (1976) model in two ways. First, this model incorporates both default risk and interest rate risk. Second, this approach explicitly allows for deviations from strict absolute priority. In developing the model architecture, our objective is to be able to provide tractable valuation models for risky debt securities. Accordingly, we present the simplest possible specification for the model rather than the most general. This has the important advantage of allowing us to derive simple closed-form expressions for both risky fixed-rate and floating-rate debt. These closed-form expressions provide a number of new insights about the properties of corporate debt prices.

We first apply our framework to value risky discount and coupon bonds. We show that the credit spreads implied by the model are consistent with many of the properties of actual credit spreads. For example, the model implies credit spreads comparable in magnitude to actual spreads and allows the term structure of credit spreads to be either monotone increasing or hump shaped. An important implication of our results is that credit spreads for firms with similar default risk can vary significantly if the assets of the firms have different correlations with changes in interest rates. This property of the model has the potential to explain why bonds with similar credit ratings but in different industries or sectors have widely differing credit spreads. Finally, we show that the properties of high-yield bonds can be very different from those of less risky debt. For example, the duration or interest rate sensitivity of a high-yield bond may actually increase as it gets closer to its maturity date.

We then derive closed-form expressions for the value of risky floating-rate debt. We find that the price of a floating-rate bond can be an increasing function of the maturity of the bond in some situations. Similarly, the value of floating-rate debt can be an increasing function of the level of interest rates. These results illustrate that the properties of floating-rate debt are fundamentally different from those of fixed-rate debt. In general, the price of a floating-rate bond need not equal its par value even on coupon payment dates because of the risk of default.

Using Moody's corporate bond yield averages, we examine whether the implications of our model are consistent with the actual properties of credit spreads. As implied by the model, we find that credit spreads are strongly negatively related to the level of interest rates. Furthermore, changes in interest rates account for the majority of the variation in credit spreads for most of the bonds in the sample. This drives home the importance of allowing for interest rate risk in addition to default risk in valuing risky debt securities. We also find that the differences in the duration of bonds across industries and sectors is consistent with the differences in correlations with changes in the interest rate. These results provide supporting evidence for the empirical implications of our valuation model.

There are a number of other articles focusing on the valuation of corporate securities that allow for both default risk and interest-rate risk. These include Ramaswamy and Sundaresan (1986), Hull and White (1992), Maloney (1992), Jarrow and Turnbull (1992a, 1992b, 1992c, 1992d), Kim, Ramaswamy, and Sundaresan (1993), Ginzburg, Maloney, and Willner (1993), Shimko, Tejima, and Deventer (1993), Genotte and Marsh (1993), and Nielsen, Saa-Requejo, and Santa-Clara (1993). This article distinguishes itself from each of these other contributions in that it is the only one that provides closed-form valuation expressions for risky coupon bonds as well as risky floating-rate debt. In addition, it is the only one that jointly allows for (a) default before the firm exhausts all its assets, (b) complex capital structures including multiple issues of debt, (c) deviations from strict absolute priority, and (d) empirical evidence supporting the implications of the model.

The remainder of this article is organized as follows. Section I presents the basic valuation framework. Section II derives a valuation model for risky fixed-rate debt and examines its implications for the risk structure of interest rates. Section III presents the valuation model for floating-rate debt. Section IV presents the results of the empirical analysis. Section V summarizes the article and makes concluding remarks.

I. The Valuation Framework

In this section, we extend the Black and Cox (1976) model to develop a simple continuous-time valuation framework for risky debt that allows for both default risk and interest rate risk. This framework is then used in later sections to derive closed-form valuation expressions for a variety of risky corporate debt securities. The basic assumptions of this framework parallel those of Black and Scholes (1973), Merton (1974), and Black and Cox (1976), and are discussed individually below.

ASSUMPTION 1: *Let V designate the total value of the assets of the firm. The dynamics of V are given by*

$$dV = \mu V dt + \sigma V dZ_1, \tag{1}$$

where σ is a constant and Z_1 is a standard Wiener process.

ASSUMPTION 2: *Let r denote the short-term riskless interest rate. The dynamics of r are given by*

$$dr = (\zeta - \beta r)dt + \eta dZ_2, \tag{2}$$

where ζ, β, and η are constants and Z_2 is also a standard Wiener process. The instantaneous correlation between dZ_1 and dZ_2 is ρdt.

This assumption about the dynamics of r is drawn from the term structure model of Vasicek (1977). Although consistent with many of the observed properties of interest rates, these dynamics can allow negative interest rates. There are several reasons, however, why this assumption may be justifiable in the context of this model. First, the probability of negative interest rates occurring is small for realistic parameter values. Second, given that the current value of r is positive, these dynamics always imply positive expected future values of r. This is important since the primary effect of r on credit spreads is through its expected future value. Note that our approach could be extended to allow for more general interest rate processes, although risky debt prices would then need to be solved for numerically.

ASSUMPTION 3: *The value of the firm is independent of the capital structure of the firm.*

This is the standard assumption that the Modigliani-Miller Theorem holds. This assumption also implies that changes in capital structure, such as payments of coupons and principal, have no effect on V. This is easily satisfied, for example, if coupons and principal payments are financed by issuing new debt. Implicit in this assumption is the notion that the capital structure of the firm is held constant over time or that the status quo is maintained.[1] This is reasonable in light of the fact that in this frictionless continuous-time framework, the firm has no incentive to alter its capital structure.[2] We allow the capital structure of the firm to consist of a variety of contingent claims including debt with different coupon rates, priorities, and maturity dates.

ASSUMPTION 4: *Following Black and Cox* (1976), *we assume there is a threshold value K for the firm at which financial distress occurs. As long as V is greater than K, the firm continues to be able to meet its contractual obligations. If V reaches K, however, the firm immediately enters financial distress, defaults on all of its obligations, and some form of corporate restructuring takes place.*

An important implication of this assumption is that default occurs for all debt contracts simultaneously. This is realistic since when a firm defaults on

[1] For a model with dynamic capital structure choice, see Fischer, Heinkel, and Zechner (1989a, 1989b).

[2] In the Leland (1994) model, the firm faces taxes and bankruptcy costs that imply an optimal capital structure. As a result, firms may have incentives to move toward the optimal capital structure in the Leland model.

a debt issue, it typically defaults on other issues because of cross-default provisions, acceleration of principal provisions, or injunctions against making coupon payments on other debt issues. Although we assume that K is a constant, which is consistent with the assumption of a stationary capital structure, we could extend the analysis to allow K to depend on time and the riskless interest rate or to follow a separate stochastic process. However, since it is the ratio of V to K, rather than the actual value of K, that plays the major role in our analysis, allowing a more general specification for K simply makes the model more complex without providing additional insight into the valuation of risky debt.[3]

This definition of financial distress is consistent with both the case where the firm is insolvent because assets of $V = K$ do not generate sufficient cash flow to meet current obligations, as well as the case where assets of $V = K$ imply a violation of minimum net worth or working-capital requirements. The distinction between flow-based and stock-based insolvency is discussed in Wruck (1990) and Kim, Ramaswamy, and Sundaresan (1992).

Since financial distress is triggered when $V = K$, a reorganization or bankruptcy is simply a mechanism by which total assets of K are allocated to the various classes of corporate claimants. There are a variety of ways in which a corporate restructuring can occur including a Chapter 7 liquidation, a Chapter 11 reorganization, a Chapter 11 liquidation, or a private debt restructuring.[4]

The traditional approach to valuing corporate securities assumes that strict absolute priority holds. However, a growing amount of evidence shows that absolute priority rules are frequently violated in corporate restructurings. For example, Franks and Torous (1989) find that absolute priority is violated in 78 percent of the bankruptcies in their sample. Similar percentages are found by Eberhart, Moore, and Roenfeldt (1990) and Weiss (1990). In addition, recent research suggests that the actual payoff to a bondholder in a reorganization depends on a host of exogenous variables such as firm size, the bargaining power of the bondholders, the existence of an equity committee, and the strength of ties between managers and shareholders.[5] Several recent articles attempt to model some of the elements of the bargaining game among corporate claimants during financial distress and incorporate them into a model for risky debt prices. These include Anderson and Sundaresan (1992), Mella and Perraudin (1993), and Leland (1994). Although insightful, these models are limited in their ability to capture the actual properties of corporate debt, since they do not allow interest rates to be stochastic.

[3] Black and Cox (1976) assume that the default threshold is of the form Ke^{-cT} rather than a constant. This time dependence of the threshold could easily be incorporated into a more general version of our model.

[4] For a discussion of these alternatives, see Franks and Torous (1989), Gilson, John, and Lang (1990), and Wruck (1990).

[5] See Weiss (1990), LoPucki and Whitford (1990), and Betker (1992).

Rather than trying to model the complex bargaining process among corporate claimants during a restructuring or bankruptcy, we take the allocation of the firm's assets as exogenously given.

ASSUMPTION 5: *If a reorganization occurs during the life of a security, the security holder receives* $1 - w$ *times the face value of the security at maturity.*

An equivalent way of specifying the payoffs in the event of a default would be to assume that the security holder receives N riskless zero-coupon bonds at the time of the default, where N equals $1 - w$ times the face amount of the debt, and where the maturity date of the riskless bonds is the same as for the original debt. This equivalent specification is consistent with typical reorganizations in which security holders receive new securities rather than cash in exchange for their original claims. We note that there are other possible specifications for the payoff in reorganization. For example, w could be allowed to depend on the remaining maturity of the bond or even to depend on the level of interest rates at the time the firm defaults.

The factor w represents the percentage writedown on a security if there is a reorganization of the firm during the life of the security. For limited liability securities, $w \leq 1$. In general, w will differ across the various bond issues and classes of securities in the firm's capital structure. When $w = 0$, there is no writedown and the security holder is unimpaired. When $w = 1$, the security holder receives nothing in a restructuring. If $w < 0$, a security holder actually benefits from a restructuring.[6] Note that nothing in this assumption precludes w from being viewed as the expected outcome from a game theoretic model of the bargaining process.

In practice, the value of w for a particular class of securities could be estimated from actuarial information. For example, Altman (1992) finds that the average writedown w for secured, senior, senior subordinated, cash-pay subordinated, and non-cash-pay subordinated debt for a sample of defaulted bond issues during the 1985 to 1991 period is 0.395, 0.477, 0.693, 0.720, and 0.805, respectively. Franks and Torous (1994) find that the average writedown w for secured debt, bank debt, senior debt, and junior debt for a sample of firms that reorganized under Chapter 11 during the 1983 to 1990 period is 0.199, 0.136, 0.530, and 0.711, respectively. Similar results are obtained by Betker (1992). The only constraint on the value of w is the adding-up constraint that the total settlement on all classes of claims cannot exceed K.[7]

Note that even when firms have many issues of debt outstanding, the bonds are usually grouped into a small handful of categories for purposes of reorganization. Thus, only two or three different values of w are usually

[6] As shown by Franks and Torous (1989) and LoPucki and Whitford (1990), this situation can actually occur. In most cases, this results from the bondholder receiving pendency interest at a rate higher than the coupon rate of the bond during the period between the default and the execution of the reorganization plan. In other cases, a settlement made on the basis of the face amount of a long-maturity low-coupon rate bond might benefit the bondholder because of the effective shortening of the maturity date.

[7] Included in these claims would be any administrative and priority claims such as wages, taxes, and debtor-in-possession financing.

necessary in valuing a firm's debt. For example, on December 1992, General Motors Acceptance Corp. had 53 outstanding long-term debt issues listed in *Moody's Bank and Finance Manual.* Of these, 42 issues were described as either not secured, or not secured and ranking pari passu with all other unsecured obligations of the company. The priority of the remaining 11 issues was not described, but would likely be the same as that of the other 42 issues, since they were listed simply as notes or debentures.

Although we assume that w is a constant, this framework could easily be extended to allow for stochastic values of w, provided that the risk of w is unsystematic. Since w represents the outcome of the bargaining process, the assumption that w is unsystematic may not be unreasonable. Because w affects payoffs linearly, allowing w to be random simply requires that we replace w with its expected value in the valuation expressions.

ASSUMPTION 6: *We assume perfect, frictionless markets in which securities trade in continuous time.*

This assumption allows us to invoke standard results to derive the fundamental partial differential equation defining the price $H(V, r, T)$ of any derivative security with payoff at time T contingent on the values of V and r. This partial differential equation is

$$\frac{\sigma^2}{2}V^2 H_{VV} + \rho\sigma\eta V H_{Vr} + \frac{\eta^2}{2}H_{rr} + rVH_V + (\alpha - \beta r)H_r - rH = H_T, \quad (3)$$

where α represents the sum of the parameter ζ and a constant representing the market price of interest rate risk. As shown by Campbell (1986), this market price of interest rate risk can be derived within a simple general equilibrium framework in which the representative investor has logarithmic preferences. The value of the derivative security is obtained by solving equation (3) subject to the appropriate maturity condition.

The value of a riskless discount bond plays an important role in the derivation of valuation expressions for corporate securities. In this framework, the value of a riskless discount bond $D(r, T)$ is given by the Vasicek (1977) model

$$D(r,T) = \exp(A(T) - B(T)r), \quad (4)$$

where

$$\begin{aligned} A(T) = &\left(\frac{\eta^2}{2\beta^2} - \frac{\alpha}{\beta}\right)T \\ &+ \left(\frac{\eta^2}{\beta^3} - \frac{\alpha}{\beta^2}\right)(\exp(-\beta T) - 1) \\ &- \left(\frac{\eta^2}{4\beta^3}\right)(\exp(-2\beta T) - 1), \end{aligned}$$

$$B(T) = \frac{1 - \exp(-\beta T)}{\beta}.$$

II. Valuing Fixed-Rate Debt

In this section, we derive valuation expressions for risky discount and coupon bonds and examine their implications for the term structure of credit spreads. Let $P(V, r, T)$ denote the price of a risky discount bond with maturity date T. The payoff on this contingent claim is 1 if default does not occur during the life of the bond, and $1 - w$ if it does. This payoff function can be expressed as

$$1 - wI_{\gamma \le T}, \tag{5}$$

where I is an indicator function that takes value one if V reaches K during the life of the bond, and zero otherwise. More formally, I takes value one if the first-passage time γ of V to K is less than or equal to T. In addition, let X denote the ratio V/K.

PROPOSITION 1: *The value of a risky discount bond is*

$$P(X,r,T) = D(r,T) - wD(r,T)Q(X,r,T), \tag{6}$$

where

$$Q(X,r,T,n) = \sum_{i=1}^{n} q_i,$$

$$q_1 = N(a_1),$$

$$q_i = N(a_i) - \sum_{j=1}^{i-1} q_j N(b_{ij}), \qquad i = 2,3,\ldots,n,$$

$$a_i = \frac{-\ln X - M(iT/n,T)}{\sqrt{S(iT/n)}},$$

$$b_{ij} = \frac{M(jT/n,T) - M(iT/n,T)}{\sqrt{S(iT/n) - S(jT/n)}},$$

and where

$$\begin{aligned} M(t,T) = {} & \left(\frac{\alpha - \rho\sigma\eta}{\beta} - \frac{\eta^2}{\beta^2} - \frac{\sigma^2}{2}\right)t \\ & + \left(\frac{\rho\sigma\eta}{\beta^2} + \frac{\eta^2}{2\beta^3}\right)\exp(-\beta T)(\exp(\beta t) - 1) \\ & + \left(\frac{r}{\beta} - \frac{\alpha}{\beta^2} + \frac{\eta^2}{\beta^3}\right)(1 - \exp(-\beta t)) \\ & - \left(\frac{\eta^2}{2\beta^3}\right)\exp(-\beta T)(1 - \exp(-\beta t)), \end{aligned}$$

$$S(t) = \left(\frac{\rho\sigma\eta}{\beta} + \frac{\eta^2}{\beta^2} + \sigma^2\right)t - \left(\frac{\rho\sigma\eta}{\beta^2} + \frac{2\eta^2}{\beta^3}\right)(1 - \exp(-\beta t)) + \left(\frac{\eta^2}{2\beta^3}\right)(1 - \exp(-2\beta t)).$$

The term $Q(X, r, T)$ is the limit of $Q(X, r, T, n)$ as $n \to \infty$. $N(\cdot)$ denotes the cumulative standard normal distribution function.

Proof: See Appendix.

This closed-form expression involves nothing more complex that the standard normal distribution function. Note that the q_i terms in equation (6) are defined recursively, which makes it straightforward to program this valuation expression and to calculate risky discount bond prices. Although $Q(X, r, T)$ is defined as the limit of $Q(X, r, T, n)$, the convergence is rapid; numerical simulations show that setting $n = 200$ results in values of $Q(X, r, T)$ and $Q(X, r, T, n)$ that are virtually indistinguishable.

Proposition 1 shows that the value of a risky discount bond depends on V and K only through their ratio X. Thus, X provides a summary measure of default risk of the firm and can be viewed as a proxy variable for the credit rating of the firm. An important implication of this is that risky debt can be valued without having to separately specify the values of V and K. This feature dramatically simplifies the practical implementation of the model. From equation (6), the price of the risky discount bond is an explicit function of X, r, and T, and depends on the parameters w, α, β, η^2, σ^2, and ρ.

This closed-form expression has an intuitive structure. The first term in equation (6) represents the value the bond would have if it were riskless. The second term represents a discount for the default risk of the bond. The discount for default risk consists of two components. The first component, $wD(r, T)$, is the present value of the writedown on the bond in the event of a default. The second component, $Q(X, r, T)$, is the probability—under the risk-neutral measure—that a default occurs. It is important to recognize that the probability of a default $Q(X, r, T)$ under the risk-neutral measure may differ from the actual probability of a default. This is because the upward drift of the actual process for V in equation (1) is μV, while the upward drift of the risk-neutral process depends on the value of r and is independent of μ.

Since X is a sufficient statistic for default risk in this model, we do not need to condition on the pattern of cash payments to be made prior to the maturity date of a bond in order to value the bond. Intuitively, this is because we assume that financial distress triggers the default of all of the firm's debt. In contrast, the traditional approach implicitly assumes that a discount bond can only default at its maturity date. Because default risk is captured by a common state variable X in this model, bonds can be valued by conditioning on X directly rather than on the default status of other bonds. An important implication of this is that coupon bonds can be valued as simple portfolios of discount bonds.[8] This value additivity feature is a major reason why this approach is significantly more tractable than the traditional approach to valuing risky fixed-rate debt securities.

The price of a risky bond is an increasing function of the default-risk variable X. This is intuitive since the higher the value of X, the further the firm is from the default threshold and the smaller the discount for default risk. Differentiation shows that bond values are decreasing functions of w. This is because an increase in w implies that the writedown on a bond in the event of financial distress is larger. Similarly, as T increases, the value of $D(r, T)$ decreases and the risk-neutral probability of a default $Q(X, r, T)$ increases. Both of these effects tend to decrease the value of the risky bond. Hence, risky bonds are decreasing functions of T.

In general, the price of a risky bond is a decreasing function of r. Furthermore, the sensitivity of the price to changes in r provides a measure of the duration of the bond. As shown by Chance (1990) and others, the duration of a risky fixed-rate bond is shorter than for an otherwise riskless bond. This property also holds for the fixed-rate bond prices implied by this model. The reason for this is that riskless interest rate r plays two roles in the valuation of risky debt. In particular, an increase in r results in a lower value for $D(r, T)$. However, an increase in r implies that the upward drift of the risk-neutral process for V is higher. This means that as r increases, V is expected to drift away from K at a faster rate, which reduces the risk-neutral probability of a default.

Another interesting implication of this model is that the duration of a risky discount bond need not be a monotone-increasing function of its maturity. For example, for a moderate level of default risk, the duration of a zero-coupon bond can first increase with T, level out, and then decrease with T. This also serves to illustrate how different the properties of risky bonds are from those of riskless bonds. In fact, for values of X and w very close to one, the effect of r on the drift can offset the bond-price effect, and a risky bond can be an increasing function of r. Thus, the duration of very risky fixed-rate debt can actually become negative. Note that this occurs only for extremely risky debt. This is shown in Figure 1, which graphs the price of a zero-coupon bond when the value of X is very close to the default threshold. The value of the zero-coupon bond is an increasing function of r for maturities less than three years. In addition, the correlation between the assets of the firm and changes in the interest rate can be shown to have a major effect on the duration of risky fixed-rate debt.

[8] Geske (1977) shows that in the Merton (1974) framework, the value of a coupon bond is related to the value of a compound option. Determining the value of this compound option requires evaluating a N-dimensional integral, where N is the number of remaining coupon payments.

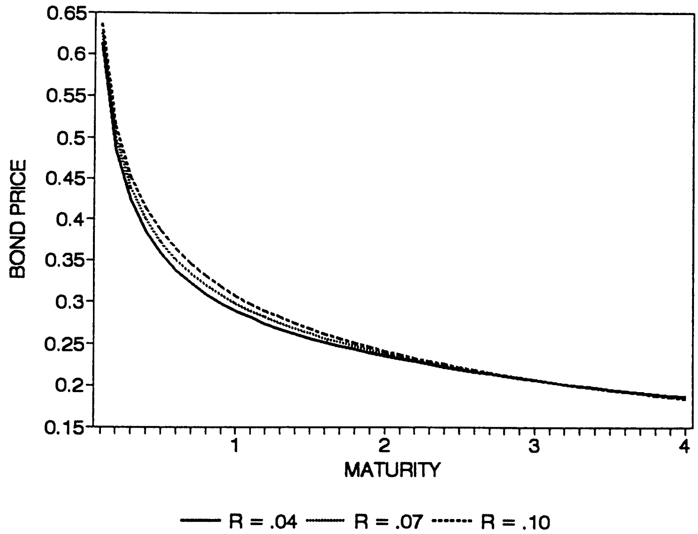

Figure 1. Discount bond prices as a function of the maturity of the bond. The parameter values used are $X = 1.05$, $w = 0.9$, $\sigma^2 = 0.04$, $\rho = -0.25$, $\alpha = 0.06$, $\beta = 1.00$, and $\eta^2 = 0.001$.

Given the explicit solution for risky fixed-rate debt, we can solve for the credit spread. This is defined as the difference between the yields of a risky and a riskless bond with identical maturity dates and coupon rates. Figure 2 graphs the term structure of credit spreads for an eight percent coupon bond for various values of X. The interest rate parameter values used in these examples are chosen to closely match the observed moments of the short-term interest rate during the past thirty years. As shown, the term structure of credit spreads can be monotone increasing as well as hump shaped. This corresponds well with recent empirical evidence by Sarig and Warga (1989), which suggests that the term structure of credit spreads is monotone increasing for bonds with high ratings, and humped shaped for bonds with low ratings. In addition, the magnitudes of the credit spreads implied by this model are consistent with the average levels observed in debt markets. For example, Figure 2 shows that the average credit spread for a ten-year eight percent coupon bond with $X = 2.0$ is about 60 basis points. This is close to the average spread of 48 basis points for Moody's Aaa-rated industrial bond yield average during the 1977 to 1992 period.

Figure 3 graphs the term structure of credit spreads for varying values of w. As expected, the credit spread is an increasing function of w. However, as the writedown w increases, the term structure of credit spreads can take on different shapes. For example, when $w = .25$, the maximum credit spread occurs for a bond with a maturity of about eight years. When $w = 0.75$, however, the maximum credit spread occurs for a bond with a maturity of

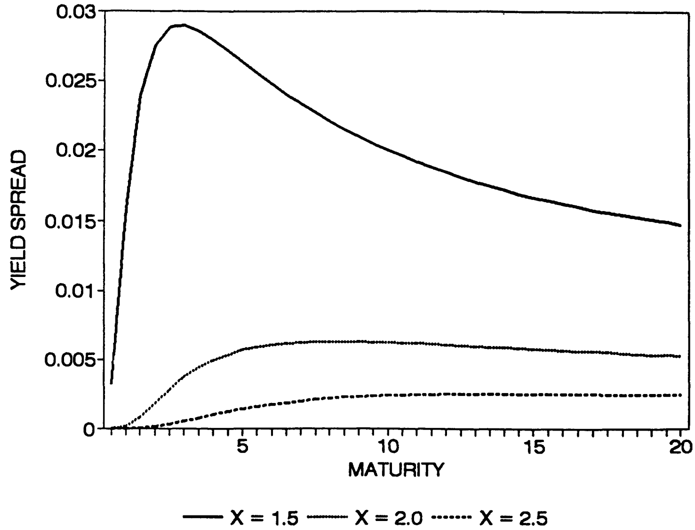

Figure 2. Credit spreads for an 8 percent bond for different values of X. The parameter values used are $r = 0.04$, $w = 0.5$, $\sigma^2 = 0.04$, $\rho = -0.25$, $\alpha = 0.06$, $\beta = 1.00$, and $\eta^2 = 0.001$.

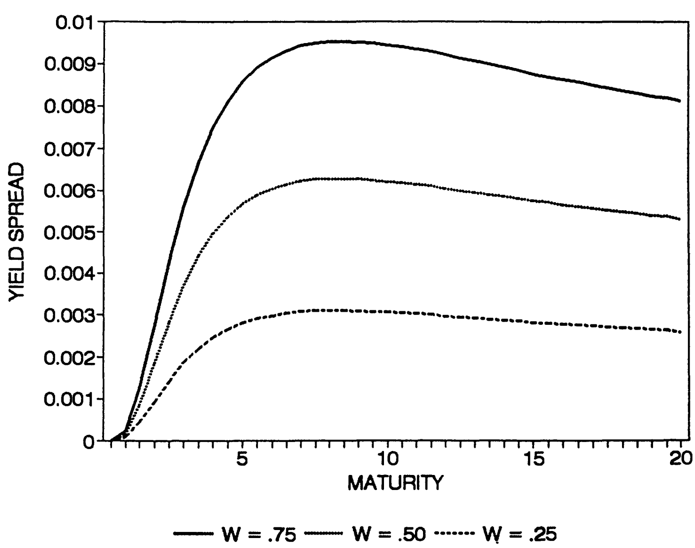

Figure 3. Credit spreads for an 8 percent bond for different values of w. The parameter values used are $X = 2.0$, $r = 0.04$, $\sigma^2 = 0.04$, $\rho = -0.25$, $\alpha = 0.06$, $\beta = 1.00$, and $\eta^2 = 0.001$.

nearly ten years. Since w is related to priority, differences in the credit spreads shown in Figure 3 can be viewed as the term structure of priority. Note that priority matters most for medium-term bonds.

The relation between credit spreads and the level of the short-term interest rate is shown in Figure 4. As discussed earlier, an increase in r tends to reduce the probability of a default because of the effect on the drift of the risk-neutral process for V. Thus, an increase in r results in a decrease in the credit spread. The magnitude of the decrease in the credit spread, however, depends on the value of ρ. This empirical implication of the model will be examined in Section V of this article.

Figure 5 graphs the credit spread for different values of the variance of the firm's assets σ^2. As σ^2 increases from 0.04 to 0.09, the maximum credit spread increases from approximately 60 basis points to approximately 300 basis points. Note that the maximum credit spreads occur at different maturities as σ^2 increases.

Figure 6 plots the relation between the term structure of credit spreads and the correlation coefficient between asset returns and changes in the interest rate. As shown, the effect of correlation can be very significant. For example, the credit spread for an 8 percent bond with a maturity of eight years widens by 27 basis points as the correlation increases from $-.50$ to $+.50$. The intuition for why the credit spread increases with ρ is that the risk-neutral distribution of future values of V depends on r. Thus, the variance of changes in the value of the firm during the life of the bond depends on the correlation between asset returns and changes in the interest rate. When ρ is positive,

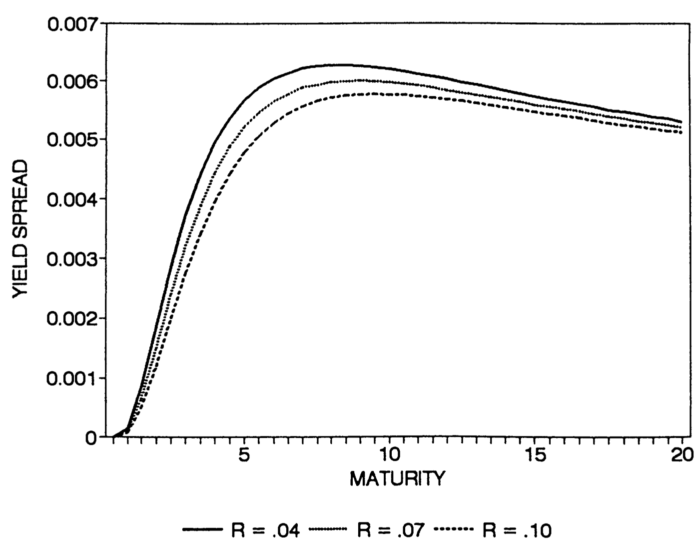

Figure 4. Credit spreads for an 8 percent bond for different values of r. The parameter values used are $X = 2.0$, $w = 0.50$, $\sigma^2 = 0.04$, $\rho = -0.25$, $\alpha = 0.06$, $\beta = 1.00$, and $\eta^2 = 0.001$.

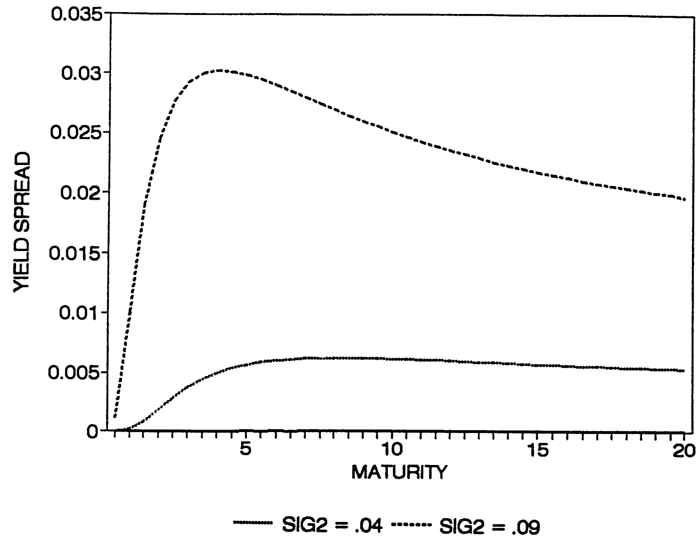

Figure 5. Credit spreads for an 8 percent bond for different values of σ^2. The parameter values used are $X = 2.0$, $r = 0.04$, $w = 0.50$, $\rho = -0.25$, $\alpha = 0.06$, $\beta = 1.00$, and $\eta^2 = 0.001$.

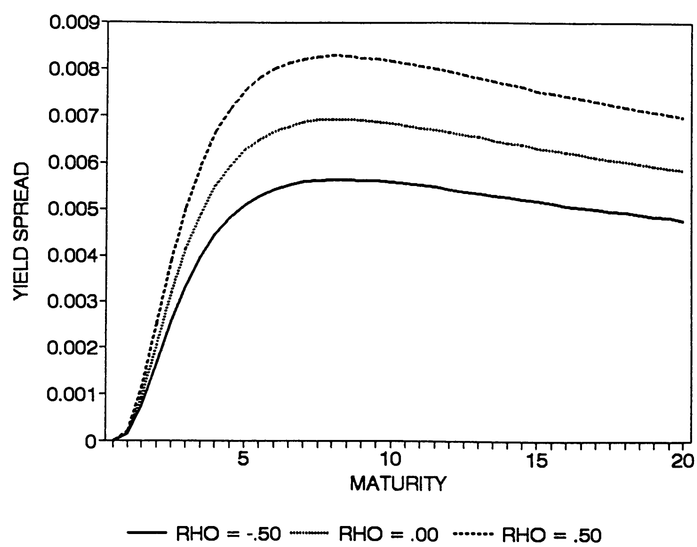

Figure 6. Credit spreads for an 8 percent bond for different values of ρ. The parameter values used are $X = 2.0$, $r = 0.04$, $w = 0.50$, $\sigma^2 = 0.04$, $\alpha = 0.06$, $\beta = 1.00$, and $\eta^2 = 0.001$.

the covariance term adds to the total variance, and therefore, increases the probability that the critical default threshold is reached during the life of the bond. These results are consistent with empirical evidence that credit spreads for Aaa-rated bond vary across sectors. For example, the average yield for Moody's Aaa-rated industrial bond index during the 1977 to 1992 period is 45 basis points less than the same measure for Aaa-rated public utility bonds. The relation between credit spreads and correlations will also be examined in Section V.

An important advantage of this model is that it can be easily implemented in practice. For example, when firms have multiple issues of debt outstanding, the value of X can be implied from the market price of the most liquid bond and then used to value the other bonds. This is similar to the familiar technique of solving for the implied variance of an at-the-money option and using it to price the remaining options. The values of σ^2 and ρ, since they are determined by the nature of the firm's assets, can be determined on the basis of historical firm or industry data, or could even be implied along with the value of X from market data. The value of r and the three term structure parameters α, β, and η^2 are easily obtained from term structure data.[9]

III. Valuing Floating-Rate Debt

In this section, we derive valuation expressions for floating-rate coupon payments. The value of a floating-rate note or bond can then be obtained by summing the values of the floating-rate coupons and the value of the terminal principal payment as given in the previous section. Let $F(X, r, \tau, T)$ represent the value of one floating-rate coupon payment to be made at time T, where the floating rate is determined at time τ, $\tau \leq T$. The payoff on this claim at time T is the value of r at time τ if default does not occur prior to T, and $(1 - w)r$ if it does. This payoff function can be expressed as

$$r(1 - wI_{\gamma \leq T}), \tag{7}$$

where I is again the indicator function. Note that the payoff received at time T is simply the value of r at time τ multiplied by the payoff function for a risky discount bond. This similarity will allow us to make direct comparisons between fixed- and floating-rate payments.

PROPOSITION 2: *The value of a risky floating-rate payment is*

$$F(X,r,\tau,T) = P(X,r,T)R(r,\tau,T) + wD(r,T)G(X,r,\tau,T), \tag{8}$$

[9] For example, see Chan et al. (1992).

where

$$R(r,\tau,T) = r\exp(-\beta\tau) + \left(\frac{\alpha}{\beta} - \frac{\eta^2}{\beta^2}\right)(1-\exp(-\beta\tau)) + \left(\frac{\eta^2}{2\beta^2}\right)\exp(-\beta T)(\exp(\beta\tau) - \exp(-\beta\tau)),$$

$$G(X,r,\tau,T,n) = \sum_{i=1}^{n} q_i \frac{C(\tau, iT/n)}{S(iT/n)} M(iT/n, T),$$

and where

$$C(\tau,t) = \left(\frac{\rho\sigma\eta}{\beta} + \frac{\eta^2}{\beta^2}\right)\exp(-\beta\tau)(\exp(\beta\min(\tau,t)) - 1) - \frac{\eta^2}{2\beta^2}\exp(-\beta\tau)\exp(-\beta t)(\exp(2\beta\min(\tau,t)) - 1).$$

The term $G(X, r, \tau, T)$ is the limit of $G(X, r, \tau, T, n)$ as $n \to \infty$. The remaining terms are as defined in Proposition 1.

Proof: See Appendix.

The value of a floating-rate coupon payment is an explicit function of X, r, τ, and T. The ratio X is again a sufficient statistic for the riskiness of the firm. Numerical simulations show that $G(X, r, \tau, T, n)$ converges rapidly to $G(X, r, \tau, T)$.

This closed-form expression for $F(X, r, \tau, T)$ parallels that for $P(X, r, T)$. From equation (7), the floating-rate coupon payoff at time T is the value of r at time τ multiplied by the payoff function for a risky discount bond. Consistent with this, the first term in equation (8) is simply the price of a risky discount bond times the expected value of the value of r at time τ under the risk-neutral process. However, since r is correlated with X under the valuation measure, the price of a risky floating-rate payment must reflect this correlation. The second term in equation (8) adjusts for this correlation through the term $C(\tau, t)$ which is the covariance of the value of r at time τ with the value of $\ln X$ at the time t of its first passage to zero. Note that the correlation between r and X will generally not equal zero even if the instantaneous correlation coefficient ρ is zero. This is because the drift term for the risk-neutral process for X depends on r, which induces correlation between r and X when measured over discrete intervals of time.[10]

This valuation expression has many important implications for the values of floating-rate securities. To illustrate, recall that the price of a fixed-rate

[10] The effects of temporal aggregation on moments of continuous-time processes are discussed in Longstaff (1989).

coupon payment is a decreasing function of T. In contrast, the value of a floating-rate coupon payment can be an increasing function of T. This is shown in Figure 7, which graphs the price of the floating-rate coupon payment as a function of T. The intuition for this property is that when r is below its long-run average value, the expected value of the payoff equation (7) is an increasing function of T since r is mean-reverting. As T increases, however, the discount factor applied to the payoff tends to reduce the value of the floating-rate payment. For small values of T, the first effect can offset the second effect, resulting in a positive relation between the value of the floating-rate payment and T. As $T \to \infty$, however, the value of the floating-rate payment approaches zero.

Another surprising implication of this model is that the value of the floating-rate coupon payment can be an increasing function of r. This is also shown in Figure 7. One reason for this is that an increase in r again has the effect of increasing the expected payment while reducing the discount factor applied to the payoff. For small values of T, the first effect can dominate the discount-factor effect. A second reason for this property is related to the correlation between changes in interest rates and the returns of the firm. When the correlation is positive, an increase in r implies that larger values of X are more likely, which decreases the default risk of the firm and leads to an increase in the value of the floating-rate coupon payment. This feature illustrates that the correlation between interest rates and the returns of the firm can play a significant role in determining the values of risky corporate debt.

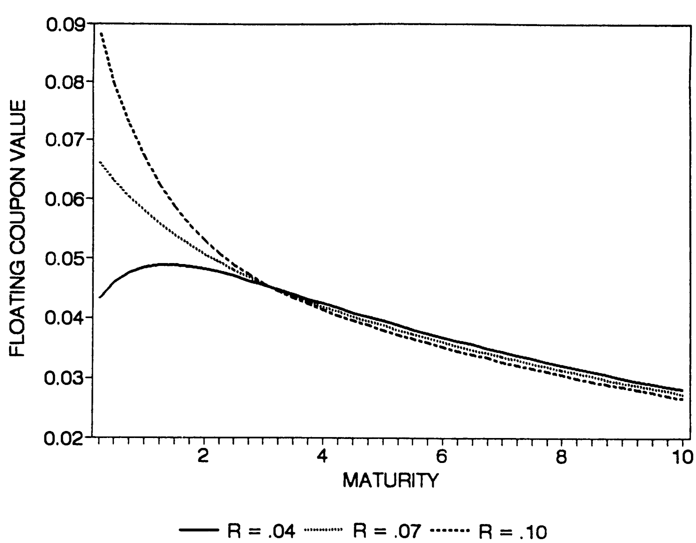

Figure 7. Values of floating-rate coupon payments for different values of r. The parameter values used are $X = 2.0$, $w = 0.50$, $\sigma^2 = 0.04$, $\rho = -0.25$, $\alpha = 0.06$, $\beta = 1.00$, $\eta^2 = 0.001$, and $\tau = T$.

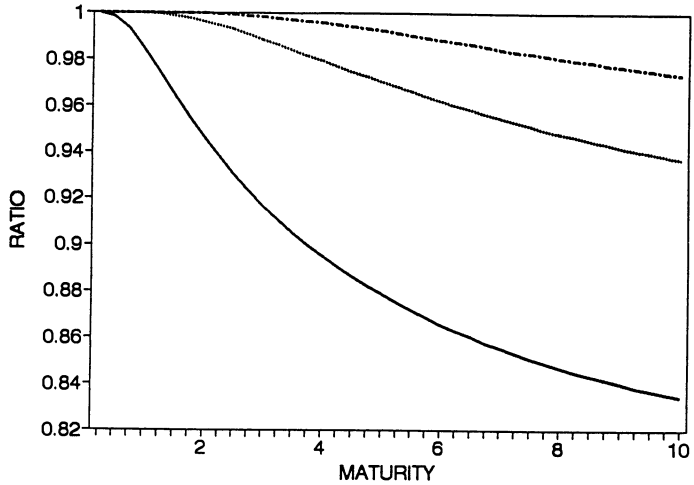

Figure 8. Ratio of the value of a risky floating-rate coupon payment to that of a riskless floating-rate coupon payment for different values of X. The parameter values used are $r = 0.04$, $w = 0.50$, $\sigma^2 = 0.04$, $\rho = -0.25$, $\alpha = 0.06$, $\beta = 1.00$, $\eta^2 = 0.001$, and $\tau = T$.

To provide some additional insights into the pricing of risky floating-rate debt, we compute the ratio of the price of a risky floating-rate coupon payment to that of a riskless floating-rate coupon payment. The ratio provides a measure of percentage size of the discount for risk. The ratio of the risky to riskless prices is shown in Figure 8 for different horizons and values of X. As illustrated, the shape of the relation between risky and riskless floating-rate payments depends critically on the value of X. The value of the ratio is also affected by the correlation coefficient ρ.

This expression for valuing floating-rate coupon payments can easily be extended to value coupon payments that are tied to a specific yield rather than to r, or to a specific yield plus a spread. For example, from equation (4), the yield on a t-maturity riskless bond is given by $B(t)r/t - A(t)/t$, which is a linear function of r. Thus, the value of a claim that pays the t-maturity yield determined at time τ as a floating-rate coupon at time T is simply $B(t)F(X, r, \tau, T)/t - A(t)P(X, r, T)/t$. Again, the value of a stream of floating-rate payments equals the sum of the values of the individual payments.

IV. Empirical Analysis

The valuation framework for risky debt presented in this article has many empirical implications for fixed-income markets. One of the most important of these is that credit spreads for corporate bonds are driven by two factors: an asset-value factor and an interest-rate factor. Furthermore, the correlation between the two factors plays a critical role in determining the properties of credit spreads. In contrast, the traditional approach implies that credit spreads depend on only an asset-value factor.

To provide some evidence about the properties of actual credit spreads for corporate bonds, we collected monthly data for Moody's industrial, utility, and railroad corporate bond yield averages for the 1977 to 1992 period as well as the corresponding yields for 10-year and 30-year Treasury bonds. Since the bonds used in the Moody's yield averages have varying maturities, we compute the average maturity of the bonds in the sample using the average-maturity data reported in *Moody's Bond Record*. We then compute credit spreads by taking the average of the 10-year and 30-year Treasury yields that matches the maturity of the corporate yield average for that month, and then subtracting the Treasury average from the corporate yield.[11] For most bonds, 190 months of data are available. For the Aaa-rated utility yields, data for ten months are missing. In addition, the railroad yield average was discontinued in 1989, resulting in a time series of 149 monthly observations for these bonds. In addition to computing credit spreads, we also calculate relative credit spreads by dividing the corporate yield by the corresponding-maturity Treasury yield.

Table I presents summary statistics for the credit spreads and relative credit spreads stratified by industry and credit rating. As expected, credit spreads increase in both absolute and relative terms as the credit rating of the bond decreases. In general, the same is true for the standard deviation of the credit spread. It is important to observe that bonds with the same credit rating but from different industries or sectors need not have similar credit spreads. This demonstrates that credit rating is not a sufficient statistic for the risk of a corporate bond.

To examine whether the properties of credit spreads are consistent with the implications of our two-factor framework, we regress changes in credit spreads on proxies for the two factors. As a proxy for the changes in the interest rate, we use changes in the 30-year Treasury yield. As a proxy for the return on the underlying assets, we use the returns computed from Standard and Poor's industrial, utility, and railroad stock indexes. Let ΔS denote the change in the credit spread. Similarly, let ΔY denote the change in the 30-year Treasury yield and let I denote the return on the appropriate equity index. The regression equation is given by

$$\Delta S = a + b\Delta Y + cI + \varepsilon, \tag{9}$$

where a, b, and c are regression coefficients.

[11] Since changes in the 10-year and 30-year Treasury yields are almost perfectly positively correlated, the empirical results are robust to the way that the 10-year and 30-year Treasury yields are weighted in computing credit spreads.

Table I

Summary Statistics for the Credit Spreads in Moody's Utility, Industrial, and Railroad Bond Yield Averages

The credit spread is the difference between the corporate yield and the yield for a Treasury bond with the same maturity. The relative spread is the ratio of the corporate yield to the yield for a Treasury bond with the same maturity. Yields and spreads are in percentage terms. The sample period is from April 1977 to December 1992.

	Mean of Credit Spread	Std. Dev. of Credit Spread	Mean of Relative Spread	Std. Dev. of Relative Spread	No. of Observations
Aaa Utilities	0.930	0.349	1.0975	0.034	180
Aa Utilities	1.276	0.431	1.1314	0.038	190
A Utilities	1.660	0.667	1.1696	0.054	190
Baa Utilities	2.077	0.758	1.2116	0.057	190
Aaa Industrials	0.481	0.373	1.0560	0.051	190
Aa Industrials	0.809	0.452	1.0888	0.059	190
A Industrials	1.231	0.580	1.1321	0.071	190
Baa Industrials	1.835	0.654	1.1972	0.084	190
Aa Railroads	0.191	0.869	1.0284	0.092	149
A Railroads	0.794	0.770	1.0887	0.088	149
Baa Railroads	1.240	0.821	1.1337	0.097	149

The two-factor model has a number of interesting implications for the coefficients in these regressions. First, the fixed-rate valuation expression in equation (6) can be shown to imply that

$$b < 0. \tag{10}$$

Thus, the model implies that credit spreads narrow as interest rates increase. The reason for this counterintuitive implication is that an increase in the interest rate increases the drift of the risk-neutral process for V, which in turn makes the risk-neutral probability of a default lower. Consequently, the credit spread is inversely related to the level of interest rates in this model. In addition, the inverse relation is more pronounced for firms with higher default probabilities.

A second implication of the model is that credit spreads are negatively related to returns on the firm's assets or equity,

$$c < 0. \tag{11}$$

The reason for this is simply that an increase in the value of a firm's assets or equity decreases the probability that the default boundary will be reached. Again, this negative relation between credit spreads and returns should be stronger for firms with higher default probabilities.

Finally, differentiating credit spreads implied by equation (6) numerically shows that the interest-rate sensitivity of credit spreads, holding X fixed, increases with the value of ρ. The intuition for why the interest-rate sensitivity of credit spreads increases with ρ is similar to the intuition why the credit spread itself increases with ρ. When ρ is negative, changes in r tend to be reversed by changes in X. Thus a change in r has less of an effect on the credit spread than when the value of ρ is zero or positive. The correlation with changes in the 30-year Treasury yield is -0.5894 for the returns on the utility stock index, -0.2652 for the returns on the industrial stock index, and -0.1609 for the returns on the railroad stock index. Since b measures the interest-rate sensitivity of credit spreads, this implies that the value of b estimated for utility spreads should be closer to zero than the value of b estimated for industrial spreads, which in turn should be closer to zero than the value of b estimated for railroad spreads. Thus, we can test this implication of the two-factor model by comparing the values of b across bonds with different values of ρ but with similar values of c.

Table II reports the regression results. The empirical results appear consistent with the implications of the two-factor model. The estimated coefficients b are negative for each of the 11 credit spreads. With the exception of the Aaa-rated utility bonds, all of the estimates of b are statistically significant. The t-statistics for the estimates of b for the industrial and railroad bonds are all in excess of twelve.

The magnitude of the estimates of b implies that the relation between credit spreads and interest rates is economically important as well as statistically significant. For example, the regression results imply that a 100-basis-point increase in the 30-year Treasury yield reduces Baa-rated utility credit spreads by 18.4 basis points, Baa-rated industrial credit spreads by 62.6 basis points, and Baa-rated railroad credit spreads by 82.3 basis points. This effect is illustrated in Figures 9, 10, and 11, which graph changes in Baa-rated credit spreads for utility, industrial, and railroad bonds against changes in the 30-year Treasury yield. The net effect of this negative relation between credit spreads and interest rates is to make the duration of corporate bonds shorter than would be the case for Treasury bonds. The reason for this is that an increase in the riskless rate is partially offset by a decline in the credit spread, implying that the change in price for a risky bond is less than for a riskless bond. As implied by the model, the coefficient b generally decreases with the credit rating of the bonds. Exceptions include the Baa-rated utility and A-rated railroad credit spreads.

Table II also shows that all of the estimates of c are negative. In addition, most of the estimates are statistically significant although not as significant as the estimates of b. With the exception of the railroad credit spreads, the estimates of c decline monotonically with the credit rating of the bonds. The economic magnitude of these estimates is also important. For example, a 10 percent return, $I = 0.10$, reduces Baa-rated utility credit spreads by 16.2 basis points, Baa-rated industrial credit spreads by 20.2 basis points, and Baa-rated railroad credit spreads by 5.9 basis points. These results are consistent with the evidence of Jones, Mason, and Rosenfeld (1984), who find that equity returns are related to prices of below-investment-grade bonds, and argue that allowing interest rates to be stochastic may improve the performance of the traditional model.

Table II

Results from Regressing Monthly Changes in Credit Spreads on Monthly Changes in the 30-year Treasury Bond Yield and the Return on the Corresponding Standard and Poor's Stock Index

The term ΔS is the change in the credit spread, ΔY is the change in the 30-year Treasury bond yield, and I is the return on the corresponding stock index.

$$\Delta S = a + b\Delta Y + cI + \varepsilon$$

	a	b	c	t_a	t_b	t_c	R^2	N
Aaa Utilities	0.00583	−0.04413	−0.54885	0.63	−1.46	−1.45	0.015	179
Aa Utilities	0.00548	−0.07725	−0.77623	0.49	−2.10	−1.68	0.025	189
A Utilities	0.00766	−0.21993	−1.25687	0.63	−5.55	−2.52	0.146	189
Baa Utilities	0.00928	−0.18386	−1.62075	0.77	−4.69	−3.28	0.107	189
Aaa Industrials	0.00822	−0.26066	−0.63411	1.09	−12.57	−3.01	0.459	189
Aa Industrials	0.01318	−0.43914	−1.23965	1.19	−14.58	−4.04	0.533	189
A Industrials	0.01750	−0.50627	−1.88404	1.41	−14.94	−5.47	0.548	189
Baa Industrials	0.01852	−0.62645	−2.02645	1.23	−15.23	−4.84	0.556	189
Aa Railroads	0.02089	−0.78428	−0.66597	1.18	−18.12	−2.15	0.694	148
A Railroads	0.01139	−0.77246	−0.17902	0.63	−17.53	−0.57	0.680	148
Baa Railroads	0.01750	−0.82324	−0.59208	1.06	−20.44	−2.05	0.742	148

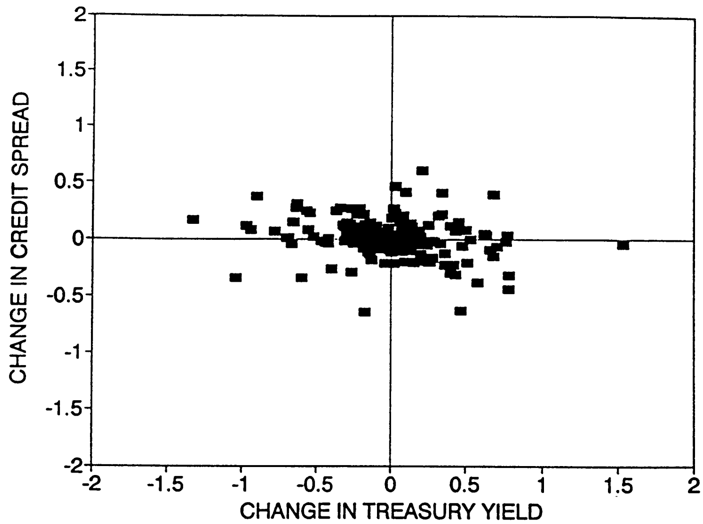

Figure 9. Plot of monthly changes in the credit spread for Moody's index of Baa-rated utility bonds as a function of monthly charges in the 30-year Treasury yield for the 1977 to 1992 period. Spreads and yields are multiplied by 100.

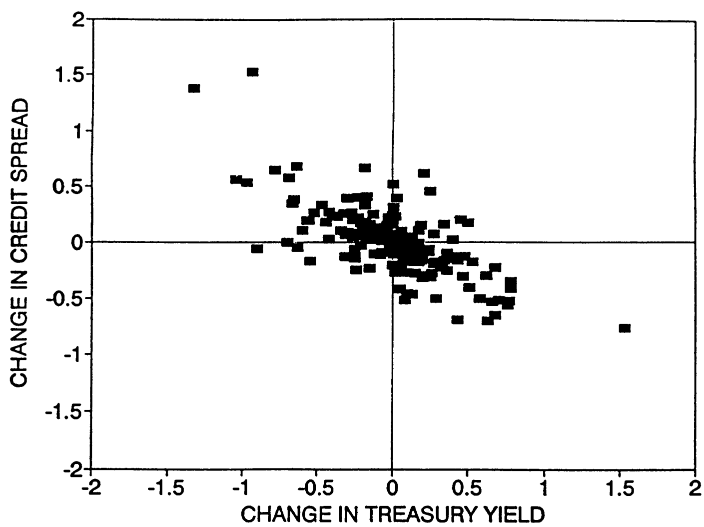

Figure 10. Plot of monthly changes in the credit spread for Moody's index of Baa-rated industrial bonds as a function of monthly changes in the 30-year Treasury yield for the 1977 to 1992 period. Spreads and yields are multiplied by 100.

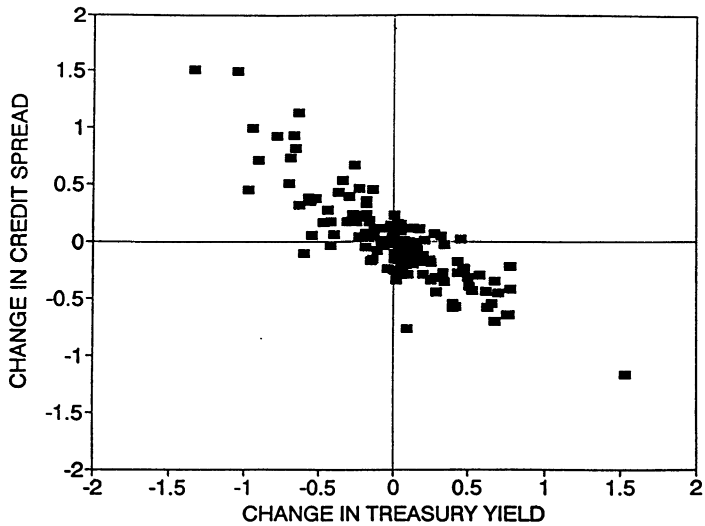

Figure 11. Plot of monthly changes in the credit spread for Moody's index of Baa-rated railroad bonds as a function of monthly changes in the 30-year Treasury yield for the 1977 to 1989 period. Spreads and yields are multiplied by 100.

These results for b and c provide clear evidence against the traditional approach to valuing risky debt in which the interest rate is assumed to be constant, and firm value is the only factor determining credit spreads. In fact, the variation in credit spreads due to changes in the level of interest rates is more important for these investment-grade bonds than the variation due to changes in the value of the firm. To see this, note that the standard deviation of monthly changes in the 30-year Treasury yield during the sample period is 36.8 basis points. Similarly, the standard deviations of monthly returns for the utility, industrial, and railroad stock indexes are 0.029, 0.036, and 0.053 respectively. Multiplying these values by the parameter estimates b and c implies that a one-standard-deviation increase in the 30-year yield reduces the Baa-rated utility credit spread by 6.8 basis points, while a one-standard-deviation positive return for the utility index reduces the credit spread by 4.7 basis points. The corresponding measures for the Baa-rated industrial credit spread are 23.1 basis points and 7.3 basis points, and the corresponding measures for the Baa-rated railroad credit spread are 30.3 basis points and 3.1 basis points.

The third implication of the two-factor model focuses on the relation between the values of b and the correlation coefficient ρ, holding c fixed. As shown in Table II, the implications of the model appear to be supported by the regression estimates. For example, the value of c is roughly comparable across Aaa-rated utility and industrial and Aa-rated railroad bonds. The corresponding ranking of b coefficients for these bonds is precisely as implied by the two-factor model. Similar results hold for the other rating categories. Casual observation of Figures 9, 10, and 11 clearly shows that the interest rate sensitivity of credit spreads is as predicted by the model. These results suggest that the correlation between asset returns and changes in the interest rate has an important effect on the interest-rate sensitivity of credit spreads, which in turn is the major source of variation in the credit spreads of these investment-grade bonds. Equivalently, these results imply that the correlation coefficient ρ is major determinant of a risky bond's duration.

Finally, it is important to acknowledge that the two-factor model does not capture all of the variation in credit spreads. This is particularly true for utility bonds where the R^2s for the regressions range from 0.015 to 0.146. For the other bonds, however, the regression R^2s are quite high and range from 0.459 to 0.742. Thus, the majority of the variation in credit spreads is captured by the two-factor model for these bonds. Note that the R^2s are generally higher for lower-rated bonds.

To provide additional insights into the properties of credit spreads, we also regress changes in relative credit spreads on percentage changes in the 30-year Treasury yield as well as the returns on the various stock indexes. Let ΔR be the change in the relative credit spread and let PY be the percentage change in the 30-year Treasury yield. Table III reports the results from estimating the regression

$$\Delta R = a + bPY + cI + \varepsilon. \tag{12}$$

The implications of the two-factor model for the regression parameters can be shown to be similar to those described earlier.

In general, the results from this regression parallel those reported in Table II. If anything, the implications of the two-factor model are more strongly supported by these results. In particular, the estimates of b now decrease monotonically as we move from higher to lower credit ratings. Similarly, the relation between b and the correlation coefficient ρ, holding c fixed, is even more striking. One major difference between the two regressions is that the R^2s in Table III are generally much higher than those in Table II. For example, the two-factor model is able to explain 65 and 77 percent of the variation in the relative spreads of Baa-rated industrial and railroad bonds. Similarly, this regression specification now allows the two-factor model to explain more than 38 percent of the variation in the relative spreads of Baa-rated utility bonds.

V. Conclusion

This article develops a simple new framework for valuing risky corporate debt that incorporates both default risk and interest-rate risk. We apply this model to derive closed-form valuation expressions for fixed-rate and floating-rate debt. An important feature of our approach is that it can be applied directly to value risky debt when there are many coupon payment dates or when the capital structure of the firm is very complex. In addition, this

Table III

Results from Regressing Monthly Changes in Relative Credit Spreads on Monthly Percentage Changes in the 30-year Treasury Bond Yield and the Return on the Corresponding Standard and Poor's Stock Index

The term ΔR is the change in the relative credit spread, PY is the change in the 30-year Treasury bond yield, and I is the return on the corresponding stock index.

$$\Delta R = a + bPY + cI + \varepsilon$$

	a	b	c	t_a	t_b	t_c	R^2	N
Aaa Utilities	0.00082	−0.19300	−0.07061	0.86	−5.77	−1.79	0.177	179
Aa Utilities	0.00078	−0.26897	−0.08993	0.72	−7.10	−2.00	0.241	189
A Utilities	0.00088	−0.40640	−0.10928	0.73	−9.67	−2.19	0.382	189
Baa Utilities	0.00106	−0.41996	−0.14890	0.89	−10.01	−2.99	0.383	189
Aaa Industrials	0.00115	−0.36209	−0.06507	1.31	−14.21	−2.70	0.522	189
Aa Industrials	0.00164	−0.53685	−0.11352	1.46	−16.44	−3.64	0.593	189
A Industrials	0.00203	−0.66144	−0.16940	1.58	−17.65	−4.72	0.625	189
Baa Industrials	0.00220	−0.82130	−0.18029	1.44	−18.56	−4.26	0.650	189
Aa Railroads	0.00279	−0.78723	−0.04965	1.64	−17.59	−1.68	0.680	148
A Railroads	0.00183	−0.83482	−0.00805	1.02	−17.53	−0.26	0.680	148
Baa Railroads	0.00253	−0.91002	−0.04776	1.61	−22.03	−1.75	0.770	148

approach allows us to relax the assumption of strict absolute priority which underlies the traditional approach to valuing risky debt.

A number of important insights about the valuation of risky debt emerge from this analysis. We show that the correlation of a firm's assets with changes in the level of the interest rate can have significant effects on the value of risky fixed-income securities. We also show that the term structure of credit spreads can have a variety of different shapes. In addition, our model implies that credit spreads are negatively related to the level of interest rates. Finally, our model has many implications for hedging the interest rate and default risk of corporate debt.

The empirical results suggest that the implications of this valuation model are consistent with the properties of credit spreads implicit in Moody's corporate bond yield averages. In particular, credit spreads are negatively related to the level of interest rates. Furthermore, differences in credit spreads across industries and sectors appear to be related to difference in correlations between equity returns and changes in the interest rate. We also find that changes in interest rates account for more of the variation in credit spreads for investment-grade bonds than changes in the value of the assets of the firm. The results provide strong evidence that both default risk and interest rate risk are necessary components for a valuation model for corporate debt.

Finally, we observe that while traditional approaches to modelling risky debt provide important conceptual insights, they have not provided practical tools for valuing realistic types of corporate securities. The primary advantage of this model is that it is easily applied to all types of corporate debt securities and, therefore, can be used to provide specific pricing and hedging results rather than just general implications. In particular, the model provides a simple theoretical benchmark against which the observed properties of risky corporate debt prices can be compared. Future research should focus on testing whether this two-factor model is able to explain the actual level of corporate bond yields using detailed cross-sectional and time-series data for individual bonds and firms.

Appendix

Proof of Proposition 1. Let $P(X, r, T) = D(r, T)(1 - wQ(X, r, T))$. Differentiation shows that equation (3) is satisfied if $Q(X, r, T)$ is the solution to

$$\frac{\sigma^2}{2} X^2 Q_{XX} + \rho\sigma\eta X Q_{Xr} + \frac{\eta^2}{2} Q_{rr} + (r - \rho\sigma\eta B(T)) X Q_X$$
$$+ (\alpha - \beta r - \eta^2 B(T)) Q_r - Q_T = 0, \tag{A1}$$

subject to the initial condition $Q(X, r, 0) = I_{\gamma \leq T}$. Using the results in Friedman (1975), $Q(X, r, T)$ is the probability that the first passage time of ln X to zero is less than T, where probabilities are taken with respect to the time-dependent processes

$$d \ln X = (r - \sigma^2/2 - \rho\sigma\eta B(T - t))dt + \sigma dZ_1, \tag{A2}$$

$$dr = (\alpha - \beta r - \eta^2 B(T - t))dt + \eta dZ_2. \tag{A3}$$

Integrating the dynamics for r from time zero to time τ implies that

$$\begin{aligned} r_\tau = r \exp(-\beta\tau) &+ \left(\frac{\alpha}{\beta} - \frac{\eta^2}{\beta^2}\right)(1 - \exp(-\beta\tau)) \\ &+ \frac{\eta^2}{2\beta^2} \exp(-\beta T)(\exp(\beta\tau) - \exp(-\beta\tau)) \\ &+ \eta \exp(-\beta\tau)\int_0^\tau \exp(\beta s)\, dZ_2. \end{aligned} \tag{A4}$$

Integrating the dynamics for ln X, substituting in for the value of r from the above equation, and evaluating the resulting double integral by applying Fubini's Theorem implies that

$$\ln X_T = \ln X + M(T,T) + \frac{\eta}{\beta}\int_0^T (1 - \exp(-\beta(T - t))\, dZ_2 + \sigma\int_0^T dZ_1. \tag{A5}$$

Thus, ln X_T is normally distributed with mean ln $X + M(T, T)$ and variance $S(T)$. Similarly, the joint bivariate distribution of ln X_t and ln X_T implies that ln X_T, conditional on ln $X_t = 0$, is normally distributed with mean $M(T, T) - M(t, T)$ and variance $S(T) - S(t)$. Let $q(0, \tau \mid \ln X, 0)$ be the first passage density of ln X to zero at time τ starting from ln X at time zero. From Buonocore, Nobile, and Ricciardi (1987) 2.2a, the first passage density is defined implicitly by the integral equation

$$N\left(\frac{-\ln X - M(t,T)}{S(t)}\right) = \int_0^t q(0, \tau \mid \ln X, 0) N\left(\frac{M(\tau,T) - M(t,T)}{S(t) - S(\tau)}\right) d\tau, \tag{A6}$$

where $\tau \leq t \leq T$. Dividing the period from time zero to time T into n equal subperiods and discretizing the above integral equation gives the following system of linear equations

$$N(a_i) = \sum_{j=1}^{i} q_i N(b_{ij}), \qquad i = 1, 2, \ldots, n. \tag{A7}$$

where

$$q_i = q(0, iT/n \mid \ln X, 0)T/n. \tag{A8}$$

These equations are easily solved as a recursive system for the q_i terms. The sum of the q_i terms provides an approximation to the value of $Q(X, r, T)$. As n increases, the approximation $Q(X, r, T, n)$ converges to the value $Q(X, r, T)$.

Proof of Proposition 2: Following the same approach as in Proposition 1 implies that the value of the floating-rate coupon can be expressed as

$$F(X, r, \tau, T) = D(r, T)E[r_\tau] - wD(r, T)E[r_\tau I_{\gamma \leq T}], \tag{A9}$$

where the expectations are taken with respect to the processes (A2) and (A3). From equation (A4), the expectation $E[r_\tau]$ is $R(r, \tau, T)$. The results in Buonocore, Nobile, and Ricciardi (1987) can also be used to show that

$$E[r_\tau I_{\gamma \leq T}] = \int_0^T E[r_\tau \mid \ln X_t = 0] q(0, t \mid \ln X, 0)\, dt. \tag{A10}$$

Standard results for the bivariate normal distribution can be used to derive the conditional expectation

$$E[r_\tau \mid \ln X_t = 0] = R(r, \tau, T) - \frac{C(\tau, t)}{S(t)} M(t, T), \tag{A11}$$

where $C(\tau, t)$ is the covariance between r_τ and $\ln X_t$. This covariance can be obtained from equations (A4) and (A5),

$$\begin{aligned} C(\tau, t) = {} & \left(\frac{\rho\sigma\eta}{\beta} + \frac{\eta^2}{\beta^2} \right) \exp(-\beta\tau)(\exp(\beta \min(\tau, t)) - 1) \\ & - \frac{\eta^2}{2\beta^2} \exp(-\beta\tau)\exp(-\beta t)(\exp(2\beta \min(\tau, t)) - 1). \end{aligned} \tag{A12}$$

Discretizing the integral in equation (A10) gives the following approximation for $E[r_\tau \mid \ln X_t = 0]$

$$\begin{aligned} & = \sum_{i=1}^{n} q_i \left(R(r, \tau, T) - \frac{C(\tau, iT/n)}{S(iT/n)} M(iT/n, T) \right), \\ & = R(r, \tau, T)Q(X, r, T) - G(X, r, \tau, T, n). \end{aligned} \tag{A13}$$

As n increases, the approximation $G(X, r, \tau, T, n)$ converges to the value $G(X, r, \tau, T)$. Substituting into equation (A9), and recalling the definition of $P(X, r, T)$, gives $F(X, r, \tau, T)$.

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