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AN ANALYTIC VALUATION FORMULA FOR UNPROTECTED AMERICAN CALL OPTIONS ON STOCKS WITH KNOWN DIVIDENDS

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Sometimes it pays to exercise an American-type call option prematurely, just prior to a cash emission by the underlying security. Such an option can be expressed as a combination of three European-type options whose valuation formulae are known.

The unprotected American call written against a dividend-paying stock is the predominant actively-traded option. On the C.B.O.E., call options have no contracted 'protection' against the stock price decline that occurs when a dividend is paid. Thus, there is an important deficiency in option pricing theory in terms of its empirical applicability. All known valuation formulae assume an absence of dividends [Black–Scholes (1973)], or a continuous dividend generating process [Merton (1973), Geske (1975)], or else require numerical solution [Schwartz (1977)]. Furthermore, the original Black–Scholes formula is known to give biased predictions of market prices [see Black (1975, p. 64)]; and the bias is widely believed to be related at least partly to the dividend problem.

This note amends the theory by presenting a simple revised formula that could apply to many empirical situations and that can be extended to more complex situations with ease.

The notation is the standard proposed by Smith (1976):

$c(S,T,X)$, the market value of a European call option,
$C(S,T,X)$, the market value of an American call option,
S, the current stock price, net of escrowed dividend (S_τ is the stock price after τ periods),
T, the time until expiration,
X, the exercise price,
r, the riskless (and constant) rate of interest, continuously compounded,
σ^2, the variance rate of the return on S.

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Assuming that the stock pays no dividend and that its price follows a lognormal diffusion process, Black and Scholes used an arbitrage argument in a no-tax world to obtain their well-known analytic formula for c (which depends only on S, T, X, σ^2, and r). Other work has relaxed the diffusion process assumption [Cox and Ross (1976), Merton (1976), and Rubinstein (1977)], the constant riskless rate assumption [Merton (1973)], and the tax assumption [Ingersoll (1976, pp. 109–112)]. Smith (1976) gives a lucid review of some of this work.

When the stock pays no dividends, Merton has shown that the American and European call options have equal value because the American option will never be exercised before maturity. When dividends are paid, however, the American call can be worth more than the European because there is a non-zero probability of early exercise.

Let the stock's dividend history be described by the following additional assumptions:

D, a dividend of known size, will be paid to each shareholder with certainty.
t is the known time until the ex-dividend instant ($t < T$). At t, the stock has just gone ex-dividend.
α is the known decline in the stock price at the ex-dividend instant as a proportion of the dividend.
No other dividend will be paid before T has elapsed.

If the dividend is certain, the total market value of the stock *cum* dividend cannot follow a lognormal process; for there would then be some chance that the dividend could not be paid. This difficulty is easily and sensibly resolved by defining the 'stock price', S, as the total market price, P, less the discounted escrowed dividend; i.e., for any $\tau < t$, $S_\tau = P_\tau - \alpha D e^{-r(t-\tau)}$ and for $\tau \geq t$, $S_\tau = P_\tau$. Note that the variance rate σ^2 applies to the process described by S.

At the instant before the stock goes ex-dividend, the American option holder observes that his option would be worth $c(S_t, T-t, X)$ an instant later if he allows it to remain unexercised. (Just after the ex-dividend date, the American and European options have equal value since no additional dividends will be paid before expiration.) If he exercises just before t, however, he will receive $S_t + \alpha D - X$. But we know from the Black–Scholes formula that c is bounded from below and is asymptotic to the lower bound for increasing stock price; i.e.,

$$\lim_{S\to\infty} c(S,T-t,X) = S_t - Xe^{-r(T-t)}.$$

Thus, if $\alpha D > X[1 - e^{-r(T-t)}]$, there exists some finite ex-dividend stock price above which the American option will be exercised just before t. Early exercise is more likely the larger the dividend, the higher the stock price relative to the exercise price (the more 'deeply-in-the-money' the option), and the shorter the time period between expiration and dividend payment dates.[1] Generally speaking whether or not an American call option will be exercised just an instant before the stock goes ex-dividend depends on the value of a European option just an instant after. Fig. 1 illustrates this with the well known chart of option price versus stock price at time t.

An example of these circumstances occurred in October 1976 when General Motors declared a three-dollar dividend to be paid to stockholders of record on November 4. The in-the-money option maturing in January ($X = \$60, T = 2$ months), was quoted for sale at near $P-X$ for a week before the ex-dividend date. On the ex-dividend day, the option closed at $S-X+75$¢. This seemed to imply an option price decline of $16\frac{1}{2}\%$ in one day (accompanying the decline in

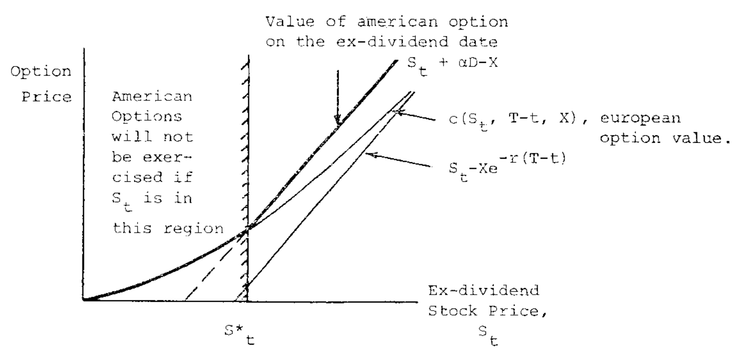

Fig. 1. American option value at the ex-dividend time.

P of about three dollars). The option price sequence was rational only if all of the outstanding options were exercised on November 3, new options having been written with the same exercise price and maturity date the next day. Only prices of newly-written options could have been quoted then. Notice that such large 'price declines' are easily forecastable but that they do not represent extraordinary profit opportunities for option writers.

Anytime *before* the ex-dividend date, the American option should reflect the probability that it will be exercised early. In fact, there is one very simple circumstance in which the unprotected option valuation formula can be given directly. The option may be so deeply in the money that the probability is nil that it won't be exercised early. Formally:

[1]For an alternative proof of the early exercise of an American call, see Smith (1976, pp. 13–14).

Proposition I. As $\text{Prob}(c_t < S_t+\alpha D-X)\rightarrow 1$, then an unprotected American option approaches the Black–Scholes valuation but with the ex-dividend date used in place of the option's contracted expiration date.[2]

The argument is rather obvious: Since the option is almost sure to be exercised an instant before t, t becomes its effective maturity. No dividend is paid before t so the Black–Scholes European formula applies, but over an interval shorter than the maturity stated on the option contract. Usually the probability will not be one, no matter how deeply-in-the-money an option may be.[3] In many practical situations, however, the probability may be so close to one that Proposition I gives a valuation whose error is within the bounds of transaction costs. Notice that this valuation formula would apply even with an uncertain dividend, provided that the dividend were known to be 'large' relative to $c-S+X$.

When Proposition I does not apply, similar reasoning can yield more general (but more complicated) results. We know the ex-dividend stock price above which the original American option will be exercised. It is the solution S_t^* to

$$c(S_t^*, T-t, X) = S_t^*+\alpha D-X. \tag{1}$$

This is the critical ingredient in the more general result. Note that S_t^* is different for each option whose contractual features are different.

Since the stock price S_t^* that separates the exercise and non-exercise regions on the ex-dividend date is known in advance,[4] a combination of hypothetical options can be constructed which match perfectly the contingencies faced by the original American option holder. Formally:

Proposition II. The value of an unprotected American call option with exercise price X, whose stock makes a single, certain dividend payment D after t periods and before the option's contracted expiration (which occurs after T periods) is the sum of the values of:

(a) a European call option on the stock with an exercise price of X and maturity T,
(b) plus a European call option on the stock with exercise price $S_t^*+\alpha D$ and maturity $t-\varepsilon$ ($\varepsilon > 0$, $\varepsilon \simeq 0$).
(c) minus a European call option on the option described under (a) with maturity of $t-\varepsilon$ and exercise price of $S_t^*+\alpha D-X$.

[2]This valuation was suggested first by Black (1975, pp. 41, 61) as the lower bound on the value of an unprotected American call.

[3]Except for option contracts with unusually small exercise prices or for liquidating dividends.

[4]Provided, of course, that the variance rate of the stock's return and the riskless rate of interest are known and constant. Schwartz (1977) mentioned this point and used the value of S^* in a numerical solution algorithm.

At the instant after the ex-dividend date, the cash receipts and net position are as follows.

For $S_t > S_t^*$,		For $S_t < S_t^*$,	
Cash receipts are		Portfolio positions are	
From	(a) 0	From	(a) Open
	(b) $S_t+\alpha D-S_t^*-\alpha D$		(b) Expired
	(c) $S_t^*+\alpha D-X$		(c) Expired
Total	$S_t+\alpha D-X$ in cash		Option on S until T with exercise price X

If the ex-dividend stock price is above S_t^*, options (b) and (c) are exercised. This provides a net cash flow of $S_t+\alpha D-X$ and leaves the investor with no open options [option (a) being taken by the exercise of (c)]. If the ex-dividend price is below S_t^*, however, options (b) and (c) are allowed to expire unexercised and (a) remains alive. No cash is transferred at the ex-dividend date in this case.

In order to calculate the value of the American option, we merely need to value the sum of its three components. Component (b) is no problem since the Black–Scholes formula applies directly. As for component (a), the Black–Scholes formula also applies because the stock price drop on the ex-dividend date is known in advance.[5]

Component (c) presents the most difficult valuation problem. Fortunately, the recent work of Geske (1976) provides a solution as his compound option formula can be applied directly. The details are given in the appendix and examples of the resulting formula are plotted in fig. 2.

Caveats and generalizations

The present modification to the Black–Scholes formula does not explain all the empirical facts. For example, the modified American call value curve always lies below the original Black–Scholes curve, as fig. 2 shows, and this makes the Black–Scholes bias worse for deep out-of-the-money options. Geske (1976) is able to explain the upward price bias of out-of-the-money options by noting that the stock of a levered firm is itself an option on the firm's assets. Perhaps a combination of these two formulations would explain better the prices of unprotected American calls on dividend-paying, levered firms. Then again, there are other candidates for explanatory variables such as transaction costs, non-lognormal processes ruling the stock price, and uncertainty in dividends. In

[5]As demonstrated by Rubinstein (1977, pp. 419–420), the value of a *European* option can also be obtained under the more general condition that the dividend *yield* is a non-stochastic function of time.

this last regard, the stock price decline (α) on the ex-dividend date might be important. This decline is related to the capital gains/ordinary tax differential on personal income and to the 85 percent exclusion from corporate taxes of dividends. Conceivably, if the present modified formula were accurate, it could be used to estimate α for the next ex-dividend date, once the dividend is announced, and the marginal shareholder tax rate could thereby be deduced. (However, the slack in prices caused by trading costs would work against the accuracy of such an estimate.)

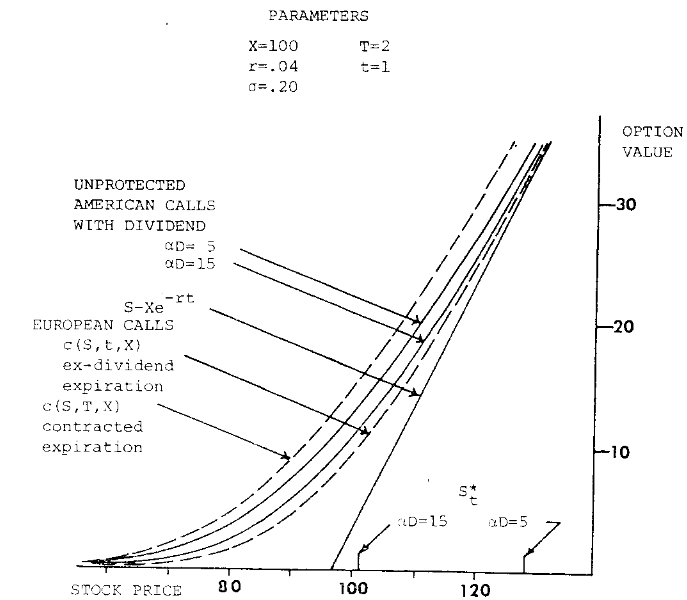

Fig. 2. Unprotected American call options with known dividends.

The basic solution can easily be generalized to cases where more than one future known cash payment will be made within the option's term. Two or more *known* successive dividends are rare for common stocks; but the generalization might apply to call options on corporate bonds or to prepayment options on standard mortgages (since the periodic cash payments are known throughout the lives of these contracts). To sketch the generalization, imagine a stock that has promised N successive known dividends. Between the $(N-1)$st and the Nth dividends, the American option on the stock would be valued by the formula in this paper. Before the $(N-1)$st dividend, a three part portfolio of options could again be constructed such that the contingencies were perfectly matched. This portfolio would consist of:

(a) a modified American option, valued by the formula in this paper, with the same parameters (X,T, plus the date and size of the Nth dividend),
(b) plus a European call on the stock with an exercise price of $S^{**}+\alpha D$ and maturity $t_{N-1}-\varepsilon$ [where S^{**} is the stock price above which the American option would be exercised at t_{N-1}, the date of the $(N-1)$st dividend],
(c) minus a European option on the option in (a) with maturity $t_{N-1}-\varepsilon$ and exercise price of $S^{**}+\alpha D-X$.

Of course, the resulting formulae would be more complex because there would now be an option (c) of an option of an option. For N payments, the final formulae would include an N-fold option of options. Conceptually, however, the generalization is straightforward. The only difficulty in obtaining the exact analytic formula seems to be the tedious algebra.

Appendix

According to Proposition II, the value of an unprotected American call option facing a known dividend payment is equal to the sum of three separate hypothetical options. This appendix presents details of the individual valuation expressions and gives the aggregate value.

Define the function q by

$$q(s,\tau,x) \equiv [\ln(s/x)+(r+\sigma^2/2)\tau]/\sigma\sqrt{\tau}.$$

Define $N(q)$ as the univariate standard normal probability distribution function (i.e., Prob $(y \leq q)$ where y is unit normal), and define $N(q,p)$ as the bivariate standard normal p.d.f. with correlation coefficient $+\sqrt{(t/T)}$.

Define S_t^* as the solution to

$$S_t^* N[q(S_t^*,T-t,X)]-X\mathrm{e}^{-r(T-t)}\,N[q(S_t^*,T-t,X)-\sigma\sqrt{(T-t)}]$$
$$-S_t^*-\alpha D+X=0.$$

The values of components (a) and (b) in Proposition II are then given by the Black–Scholes formulae,

$$c_a = SN[q(S,T,X)]-N[q(S,T,X)-\sigma\sqrt{T}]\mathrm{e}^{-rT}X,$$

and

$$c_b = SN[q(S,t,S_t^*+\alpha D)]-(S_t^*+\alpha D)\,N[q(S,t,S_t^*+\alpha D)-\sigma\sqrt{t}]\mathrm{e}^{-rt},$$

K

and the value of component (c) is given by the Geske formula

$$c_c = SN[q(S,t,S_t^*),\, q(S,T,X)]-XN[q(S,t,S_t^*)-\sigma\sqrt{t},q(S,T,X)$$
$$-\sigma\sqrt{T}]\mathrm{e}^{-rT}-(S_t^*+\alpha D-X)\,N[q(S,t,S_t^*)-\sigma\sqrt{t}]\mathrm{e}^{-rt}.$$

The unprotected American call option would then have the value

$$c_a+c_b-c_c.$$

References

Black, F., 1975, Fact and fantasy in the use of options, Financial Analysts Journal 31, 36–41, 61–72.
Black, F. and M. Scholes, 1973, The pricing of options and corporate liabilities, Journal of Political Economy 81, 637–654.
Cox, J.C. and S. Ross, 1976, The valuation of options for alternative stochastic processes, Journal of Financial Economics 3, 145–166.
Geske, R., 1975, The pricing of options with stochastic dividend yield, Working Paper (University of California, Berkeley, CA).
Geske, R., 1976, The valuation of compound options, Working Paper (University of California, Berkeley, CA).
Ingersoll, J.E., Jr., 1976, A theoretical and empirical investigation of the dual purpose funds, Journal of Financial Economics 3, 83–123.
Merton, R.C., 1976, Option pricing when underlaying stock returns are discontinuous, Journal of Financial Economics 3, 125–144.
Merton, R.C., 1973, The theory of rational option pricing, Bell Journal of Economics and Management Science 4, 141–183.
Rubinstein, M., 1977, The valuation of uncertain income streams and the pricing of options, Bell Journal of Economics 7, 407–425.
Schwartz, E.S., 1977, The valuation of warrants: Implementing a new approach, Journal of Financial Economics 4, 79–93.
Smith, C.W., Jr., 1976, Option pricing: A review, Journal of Financial Economics 3, 3–51.