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Determinants of GNMA Mortgage Prices

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This paper contrasts three different arbitrage-based models for the pricing of GNMA securities, and analyzes the effect of different assumptions about the call policy pursued by the issuers of the underlying mortgages. Both the nature of the interest-rate uncertainty captured by the model and the assumed call policy have a major effect on the yield differentials predicted between GNMA securities and Treasury Bonds.

Although GNMA mortgage-backed securities, like US Treasury Bonds, are backed by the full faith and credit of the US Treasury, they trade at yields that differ systematically from those of Treasury Bonds of the same coupon and maturity. Several studies, [10][12][13][11][8], have analyzed these yield differentials on the basis of the particular features of the GNMA security, notably the amortization and call provisions. These studies have for the most part relied on a simple model of the yield curve that determines the yields on all default-free securities as a function of a single-state variable, the yield on an instantaneously maturing bond, and a single market price of risk parameter. The basis of the model is a no-arbitrage condition in a continuous time setting.

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In the first part of this paper, we demonstrate the principles underlying all arbitrage-based models of the yield curve, and show how the general model may be specialized to yield the Brennan-Schwartz, [2][3][4][5][6], two-interest rate model, which may in turn be restricted to yield the single-interest rate models employed in previous studies of GNMA securities. In the second part of the paper we use parameter estimates of the Brennan-Schwartz model derived from an earlier detailed study of the US Treasury Bond market [5] to arrive at estimates of the pricing errors likely to arise from use of the simple single-state variable models. We then offer some new estimates of the equilibrium yield differentials between GNMA securities and Treasury Bonds under alternative yield curve scenarios and assumptions about issuer call policy. It appears that assumptions about call policy are critical to the value estimates. We conclude by suggesting how the estimation of an empirical call frequency function could be used to arrive at a more accurate pricing model.

ALTERNATIVE MODELS FOR PRICING INTEREST DEPENDENT CLAIMS

Arbitrage Pricing in Continuous Time Models[1]

Consider a frictionless economy in which securities may be traded continuously in perfectly competitive markets. The state of the economy at time t is assumed to be described by an m-vector of state variables, $\underline{X}(t) \equiv (X_1(t), \ldots, X_m(t))$, which follow a joint stochastic process of the general type:

$$dX_i = \beta_i \, dt + \eta_i \, dz_i \tag{1}$$

where dz_i is the increment of a Wiener process and $E[dz_i] = 0, dz_i^2 = dt$, $dz_i \, dz_j = \rho_{ij} \, dt$. The parameters of this stochastic process may depend upon the current values of the state variables and on calendar time.

Since X constitutes a complete description of the current state of the economy, it follows that the value of any security j may be written as a function of the state variables and time $G^j(\underline{X};t)$. Similarly the instantaneous payout rate on security j may be written as $C^j(\underline{X};t)$.

The instantaneous return to the holder of the security is given by the sum of the payout $C^j dt$ and the instantaneous price change dG^j. Using Ito's Lemma to derive the latter, the instantaneous return from holding the security may be written as

$$dG^j + C^j dt = G^j \mu^j \, dt + \sum_{i=1}^{m} G^j_i \eta_i \, dz_i \tag{2}$$

where

$$\mu^j = \left[\sum_{i=1}^{m} G^j_i \beta_i + 1/2 \sum_{i=1}^{m} \sum_{k=1}^{m} G^j_{ik} \rho_{ik} \eta_i \eta_k + G^j_t\right] / G^j \tag{3}$$

and

$$G^j_i \equiv \partial G^j / \partial X_i, \; G^j_t \equiv \partial G^j / \partial t, \text{ etc.}$$

Now let r denote the instantaneously riskless interest rate, and consider a portfolio P of m arbitrarily chosen securities with a long position of b_j units of security j (j=1,...,m) and a short position of $\sum_{j=1}^{m} b_j G^j$ in the riskless security. The net investment in this portfolio is zero and the instantaneous change in its value is given by

$$dP = \sum_{j=1}^{m} b_j (dG^j + C^j dt) - \sum_{j=1}^{m} b_j G^j r \, dt \tag{4}$$

$$= \sum_{j=1}^{m} b_j G^j (\mu^j - r) dt + \sum_{j=1}^{m} \sum_{i=1}^{m} b_j G^j_i \eta_i \, dz_i \tag{5}$$

If the portfolio composition b_j (j=1,...,m) is chosen so that

$$\sum_{j=1}^{m} b_j G^j_i \eta_i = 0 \qquad (i=1,\ldots,m) \tag{6}$$

the instantaneous change in the value of the portfolio will be non-stochastic, and to avoid arbitrage profits it is necessary that the return on the portfolio be identically zero so that

$$\sum_{j=1}^{m} b_j G^j (\mu^j - r) = 0 \tag{7}$$

(6) and (7) describe a system of (m+1) linear equations in the m unknowns b_j (j=1,...,m), and the system will have a solution only if the equations are linearly dependent so that there exist values of λ (i=1,...,m) such that

$$G^j (\mu^j - r) = \sum_{i=1}^{m} \lambda_i G^j_i \eta_i \qquad (j=1,\ldots,m) \tag{8}$$

Moreover, since the securities included in the portfolio were selected arbitrarily, equation (8) must hold for all securities for the same values of λ (i=1,...,m) although these values may be state dependent.

[1] This section borrows heavily from Cox, Ingersoll and Ross [9].

Equation (8) bears a striking resemblance to the Ross [19] Arbitrage Pricing Theory and, like that theory, expresses the risk premium on a security as a linear function of the proportional sensitivities of the security value to each of the state variables, G_i^j/G^j.

Nothing further may be said about the nature of λ_i, the 'market price of risk' for state variable i, without a more complete description of the economy, including investor tastes and investment opportunities.[2] However to obtain an operational model it may be necessary to make explicit assumptions about λ_i as we shall see below.

Substituting for μ^j in (8) from (3) and dropping the superscript, the no-arbitrage condition (8) is seen to imply that the value of any security must satisfy the partial differential equation.

$$1/2 \sum_{i=1}^{m} \sum_{k=1}^{m} G_{ik} \rho_{ik} \eta_i \eta_k + \sum_{i=1}^{m} G_i (\beta_i - \lambda_i \eta_i) + G_t + C - rG = 0 \qquad (9)$$

This equation, which has been referred to by Cox, Ingersoll and Ross [9] as the Fundamental Partial Differential Equation for Contingent Claims, suffices in principle to determine the value of any security once the relevant boundary conditions for the security are appended and the payout function $C(\cdot)$ defined. In practice, as we shall see below, there will remain the difficult problem of choosing the state variables that are relevant for the value of any particular security. However, before turning to this issue we will consider the payout functions for Treasury and GNMA securities and the boundary conditions that may be inferred from the characteristics of the securities.

Payouts and Terminal Conditions for GNMA Securities and Treasury Bonds

The Treasury Bond The Treasury Bond is assumed to mature at time T, to have a par value of unity and to pay a coupon continuously at the rate c. Therefore the payout function is

$$C(\underline{X};t) = c \qquad (10)$$

and the value of the bond at maturity satisfies

$$G(\underline{X};T) = 1 \qquad (11)$$

The GNMA Security The GNMA security is essentially a government guaranteed portfolio of mortgages with identical term and coupon rate. The mortgages are freely callable and different valuation

[2] See Merton [17] for a similar equation or Cox, Ingersoll and Ross [9] for a complete general equilibrium model

models are appropriate depending upon the assumptions made about the call policy followed by the issuers of the mortgages.

We consider first the case of an 'optimal' call policy that is chosen to minimize the value of the mortgage (the issuer's liability) and hence the value of the GNMA security itself. If the issuers of all the mortgages underlying a given GNMA security pursue an optimal call policy, then the whole of the security will be called for redemption at the same time. Otherwise partial redemptions are likely; we will say more about these below.

Let F(t) represent the principal amount outstanding at time t under a GNMA security which also has a coupon rate c. Then for a T year fully amortizing security with continuous payments the total payout rate is

$$C(\underline{X};t) = \frac{cF(0)}{1 - e^{cT}} \qquad (12)$$

where F(0) is the original principal amount, and the principal outstanding at time t is

$$F(t) = \quad F(0) \left(\frac{1 - e^{c(T-t)}}{1 - e^{cT}}\right) \qquad (13)$$

Since the security is fully amortizing, its value at maturity satisfies the boundary condition

$$G(\underline{X};T) = F(T) = 0 \qquad (14)$$

A call policy can be specified as a set of state variable vectors $\hat{\underline{X}}(t)$, t ε (O,T), such that the security is called when $\underline{X}(t) \in \hat{\underline{X}}(t)$. This implies that under a given call policy the value of the security satisfies

$$G(\underline{X};t) = F(t) + P(\underline{X},t) \qquad \text{for } \underline{X} \in \hat{\underline{X}}(t) \qquad (15)$$

where $P(\cdot)$ is the penalty for early prepayment.

For an optimal, value-minimizing, call policy the security value also satisfies the Merton-Samuelson [18] high contact condition.

$$G_i(\underline{X};t) = P_i(\underline{X},t) \qquad \text{for } \underline{X} \in \hat{\underline{X}}(t), \; i=1,\dots,m \qquad (16)$$

Thus, under the assumption that the mortgages are called optimally the value of the GNMA security satisfies the partial differential equation (9), where the payout function is given by (12), subject to the boundary conditions (14), (15) and (16).

Following Dunn and McConnell [13], we consider next the case of a suboptimal call policy. We assume first that the rate at which the outstanding principal is called for redemption can be written as a function of the state variables and time, $\pi(\underline{X};t)$ per unit of outstanding mortgages. Let f denote the fraction of the underlying mortgages that remains outstanding. Then

$$df = -\pi(\underline{X};t)f\,dt \tag{17}$$

and the value of the GNMA security is now a function of f also. We write it as $H(\underline{X};t,f)$ and note that

$$H(\underline{X};t,f) = fH(\underline{X};t,1) \equiv h(\underline{X};t) \tag{18}$$

The cash flow to the holder of the security is now

$$C(\underline{X};t,f) = \frac{cF(0)}{1-e^{cT}} + \pi fF(t) \tag{19}$$

Treating f as the (m+1) state variable the partial differential equation (9) becomes

$$1/2 \sum_{i=1}^{m}\sum_{k=1}^{m} H_{ik}\rho_{ik}\eta_i\eta_k + \sum_{i=1}^{m} H_i(\beta_i - \lambda_i\eta_i) + H_t - \pi H \tag{20}$$

$$+ \frac{cF(0)}{1-e^{cT}} + \pi fF - rH = 0$$

Using the homogeneity property (18), this is equivalent to

$$1/2 \sum_{i=1}^{m}\sum_{k=1}^{m} h_{ik}\rho_{ik}\eta_i\eta_k + \sum_{i=1}^{m} h_i(\beta_i - \lambda_i\eta_i) + h_t \tag{21}$$

$$+ C + \pi(F-h) - rh = 0$$

where $C = cF(0)/(1-e^{ct})$. The value of the GNMA security when none of the original mortgages have been called (f=1) is given by the solution to (21) subject to

$$h(\underline{X};T) = 0 \tag{22}$$

This result can then be scaled using (18) to take account of calls that have already been made (f<1).

Equation (21) can also be derived by assuming that calls are stochastic so that $\pi(\cdot)$ represents the intensity of a Poisson process governing the event of call of all of the mortgages underlying the security. This is the approach followed by Brennan and Schwartz [3] and by Dunn and McConnell [12] [13].

It is also possible of course to assume that a part of the GNMA security will be called optimally and the rest autonomously at the rate $\pi(\cdot)$. In this case the two parts can be valued separately according to the principles described here, and the results summed.

The Brennan-Schwartz Two-State Variable Model

In order to implement the general valuation model described in the first section of this article it is necessary to specify a limited number of state variables that are relevant to the pricing of the security under consideration. The more state variables are included, the more realistic will be the model; however, it will also be less tractable. In a series of papers Brennan and Schwartz have proposed and developed a two-state variable model that is applicable to the pricing of interest-dependent claims such as default-free bonds and the options written on them.

The fundamental assumption underlying this model is that the prices of all default-free bonds at any moment in time can be expressed in terms of the values of two, possibly unknown, stochastic factors, X_1 and X_2, which follow a stochastic process of the type (1). Then the price of a bond with continuous coupon rate C, face value of unity and maturity τ, can be written as $B(X_1, X_2, \tau, C)$.

The instantaneously riskless rate, the "short rate," is the yield on the currently maturing discount bond and is defined by

$$r(X_1, X_2) = \lim_{\tau \to 0} \frac{-\ln B(X_1, X_2; \tau, 0)}{\tau} \tag{23}$$

Similarly, the "consol rate" is defined as the yield on a bond whose maturity is infinite

$$l(X_1, X_2) = \frac{c}{B(X_1, X_2, \infty, c)} \tag{24}$$

If equations (23) and (24) can be inverted and the state variables X_1 and X_2 expressed as twice differentiable functions of the potentially observable interest rates, r and l, then, as pointed out by Cox, Ingersoll and Ross [9], bond prices may be expressed as functions of the state variables proxies, r and l, and the value of any default-free security may be expressed simply as $B(r,l,t)$ for a given payout function $C(r,l,t)$ and boundary conditions.

The state variable proxies r and l are assumed to follow the stochastic process

$$dr = \beta_1(r,l,t)dt + \eta_1(r,l,t)dz_1 \tag{2}$$

$$dl = \beta_2(r,l,t)dt + \eta_2(r,l,t)dz_2$$

where $dz_1 \, dz_2 = \rho dt$. This conforms to the general class of processes (1), so that under the assumptions of this section the value of any default-free claim satisfies the following specialization of the partial differential equation (9):

$$1/2\, B_{rr}\eta_1^2 + B_{rl}\rho\eta_1\eta_2 + 1/2 B_{ll}\eta_2^2 + B_r(\beta_1 - \lambda_1\eta_1) + B_l(\beta_2 - \lambda_2\eta_2) + B_t + C - rB = 0. \quad (26)$$

Equation (26) contains two undetermined 'market price of risk' parameters, λ_1 and λ_2. However, the value of a consol bond paying a continuous coupon at the rate of \$1 per period is l^{-1}, and its derivatives with respect to r and l are readily computed. Substitution of these derivatives for the value of the consol bond into (26) yields the following expression for λ_2, the market price of consol rate risk:

$$\lambda_2(r,l,t) = -\frac{\eta_2}{l} + (\beta_2 - l^2 + rl)/\eta_2 \quad (27)$$

Finally, substituting from (27) in (26) the Brennan-Schwartz two-state variable equation for valuing interest-dependent claims is

$$1/2\, B_{rr}\eta_1^2 + B_{rl}\rho\eta_1\eta_2 + 1/2 B_{ll}\eta_2^2 + B_r(\beta_1 - \lambda_1\eta_1) + B_l(\eta_2^2/l + l^2 - rl) + B_t + C - rB = 0 \quad (28)$$

Single-State Variable Models

Prior analyses of the pricing of the GNMA security have posited an even simpler model of the yield curve than that described in the previous section; in this model it is assumed that the values of all default-free securities may be expressed in terms of a single stochastic state variable that may be taken as the current short rate [8] [11] [12] [13]. This assumption implies that once the short rate is known the whole yield curve is determined, and hence does not admit the possibility that for the same value of the short rate the yield curve should sometimes slope up and sometimes slope down. In contrast, the Brennan-Schwartz two-state variable model, by allowing for independent variation in the consol rate, does admit such a possibility.

In fact the single-state variable model can be derived from the Brennan-Schwartz model by restricting the stochastic process (25) by the assumptions[3]

$$\beta_1(r,l) = \gamma(r)$$

$$\eta_1(r,l) = \zeta(r)$$

When these restrictions are imposed in equation (28), it is seen that the stochastic state variable l enters the equation only when multiplied by a partial derivative of B with respect to l. Since l does not enter the boundary conditions for either Treasury Bonds or GNMA securities, it follows that the values of both securities may be written as functions only of r and t, F(r,t), where F(r,t) satisfies the restricted version of (28):

$$1/2\, F_{rr}\eta_1^2 + F_r(\gamma - \lambda_1\zeta) + F_t + C - rF = 0. \quad (29)$$

(29) is the standard partial differential equation of single-state variable models for default-free securities, including bonds and the options written on them. We shall refer to this as the 'short rate model'.

It is also possible to derive the Black-Scholes [1] partial differential equation for pricing options from the two-state variable model by requiring that the short rate remain constant by constraining the stochastic process so that $\beta_1 = \eta_1 = 0$. With these constraints equation (28) becomes

$$1/2\, B_{ll}\eta_2^2 + B_l(\eta_2^2/l + l^2 - rl) + B_t + C - rB = 0. \quad (30)$$

To see the equivalence of (30) to the Black-Scholes-Merton partial differential equation for the value of an option on an asset with payouts, consider the change of variable

$$L = C/l \quad (31)$$

L will be recognized as the market value of a consol bond paying a continuous coupon at the rate C. Then, defining $B(l,t) \equiv H(L,t)$ as the value of an option on the consol bond, it is easily shown that

$$B_l = -L\, H_L/l \quad (32)$$

$$B_{ll} = -L^2 H_{LL}/l^2 + 2H_L L/l^2 \quad (33)$$

$$B_t = H_t \quad (34)$$

Using Ito's Lemma it follows from (32) that $\eta_2 = l\sigma_L$ where σ^2_L is the variance of the rate of return on the consol. Making the appropriate substitutions in (28) and setting C = 0 yields the equivalent partial differential equation:

$$1/2\, H_{LL} L^2\sigma^2_L + H_L(rL - C) + H_t - rH = 0. \quad (35)$$

This is the Black-Scholes-Merton equation for the value of an option on a security with payout rate C. It follows therefore that use of the Black-Scholes-Merton equation (35) to value debt options is equivalent to using the Brennan-Schwartz equation (28) under the

[3] Henceforth we shall assume that the model parameters are time independent

assumptions that the underlying asset is a consol bond and the instantaneously riskless interest rate is nonstochastic.

EMPIRICAL ESTIMATES

Parameter Estimates for the Brennan-Schwartz Model[4]

The coefficients of the partial differential equation (28) that underlies the Brennan-Schwartz model depend on two types of function: β_1, β_2, η_1, η_2 and ρ that derive from the stochastic process (25), and $\lambda_1(\cdot)$ the market price of risk function. We shall take up first the estimation of the stochastic process and then the estimation of $\lambda_1(\cdot)$.

The specific form of the stochastic process that was assumed for purposes of estimation was

$$dr = [a_1 + b_1(l-r)]\,dt + r\sigma_1 dz_1 \qquad (36)$$
$$dl = l(a_2 + b_2 r + c_2 l)dt + l\sigma_2 dz_2.$$

This formulation presupposes that the scale of the unanticipated increment in each of the interest rates is proportional to the current value of that rate, so that interest-rate volatility increases with the level of rates. The coefficient of dt in the short-rate equation reflects the essence of expectations-based theories of the term structure, which is that long rates are based upon expectations about future short-interest rates: if such expectations are rational, the short rate will have a tendency to regress towards the current value of the long rate so that $b_1>0$. The coefficient of dt in the consol rate equation was obtained by treating $\lambda_2(\cdot)$, the market price of consol rate risk, as a linear function of r and l and solving equation (27) for $\beta_2(\cdot)$: it should be observed that $\beta_2(\cdot)$ does not enter the partial differential equation (28) and hence will not affect our estimates of security values.

The equation system (36) was estimated in discrete form using monthly data from the CRSP U.S. Government Bond file for the period 1958-1979. r was taken as the annualized yield to maturity on the US Government Treasury Bill whose maturity was closest to 30 days on the last trading day of each month. The consol rate, l, was approximated by the annualized yield to maturity on the highest yielding US Government Bond with a maturity exceeding 20 years; if no such bond was available in a particular year, then the highest-yielding bond with a maturity of more than 15 years was used instead.

[4] A more complete description is available in Brennan and Schwartz [5].

The process actually estimated was

$$dr = [-.0887 + .1102(l-r)]dt + .1133r\,dz_1 \qquad (37)$$
$$\quad\;\;(.0526)\;(.0301)$$

$$dl = [\;.0089 + .0036r - .0037l\,]\,dt + .0298l\,dz_2 \qquad (38)$$
$$\quad\;\;(.0069)\;(.0017)\;(.0020)$$

and $\rho = .21$, where standard errors are in parentheses.

$\lambda_1(\cdot)$, the market price of short-term interest rate risk, was assumed to be an intertemporal constant for purposes of estimation. The details of the estimation are presented in Brennan and Schwartz [6]. The principle employed was to solve the partial differential equation (28) with C=0 and boundary condition B(r,l,T) = 1. The resulting present value factors were then used to value the bonds represented each month on the CRSP Government Bond File, and by minimizing the price prediction errors an estimate of -0.45 was obtained for λ_1.

Alternative Predictors of Debt Call Option Values

A GNMA security differs from a Treasury Bond both because it is an amortizing security and because it is callable. It is a relatively straightforward task to determine in principle the yield differential due to the amortizing feature since an amortizing security is equivalent to a portfolio of non-amortizing securities. Therefore the critical element of models for predicting the GNMA security-Treasury Bond yield differential is the model for valuing the call feature. For this reason it is interesting to compare the call option values predicted by the full Brennan-Schwartz model and by the two single-state variable models obtained by imposing restrictions on the stochastic process — the 'short-rate' model and the Black-Scholes model.

Table 1[5] reports the values of call options on a 20-year bond predicted by the three models. The values were computed as follows. For the Brennan-Schwartz model bond values were estimated for each value of (r,l,t) by solving the partial differential equation (28) subject to the boundary condition (11) and payout function (10), using the parameter values whose estimation was described in the section previous to this one. The resulting bond values were used in the boundary conditions for the call option whose value, V(r,l,t), also satisfies (28). Then for r = 10% the values of l corresponding to different bond values were determined and the value of the option for that value of l and r = 10% was found.

The Black-Scholes values were found by solving (28) with the restriction $\beta_1 = \eta_1 = 0$ and C = 0, taking l as the yield to maturity on the

[5] This is Table A-1 of Brennan and Schwartz [7].

underlying security whose value determines the boundary condition for the call option.

For the short-rate model the specific forms of the functions $\gamma(r)$ and $\zeta(r)$ employed to represent the drift and instantaneous standard deviation of the stochastic process for r are

$$\gamma(r) = .1102(\bar{r} - r)$$

$$\zeta(r) = .1133(r)$$

The speed-of-adjustment coefficient and standard deviation parameter are taken from the estimated short-rate process (31). Instead of pre-specifying the target interest rate, $\bar{r}$, both the underlying security and the call option were valued for a range of values of $\bar{r}$ by solving the differential equation (28) with $\eta_2 = 0$, subject to the appropriate boundary conditions. To determine the option values reported in Table 1, the value of $\bar{r}$ was found which was consistent with the given bond price and r = 10%; the relevant values of $\bar{r}$ are reported in the bottom line of the table. The option value was then determined for these values of r and $\bar{r}$.

It is apparent from Table 1 that both single-state variable models, the Black-Scholes model and the short-rate model, substantially understate the value of short-term call options on 20-year bonds relative to the Brennan-Schwartz two-state variable model. Similar, though less pronounced biases were found for options on 5-year bonds. The cause of the bias appears to be the reduced level of uncertainty about future bond prices that results from ignoring the stochastic nature of one of the two-state variables.

Since σ_2 is not equal to zero, these results suggest that estimates of the equilibrium yield differential between GNMA securities and Treasury Bonds based on single-state variable models, which assume σ_2 equal to zero, are likely to understate the true differentials.[6] This impression is confirmed by the results we obtain by applying the two-state variable to the valuation of GNMA securities to which we now turn.

Estimates of GNMA Security-Treasury Bond Price and Yield Differentials

In this section we report the prices and promised yields of 8% 30-year GNMA securities and Treasury Bonds for different values of the state variables describing the yield curve. The parameter values

[6] See for example Buser and Hendershott [8] who estimate the stochastic process for r assuming a two-state variable model by including the long rate as the target towards which the short rate adjusts, and yet ignore stochastic variation in l in pricing GNMA securities. This corresponds precisely to what we did in implementing the short-rate

TABLE 1

Values of Call Option on 10%, 20-Year Bond

Option Maturity	6 months					12 months					24 months				
Bond Price	\$900	\$950	\$1000	\$1050	\$1100	\$900	\$950	\$1000	\$1050	\$1100	\$900	\$950	\$1000	\$1050	\$1100
Option Values															
Brennan-Schwartz	2.8	10.7	23.1	61.6	104.1	6.3	17.2	33.0	68.9	108.1	13.1	26.9	45.4	78.4	114.1
Black-Scholes	1.5	7.3	16.3	57.2	100.0[2]	4.5	13.5	26.4	61.9	100.0[2]	10.8	22.7	38.0	68.2	101.3
'Short-rate'	0.0	0.0	23.9	52.5	102.2	0.0	0.0	25.4	55.0	104.1	0.0	0.0	27.5	58.4	106.7
Target rate: $\bar{r}$%[1]	(8.9)	(8.3)	(7.7)	(7.2)	(6.7)	(8.9)	(8.3)	(7.7)	(7.2)	(6.7)	(3.9)	(8.3)	(7.7)	(7.2)	(6.7)

Exercise Price: \$1000 plus accrued Interest

Short Rate (r): 10%

[1] This is the unique target rate for the short rate which yields the bond price at the head of the column when r = 10%

[2] Optimal to exercise.

for the model are those whose estimation was described in the previous section, and the bonds are assumed to have a par value of $100.

The Treasury Bond values were found by solving numerically the partial differential equation (28) subject to (10) and (11). The GNMA security values were found for three different assumptions about mortgage calls. Values under the optimal call policy were found by solving (28) subject to (12), (14), (15) and (16). Two different assumptions about suboptimal call policy were made.

First, following Dunn and McConnell [12][13], it was assumed that there is an autonomous probability rate $\pi(r,l,t)$ that the whole security will be called in year t; and that in addition the security is called if it is optimal to do so, taking account of the probability of future autonomous call. We refer to this assumption as the optimal call policy with autonomous calls. The probability rates of autonomous call used by Dunn and McConnell and adopted by us reflect FHA experience and depend only on time and not on the state variables r and l. Moreover, since the FHA experience reflects total calls, both optimal and suboptimal, the Dunn and McConnell assumption overstates the probability of call. Therefore our second assumption is that calls are purely autonomous. It should be stressed that neither of these assumptions is entirely satisfactory: presumably the autonomous call rate depends on r and l since the values of these variables will determine the net cost or benefit of a call that will be taken into consideration by the mortgage issuer in making his call decision. Unfortunately we have no information about the form of the empirical autonomous call rate $\pi(r,l,t)$.

The GNMA security was valued under the assumption of an optimal call policy with an autonomous call rate by solving (21) subject to (22) and the boundary condition arising from the optimal call policy:[7]

$$h_r(r,l,t) = h_l(r,l,t) = 0, \text{ for } h(r,l,t) = F(t) \qquad (39)$$

was taken from Dunn and McConnell [12]. To take account of autonomous calls only, the security was valued without imposing condition (39).

The results are summarized in Tables 2, 3, 4 and 5. Referring to Tables 2 and 3 we note first that under the optimal call policy the value of the GNMA security may exceed as well as fall short of that of the Treasury Bond: the higher is the long rate the more likely it is

[7] The prepayment penalty was assumed to be zero, and other costs and benefits of prepayments were ignored. For a detailed discussion of these see Hendershott and Hu [5] and Hendershott, Hu and Villani [16].

TABLE 2

Prices of 8%, 30-Year Treasury Bond and GNMA Securities per $100 of Principal

		Treasury Bond	GNMA Security: Optimal Call Policy	GNMA Security: FHA Call Experience (1)	GNMA Security: FHA Call Experience (2)
r = 6%	ℓ = 6%	123.2	100.0*	100.0*	115.4
	8%	96.7	92.6	96.6	99.7
	10%	78.9	79.9	86.6	87.4
	12%	66.3	69.1	77.3	77.7
r = 8%	ℓ = 6%	122.8	99.6	99.8	114.4
	8%	96.5	91.9	95.6	98.9
	10%	78.8	79.6	86.0	86.9
	12%	66.3	68.9	76.9	77.2
r = 10%	ℓ = 6%	122.4	98.7	99.0	113.4
	8%	96.3	91.1	94.6	98.2
	10%	78.7	79.2	85.4	86.3
	12%	66.3	68.7	76.5	76.8
r = 12%	ℓ = 6%	122.0	97.6	98.0	112.4
	8%	96.1	90.4	93.6	97.4
	10%	78.6	78.8	84.7	85.7
	12%	66.2	68.5	76.0	76.3

(1) Optimal Call Policy with Autonomous Call Rate
(2) Autonomous Calls only
*Optimal to Call

that the GNMA security will trade at a premium relative to the bonc Since under an optimal call policy the call feature can only decreas the value of the security, it is apparent that this phenomenon can onl be due to the amortizing feature of the GNMA. At high interest rate the probability of call is slight and the call feature has little influenc on value; on the other hand since both securities are trading belov par the shorter average term of the GNMA due to the amortizatio feature causes it to trade at a premium. This effect is enhanced unde the suboptimal call policies since autonomous calls serve to furthe reduce the average maturity of the GNMA. It is interesting to not that in the simulations reported in the pioneering work by Dunn anc McConnell [12] [13] the GNMA price never exceeds the bond pric This is because the single-state variable model they employ permit too little flexibility in the choice of yield curve scenarios: indeed thei parameters were chosen so that the long-run mean of the instantaneously riskless rate, r is 5.6%, effectively excluding the possibilit of high yields on long-term bonds. The highest 30-year yield gener

TABLE 3

Price Differentials between 8%, 30-Year Treasury Bonds and GNMA Securities per $100 of Principal

(Numbers in parentheses indicate that the Treasury Bond price exceeds that of the GNMA Security)

		GNMA Security		
		Optimal Call Policy	FHA Call Experience (1)	(2)
r = 6%	ℓ = 6%	(23.2)	(23.2)	(7.2)
	8%	(4.1)	(0.1)	1.0
	10%	1.0	7.7	8.5
	12%	2.8	11.0	11.3
r = 8%	ℓ = 6%	(23.2)	(23.0)	(8.4)
	8%	(4.6)	(0.9)	2.4
	10%	0.8	7.2	8.1
	12%	2.6	10.6	10.9
r = 10%	ℓ = 6%	(23.7)	(23.4)	(9.0)
	8%	(5.2)	(1.9)	1.9
	10%	0.5	6.7	8.6
	12%	2.4	10.2	10.5
r = 12%	ℓ = 6%	(23.4)	(24.0)	(9.5)
	8%	(5.7)	2.5	1.3
	10%	0.2	6.1	7.1
	12%	0.3	9.8	10.1

(1) Optimal Call Policy with Autonomous Calls
(2) Autonomous Calls only

ated by their model was 8.72% and this was for a short rate of over 20%.

Dunn and McConnell found that the Treasury Bond value was more sensitive to the level of interest rates than was the value of GNMA security. This is to be expected in view of both the call feature and the amortization of the GNMA: a similar effect is apparent in Tables 2 and 3 if the level of interest rates is measured by the long rate, l. However, quite the opposite relation holds for the short rate r: the Treasury Bond is less sensitive to this variable than the GNMA. The reason for this is the shorter average term of the GNMA which causes it to behave more like a short-term security. However, the values of both the GNMA and the bond are much less sensitive to r than to l.

It is of interest also to compare the size of the bond-GNMA price differentials predicted by the two models under similar yield curve scenarios. Dunn and McConnell find that when both r and the 30-

year yield are 8% the differential is $2.87 per $100 of par value. Our model predicts a differential of $4.60 for r = l = 8%. As was discussed in the previous section, the reason for this is the much greater uncertainty about the future course of interest rates that is inherent in the two-state variable model: the value of the call option increases with uncertainty. It is worth noting that in the simulations of Dunn and McConnell the 30-year rate ranges only from 7.5% to 8.9% as the short rate varies from zero to 21%.

Tables 4 and 5 compare the promised yield to maturity on the bond and GNMA security. Not too much significance can be attached to these numbers since with the possibility of early call there is no simple relation between promised and expected yields. Nevertheless, it is of interest to note that under the optimal call policy the promised yield on the GNMA always exceeds that on the Treasury Bond even though, as we have seen its price may sometimes be higher than that of the bond: the range of the promised yield differential is 0.3% to 1.8% depending upon the yield curve scenario. However with the suboptimal call policies the promised yield on the GNMA, which does not take account of calls, may be above or below that of the bond: it is most likely to be below when the long rate is high and the securities

TABLE 4

Promised Yields on 8% 30-Year Treasury Bond and GNMA Securities

		Treasury Bond	GNMA Security		
			Optimal Call Policy	FHA Call Experience (1)	(2)
r = 6%	ℓ = 6%	6.3%	8.0%	8.0%	6.6%
	8%	8.3	8.8	8.3	8.0
	10%	10.3	10.5	9.5	9.5
	12%	12.2	12.4	11.0	10.9
r = 8%	ℓ = 6%	6.3	8.0	8.0	6.6
	8%	8.3	8.9	8.4	8.1
	10%	10.3	10.5	9.6	9.5
	12%	12.2	12.4	11.0	11.0
r = 10%	ℓ = 6%	6.3	8.1	8.1	6.7
	8%	8.3	9.0	8.5	8.2
	10%	10.3	10.6	9.7	9.6
	12%	12.2	12.5	11.0	11.0
r = 12%	ℓ = 6%	6.4	8.2	8.2	6.8
	8%	8.4	9.0	8.7	8.3
	10%	10.3	10.7	9.8	9.7
	12%	12.2	12.5	11.2	11.1

(1) Optimal Call Policy with Autonomous Call Rate
(2) Autonomous Calls only

TABLE 5

Promised Yield Differentials between 8%, 30-Year Treasury Bonds and GNMA Securities

(Numbers in parentheses indicate that GNMA yield is less than Treasury Bond Yield)

		GNMA Security		
		Optimal Call Policy	FHA Call Experience (1)	(2)
r = 6% ℓ =	6%	1.7%	1.7%	0.3%
	8%	0.5	0.0	(0.3)
	10%	0.2	(0.8)	(0.8)
	12%	0.2	(1.2)	(1.3)
r = 8% ℓ =	6%	1.7	1.7	0.3
	8%	0.6	0.1	(0.2)
	10%	0.2	(0.7)	(0.8)
	12%	0.2	(1.2)	(1.2)
r = 10% ℓ =	6%	1.8	1.8	0.4
	8%	0.7	0.2	(0.1)
	10%	0.3	(0.6)	(0.7)
	12%	0.3	(1.2)	(1.2)
r = 12% ℓ =	6%	1.8	1.8	0.4
	8%	0.6	0.3	(0.1)
	10%	0.4	(0.5)	(0.6)
	12%	0.3	(1.0)	(1.1)

(1) Optimal Call Policy with Autonomous Calls
(2) Autonomous Calls only

are selling below par because of the shorter average term of the GNMA. Finally, we note that the promised yield on the GNMA declines as autonomous calls are introduced and as optimal calls are eliminated.

Returning to Tables 2 and 3, we find that the GNMA security value is quite sensitive to the assumption made about call policy; for example when r = l = 8% the equilibrium value of the GNMA is less than that of the bond by $4.60 under the optimal call policy; under the optimal call policy with autonomous redemptions the difference falls to $0.90 with autonomous calls only, the GNMA value actually exceeds the bond value by $2.40. Thus the assumption about call policy may change the value of the GNMA by as much as $7.00 per $100 of face value.

It seems that if further progress is to be made in devising useful predictive models of GNMA prices not only will it be necessary to [illegible] the simple single-state variable model in favour of two-state

variable or even more complex models, it will also be necessary to model more precisely the function $\pi(r,l,t)$ predicting mortgage calls. We will conclude therefore with a suggestion as to how this problem may be approached.[8]

An important advance in estimation of an empirical call frequency function that recognizes the effect of interest rates on calls has been made in a paper by Green and Shoven [14]. These authors posit that the probability of a call depends both upon the age of the mortgage and upon the exercise value of the call option (normalized by the original face value of the mortgage). They employ a proportional hazards model that presupposes that the probability of call can be factored into two functions, one of which depends only on the age of the mortgage, and the other which depends only on the exercise value of the call. In our notation the call frequency function is written as

$$\pi(r,l,t) = \lambda(t)e^{\beta X(r,l,t)} \qquad (40)$$

where $X(t) = [B(r,l,t) - F(t)]/F(T)$ and $B(r,l,t)$ is the present value of the remaining payments on the mortgage assuming no prepayments. $X(\cdot)$ is the (normalized) exercise value of the call option. Green and Shoven estimate the parameters, $\lambda(t)$, β, from a large sample of Californian mortgages. $B(r,l,t)$ is approximated by the value of the remaining mortgage payments discounted at the current (30-year) mortgage rate.

It is the expected call frequency that is relevant in valuing a GNMA security and insofar as the parameters of the empirical call frequency function can be taken as stationary the GNMA may be more accurately valued by substituting expression (4) for π in the partial differential equation (21). Clearly, the next step is to compare the price predictions of such an empirically based model with observed GNMA prices.

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[8] Dunn and McConnell [12] allow the prepayment probability rate to depend upon the level of interest rates.

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