GRADE SELECTION AND BLENDING TO OPTIMIZE COST AND QUALITY

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In many chemical process applications, a large mix of products is produced by blending them from a much smaller set of basic grades. The basic grades themselves are typically produced on the same process equipment and inventoried in batches. Decisions that arise in this process include selecting the set of basic grades, determining how much of each basic grade to produce, and how to blend basic grades to meet final product demand. We model this problem as a nonlinear mixed-integer program, which minimizes total grade inclusion, batching, blending, and quality costs subject to meeting quality and demand constraints for these products. Heuristics and lower bounds are developed and tested. The methods are applied to data from Europe's leading manufacturer of wheat- and starch-based products. Our results suggest that this model could potentially reduce annual costs by a minimum of 7%, translates to annual savings of around $5 million.

1. INTRODUCTION

In many chemical process environments, a very large set of final products may be made available to customers where the products differ primarily in the mix of one or more ingredients. At the basic commodity level, there may be just one key ingredient, and the products vary in the concentration of this ingredient. The products are themselves often used by customers as feedstock or inputs to other processes or are blended into other formulations. Examples include products such as asphalt, coal, fertilizers, animal feed, paint, petrochemicals, edible oils, and sugars (such as dextrose, fructose, or glucose).

Although the number of products sold may be quite large (in the hundreds in the case of glucose solutions, for example), the products can be made by blending them from a small set of "basic grades." There are obvious cost advantages to producing and storing only a small set of grades, especially since the production processes utilized are often semi-continuous in nature, involving very significant costs related to batching and storage, as well as quality control and yield management. In this paper we consider the problem of selecting basic grades to be manufactured so as to produce a given set of products while minimizing total grade inclusion, batching, blending, and quality costs. "Quality" here refers to conformance to product specifications. Quality is affected by the choice of grades used to blend a product, and hence by the set of available grades. The costs of quality are the expected costs of correcting nonconforming product. Costs of grade inclusion vary by grade, depending on the concentration of ingredients. Batching costs depend on the volume to be produced, since there are economies of scale in batch production; blending costs depend on the grade to be used in forming the final product. We consider the class of problems where there is just one ingredient that varies in concentration across the grades and products. In principle, the model is capable of extension to the case of multiple ingredients and attributes. We also assume that the products can be made from grades by blending them in the right proportions.

This problem is related to the traditional blending problem, in which, given a set of basic grades, the objective is to mix these grades and form products to minimize blending costs subject to meeting the quality and demand requirements of the products. This problem formulated as the nut mix problem (Charnes et al. 1953), and the sausage-blending problem (Steuer 1986) was one of the earliest, simplest and most widely understood applications of linear programming. This basic model has been refined and applied across a variety of industries. Notable applications can be found in the petrochemical, agricultural, fertilizer, coal, and asphalt industries.

Application in the petrochemical industry centers on the gasoline-blending problem. Rigby et al. (1995) discuss successful implementation of such models at Texaco. Glen (1988) considers the blending problem in the agriculture industry, in which crop nutrient requirements are met by blending a pre-existing set of mixtures. Ashayeri et al. (1994) address the formulation of the blending problem at a chemical fertilizer plant in which fertilizers are produced by blending various types of raw materials. Candler (1991) addresses the coal-blending problem, in which different grades of coal are mixed to minimize blending costs and the probability of rejection by a customer. Finally, Martin and Lubin (1985) address the blending problem in the asphalt-processing industry in which a particular product of asphalt is produced by blending a combination of a few basic "flux" grades of asphalt.

The problem considered here differs from the problems addressed by those papers in several respects. First,

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our formulation is more general, as it considers the joint production-blending problem, which, in addition to determining which basic grades to blend to form a product (the blending problem), also determines how to produce these basic grades (the production problem). This choice is based on minimizing grade inclusion, batching, blending, and quality costs subject to quality and demand constraints on the products. Thus, it directly links the production problem with the blending problem. This connection is of practical importance. Second, it explicitly develops a quality model that calculates quality costs as a function of the blend used. Third, much of the literature to date addresses problems that are either of small size or of a structure simple enough to be solved using commercially available math programming software. In contrast, the problem we consider here is more complex. In our computational experience, we observe that powerful commercial software tools cannot generate feasible solutions to even small problems. Consequently, we develop bounds and heuristics designed to solve large-sized practical problems to near optimality. Finally, to the best of our knowledge, this is the first investigation of the applicability of these methods for the glucose-processing industry. We validate this model using data from a large European company.

This paper is organized as follows: In the next section we formulate the problem of grade selection, grade production, and assignment of grades to products as a nonlinear mixed-integer program. In §3, expressions for product conformance are derived under certain realistic assumptions. Subsequently, we derive some basic properties of the model. A problem decomposition and lower bounds are developed in §4. In §5, upper bounds and heuristics are developed. We report computational results in §6. In §7, we describe an application of this method to data from the food-processing industry. In the concluding section, we summarize our work and suggest future research directions.

2. MODEL FORMULATION

Consider a manufacturing facility producing n products and let $j \epsilon J = (1, \ldots, n)$ index the set of products. These products are made by producing m basic grades and blending one or more of these grades indexed by $i \epsilon I = (1, \ldots, m)$ to achieve a prespecified level a_j of a key ingredient for the jth product. In all other attributes, these products are identical. For instance, in sugar solutions, this ingredient is typically the sugar level measured by the dextrose equivalence (DE) scale. To choose the basic grades and their blending quantities, define the variables

$$y_i = \begin{cases} 1, & \text{if grade } i \text{ is chosen as a basic grade,} \\ 0, & \text{otherwise,} \end{cases}$$

d_{ij} = Quantity of basic grade i used in the blending of product j.

We are given:

K_i = Fixed cost for including product i as a basic grade (\$),
S_i = Setup cost for producing a batch of basic grade i (\$),
h_i = Holding cost per unit of basic grade i (\$/unit time),
C_{ij} = Cost for blending (producing and mixing) basic grade i to form product j (\$/unit),
D_j = Annual demand for product j (units).

In this model, "quality" refers to conformance to product specifications. Quality is affected by the choice of basic grades used to blend the product, and hence, by the set of available grades. The costs of quality are the expected costs of correcting nonconforming product. Define vector $d_j = (d_{1j}, d_{2j}, \ldots, d_{mj})$ and $\gamma_j(d_j)$ a scalar function of vector d_j representing these costs for product j when it is blended using the basic grades and quantities represented by d_j. This function is derived in the next section, under certain assumptions. The Basic Grade Selection Problem (BGSP) can be represented by the following nonlinear mixed integer program:

$$\text{(BGSP)} \quad \min Z = \sum_i K_i y_i + \sum_i \sqrt{2S_i h_i \sum_j d_{ij}} + \sum_i \sum_j C_{ij} d_{ij} + \sum_j \gamma_j(d_j)$$

Subject to

$$d_{ij} \leqslant D_j y_i \qquad \forall i, j \tag{1}$$

$$\sum_i a_i d_{ij} = a_j \sum_i d_{ij} \qquad \forall j \tag{2}$$

$$\sum_i d_{ij} = D_j \qquad \forall j \tag{3}$$

$$d_{ij} \geqslant 0 \qquad \forall i, j \tag{4a}$$

$$y_i \in \{0, 1\} \qquad \forall i. \tag{4b}$$

The objective function Z consists of the fixed cost of including grades for blending, the batching costs (consisting of holding and setup costs) associated with producing these grades, the costs of blending (producing and mixing) basic grades to form products, and the costs of quality for the products when they are blended using basic grades. In deriving the batching costs, we assume that grades are produced and inventoried in lot sizes derived from their basic economic order quantity. Consequently, given a set of blending choices $\{d_{ij}\}$, we would produce $Q_i = \sqrt{2S_i \sum_j d_{ij}/h_i}$ units of grade i, resulting in holding and setup costs of $\sqrt{2S_i h_i \sum_j d_{ij}}$.

Constraint (1) ensures that products can be blended only from chosen basic grades. Constraint (2) enforces that the blending procedure for each product results in its targeted ingredient level. Constraint (3) ensures that demand for all products are met, while nonnegativity and 0-1 integrality of decision variables are imposed by Constraints (4a) and (4b).

PROPOSITION 1. *BGSP is NP-hard.*

PROOF. The uncapacitated plant location problem can be derived as a special instance of the BGSP by setting the

holding and setup costs for basic grade production, the coefficients $\{a_i\}$, and the conformance costs of quality to zero. Since it is known that the plant location problem is NP-hard, the reduction establishes that BGSP is NP-hard. □

3. ESTIMATION OF QUALITY COSTS

To estimate $\gamma_j(d_j)$, the expected costs of correcting nonconforming product when d_{ij} units of basic grade i is used to blend product j, we introduce the following variables:

$$x_{ij} = \begin{cases} 1, & \text{if basic grade } i \text{ is used to blend product } j \\ 0, & \text{otherwise,} \end{cases}$$

$x_j = (x_{1j}, x_{2j}, \ldots, x_{mj})$: The vector of basic grades used to make product j,

$d_j = (d_{1j}, d_{2j}, \ldots, d_{mj})$: The vector of basic grade quantities used to make product j.

$P_j(x_j)$ = Probability of non-conformance of product j as a function of x_j.

We are given:

q_j = The blending batch size of product j.

R_j = The rework cost per batch of product j.

We calculate $\gamma_j(d_j)$, the total expected costs of nonconformance associated with the jth product, as $\gamma_j(d_j) = R_j P_j(x_j) D_j / q_j$. To calculate $P_j(x_j)$, we assume that the customer-specified product quality tolerances are specified as a fraction ψ_j of attribute level a_j. Thus, the upper specification limit (USL) is $a_j(1+\psi_j)$, and the lower specification limit (LSL) is $a_j(1-\psi_j)$. Typically, during blending, a fixed volume q_{ij} of grade i is mixed with other grades to form product j. Blending errors occur due to variation in composition or volumes of the grades constituting the blend. In this problem we assume that the composition of the basic grade i is controllable and measurable. Thus, it can be fixed at a_i without composition errors. In our application, we consider continuous flow processes in which grades are fluids mixed to form a product. Errors in mixing volumes can occur due to startup and shutdown times of the mixing valves and, consequently, are additive and independent of the volume of the blend. However, it is conceivable that in product-blending applications involving discrete components (such as scrap steel mixing), such errors can be multiplicative and depend on the volume of the blend. We will consider this case in future work. We assume that blending errors ϵ_{ij} are normally distributed with mean 0 and standard deviation σ_{ij}. Let $\Phi(.)$ denote the cumulative distribution function of the standard normal variate.

To estimate the value of $P_j(x_j)$, define $\tilde{q}_{ij} = q_{ij} + \epsilon_{ij}$ as the variable representing the volume distribution of basic grade i used to blend product j. We assume that by using appropriate topping off mechanisms, the total volume of the product is made exactly equal to q_j so that and $q_j = \sum_i \tilde{q}_{ij}$. However, as

$$a_j = \frac{\sum_i a_i x_{ij} q_{ij}}{\sum_i q_{ij}} = \frac{\sum_i a_i x_{ij} q_{ij}}{q_j},$$

even a small deviation in the basic grade volume used in the blend can induce significant variation in the composition of the product. The resulting composition distribution of product j can be expressed as

$$\tilde{a}_j = \frac{\sum_i a_i x_{ij} q_{ij}}{q_j} + \frac{\sum_i a_i x_{ij} \epsilon_{ij}}{q_j}.$$

Note that our assumption of normal errors implies that $\tilde{a}_j$ is normally distributed with mean a_j and standard deviation $\sqrt{\sum_i a_i^2 x_{ij} \sigma_{ij}^2}/q_j$. By definition, $P_j(x_j) = P[\tilde{a}_j \geqslant \text{USL}] + P[\tilde{a}_j \leqslant \text{LSL}] = P[\tilde{a}_j \geqslant a_j(1+\psi_j)] + P[\tilde{a}_j \leqslant a_j(1-\psi_j)]$. Thus

$$P_j = 1 - \Phi\left(\frac{a_j \psi_j q_j}{\sqrt{\sum_i a_i^2 x_{ij} \sigma_{ij}^2}}\right) + \Phi\left(\frac{-a_j \psi_j q_j}{\sqrt{\sum_i a_i^2 x_{ij} \sigma_{ij}^2}}\right)$$
$$= 2\Phi\left(\frac{-a_j \psi_j q_j}{\sqrt{\sum_i a_i^2 x_{ij} \sigma_{ij}^2}}\right).$$

We now establish results that simplify the problem (BGSP).

PROPOSITION 2. *Let* $x_{ij} = \lceil \frac{d_{ij}}{D_j} \rceil$, *where* $0 \leqslant d_{ij} \leqslant D_j$. *Then,*

$$P_j(d_j) = 2\Phi\left(\frac{-\psi_j a_j q_j}{\sum_i a_i^2 x_{ij} \sigma_{ij}^2}\right)$$

is a concave function of vector $d_j = (d_{1j}, d_{2j}, \ldots, d_{mj})$.

PROOF. Let $d_j^{(1)} = (d_{1j}^{(1)}, d_{2j}^{(1)}, \ldots, d_{mj}^{(1)})$ and $d_j^{(2)} = (d_{1j}^{(2)}, d_{2j}^{(2)}, \ldots, d_{mj}^{(2)})$ represent two vectors of basic grade quantities used to make the jth product. Define $d_j^{(3)} = \lambda d_j^{(1)} + (1-\lambda) d_j^{(2)}$ for $0 \leqslant \lambda \leqslant 1$. To establish $P(d_j)$ is concave in d_j, we need to show that $\lambda P_j(d_j^{(1)}) + (1-\lambda) P_j(d_j^{(2)}) \leqslant P_j(d_j^{(3)})$.

Let $x_j^{(1)} = (x_{1j}^{(1)}, x_{2j}^{(1)}, \ldots, x_{mj}^{(1)})$ represent the vector of basic grades used to make product j corresponding to d_j^1. Note that the ith component of $x_j^{(1)}$ represented by $x_{ij}^{(1)} = 1$ iff $0 < d_{ij}^{(1)} \leqslant D_j$ and $x_{ij}^{(1)} = 0$ otherwise. Similarly, let $x_j^{(2)} = (x_{1j}^{(2)}, x_{2j}^{(2)}, \ldots, x_{mj}^{(2)})$ represent the vector of basic grades used to make product j corresponding to $d_j^{(2)}$. Here again the ith component of $x_j^{(2)}$ represented by $x_{ij}^{(2)} = 1$ iff $0 < d_{ij}^{(2)} \leqslant D_j$ and $x_{ij}^{(2)} = 0$ otherwise. Now consider $x_j^{(3)} = (x_{1j}^{(3)}, x_{2j}^{(3)}, \ldots, x_{mj}^{(3)})$, the vector of basic grades used to make product j corresponding to $d_j^{(3)}$. It is important to recognize that here the ith component of $x_j^{(3)}$ represented by $x_{ij}^{(3)} = 1$ iff $0 < d_{ij}^{(1)} \leqslant D_j$ *or* $0 < d_{ij}^{(2)} \leqslant D_j$ and $x_{ij}^{(3)} = 0$ iff $d_{ij}^{(1)} = 0$ *and* $d_{ij}^{(2)} = 0$. Thus, $\sum_i x_{ij}^{(3)} \geqslant \sum_i x_{ij}^{(1)}$ and $\sum_i x_{ij}^{(3)} \geqslant \sum_i x_{ij}^{(2)}$. Since $a_j, \psi_j, q_j, a_i, \sigma_{ij} \geqslant 0\ \forall i, j$, $\frac{-\psi_j a_j q_j}{\sum_i a_i x_{ij}^{(3)} \sigma_{ij}} \geqslant \frac{-\psi_j a_j q_j}{\sum_i a_i x_{ij}^{(1)} \sigma_{ij}}$ and $\frac{-\psi_j a_j q_j}{\sum_i a_i x_{ij}^{(3)} \sigma_{ij}} \geqslant \frac{-\psi_j a_j q_j}{\sum_i a_i x_{ij}^{(2)} \sigma_{ij}}$. As the cumulative distribution function of the standard normal variate $\Phi(t)$ is nondecreasing for any real t,

$$P(d_j^{(3)}) = 2\Phi\left(\frac{-\psi_j a_j q_j}{\sum_i a_i x_{ij}^{(3)} \sigma_{ij}}\right) \geqslant 2\Phi\left(\frac{-\psi_j a_j q_j}{\sum_i a_i x_{ij}^{(1)} \sigma_{ij}}\right) = P(d_j^{(1)})$$

and

$$P(d_j^{(3)}) = 2\Phi\left(\frac{-\psi_j a_j q_j}{\sum_i a_i x_{ij}^{(3)} \sigma_{ij}}\right) \geqslant 2\Phi\left(\frac{-\psi_j a_j q_j}{\sum_i a_i x_{ij}^{(2)} \sigma_{ij}}\right) = P(d_j^{(2)}).$$

Thus, $\lambda P_j(d_j^{(1)}) + (1-\lambda) P_j(d_j^{(2)}) \leqslant P_j(d_j^{(3)})$, implying that $P(d_j)$ is concave in d_j. □

PROPOSITION 3. *In the optimal solution to (BGSP), for any j, at most two of the $\{d_{ij}\}$ will be positive.*

PROOF. Note that (BGSP) can be written as

$$\text{Min} \sum_i \Psi_i\left(\sum_j d_{ij}\right) + \sum_i \sqrt{2S_i h_i \sum_j d_{ij}} + \sum_i \sum_j C_{ij} d_{ij} + \sum_j \gamma_j(d_j)$$

Subject to

$$\sum_i a_i d_{ij} = a_j \sum_i d_{ij} \quad \forall j \tag{2}$$

$$\sum_i d_{ij} = D_j \quad \forall j \tag{3}$$

$$d_{ij} \geqslant 0 \quad \forall i, j, \tag{4a}$$

where

$$\Psi_i\left(\sum_j d_{ij}\right) = \begin{cases} K_i & \text{if } \sum_j d_{ij} > 0 \\ 0 & \text{otherwise} \end{cases}$$

$\gamma_j(d_j) = R_j P_j(x_j) D_j / q_j$, and $P_j(x_j) = 2\Phi(\frac{-a_j \psi_j q_j}{\sqrt{\sum_i a_j^2 x_{ij} \sigma_{ij}^2}})$, where $x_{ij} = \lceil \frac{d_{ij}}{D_j} \rceil$.

By the previous proposition, the objective function of this problem is concave, and the optimal solution must be attained at an extreme point of the feasible region. Any extreme point can have at most two d_{ij} positive, as for a given j, we have exactly two constraints for this problem. Hence, the proposition follows. □

We use Proposition 3 to reformulate BGSP. First define γ_j as

$$\gamma_j = \frac{1}{2} \sum_i \sum_s Q_{isj} W_{isj}$$

where:

$$Q_{isj} = P_{isj} R_j D_j / q_j$$

$$P_{isj} = 2\Phi\left(\frac{-\Psi_j a_j q_j}{\sqrt{a_i^2 \sigma_{ij}^2 + a_s^2 \sigma_{sj}^2}}\right); s \neq i, \forall i, s$$

$$P_{isj} = 0, \qquad s = i, \forall i, s$$

and

$$W_{isj} \geqslant x_{ij} + x_{sj} - 1, W_{isj} \geqslant 0; s \neq i, \ \forall i, s \tag{5}$$

$$d_{ij} \leqslant D_j x_{ij} \quad \forall i, j \tag{6}$$

$$\sum_i x_{ij} \leqslant 2 \quad \forall j \tag{7}$$

$$x_{ij} \in \{0, 1\}. \tag{8}$$

Constraint (5) ensures that indicator variable W_{isj} assumes the value 1 if basic grades i and s are used to produce product j or 0 otherwise. Constraint (6) implies that products are blended only from assigned basic grades. Proposition 3 is enforced by Constraint (7). The 0-1 integrality of x_{ij} is imposed by Constraint (8). Substituting these costs in (BGSP), we get

$$\text{(BGSP)} \quad Z = \text{Min} \sum_i K_i y_i + \sum_i \sqrt{2S_i h_i \sum_j d_{ij}} + \sum_i \sum_j C_{ij} d_{ij} + \frac{1}{2} \sum_j \sum_i \sum_s Q_{isj} W_{isj}$$

Subject to: (1) to (8).

Since the BGSP is NP-hard, it is unlikely that we can solve large real problems to optimality. We confirm this in our numerical experiments. Consequently, it is crucial that we develop robust heuristics and efficient lower bounds. Proposition 3 is particularly important in computing lower bounds for this problem, a question we consider next.

4. PROBLEM DECOMPOSITION AND LOWER BOUNDS

If basic grades for blending are chosen *a priori*, Proposition 3 reduces the complexity in choosing the best blending combination for any product. To recognize this fact, consider the jth product with ingredient level a_j. Let k grades have attributes greater than this level. Proposition 3 reduces the potential blending combinations for this product from $(\sum_{t=1}^{n-k-1} (\frac{n-k}{t} - 1))(\sum_{t=1}^{k} \binom{k}{t})$ to $(n-k-1)k$. However, to apply this proposition, this problem needs to be decomposable by product. While the constraints in this problem can be decomposed by product, the objective function is not separable in j due to the term $\sum_i \sqrt{2S_i h_i \sum_j d_{ij}}$. This term can be linearized using standard techniques (e.g., Bradley et al. 1977, pp. 602–608). However, such techniques increase the computational complexity of this problem, and consequently are unsuitable for application to even small problems.

To address this problem, we replace $\sum_i \sqrt{2S_i h_i \sum_j d_{ij}}$ by $\sum_i \sqrt{2S_i h_i B_i}$ and introduce the following constraints:

$$B_i \geqslant \sum_j d_{ij} \quad \forall i, \tag{9}$$

$$B_i \leqslant y_i \sum_j D_j \quad \forall i, \tag{10}$$

$$\sum_i B_i = \sum_j D_j = D. \tag{11}$$

Note that (3), (4a), (4b), (9), and (10) imply (1). We next introduce $\mu \geqslant 0$, a vector of Lagrange multipliers associated with Constraints (9). Relaxing Constraints (9) decomposes the problem into the following subproblems:

$$\text{(OP)} \quad V(\mu) = \underset{B_i, y_i}{\text{Min}} \sum_i \left\{K_i y_i + \sqrt{2S_i h_i B_i} - \mu_i B_i\right\}$$

Subject to: (4b), (10), (11), and

$$B_i \geqslant 0 \quad \forall i \tag{12}$$

$$\text{(IP)} \quad W(\mu) = \underset{d_{ij},\, w_{isj}}{\text{Min}} \sum_i \sum_j (\mu_i + C_{ij}) d_{ij} + 0.5 \sum_j \sum_i \sum_{s \neq i} Q_{isj} W_{isj}$$

Subject to: (2) to (4a), (5) to (8)

$$\text{(RBGSP)} \quad Z(\mu) = V(\mu) + W(\mu).$$

The Lagrangean dual corresponding to (RBGSP) is given by

$$\text{(DBGSP)} \quad Z = \underset{\mu \geqslant 0}{\text{Max}}\, V(\mu) + W(\mu).$$

The Relaxed Basic Grade Selection Problem (RBGSP) consists of an outer problem (OP) and an inner problem (IP). The outer problem can be thought of as determining the selection and production quantities for basic grades. The inner problem then assigns grades to products while considering blending and quality costs. Of course, the actual solution values for the decision variables may be of little significance. The dual variables μ may be interpreted as the variable costs of using basic grades for blending in the inner problem, which becomes a "credit" for the outer problem.

A variety of standard methods could be used to solve the dual problem (DGBSP). However, we show that the optimal dual vector μ^* can be determined *a priori*, and the lower bound can be computed by solving the inner problem (IP) just once. We first note some properties of the problems (IP) and (OP). The solution to (OP) is defined by Proposition 4.

PROPOSITION 4. *$B_{i^*} = D$ for i^*, and $B_k = 0$ for all $k \neq i^*$, where $i^* = \text{argmin}_{i=1,2,\ldots,n}\{P_i D - \mu_i D\}$, where $P_i = \frac{K_i + \sqrt{2S_i h_i D}}{D}$.*

PROOF. Problem (OP) can be rewritten as

$$\text{Min} \sum_i \left\{ K_i \delta(B_i) + \sqrt{2S_i h_i B_i} - \mu_i B_i \right\}$$

$$\text{s.t.} \sum_i B_i = D$$

$$B_i \geqslant 0, \quad \forall i$$

where $\delta(x)$ is the Kronecker delta function ($\delta(x) = 0$ if $x = 0$, $\delta(x) = 1$ if $x > 0$). The feasible region has the extreme points given by: $\{B_i = D$ for i, and $B_k = 0$ for all $k \neq i$, for each $i = 1, 2, \ldots, n\}$. Since (OP) involves minimization of a concave function, the result follows. □

PROPOSITION 5. *The value $W(\mu)$ of the inner problem is nondecreasing in each μ_i.*

PROOF. Follows directly by inspection of $W(\mu)$. □

PROPOSITION 6. *The solution μ^* to (DBGSP) must be of the form $P_i D - \mu_i^* D = P_k D - \mu_k^* D$ for all i, k.*

PROOF. Suppose not, then

$$\exists i, k \ni P_i D - \mu_i^* D < P_k D - \mu_k^* D \tag{a}$$

This implies that $B_k = 0$. But then, μ_k^* could be increased until equality holds in (a), since the value $V(\mu)$ of (OP) would be unchanged, while the value $W(\mu)$ of (IP) would be nondecreasing. □

PROPOSITION 7. *The optimal solution to (DBGSP) is $\mu_i^* = P_i$, and the optimal lower bound is given by $LB^* = W(P)$*

PROOF. The preceding proposition implies that the components of μ^* are of the form $\mu_i^* = P_i + m^*$ for some scalar m^*, which does not depend on i. By substituting μ_i^* in the problems (IP) and (OP), we see that $V(\mu^*) = -m^* D$, $W(\mu^*) = W(P) + m^* D$ and therefore $LB^* = V(\mu^*) + W(\mu^*) = W(P)$. □

An Alternative Lower Bound

Numerical tests show the lower bound LB to be poor under certain circumstances. We now develop an alternative scheme based on the cost approximation that appears in the previous bound. Note that the expression for μ_i is essentially the average cost per unit for producing D units of grade i. However, for many grades, there will never be a reason to produce the quantity D. In addition, it can be observed from Propositions 4 and 7 that this choice affects the quality of the bound.

In deriving the alternate lower bound, we replace (10) by a tighter constraint (13) defined as $B_i \leqslant \overline{B}_i y_i \ \forall i$, where $\overline{B}_i$ represents the maximum quantity of grade i that can conceivably be used (we compute $\overline{B}_i$ by the procedure described in the appendix). We next define problem ($\overline{\text{OP}}$) as

$$(\overline{\text{OP}}) \quad \overline{V}(\mu) = \sum_i \overline{V}_i(\mu_i),$$

where

$$\overline{V}_i(\mu) = \underset{y_i}{\text{Min}}\, K_i y_i + \sqrt{2S_i h_i B_i} - \mu_i B_i$$

s.t. (4b), (12)

$$B_i \leqslant \overline{B}_i y_i. \tag{13}$$

Note that Constraint (11) is dropped from this formulation. The lower bound based on this subproblem is given by

$$\overline{LB}(\mu) = \overline{V}(\mu) + W(\mu),$$

$$\overline{LB}^* = \underset{\mu \geqslant 0}{\text{Max}}\, \overline{LB}(\mu).$$

It cannot immediately be said that this formulation will necessarily result in a higher bound than the previous one, since the feasible set for this new subproblem is not contained in the feasible set of the previous version (i.e., the bounds are not comparable). However, we show that the new formulation does in fact result in a tighter bound.

PROPOSITION 8. $\overline{V}(\mu) \leqslant 0$.

PROOF. The solution B_i^* to subproblem $(\overline{\mathrm{OP}})$ is reached at an extreme point, since the objective function is concave. It is given by

$$B_i^* = \begin{cases} \overline{B}_i, & \text{if } K_i + \sqrt{2S_i h_i \overline{B}_i} - \mu_i \overline{B}_i \leqslant 0, \\ 0 & \text{otherwise.} \end{cases}$$

Hence $\overline{V}_i(\mu_i) \leqslant 0$ and the result follows. □

PROPOSITION 9. *Define vector* $\bar{\mu}^*$ *by* $\bar{\mu}_i^* = \frac{K_i + \sqrt{2S_i h_i \overline{B}_i}}{\overline{B}_i}, \forall i$. *Then* $\overline{LB}^* = W(\bar{\mu}^*)$.

PROOF. For $\mu_i \geqslant \bar{\mu}_i^*$, by Proposition 8, $\overline{B}_i^* = \overline{B}_i$, $\overline{V}_i(\mu_i) = K_i + \sqrt{2S_i h_i \overline{B}_i} - \mu_i \overline{B}_i \leqslant 0$, and $\frac{\partial \overline{V}(\mu)}{\partial \mu_i} = -B_i$. For $\mu_i \leqslant \bar{\mu}_i^*$, $B_i^* = 0$, $\overline{V}(\mu) = 0$, and $\frac{\partial \overline{V}(\mu)}{\partial \mu_i} = 0$. For the problem (IP), we note that the objective function is given by

$$\begin{aligned} W(\mu) &= \sum_i \sum_j (\mu_i + C_{ij}) d_{ij}^* + \frac{1}{2} \sum_j \sum_i \sum_{s \neq i} Q_{isj} W_{isj} \\ &= \sum_i \sum_j C_{ij} d_{ij}^* + \sum_i \sum_j \mu_i d_{ij}^* + \frac{1}{2} \sum_j \sum_i \sum_{s \neq i} Q_{isj} W_{isj} \\ &\leqslant \sum_i \sum_j C_{ij} d_{ij}^* + \sum_i \sum_j \mu_i \overline{B}_i + \frac{1}{2} \sum_j \sum_i \sum_{s \neq i} Q_{isj} W_{isj}. \end{aligned}$$

Thus,

$$W(\mu) \leqslant W(0) + \sum_i \mu_i \overline{B}_i,$$

and also

$$\frac{\partial W(\mu)}{\partial \mu_i} = \sum_j d_{ij}^* \leqslant \overline{B}_i.$$

Hence $0 \leqslant \frac{\partial(\overline{LB})}{\partial \mu_i} \leqslant B_i$ for $\mu_i \leqslant \bar{\mu}_i^*$, and $\frac{\partial(\overline{LB})}{\partial \mu_i} \leqslant 0$ for $\mu_i \geqslant \bar{\mu}_i^*$. This implies that $\overline{LB}(\mu)$ is maximized at $\mu_i = \bar{\mu}_i^*$. Applying the same reasoning used to prove Proposition 7, we see that $\overline{LB}^* = W(\bar{\mu}^*)$. □

PROPOSITION 10. $\overline{LB}^* \geqslant LB^*$.

PROOF. Follows directly from Propositions 7 and 9.

The inner problem (IP) is solved as follows:

Step 1. Consider the jth product and define sets $P_j = \{i \epsilon I : a_i < a_j\}$ and $Q_j = \{i \epsilon I : a_i > a_j\}$. A feasible blending pair is any two-tuple (p, q) such that $p \epsilon P_j$ and $q \epsilon Q_j$. Observe that we would have $|P|(n - |Q|)$ possible pairs. For each pair (p, q), set

$$d_{pj} = \left| \frac{a_q - a_j}{a_p - a_q} \right| D_j \quad \text{and} \quad d_{qj} = \left| \frac{a_p - a_j}{a_p - a_q} \right| D_j$$

and compute the associated costs $c_{pj} d_{pj} + c_{qj} d_{qj} + Q_{pqj}$. Let $Z_{(p^*, q^*)}$ represent the minimum of these costs achieved by the pair (p^*, q^*). If the ingredient level for this product does not coincide with the ingredient level of any basic grade, blending is required. The product is assigned to be blended from these grades in the calculated quantity. Else go to step 2.

Step 2. If the ingredient level for the jth product corresponds to the ingredient level of the ith basic grade and if we choose to produce this product directly, blending is not required. Consequently, the cost of nonconformance due to the blending process is 0. We set $d_{ij} = D_j$ and the cost of producing this product is $c_{ij} D_j$. However, if $c_{ij} D_j \geqslant Z_{(p^*, q^*)}$, we will not include this grade and produce this product from its best blend. This procedure is repeated across all products to solve the inner problem.

5. UPPER BOUNDS AND HEURISTIC SOLUTIONS

In general, the solution provided by the lower bound may not be feasible due to the violation of Constraint (1). To achieve this feasibility, we develop a Lagrangean heuristic. We also suggest alternative heuristics. These procedures provide upper bounds on the BGSP.

The Lagrangean Heuristic

In this heuristic, we use the solution provided by the Lagrangean dual (i.e., the lower bound) and induce feasibility. We consider the basic grades chosen by the outer problem, and include the basic grades representing the extremes in the ingredient level if necessary. Let F represent the index set of these grades. Consider the jth product and define sets $P_j' = \{i \epsilon F | a_i < a_j\}$ and $Q_j' = \{i \epsilon F | a_i > a_j\}$. Let $a_s \sigma_{sj} = \min\{a_i \sigma_{ij} | i \epsilon P'\}$ and $a_t \sigma_{tj} = \min\{a_i \sigma_{ij} | i \epsilon Q'\}$. Note that from the structure of $P_j(x_j)$, this choice would minimize the probability of nonconformance of product j. This structure also implies that the obvious choice of blending two "closest" grades may not be optimal even from a strictly quality perspective, as additive quantity errors imply that the probability of nonconformance is increasing in a_i. Thus, blending with the two "closest" grades will result in a higher probability of nonconformance than the case when we choose two grades such that one of them has the lowest possible a_i and the other the smallest possible a_i that is greater than the product attribute level j. Product j, if blended, would be produced by the feasible pair (s, t) in quantities

$$d_{sj} = \left| \frac{a_t - a_j}{a_s - a_t} \right| D_j \quad \text{and} \quad d_{tj} = \left| \frac{a_s - a_j}{a_s - a_t} \right| D_j.$$

The costs of this assignment is $Z_{(s,t)} = c_{sj} d_{tj} + c_{tj} d_{tj} + Q_{stj}$. If the ingredient level of this product does not coincide with the ingredient level of any basic grade, we blend this product using this combination. If the ingredient level coincides with the ith basic grade, we blend this product only if $c_{ij} D_j \geqslant Z_{(s,t)}$. Otherwise, the demand for this product is directly satisfied from this basic grade. This procedure is repeated across all products.

We also developed several other heuristics to choose basic grades. In all these procedures, to guarantee feasibility, we first select basic grades representing the extremes in ingredient levels. Additional grades are then selected by

the appropriate heuristic For any given set of basic grades, the inner problem is now solved by the method used to compute lower bounds restricted to this set. The total costs are recomputed with addition of each individual grade, and the process is stopped when the increase in costs of adding the grade is greater than the reduction in costs associated with this addition. Among the several heuristics we tested to choose basic grades, we found that the following approaches worked well in our computational experience and application:

1. Demand Heuristic. We include products as basic grades in decreasing order of total product demand D_j.

2. Setup/Demand Heuristic: Products are included as basic grades in increasing order of K_j/D_j.

A detailed description of the other heuristics tested, which included grade selection in increasing order of production setup costs, blending costs, and variance in the blending quantity errors, can be found in Karmarkar and Rajaram (1998). While these heuristics did not give the best results in our application data sets, they may be of interest in applications where a crucial parameter such as production setup costs, blending costs, or blending errors could be dominant.

6. COMPUTATIONAL RESULTS

We have tested all the schemes designed to calculate lower bounds and upper bounds (described in §4 and §5) for this problem on data provided by a manufacturer of glucose. This data set consisted of all the input parameters required for this problem for 200 kinds of glucose products produced at this plant. The parameters included grade inclusion, batching (setup and holding), blending (production and mixing) costs, product attribute levels, and annual demand, and finally, all the parameters required to calculate the costs of quality described by the model in §3.

Recall that the objective function Z of the basic grade selection problem consists of the following costs:

1. Total cost of inclusion and batching costs of basic grade
$$\left(\sum_i K_i y_i + \sum_i \sqrt{2S\,h_i \sum_j d_{ij}}\right).$$
2. Total cost of blending basic grades to form products $(\sum_i \sum_j C_{ij} d_{ij})$.
3. Total cost of quality due to improper blending $(\frac{1}{2} \sum_j \sum_i \sum_s Q_{isj} W_{ij})$.

To analyze the relative proportions of these costs, we defined the following measures:

$$R_1 = \frac{\sum_{j=1}^{200} \left(\frac{(K_j + \sqrt{2h_j S_j D_j}}{C_j D_j} \right)}{200};$$

$$R_2 = \frac{\sum_{j=1}^{200} \left(\frac{R_j}{\overline{C}_j} \right)}{200},$$

where $\overline{C}_j = \dfrac{\sum_{i=1}^{200} C_{ij}}{200}$.

Here, we first calculate the ratios of grade inclusion and batching cost to the total blending cost for each product. We then define R_1 as the average of these ratios across the 200 products. Next, we calculate the ratio of unit rework costs to the unit blending costs for each product and similarly define R_2 as the average of these ratios across these products. For the reference data set, $R_1 = 4.1$ and $R_2 = 2.9$.

We tested how sensitive our bounding techniques were to the scale of these cost parameters. To perform this analysis across the 200 products, we first scaled the grade inclusion and batching cost parameters by factors 1/3, 1/2, 2, and 3 (i.e., changing R_1 by these factors), then scaled the rework cost by the same factors (i.e., changing R_2 by the same factors), and finally scaled the blending costs by these factors. Our scaling factors were chosen by roughly estimating such costs across industries such as petrochemicals, food processing, coal production, and chemical fertilizers, based upon informal discussions with managers in these industries. This scaling in effect ensured that the cost proportions of these data were representative across this spectrum of industries. Note that our scaling procedure results in 64 (i.e., $4 \times 4 \times 4$) additional data sets of the 200-product problem.

While we could generate feasible solutions for this problem using the heuristics we proposed and assess the quality of these solutions using the lower bounds developed, it was important to compare these bounds with an alternate set of bounds derived using existing commercial software or any other method. This comparison is needed to better understand whether the gaps between the heuristics and bounds were due to the quality of the heuristics or the lower bounds we employed. We first developed a program to compute optimal solutions by exhaustive enumeration. It can be shown that the number of feasible solutions is of the order $2^n O(n^3)$. Consequently, this is not a viable method for large problems. We also elected to use GAMS (Brooke et al. 1992) to develop these alternate bounds. We tried to solve the reference (200 product) problem using the DICOPT, a nonlinear mixed-integer program solver in GAMS to solve the problem to optimality. We found that GAMS was unable to generate a feasible solution to this problem. To generate lower bounds, we linearized the square root function $\sum_i \sqrt{2S_i h_i \sum_j d_{ij}}$ using standard techniques (for example, see Bradley et al. 1977). To derive an upper bound for this problem, we approximated this function by a tangent at the point $\sum_j d_{ij} = (\sum_j D_j)/2$. We tried to solve the resulting mixed-integer programs using OSL, the recommended solver for this class of problems in GAMS. However, OSL was unable to generate feasible solutions for the problem.

Based on this experience, we used the reference data to extract smaller problems (with n products, $n < 200$). We arranged products in increasing order of attribute level a_i and always included products with the maximum and minimum attribute levels to ensure feasibility. To choose the remaining $n-2$ products, we first formed $n-2$ clusters, each with around $200/(n-2)$ products. A product with an

Table 1. Percentage gaps from tightest lower bound in the reference data.

Method	Problem Size $n = 5$	Problem Size $n = 7$	Problem Size $n = 10$	Problem Size $n = 15$	Problem Size $n = 100$	Problem Size $n = 200$
Optimal Solutions:						
Exhaustive Enumeration	*	*	–	–	–	–
Upper Bounds:						
1. GAMS	3.5	3.5	–	–	–	–
2. Demand heuristic	0.2	1.0	1.2	1.5	9.4	8.2
3. Setup/demand heuristic	0.2	1.0	1.2	1.5	8.3	4.2
Lower Bounds:						
1. GAMS	0.2	0.2	–	–	–	–
2. Lagrangean 1	35.4	28.6	24.7	18.4	5.1	3.0
3. Lagrangean 2	2.0	0.8	*	*	*	*

*: Value chosen to be the lower bound.
–: Technique was unable to find a feasible solution.

attribute level representing the midpoint for the products in a given cluster was then chosen to represent a given cluster. We used this procedure to extract data corresponding to "realistic" 5, 7, 10, 15, and 100 product problems.

Tables 1 and 2 summarize some salient results from our computational tests. In any table, a row represents the solution technique used. These included the exhaustive enumeration solution, the upper bounds generated by the GAMS, and the two heuristics and the lower bounds based on GAMS and our two techniques. Columns in the table represent the problem size (represented by the number of products) of the basic grade selection problem. The numbers in the body of the table describe the percentage gap of the technique from a reference point, if that technique was successful in generating a solution for the given problem. Since the five product problem was solved to optimality by exhaustive enumeration, this was used as the reference point for this case. However, this procedure and GAMS were unable to generate optimal solutions for problems with more than seven products. Consequently, for the remaining problems, the tightest lower bound (i.e., the maximum value of the lower bounds obtained) was used as this reference.

Our computational results have been quite encouraging. Consider Table 1, representing percentage gaps for the reference data. When we use the demand heuristic to solve the 200-product problem, the gap from the lower bound was 8.2%. The corresponding gap with the setup/demand

Table 2. Average percentage gaps from tightest lower bound across 65 data sets.

Method	Problem Size $n = 5$	Problem Size $n = 7$	Problem Size $n = 10$	Problem Size $n = 15$	Problem Size $n = 100$	Problem Size $n = 200$
Optimal Solutions:						
Exhaustive Enumeration	*	*	–	–	–	–
Upper Bounds:						
1. GAMS	3.3	3.5	–	–	–	–
2. Demand heuristic	0.7	1.1	1.8	1.9	9.8	8.8
3. Setup/demand heuristic	0.7	1.1	1.8	1.9	7.5	5.6
Lower Bounds:						
1. GAMS	0.1	0.1	–	–	–	–
2. Lagrangean 1	35.4	28.7	24.7	19.1	6.4	4.0
3. Lagrangean 2	2.0	0.7	*	*	*	*

*: Value chosen to be the lower bound.
–: Technique was unable to find a feasible solution.

heuristic was around 4%. Across all the problems these gaps varied from 0.2% to 15% for the demand heuristic and 0.2% to 12% for the setup demand heuristic. Detailed results for these cases can be found in Karmarkar and Rajaram (1998).

We also computed the average percentage gaps for all the computational techniques based upon the reference and 64 data sets for a problem with given number of products. These results, summarized in Table 2, suggest that average gaps for the demand heuristic ranged from 1.1% to 9.8%, while the corresponding gap for the setup/demand heuristic ranged from 0.7% to 7.5%. This suggests that these heuristics provide a strong basis to address this problem.

We wanted to better understand the circumstances under which percentage gaps increase. This could provide us with insights to improve the heuristic and lower bounds. We observed from our analysis of the 200-product problem that these gaps are uniformly higher when grade inclusion and batching or rework costs are higher, or while blending costs are lower than the reference case. Conversely, the gaps are significantly lower when grade inclusion and batching or rework costs are lower, or while blending costs are higher than the reference case. It is important to note that these gaps were reduced largely because the lower bounds became tighter. For instance, the average solution provided by the lower bounds increased by an average of 2%, while the average solution provided by the heuristics decreased only slightly, by around 0.5%. This indicates that the heuristics are fairly stable across a wide range of parameter variation, while there could still be potential to improve the lower bounds. This conjecture is supported when we consider the five-product problem in the tables and observed that, on an average, the solution provided by both these heuristics is only 0.7% above the exhaustive enumeration based optimal solution, while the lower bounds are around 2% lower than this optimal solution. In the final analysis, the real measure of performance of the heuristics is the quality of the decisions based on its solution, a question we consider in the application.

7. APPLICATION

We have applied the methods in this paper to production data from Europe's leading manufacturer of made-to-order wheat- and corn-based starch products, such as glucose, sorbitol, dextrose, and gluten. These products are used extensively as components in the food-processing industries (for example, by breweries, confectioneries, and bakeries), consumer product industries (for example, cosmetics and toothpaste), and other industries such as paper, pharmaceuticals, textiles, and specialty chemicals. Rather than produce small runs for each customer and incur extensive downtimes due to switchovers, a few types of basic grades are produced across a range of product attribute levels. Customer requirements are met by blending these basic grades, thus enabling long production campaigns, without compromising the ability to accurately meet customer demand.

Table 3. Percentage gaps of heuristics from tightest lower bound and cost savings when applied to refinery data.

Refinery site	Percentage gap using setup/demand Heuristic	Cost reduction (%)	Cost reduction (Million $)
1	2	8	0.7
2	1	9	0.8
3	1	7	0.4
4	3	8	0.5
5	4	7	0.6
6	2	8	1
7	1	10	1

We tested our model on data provided to us from refining processes, because refined products accounted for a large part of total profits. The data available to us included 1997 data on all the parameters needed to use our model from seven refineries located in five countries. Note that we used the largest data set (based on a refinery producing 200 products) as the reference problem in the computational analysis described in the previous section.

We solved the basic grade selection problem for the data from these seven refineries using all the heuristics proposed by Karmarkar and Rajaram (1998). We used the Lagrangean-based methods described in §4 to compute lower bounds. The lowest gaps were provided when we used the alternate lower bound technique, and the setup/demand heuristic. These results, summarized in Table 3, indicate that solutions provided by this heuristic were within 4% of the lower bounds.

We used the solution provided by the setup/demand heuristic at a given refinery and calculated the total grade inclusion, batching, blending, and rework costs that would have resulted, had the basic grades, production, and blending quantities suggested by our method been implemented. We compared our costs with actual annual costs at these refineries and found that our method would have reduced total costs by at least 7% in all these sites. Had this approach been implemented, the total annual cost savings at all these plants would have been around $5 million. Individual percentage and absolute cost savings for these refineries are also summarized in Table 3. We note that our solution had a fewer number of products as basic grades, and they were different from those currently in use at these processes.

8. SUMMARY AND FUTURE RESEARCH

In this paper, we consider the basic grade selection problem. Given a set of final products, we choose a smaller set of products known as basic grades and blend these grades to meet product demand. Decisions that arise in this process include selecting products to be basic grades, determining how much of each basic grade to produce, and how to blend basic grades to meet final product demand. We model this problem as a nonlinear mixed-integer program, in which these decisions are

made to minimize total grade inclusion, batching, blending, and quality costs subject to meeting quality and demand constraints for these products. Heuristics and lower bounds are developed for this problem. Extensive computational analysis reveals that these heuristics provide an intuitive and effective way to solve the problem.

We applied this model to assess the validity of basic grade selection, production, and blending procedures to data from seven refineries of Europe's leading manufacturer of wheat- and corn-based products. Had the solution provided by the setup/demand heuristic been implemented at these refineries, total grade inclusion, batching, blending, and quality costs would have been reduced by at least 7% at all these refineries, which would have resulted in annual savings of $5 million.

Our future research will address some important extensions to this problem. One such case arises when a product is specified by a vector of attributes rather than a single attribute. While this does not occur in our application, multiattribute product specifications are common for petrochemical products. For instance, gasoline blends are usually specified in terms of octane, viscosity, and density levels. In principle, such specifications can be incorporated in our model by redefining a_{ip} as the pth specification of the ith product and rewriting Constraint (2) as $\sum_i a_{ip} d_{ij} = a_{jp} \sum_i d_{ij}$ $\forall j, p$ and making the necessary changes in the quality model. However, solving the inner problem is now complicated as Proposition 3 is not necessarily true.

Another class of extensions arises from different quality-cost models. In this paper, we consider the case of known input composition and uncertainty in volumes due to additive errors. We are currently studying scrap steel blending in which the composition of the blend is uncertain, and the uncertainty in volumes are caused by multiplicative errors, which are dependent on the size of the blending batch. Finally, extensions to this model arise when different production cost functions are used to represent the cost for producing basic grades.

APPENDIX

Procedure for Computing $\overline{B}_i$

Let product j be produced by blending basic grades i and k. Also, let $J_i = \{1, 2, \ldots, n\} \backslash \{i\}$ be the set of products made using basic grade i along with some other grade. Define the following variables:

$\overline{B}_i$ = maximum amount of grade i required to produce products in J_i.

$\overline{D}_i$ = demand for product i.

Without loss of generality, assume that grades are indexed in nondecreasing order of a_i. Define the following index sets:

$$K_i^- = \{1, 2, \ldots, j-1\},$$

$$K_i^+ = \{j+1, j+2, \ldots, n\},$$

$$J_i^- = \{j | a_j < a_I\},$$

$$J_i^* = \{j | a_j > a_i\}.$$

We calculate $\overline{B}_i$ using the following equation:

$$\overline{B}_i = d_i + \sum_{j=1}^{i-1} \max_{k \in K_i^-} \left\{ \frac{|a_k - a_j|}{|a_k - a_i|} * d_j \right\} + \sum_{j=i+1}^{n} \max_{k \in K_i^+} \left\{ \frac{|a_k - a_j|}{|a_k - a_i|} * d_j \right\}.$$

This upper bound consists of three components. The first term represents the demand for the product if it is used as a basic grade and not blended. The second term of the equation represents the maximum quantity of basic grade i that will be used to blend product j when $a_i > a_j$ calculated across set K_i^- for which $a_j > a_k$ $\forall k \epsilon K_i^-$. This amount is summed for all products in the set J_i^- to calculate the total maximum usage for basic grade i for products whose attribute level is lower than that of this grade. In a similar manner, the third term calculates the total maximum usage for basic grade i for products whose attribute level is higher than that of the ith basic grade. Thus, $\overline{B}_i$ calculated by this equation represents the maximum quantity of basic grade that would be used for blending, providing a reasonable upper bound on B_i.

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