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Robust Controls for Network Revenue Management

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Revenue management models traditionally assume that future demand is unknown but can be described by a stochastic process or a probability distribution. Demand is, however, often difficult to characterize, especially in new or nonstationary markets. In this paper, we develop robust formulations for the capacity allocation problem in revenue management using the maximin and the minimax regret criteria under general polyhedral uncertainty sets. Our approach encompasses the following open-loop controls: partitioned booking limits, nested booking limits, displacement-adjusted virtual nesting, and fixed bid prices. In specific problem instances, we show that a booking policy of the type of displacement-adjusted virtual nesting is robust, both from maximin and minimax regret perspectives. Our numerical analysis reveals that the minimax regret controls perform very well on average, despite their worst-case focus, and outperform the traditional controls when demand is correlated or censored. In particular, on real large-scale problem sets, the minimax regret approach outperforms by up to 2% the traditional heuristics. The maximin controls are more conservative but have the merit of being associated with a minimum revenue guarantee. Our models are scalable to solve practical problems because they combine efficient (exact or heuristic) solution methods with very modest data requirements.

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1. Introduction

The field of revenue management (RM) originated in the airline industry as a way to efficiently allocate fixed capacity to different classes of customers. Since then, its scope has expanded, combining capacity rationing with pricing tactics, and the concept has been applied to a variety of industries such as hotels, rental cars, and media. See Talluri and van Ryzin (2004) for an overview of the field.

The decision to accept or reject an incoming customer is often made without knowing future demand. Traditional RM models assume that future demand is unknown but can be characterized by either a stochastic process representing the customer arrival process (the *dynamic* models) or a probability distribution representing the aggregate number of customers (the *static* models).

Accurate forecasting is key to effective RM. The best forecasts are typically obtained by gathering demand information from different sources (e.g., historical sales data, recent bookings, competitive environment), interpreting it carefully (e.g., sales data are only censored demand data), and combining alternative forecasting methods such as time series and regression models (Boyd and Bilegan 2003). In general, quantitative forecasting methods are favored in stable business environments with large amounts of historical data, whereas simple forecasting methods, based on expert judgment or single-point forecast estimation, are usually prevalent in new or nonstationary business environments. Given the reliance of RM models on demand forecasting, one may wonder if RM is really effective with limited information about demand.

In this paper, we investigate the problem of allocating fixed network capacity to different classes of customers with limited demand information. Instead of assuming knowledge about a demand stochastic process, as is traditionally done, we only assume that the demand lies in a *polyhedral uncertainty set* and make no assumption about the sequence of arrivals. This representation of uncertainty captures the stochastic nature of demand while remaining simple to estimate. In particular, forecasting experts often characterize demand with three points, namely the base-case, worst-case, and best-case scenarios, which naturally leads to an interval representation of demand uncertainty.

We consider the *maximin* and the *minimax regret* decision-making criteria. The maximin criterion guarantees a minimum level of profit and is more appropriate for risk-averse decision makers. In contrast, the minimax regret criterion minimizes the opportunity cost from not knowing the demand and gives rise to less conservative recommendations.

The booking policy considered in this paper generalizes the following controls, which are frequently used in practice: *nested booking limits*, *partitioned booking limits*, *displacement-adjusted virtual nesting* (DAVN), and *fixed bid prices*. We develop a mixed-integer formulation for computing the worst-case performance (minimum revenue or maximum regret) of any booking policy. We then show that the minimax regret booking controls under interval uncertainty for the single-resource RM problem and for a bundle RM problem can be obtained either in closed form or by solving a linear optimization problem (LP). Moreover, we show that in these cases a type of DAVN booking policy is robust, both from a maximin and a minimax regret perspective. For more general networks, we develop a simple heuristic that computes the minimax regret partitioned booking limits. Our numerical study suggests that the minimax regret controls outperform the traditional open-loop controls, even with simple interval uncertainty sets, especially when demand is correlated or censored. In particular, on real large-scale problem sets, the minimax regret approach outperforms by up to 2% the traditional heuristics. The maximin controls are, in contrast, more conservative but provide a minimum revenue guarantee. Our approach is scalable to solve large network RM problems as it combines efficient solution procedures with very modest data requirements.

1.1. Literature Review

The single-resource capacity control problem was introduced by Littlewood (1972) with two classes of customers arriving sequentially, and was subsequently extended to multiple classes of customers. With sequential arrivals, the optimal booking policy is known to be nested (Talluri and van Ryzin 2004). Nested booking limits can be computed either optimally or with the expected marginal seal revenue (EMSR) heuristics EMSR-a and EMSR-b (Belobaba 1987, 1992).

Network RM is significantly more complex and little is known about the optimal policy. Consequently, network RM is often solved heuristically, either by approximating the revenue-to-go function in the dynamic program (Bertsimas and Popescu 2003, Adelman 2007) or by restricting the set of feasible policies. Commonly used controls are partitioned booking limits, nested booking limits, and bid prices. Although these controls have traditionally been obtained with mathematical programming formulations (Talluri and van Ryzin 2004), dynamic programming and optimal control formulations have recently received considerable attention (Adelman 2007, Topaloglu 2009, Talluri 2008, Akan and Ata 2006). The controls can then be fine-tuned with a stochastic gradient algorithm based on demand samples (Bertsimas and de Boer 2005, van Ryzin and Vulcano 2008).

Traditional RM models, which assume probabilistic demand information, are very sensitive to forecasting errors (Weatherford and Belobaba 2002). Instead of estimating a demand probability distribution from historical data, data-driven optimization methods directly use the historical sales data to optimize booking controls with a stochastic gradient algorithm (van Ryzin and McGill 2000, Bertsimas and de Boer 2005, van Ryzin and Vulcano 2008). Nonparametric optimization methods have also been widely applied in inventory management (Godfrey and Powell 2001, Levi et al. 2007, Huh and Rusmevichientong 2009) and pricing (Erken and Maglaras 2006, Rusmevichientong et al. 2006, Besbes and Zeevi 2009).

In new or nonstationary business environments, however, historical data are either unavailable or of little value. In fact, the only demand information that is available, if any, often comes from industry experts, based on their judgment and experience. Decisions must therefore be made under limited demand information using robust optimization. Robust optimization methods, rejuvenated by Ben-Tal and Nemirovski (1999) and Bertsimas and Sim (2004), have been widely used in inventory control with the maximin criterion (Scarf 1958, Gallego and Moon 1993, Gallego et al. 2001, Bertsimas and Thiele 2006, Ben-Tal et al. 2005) and the minimax regret criterion (Yue et al. 2006, Perakis and Roels 2008). In RM, Birbil et al. (2009), Ball and Queyranne (2009), and Lan et al. (2008) analyzed robust booking limits for a single resource. Birbil et al. (2009) developed efficient algorithms to compute the maximin booking limits, partitioned or nested, under ellipsoidal uncertainty. Ball and Queyranne (2009) studied nested booking limits using the competitive ratio with no information about the demand, and Lan et al. (2008) generalized their results to interval uncertainty. Although they also cover the minimax regret, our study is more general because it also addresses network problems, general polyhedral uncertainty sets, and general open-loop booking limit controls.

1.2. Outline

This paper is organized as follows. Section 2 reviews the classical network RM problem. In §3, we introduce the decision-theoretic framework and propose a general mixed-integer formulation for evaluating the maximum regret or the minimum revenue. Section 4 then analyzes particular network examples under interval uncertainty. Numerical examples in §5 compare the performance of the proposed policies to common heuristics. Finally, §6 provides concluding remarks. All proofs appear in the online appendix.

1.3. Notations

We begin by introducing some notational conventions. Vector (resp. matrices) are denoted in small (resp. capital) bold letters. For a vector $\mathbf{x}$, x_j denotes its jth component; similarly, for a matrix $\mathbf{A}$, $\mathbf{A}_j$ represents the jth column, $\mathbf{a}_i$ the ith row, and a_{ij} the ith row, jth column element. All vectors are column vectors, and $\mathbf{x}'$ is the vector transpose. The function $\min\{\mathbf{x}, \mathbf{y}\}$ takes the componentwise minimum of vectors $\mathbf{x}$ and $\mathbf{y}$, and the function $\mathbf{x}^+$ takes the componentwise maximum of $\mathbf{x}$ and $\mathbf{0}$. Let $\mathbf{1}$ be a vector of ones. Finally, we denote by $\mathbb{1}(.)$ the indicator function.

2. Network Revenue Management

We first introduce the dynamic network RM problem. See Talluri and van Ryzin (2004) for details. Consider a network with K resources and N products differentiated by origin–destination (OD) and fare classes. Customers arrive according to a certain stochastic process over a finite time interval; let d_j be the random total demand for product j. There are $\mathbf{c}$ units available of resources. Each product j has a unit revenue r_j and consumes $\mathbf{A}_j$ units of resources.

We seek a policy π that maximizes the expected revenues $\mathbf{r}'E[\mathbf{x}^\pi]$, where $\mathbf{x}^\pi$ is the vector of total number of accepted requests when policy π is in use. The policy needs to satisfy (almost surely, denoted by a.s.) the capacity constraints, i.e., $\mathbf{A}\mathbf{x}^\pi \le \mathbf{c}$. The accepted requests are nonnegative and cannot exceed the total demand, i.e., $\mathbf{0} \le \mathbf{x}^\pi \le \mathbf{d}$. In addition, the policy is required to be nonanticipating. That is, the acceptance/rejection decision at each time t should be based only on the information acquired up to time t. Let Π be the set of nonanticipating policies. The problem can then be formulated as follows:

$$\begin{aligned} \sup_{\pi\in\Pi}\ & \mathbf{r}'E[\mathbf{x}^\pi], \\ \text{s.t.}\ & \mathbf{A}\mathbf{x}^\pi \le \mathbf{c} \quad \text{(a.s.)}, \\ & \mathbf{0} \le \mathbf{x}^\pi \le \mathbf{d} \quad \text{(a.s.)}. \end{aligned} \tag{1}$$

With deterministic demand, problem (1) reduces to a deterministic linear program (DLP).

When the arrival process is discrete, problem (1) can be formulated as a dynamic program: At each (discrete) time t, given the level of available capacity, one needs to decide whether to accept or reject the requests arriving in period t in order to maximize the total expected revenues until the end of the time horizon. The dynamic program formulation highlights the structure of the optimal policy: a request for product j at time t is accepted if and only if its fare r_j exceeds the opportunity cost from consuming $\mathbf{A}_j$ units of capacity, given the level of available capacity at time t.

However, the dynamic program is rarely solved to optimality in practice due to its large size. Network RM is therefore usually solved by either approximating the revenue-to-go function (Bertsimas and

Popescu 2003, Adelman 2007) or by considering a subset of feasible policies, such as the following:

- *Partitioned booking limits* restrict the number of requests that can be accepted for each product. The total capacity is effectively partitioned into N buckets. With partitioned booking limits, problem (1) simplifies to a stochastic nonlinear optimization problem or, if the random demands are replaced by their means, to a DLP.
- With a single resource, products are naturally ordered by fare and it is optimal to use *nested booking limits*. If $r_1 \geq r_2 \geq \cdots \geq r_N$, the booking limit on nest $\{j, \ldots, N\}$ limits the number of accepted requests for any product in $\{j, \ldots, N\}$. Nested booking limits can be computed either optimally or using the EMSR heuristics (Belobaba 1987, 1992). This idea can be extended to network RM by nesting together products that share the same OD pair (Curry 1990, Chi 1995).
- *DAVN* generalizes the idea of nesting to a network environment. The network is decomposed into K single-resource problems. For each resource, the fares are adjusted to account for network effects, e.g., using the shadow prices of the capacity constraints of the DLP. Products are then ordered by adjusted fare and grouped into different nested buckets (see Bertsimas and de Boer 2005 for details). An incoming request is accepted if there is sufficient capacity and if, on every resource, the booking limit of the associated bucket has not been reached.
- *Bid prices* associate a price p_k with each resource k in every period and for every vector of capacity consumption. A request for product j is accepted at time t if and only if the collected fare r_j exceeds the implicit cost of consuming the resources, $\mathbf{A}_j'\mathbf{p}$. Bid price controls are in general not optimal (Talluri and van Ryzin 1998), yet they are widely used in practice. Usually, the shadow prices of the DLP or of a randomized linear program (RLP) (Talluri and van Ryzin 1999) are chosen as bid prices, but more recent methods use approximate dynamic programming (Adelman 2007, Topaloglu 2009).

In this paper, we focus on *open-loop booking limit controls*. For every set of products $S \in \mathcal{S}$, we define a booking limit y_S. Denoting by x_j the realized sales of product j, the booking limit control ensures that $\sum_{j\in S} x_j \leq y_S$ for any set $S \in \mathcal{S}$. We further assume that the controls are open-loop, i.e., state independent. These controls generalize the partitioned booking limits (take $\mathcal{S} = \bigcup_{j=1}^N \{j\}$), the nested booking limits on a single resource (take $\mathcal{S} = \bigcup_{j=1}^N \{j, j+1, \ldots, N\}$), the DAVN booking limits (take $\mathcal{S} = \bigcup_{k=1}^K \bigcup_{j=1}^{N^{\max}} \{B_j^k, B_{j+1}^k, \ldots, B_{N^{\max}}^k\}$ where B_j^k is the set of products in bucket j on resource k and $N^{\max}$ is the number of such buckets), and the fixed bid prices (take $\mathcal{S} = \bigcup_{j=1}^N \{j\}$, and set $y_j = \infty$ if $r_j \geq \mathbf{p}'\mathbf{A}_j$, where $\mathbf{p}$ is the vector of bid prices, and $y_j = 0$ otherwise). They cannot, however, model bid price tables and theft nesting, which are state-dependent controls (Talluri and van Ryzin 2004).

3. Robust Problem Formulation

In this section, we formulate the general minimax regret and maximin revenue network RM problems. We first introduce the decision-theoretic framework, then offer guidelines to build demand uncertainty sets with limited demand information, and finally propose general mixed-integer formulations for computing the maximum regret or the minimum revenue for any given static booking limit policy.

3.1. Decision-Theoretic Framework

We assume only partial information about demand. Specifically, we assume that the aggregate demand $\mathbf{d}$ belongs to a polyhedral uncertainty set $\mathcal{P}$ and make no assumption about the arrival sequence. For instance, if the demand for product j is known to belong to the interval $[l_j, u_j]$ for any j, $1 \leq j \leq N$, then $\mathcal{P} = \{\mathbf{d}\colon \mathbf{l} \leq \mathbf{d} \leq \mathbf{u}\}$. Let $\mathcal{D}_{\mathcal{P}}$ be the set of multivariate stochastic processes such that $\mathbf{d} \in \mathcal{P}$, and let $\mathcal{F}$ be the feasible decision set, assumed to be compact. We denote by $R(\mathbf{y}, \mathbf{D})$ the revenue associated with a decision $\mathbf{y} \in \mathcal{F}$ when the demand process $\mathbf{D}$ is realized.

Because the expected utility maximization criterion has no meaning in a distribution-free environment, different decision criteria need to be considered. In this paper, we use the maximin and the minimax regret criteria, and refer to Ball and Queyranne (2009) and Lan et al. (2008) for an analysis of the competitive ratio.

- The *maximin* criterion selects the decision that maximizes the worst-case revenue:

$$\varphi^* = \max_{\mathbf{y}\in\mathcal{F}} \varphi(\mathbf{y}), \tag{2}$$

where the worst case is taken over all demand processes under consideration, i.e.,

$$\varphi(\mathbf{y}) = \min_{\mathbf{D}\in\mathcal{D}_{\mathcal{P}}} R(\mathbf{y}, \mathbf{D}). \tag{3}$$

• The *minimax regret* criterion selects the decision that minimizes the maximum regret, that is,

$$\rho^* = \min_{\mathbf{y}\in\mathcal{F}} \rho(\mathbf{y}), \tag{4}$$

where the regret $\rho(\mathbf{y})$ is defined as the maximum additional revenue, over all demand processes from $\mathcal{D}_{\mathcal{P}}$, that could have been obtained with full information about the demand, i.e.,

$$\rho(\mathbf{y}) = \max_{\mathbf{D}\in\mathcal{D}_{\mathcal{P}}} \left\{ \max_{\mathbf{z}\in\mathcal{F}} R(\mathbf{z}, \mathbf{D}) - R(\mathbf{y}, \mathbf{D}) \right\}. \tag{5}$$

The rationales behind these two approaches are fundamentally different. The maximin criterion guarantees a minimum revenue over all possible demand scenarios and is therefore more suitable to risk-averse decision makers. Because the maximin criterion may lead to conservative decisions, Savage (1951) introduced the minimax regret criterion to improve the average quality of decisions under uncertainty. Its ability to lead to "good" decisions "on average" depends on the validity of the premise that a decision that is equidistant from all extreme, worst-case scenarios (in terms of revenue differences) is likely to perform well. Consequently, the performance of the two criteria must be evaluated along different dimensions. Whereas the maximin criterion always gives the best worst-case revenue, the minimax regret may not lead to the best "average" revenue. Moreover, the minimax regret may not perform well in the worst case, and the maximin criterion may not perform well on average, although there are counterexamples (e.g., Scarf 1958).

3.2. Building Demand Uncertainty Sets

We next offer guidelines about how to build uncertainty sets from limited demand information. In general, the characterization of $\mathcal{P}$ will depend on the decision-making criterion.

Under the maximin criterion, a decision maker is concerned about her worst-case revenue. Instead of maximizing the worst-case revenue over $\mathcal{P}$, a decision maker may wish to consider a smaller subset of demand realizations, denoted by $\mathcal{P}(\eta) \subseteq \mathcal{P}$, such that $\Pr[\mathbf{d} \in \mathcal{P}(\eta)] \geq \eta$, so as to make less conservative decisions (Ben-Tal and Nemirovski 1999, Bertsimas and Sim 2004). If the maximin revenue is optimized over $\mathcal{P}(\eta)$, the actual revenue will be greater than the maximin bound with probability η. There is therefore a fundamental trade-off between minimum revenue and probabilistic guarantee, similar to the value-at-risk criterion.

Under the minimax regret criterion, the decision maker is concerned about the "average" performance of the robust control. There is a priori no guarantee that a minimax regret control optimized over $\mathcal{P}$ will perform better or worse than if it were optimized over $\mathcal{P}(\eta)$. Moreover, the notion of probabilistic guarantee on the maximum regret has little managerial value given that the maximum regret is never measured. A decision maker is thus free to choose any uncertainty set based on her experience and expertise so as to improve her confidence in obtaining a good solution. Based on past research, we next offer some guidelines to help a decision maker build an uncertainty set:

• Set the interval boundaries l_j and u_j at the twenty-fifth and seventy-fifth percentiles rather than the endpoints of the distribution because (i) they require less experience to be estimated, (ii) they are less sensitive to censored demand data, and (iii) they make the results more robust to the actual shape of the distribution or the location of the mode (Moder and Rodgers 1968).

• Build the uncertainty set $[l_j, u_j]$ around the median, rather than around the mean, because it is generally more informative (Perakis and Roels 2008).

• If the mean μ_j, standard deviation σ_j, and pairwise correlation ρ_{ij} are known for all products i, j, model the demand pairwise correlation with linear constraints as follows:

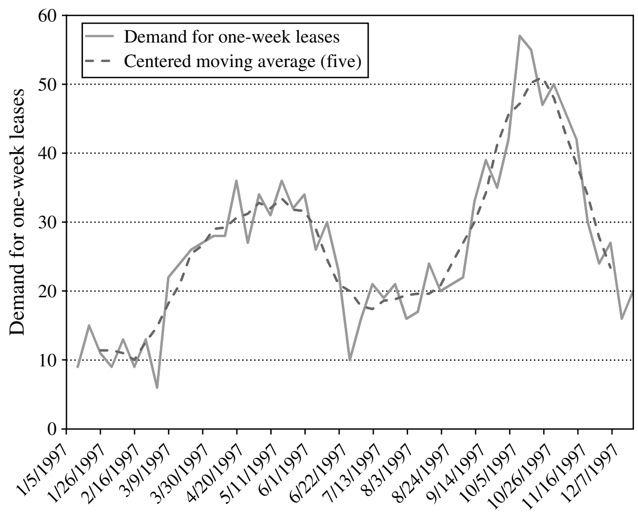
$$\left|\frac{d_i - \mu_i}{\sigma_i}\right| + \left|\frac{d_j - \mu_j}{\sigma_j}\right| \leq \sqrt{2(1+\rho_{ij})}\,\Phi^{-1}(\eta), \tag{6}$$

where $\Phi(x)$ denotes the standard normal distribution. This type of inequalities can be generalized to an arbitrary number of products (Bertsimas and Sim 2004).

We also recommend adopting consistent rules across all products, so that more volatile demands are associated with larger intervals.

3.2.1. Illustration. Using the Transportation National Group (TNG) case study (van Ryzin 1998), we

Figure 1 **Weekly Demand for One-Week Leases in the Transportation National Group Case Study (van Ryzin 1998)**



illustrate how to build uncertainty sets, with the objective of maximizing the revenue a fleet of trailers during a period of one year. Exhibit 5 in the case provides a branch weekly demand data for the year 1997 by lease duration (1, 4, 8, or 16 weeks). Figure 1 illustrates that demand is seasonal, with two peak periods in May and October. The revenue optimization cannot, therefore, be decoupled because demand seasonalities need to be accounted for. The challenge is thus to maximize revenue with only one data sample.

To proceed, we first assume that the demand in the following year will follow the same pattern as in 1997. Figure 1 shows that there is, however, a lot of noise around the underlying trend; it is therefore not reasonable to assume that future demand will exactly replicate the 1997 demand data. Instead, we consider a central moving average of five-weeks (MA(5)), which smooths the data while still capturing the seasonal cycles; moreover, this estimate turns out to be unbiased in this example.

We next assume that demand is homoscedastic around the MA(5) trend, because there is no significant change in variability between the two semesters. This simplifying assumption allows us to obtain more accurate estimates of the range of uncertainty. For every j-week lease requested at time t, we denote by $\underline{d}_{jt}$ the demand in 1997, and by MA_{jt} its corresponding moving average value. Denoting by L_j and U_j the twenty-fifth and seventy-fifth quantiles of the series of the differences $\{(\underline{d}_{jt} - MA_{jt})\}_{t=1,\ldots,50}$, we set the interval uncertainty for j-week lease requests at time t to $[l_{jt}, u_{jt}] = [MA_{jt} + L_j, MA_{jt} + U_j]$. That is, we assume that

$$MA_{jt} + L_j \leq d_{jt} \leq MA_{jt} + U_j \qquad j \in \{1, 4, 8, 16\},\ t = 1, \ldots, 50. \tag{7}$$

Accordingly, we build the uncertainty set as the interval set $\mathcal{P} = \{\mathbf{d}\colon (7) \text{ holds}\}$. Additional constraints could model the pairwise correlation between the demand of successive periods and between the demand for different durations, similar to (6). We show in §5, however, that the performance of the minimax regret controls is relatively insensitive to the choice of the uncertainty set.

3.3. Minimum Revenue and Maximum Regret

We next formulate and characterize the general minimum revenue and maximum regret network RM problems (3) and (5) as mixed-integer optimization problems (MIPs) for a given booking limit policy $\mathbf{y}$ and characterize their complexity.

Let $\mathbf{x}$ be the realized sales under a booking policy $\mathbf{y}$ when the demand is equal to $\mathbf{d}$. Clearly, the sales must be feasible, i.e., $\mathbf{x}$ must satisfy the following constraints (see (1)):

$$\begin{aligned} &\mathbf{A}\mathbf{x} \leq \mathbf{c} \\ &\mathbf{0} \leq \mathbf{x} \leq \mathbf{d} \\ &\sum_{j\in S} x_j \leq y_S \quad S \in \mathcal{S}. \end{aligned} \tag{8}$$

In fact, the sales for product j, x_j equal the demand d_j unless the demand is censored, which happens when either one runs out of capacity, i.e., when $\sum_{i=1}^N a_{ki} x_i = c_k$ for some k, $1 \leq k \leq K$, such that $a_{kj} > 0$, or one of the booking limits has been reached, i.e., when $\sum_{j\in S} x_j = y_S$ for some $S \in \mathcal{S}$ with $j \in S$. Accordingly, the realized sales under the booking policy $\mathbf{y}$ must satisfy

$$x_j = \min\left\{d_j, \min_{k=1,\ldots,K} \frac{1}{a_{kj}}\left\{c_k - \sum_{i=1,\, i\neq j}^{N} a_{ki} x_i\right\}, \min_{S\in\mathcal{S}:\, j\in S}\left\{y_S - \sum_{i\in S,\, i\neq j} x_i\right\}\right\} \quad j = 1, \ldots, N. \tag{9}$$

Based on this observation, we are now ready to present an MIP formulation for (3) and (5).

Proposition 1. *For any polyhedral uncertainty set $\mathcal{P}$ and any booking policy $\mathbf{y}$, the maximum regret $\rho(\mathbf{y})$ is equal to the optimal value of the following MIP*:

$$
\begin{aligned}
\max_{\mathbf{z},\mathbf{x},\mathbf{d},\boldsymbol{\alpha},\boldsymbol{\beta},\boldsymbol{\gamma}} \quad & \mathbf{r}'\mathbf{z}-\mathbf{r}'\mathbf{x}\\
\text{s.t.}\quad & \mathbf{d}\in\mathcal{P}\\
& \mathbf{A}\mathbf{z}\le\mathbf{c}\\
& \mathbf{0}\le\mathbf{z}\le\mathbf{d}\\
& \mathbf{A}\mathbf{x}\le\mathbf{c}\\
& \sum_{j\in S}x_j\le y_S \quad S\in\mathcal{S}\\
& \mathbf{0}\le\mathbf{x}\le\mathbf{d}\\
& \mathbf{d}\le\mathbf{x}+M(\mathbf{1}-\boldsymbol{\alpha})\\
& \sum_{j\in S}x_j\ge \beta_S y_S \quad S\in\mathcal{S}\\
& \mathbf{a}_k'\mathbf{x}\ge c_k\gamma_k \quad k=1,\ldots,K\\
& \sum_{k=1:\,a_{kj}>0}^{K}\gamma_k+\alpha_j+\sum_{S:\,j\in S}\beta_S\ge 1, \quad j=1,\ldots,N\\
& \boldsymbol{\alpha}\in\{0,1\}^N,\quad \boldsymbol{\beta}\in\{0,1\}^{|\mathcal{S}|},\quad \boldsymbol{\gamma}\in\{0,1\}^K,
\end{aligned}
\tag{10}
$$

where $M\ge\{\max_j d_j\colon \mathbf{d}\in\mathcal{P}\}$. *Similarly, for any polyhedral uncertainty set $\mathcal{P}$ and any booking policy $\mathbf{y}$, the minimum revenue $\varphi(\mathbf{y})$ is equal to the negative of the optimal value of* (10) *when* $\mathbf{z}=\mathbf{0}$.

The first constraint in (10) requires the demand vector to belong to the polyhedral uncertainty set $\mathcal{P}$. The next two constraints show that, for any demand $\mathbf{d}\in\mathcal{P}$, the perfect hindsight booking limits $\mathbf{z}$ are partitioned and can be obtained by solving a DLP. The next three constraints ensure that the sales $\mathbf{x}$ are feasible, similar to (8). Finally, the last set of constraints, which involves the binary variables $\boldsymbol{\alpha}$, $\boldsymbol{\beta}$, and $\boldsymbol{\gamma}$, ensures that (9) is satisfied. Because of the forcing constraint $d_j\le x_j+M(1-\alpha_j)$, $\alpha_j=1$ only if $x_j=d_j$. Similarly, $\beta_S=1$ only if $\sum_{j\in S}x_j=y_S$, and $\gamma_k=1$ only if $\mathbf{a}_k'\mathbf{x}=c_k$. The joint constraint $\sum_{k=1:\,a_{kj}>0}^{K}\gamma_k+\alpha_j+\sum_{S:\,j\in S}\beta_S\ge 1$ guarantees that at least one of these three scenarios occurs, for every $j=1,\ldots,N$, so as to satisfy (9).

Moreover, constraints on the sequence of arrivals can be added to (10) without additional binary variables. Suppose that product j is known to always arrive before product i and that, in (9), the sales of j are restricted by a capacity or booking limit constraint involving both i and j. Then, the sales of product i must be equal to zero. Formally,

$$
x_i\le u_i\Bigg(\alpha_j+\sum_{\substack{k=1\\ a_{kj}>0,\,a_{ki}=0}}^{K}\gamma_k+\sum_{S\in\mathcal{S}:\,j\in S,\,i\neq S}\beta_S\Bigg). \tag{11}
$$

The MIP formulation involves $N+|\mathcal{S}|+K$ binary variables, where the cardinality of $\mathcal{S}$ depends on the type of the booking limit policy. With partitioned booking limits, nested booking limits for products with the same OD pair, and bid prices, $|\mathcal{S}|\le N$; with DAVN booking limits, $|\mathcal{S}|\le KN^{\max}$, where $N^{\max}$ is the maximum number of buckets per resource.

The next proposition characterizes the complexity of the maximin and minimax regret network RM problems (2) and (4): Not only is the inner problem NP-hard, but the outer problem is also nonconvex, similar to the optimization of nested booking limits on a sample path (van Ryzin and Vulcano 2008).

Proposition 2.

(a) *Evaluating $\rho(\mathbf{y})$ on a single resource with partitioned booking limits under interval uncertainty is NP-hard*.

(b) *Evaluating $\varphi(\mathbf{y})$ on a single resource with partitioned booking limits under interval uncertainty together with a lower bound on the total demand is NP-hard*.

(c) $\rho(\mathbf{y})$ *may not be quasiconvex, and* $\varphi(\mathbf{y})$ *may not be quasiconcave*.

Despite Proposition 2, we show in the next section that networks arising in RM have a special structure, which gives rise to efficient solution methods under interval uncertainty.

4. Tractable Cases Under Interval Uncertainty

Under interval uncertainty, the maximin revenue and the minimax regret network RM problems become either computationally tractable or can be solved with efficient heuristics. We first consider two particular problem instances under interval uncertainty, namely, a single-resource network and a serial network with a product bundle, and show that the robust controls can be obtained either in closed form or by solving linear optimization problems. Moreover, we show that the most robust booking policies are similar to the DAVN policy. We then present a heuristic for choosing partitioned booking limits in a general network RM problem under interval uncertainty.

4.1. Single-Resource Revenue Management

We first consider the minimax regret version of the classic multiproduct single-resource RM problem when $\mathcal{P} = \{\mathbf{d}\colon \mathbf{l} \le \mathbf{d} \le \mathbf{u}\}$. In a single-resource RM problem, N products consume the same resource with capacity c in unit amounts. For instance, airlines sell similar seats at different fares on the same flight. Without loss of generality, we assume that $u_j \le c$ for all $j = 1, \ldots, N$.

4.1.1. Maximin Booking Limits. To characterize the maximin controls, we initially characterize the worst-case demand scenario when no control is applied, i.e., when $\mathcal{S} = \varnothing$. We demonstrate that the classic assumption that low-fare customers book first corresponds in fact to the worst-case demand sequence. This assumption is therefore robust.

Moreover, we show that the revenue is minimized when all demands are equal to their lower bounds, or when they are all equal to their upper bound. Hence, even though we make no assumption about the demand correlation when $\mathcal{P} = \{\mathbf{d}\colon \mathbf{l} \le \mathbf{d} \le \mathbf{u}\}$, the worst-case demand scenarios when $\mathcal{S} = \varnothing$ correspond to cases where demand is perfectly correlated.

Lemma 1. *For a single-resource RM problem with interval uncertainty, the revenue when $\mathcal{S} = \varnothing$ is minimized when low-fare customers book first and when $d_j = l_j$ for $j = 1, \ldots, N$ or when $d_j = u_j$ for all $j = 1, \ldots, N$.*

The worst-case demand scenario $\mathbf{d} = \mathbf{l}$ is intuitive: with lower demand, there are fewer opportunities to sell, reducing the revenue. The second worst-case demand scenario, when $\mathbf{d} = \mathbf{u}$, exploits the worst-case sequence of arrivals. If the low-fare demand is higher, more units of capacity will be sold at a low fare, and there will be less capacity available to satisfy the high-fare demand. Although no control can improve the revenue under the first worst-case scenario, partitioned booking limits can actually increase the revenue achieved under the second demand scenario. As summarized in the next proposition, the best partitioned booking limits protect capacity to cover the deterministic demand for the high-fare products. The proof is omitted.

Proposition 3. *Under interval uncertainty, the maximin booking limits are partitioned, that is, $x_j \le y_j$ for all $j = 1, \ldots, N$ with $\sum_{j=1}^{N} y_j = c$, and they are equal to $y_j = \min\{l_j, (c - \sum_{i<j} l_i)^+\}$ for $j = 1, \ldots, N$. Under this control, the revenue is minimized when low-fare customers book first and when $d_j = l_j$ for all $j = 1, \ldots, N$.*

4.1.2. Minimax Regret Booking Limits. We next analyze the worst-case demand scenario under the minimax regret criterion. As in Lemma 1, the worst-case sequence is when low-fare customers book first. There is, however, only one worst-case demand scenario, namely, when $\mathbf{d} = \mathbf{u}$.

Lemma 2. *For a single-resource RM problem with interval uncertainty, the regret when $\mathcal{S} = \varnothing$ is maximized when low-fare customers book first and when $d_j = u_j$ for all $j = 1, \ldots, N$.*

We next show that the celebrated nested booking limits policy minimizes the maximum regret. Therefore, the nested booking limit policy is also robust, in addition to being optimal under full probabilistic information about the demand when low-fare customers book first (Talluri and van Ryzin 2004).

Proposition 4. *For a single-resource RM problem with interval uncertainty, the minimax regret booking limits are nested, that is,*

$$\sum_{j\in S} x_j \le y_S \quad \forall S \in \mathcal{S} = \big\{\{N\}, \{N-1, N\}, \ldots, \{2, \ldots, N\}\big\}.$$

In the absence of control, i.e., when $\mathcal{S} = \varnothing$, the maximum regret is attained when all demands are equal to their upper bounds, as if they were perfectly correlated (Lemma 2). In contrast, under a nested booking limit policy, the worst-case demand vectors are negatively correlated, with the demand for high-fare products being equal to its lower bound, whereas the demand for low-fare products is equal to its upper bound. The worst-case sequence of arrivals remains however unchanged, with low-fare customers booking first.

Lemma 3. *For a single-resource RM problem with interval uncertainty, the regret with nested booking limit controls is maximized when low-fare customers book first and when $d_j = l_j$ for $j = 1, \ldots, t-1$ and $d_j = u_j$ for $j = t, \ldots, N$, for any t, $1 \le t \le \min\{N, \min\{k\colon y_{\{k,\ldots,N\}} - y_{\{k+1,\ldots,N\}} < l_k\}\}$.*

A corollary of Lemma 3 is that the spare capacity from a nest $S \in \mathcal{S}$, $y_S - \sum_{j\in S} x_j$, will never be utilized to serve the demand for higher-fare products $j \notin S$. Hence, even though nested booking limits are regret-minimizing (Proposition 4), the nesting property will never be exploited in the worst case.

Furthermore, Lemma 3 shows that at most N worst-case demand scenarios are relevant for selecting the minimax regret nested booking limits. Because of the special structure of the single-resource network and of the interval uncertainty set, the number of worst-case scenarios can thus be dramatically reduced, from $2^{N+|\mathcal{S}|+K} = 2^{2N}$ in (10) to at most N. As a result, evaluating the maximum regret becomes computationally tractable. In fact, the next proposition gives a closed-form expression for the minimax regret nested booking limits. Lan et al. (2008) independently proved the same result, using a competitive analysis argument.

Proposition 5. *Under interval uncertainty, the nested booking limits that minimize the maximum regret for a single-resource RM problem equal to*

$$y_{\{1,\dots,N\}} = c,$$

$$y_{\{j+1,\dots,N\}} = \left(y_{\{j,\dots,N\}} - \frac{1}{r_j}(g_j - g_{j+1})\right)^+ \qquad 1 \le j \le N-1, \tag{12}$$

in which, for $t = 1, \dots, N$,

$$g_t = \sum_{j=t}^{N} r_j \min\left\{u_j, \left(c - \sum_{i=1}^{t-1} l_i - \sum_{i=t}^{j-1} u_i\right)^+\right\}.$$

As a result, the minimax regret single-resource RM problem can be solved efficiently under interval uncertainty. Moreover, this approach can be extended to a network environment where booking limits are set only on products sharing the same OD pair. In that case, the minimax regret network RM problem can be formulated as an LP.

The next corollary shows that the minimax regret booking limits always protect the deterministic demand for high-fare products before allocating capacity to lower-fare products. That is, if $y_{\{j,\dots,N\}} - y_{\{j+1,\dots,N\}} > 0$ for some j, then $y_{\{i,\dots,N\}} - y_{\{i+1,\dots,N\}} \ge l_i$ for all $i < j$. In particular, when $\mathbf{l} = \mathbf{u}$, the minimax regret booking limits coincide with the booking limits obtained with the DLP.

Corollary 1. *Under interval uncertainty, the nested booking limits that minimize the maximum regret for a single-resource RM problem protect the deterministic demand. That is, if* $y_{\{j+1,\dots,N\}} > 0$,

$$\min\left\{u_j, \left(c - \sum_{i<j} l_i\right)^+\right\} \ge y_{\{j,\dots,N\}} - y_{\{j+1,\dots,N\}} \ge \min\left\{l_j, \left(c - \sum_{i<j} l_i\right)^+\right\} \qquad 1 \le j \le N-1.$$

According to Corollary 1, the minimax regret booking limits, if they are positive on all products, will be larger than the maximin booking limits and will therefore be associated with the same minimum revenue guarantee as the maximin booking limits (see Proposition 3). As a result, the nested booking limits (12) are robust, both from a minimax regret and a maximin standpoints.

In the case of complete uncertainty, i.e., when $l_j = 0$ and $u_j \ge c$ for all $j = 1, \dots, N$, the minimax regret booking limits simplify to

$$y_{\{j+1,\dots,N\}} = \left(y_{\{j,\dots,N\}} - \frac{r_j - r_{j-1}}{r_j} c\right)^+ \quad 1 \le j \le N-1,$$

with $y_{\{1,\dots,N\}} = c$. Therefore, if the total capacity is ample and if the spread between fares is small, every product but the last one is allocated some fraction $(1 - r_{j-1}/r_j)$ of the total capacity, in the same spirit as Littlewood's (1972) formula and EMSR-a (Belobaba 1987).

With only two products (Littlewood 1972), the optimal booking limit on the low-fare product equals

$$y_{\{2\}} = c - u_1 + \begin{cases} 0 & \text{if } u_2 \le c - u_1 \\ \dfrac{r_2}{r_1}(u_2 + u_1 - c) & \text{if } c - u_1 \le u_2 \le c - l_1 \\ \dfrac{r_2}{r_1}(u_1 - l_1) & \text{if } u_2 \ge c - l_1, \end{cases}$$

and is therefore nondecreasing with r_2, nonincreasing with r_1, and nondecreasing with c as in Littlewood's (1972) formula. In contrast to Littlewood's (1972) formula, the minimax regret booking limit depends on the variability of the low-fare demand because the minimax regret controls must be determined before any demand realization, whereas Littlewood's (1972) formula is obtained from dynamic optimization, as if the demand for low-fare products were observed before choosing the booking limit (Talluri and van Ryzin 2004).

Moreover, $y_{\{2\}}$ is nonincreasing with l_1 and with u_1. With probabilistic demand information, the optimal booking level is smaller for stochastically larger distributions for the high-fare demand, according to the first order. Similarly, one can interpret an increase in l_1 or u_1 as making the high-fare demand stochastically larger, as if the high-fare demand followed a two-point distribution.

4.2. Bundle Revenue Management

We now consider the minimax regret RM problem for bundles under interval uncertainty. Consider a serial network with K resources, each with capacity c_k and $K+1$ products. For all $j=1,\ldots,K$, product j consumes only one unit of resource j. By contrast, product 0 consumes one unit of each resource and is therefore referred to as a bundle. For instance, car rental companies offer weekly rates that are significantly cheaper than daily rates, or hotels offer weekend specials. We assume that $r_j < r_0$ for all $j=1,\ldots,K$, for otherwise there would be no demand for products $j=1,\ldots,N$. Moreover, we assume, without loss of generality, that $u_j \le c_j$ for all $j=1,\ldots,K$ and $u_0 \le \min_{k=1,\ldots,K} c_k$.

Assuming first no control, the next lemma characterizes the worst-case demand scenarios. Unlike single-resource RM problems, there exist multiple worst-case arrival sequences. In particular, product 0 can come either first or last. Therefore, if companies are able to control the sequence of arrivals without affecting their demand, they may benefit from restricting the promotional offer to either advance sales or last-minute sales. The next lemma thus highlights the importance of accurately estimating the sequence of arrivals in RM. Although assumptions about the sequence of arrivals (e.g., low-fare customers book first, or that the product mix remains constant over time) are often made without justification, Lemma 4 shows that the revenue collected under an RM policy is highly sensitive to this type of assumption.

Similar to single-resource RM, the worst-case demand scenarios are positively or negatively correlated, even though the interval uncertainty set makes no assumption about demand correlation.

LEMMA 4. *For a bundle RM problem with interval uncertainty, the regret (resp. revenue) when $\mathcal{S}=\varnothing$ is maximized (resp. minimized) in either of the following demand scenarios:*

1. *The bundle is requested first, with $d_0=u_0$, $d_j=u_j$ (resp. $d_j=l_j$) for all $j=1,\ldots,K$, provided $\sum_{j\in\mathcal{T}} r_j \ge r_0$, where $\mathcal{T}=\{1\le j\le K\colon d_j+d_0\ge c_j\}$.*
2. *The bundle is requested last, with $d_i=u_i$ for some i, $1\le i\le K$, $d_j=l_j$ for all $j=1,\ldots,K$, $j\ne i$, and $d_0=u_0$ (resp. $d_0=l_0$), provided that $\sum_{j\in\mathcal{T}} r_j < r_0$ where $\mathcal{T}=\{1\le j\le K\colon d_j+d_0\ge c_j\}$.*

We next show that both the maximum regret and the minimum revenue objectives can be improved by imposing a booking limit and a protection level on the demand for the bundle. In fact, as discussed in the online appendix, the DAVN booking limits consist of either a booking limit or a protection level on the demand for product 0, depending on whether $\sum_{j=1}^{K} r_j \mathbb{1}\{d_j+d_0>c_j\}$ is larger than or smaller than r_0. In contrast, when demand is uncertain, both controls are needed.

The structural similarity between the DAVN policy and the robust policy, both for the single-resource and the bundle RM problems, shows that the DAVN policy is robust, provided that the values of the booking limits are properly chosen. This last caveat is nevertheless secondary; in fact, we show in §5 that the performance of booking policies mostly depends on their structure, rather than on the actual values of the booking limits. The similarity between the DAVN policies and the robust policies presented in Proposition 4 and Lemma 5 thus partly explains the excellent performance of the DAVN booking policy in practice (Smith et al. 1992).

LEMMA 5. *For a bundle RM problem with interval uncertainty, the maximum regret (resp. minimum revenue) when $\mathcal{S}=\varnothing$ can be reduced with the following controls:*

$$x_0 \le y_0$$
$$x_0 \ge \bar{y}_0.$$

Under these controls, the realized revenue is minimized in any of the first two following scenarios and the regret is *maximized in any of the following demand scenarios*:

1. *The bundle is requested first, with $d_0 = u_0$ and $d_j = l_j$ for all $j = 1, \ldots, K$.*
2. *Products $j = 1, \ldots, K$ are requested first with $d_i = u_i$ for some product i, $1 \le i \le K$ and $d_j = l_j$ for all $j = 1, \ldots, K$, $j \neq i$, and the bundle is requested last with $d_0 = u_0$.*
3. *The bundle is requested first, with $d_j = u_j$ for $j = 0, \ldots, K$.*
4. *Products $j = 1, \ldots, K$ are requested first, with $d_j = u_j$ for all $j = 1, \ldots, K$, and the bundle is requested last with $d_0 = l_0$.*

Under this joint booking limit and protection-level policy, the maximin and the minimax regret RM problem with bundles can be solved as LPs. By Lemma 5, at most $K + 3$ demand scenarios must be considered to evaluate the maximum regret, instead of the worst-case estimate of 2^{2K+3} given by (10), giving rise to a computationally efficient solution method.

Proposition 6. *For a bundle RM problem with interval uncertainty, the maximin and the minimax regret controls can be obtained by solving at most $(2K+3)(2K+2)/2$ LPs. In particular, if $c_j - u_j \le \bar{y}_0 \le y_0 \le c_j - l_j$ for all $j = 1, \ldots, K$ and $l_0 \le \bar{y}_0 \le y_0 \le u_0$, the minimax regret controls solve the following LP*:

$$
\begin{aligned}
\min\ & \rho \\
\text{s.t.}\ & \rho \ge g_t - \sum_{j=1}^{K} r_j x_j^t + r_0 x_0^t \quad t = 0, \ldots, K+2 \\
& c_j - u_j \le y_0 \le c_j - l_j \quad j = 1, \ldots, K \\
& c_j - u_j \le \bar{y}_0 \le c_j - l_j \quad j = 1, \ldots, K \\
& l_0 \le \bar{y}_0 \le y_0 \le u_0,
\end{aligned}
$$

in which $g_t = \max_{\mathbf{z}} \sum_{j=0}^{K} r_j z_j$ subject to $z_0 + z_j \le c_j$ for all $j = 1, \ldots, K$ and $0 \le z_j \le d_j^t$ for all $j = 0, \ldots, K$, where d_j^t correspond to the tth demand scenario described in Lemma 5 ($t = 0$ is the first scenario, $t = 1, \ldots, K$ is the second set of scenarios, $t = K+1$ is the third scenario, and $t = K+2$ is the fourth scenario).

In case of full uncertainty, i.e., when $l_j = 0$ and $u_j = \infty$ for all $j = 0, \ldots, K$, with identical capacities $c_k = c$ for all $k = 1, \ldots, K$, the LP can be solved in closed form. The maximum regret is equal to

$$
\rho(y_0, \bar{y}_0) = \max\left\{ r_0(c - y_0), \left(r_0 - \min_{j=1,\ldots,K} r_j\right)(c - \bar{y}_0), \left(\sum_{j=1}^{K} r_j - r_0\right) y_0, \left(\sum_{j=1}^{k} r_j\right) \bar{y}_0 \right\},
$$

in which the four terms correspond to the four worst-case demand scenarios described in Lemma 5. The minimax regret booking level thus equals $y_0 = \min\{1, r_0/(\sum_{j=1}^{K} r_j)\}c$. On the other hand, the minimax regret protection level equals $\bar{y}_0 = c(r_0 - r_i)/(\sum_{j=1;\, j\neq i}^{K} r_j + r_0)$, where $r_i = \min_{j=1,\ldots,K} r_j$. Similar to the Littlewood's (1972) formula and, more generally, to the newsvendor problem, the booking limit and protection level are defined by the ratio of underage cost to the sum of underage and overage costs. Interestingly, because $(\sum_j r_j - r_0) y \le (\sum_j r_j) y$ and $r_0(c - y) \le (r_0 - \min_j r_j)(c - y)$, there is some flexibility in the choice of the booking limit y_0. Hence, the worst-case sequence according to which the bundle arrives last has a more severe impact on the maximum regret than when the bundle arrives first.

If $r_0 < \sum_{j=1}^{K} r_j$, the policy proposed in Lemma 5 is not the best possible. It would indeed be optimal to nest the set of products $\{1, \ldots, K\}$ with the bundle and define the following protection level: $x_0 + \min_{j=1,\ldots,K} x_j \ge \bar{y}_0$. Alternatively, one could use the following booking limits: $x_i \le y_i + \min_{j=1,\ldots,K;\, j\neq i} x_j$, for all $i = 1, \ldots, K$. These controls are, however, state dependent, in contrast to the traditional booking limit control $\sum_{j\in S} x_j \le y_S$. In particular, classes can be opened after being closed, even though the policy is static, i.e., is not reoptimized as requests occur. Consequently, no open-loop policy is, in general, optimal for network RM.

4.3. Partitioned Booking Limits

In this section, we consider a general network but focus on partitioned booking limits, i.e., $x_j \le y_j$ for all $j = 1, \ldots, N$. We first show that the maximin partitioned booking limits can be obtained by solving an LP. We then present an LP-based heuristic for obtaining the minimax regret partitioned booking limits.

4.3.1. Maximin Controls. Under interval uncertainty, the maximin partitioned booking limits solve the following DLP:

$$
\begin{aligned}
\max_{\mathbf{y}}\ & \mathbf{r}'\mathbf{y} \\
\text{s.t.}\ & \mathbf{A}'\mathbf{y} \le \mathbf{c} \qquad (13) \\
& \mathbf{0} \le \mathbf{y} \le \mathbf{l}.
\end{aligned}
$$

Increasing y_j above l_j would never pay off because the adversary can always choose $d_j = l_j$; however, the unused capacity $y_j - l_j$ could be used to serve the (deterministic) demand for another product, potentially leading to an increase in revenue.

4.3.2. Minimax Randomized Regret. Finding the minimax regret partitioned booking limits is, in general, NP-hard (Proposition 2). In this section, we relax the maximum regret problem (10) by assuming that the booking limits $\mathbf{z}$ can be randomized and that the capacity constraint must hold only in expectation, that is, $\mathbf{A}E[\mathbf{z}] \leq \mathbf{c}$. Under this assumption, the minimax regret network RM problem (4) is relaxed to

$$\min_{\mathbf{y}:\, \mathbf{Ay}\leq\mathbf{c},\, \mathbf{y}\geq\mathbf{0}} \left\{ \max_{\mathbf{Z}\in\mathscr{Z}} \max_{\mathbf{D}\in\mathscr{D}_{\mathscr{P}}} E_{\mathbf{Z}}[R(\mathbf{Z},\mathbf{D}) - R(\mathbf{y},\mathbf{D})] \right\}, \tag{14}$$

where $\mathscr{Z}$ is the set of nonnegative multivariate distributions such that $\mathbf{A}E[\mathbf{z}] \leq \mathbf{c}$. We call (14) the *minimax randomized regret*. Because $\mathbf{Az} \leq \mathbf{c}$ (a.s.) $\Rightarrow \mathbf{A}E[\mathbf{z}] \leq \mathbf{c}$, the minimax randomized regret is an upper bound on the minimax regret (4). The next proposition shows that the minimax randomized regret problem can be efficiently solved.

Proposition 7. *Under interval uncertainty, the minimax randomized regret network RM problem* (14) *with partitioned booking limits can be formulated as the following LP*:

$$\begin{aligned} \min_{\mathbf{p},\mathbf{q},\mathbf{y}} \quad & \mathbf{p}'\mathbf{c} + \mathbf{q}'\mathbf{1} \\ \text{s.t.} \quad & \mathbf{Ay} \leq \mathbf{c} \\ & \mathbf{p}'\mathbf{AU} + \mathbf{q}' \geq \mathbf{r}'(\mathbf{U} - \mathbf{Y}) \\ & \mathbf{p}'\mathbf{AL} + \mathbf{q}' \geq \mathbf{0} \\ & \mathbf{p}'\mathbf{AL} + \mathbf{q}' \geq \mathbf{r}'(\mathbf{L} - \mathbf{Y}) \\ & \mathbf{q}' \geq -\mathbf{r}'\mathbf{L} \\ & \mathbf{q}' \geq -\mathbf{r}'\mathbf{Y} \\ & \mathbf{y} \leq \mathbf{u} \\ & \mathbf{p}, \mathbf{y} \geq \mathbf{0}. \end{aligned} \tag{15}$$

where $\mathbf{U}$, $\mathbf{L}$, *and* $\mathbf{Y}$ *are diagonal matrices in which the diagonal elements correspond to* $\mathbf{u}$, $\mathbf{l}$, *and* $\mathbf{y}$, *respectively.*

Despite the regret randomization, problem (15) simplifies to the DLP when demand is deterministic. Under interval uncertainty, problem (15) has $(K+2N)$ variables and $(K+6N)$ constraints, whereas the DLP has only N variables and $K+N$ constraints. The larger size of the problem is the price to pay to capture demand stochasticity.

The variables $\mathbf{p}$ in (15) are the dual variables associated with the constraint $\mathbf{A}E[\mathbf{z}] \leq \mathbf{c}$. In fact, the optimal value of p_k measures the additional revenue that could be obtained if, in addition to knowing the demand distributions, the malevolent adversary were also given an additional unit of capacity c_k. Therefore, the optimal value of $\mathbf{p}$ can be used as a proxy for the marginal value of capacity. Although they are obtained from an LP, the variables $\mathbf{p}$ capture the stochastic nature of the demand, in contrast to the dual values of DLP.

5. Numerical Examples

5.1. Single-Resource Example

We first consider the single-resource four-product example in Talluri and van Ryzin (2004, pp. 48–50). Each product j is characterized by a fare r_j, a mean demand μ_j, and a standard deviation σ_j (see Table 1).

5.1.1. Policy Comparison. Consistent with the discussion in §3.2, we set the range $[l_j, u_j]$ equal to half the interquantile (twenty-fifth–seventy-fifth quantile) range. Table 1 compares the nested protection levels obtained with the minimax regret, the maximin, and EMSR-b (assuming normal distributions; see Belobaba 1992) when there are 119 seats in the aircraft. The maximin booking limits, obtained from Proposition 3, are used in a nested fashion. Moreover, the extra capacity $c - \sum_{j=1}^{N} l_j$ is randomly allocated among the different products. In contrast to the maximin booking limits, which protect little capacity for the high-fare products, the minimax regret booking limits, obtained from Proposition 5, are very

Table 1 **Problem Data and Nested Booking Limits for the Single-Resource Example from Talluri and van Ryzin (2004)**

j	r_j	Demand statistics μ_j	σ_j	Nested booking limits: Minimax regret	Maximin	EMSR-b
1	1,050	17.3	5.8	119	119	119
2	567	45.1	15.0	103	107	102
3	534	39.5	13.2	68	74	68
4	520	34.0	11.3	34	45	35
Minimum revenue				59,797	59,797	59,797
Maximum regret				3,683	5,911	3,750

similar to EMSR-b. The last two rows of Table 1 compare the minimum revenue and the maximum regret guarantees obtained with the different policies by solving (10). Consistently with Corollary 1, the minimax regret booking policy is also robust from a maximin revenue perspective. Note that the minimum revenue guarantee is low, compared to the optimal value of the DLP, equal to 73,723. In addition, the respective maximum regret performance guarantees are all small, indicating that all policies are expected to perform well on average.

5.1.2. Performance Simulation. To measure the robustness of our policies, we generate 1,000 demand scenarios and simulate the airline booking process under the proposed policies over 150 days. Similarly to de Boer et al. (2002) and Bertsimas and de Boer (2005), we model the arrival process as a nonhomogeneous Poisson process. The arrival intensity for product j in period t satisfies the following relationship:

$$\lambda_j(t) = B_j(t)G_j, \tag{16}$$

where $B_j(t)$ follows a standardized beta distribution, and G_j follows a gamma distribution. The choice of the beta distribution for modeling the arrival pattern is motivated by its flexible shape (e.g., increasing, decreasing, unimodal, bimodal) while the factor G_j creates some correlation between the demands for product j across all booking periods.[1] Under this demand model, the total number of booking requests follows a negative binomial distribution.

Table 2 displays the 90% confidence intervals for the mean revenues generated with the maximin, the minimax regret, and EMSR-b when capacity is varied from 80 to 150, creating demand factors (DFs) from 1.7 to 0.9, under standard nesting.

In general, the robust approaches perform almost as well as, and sometimes better than, EMSR-b. Their good performance is remarkable despite their worst-case focus, especially for the maximin approach. One should nevertheless note that the performance of the maximin booking limits would have been significantly lower if they had not been nested.

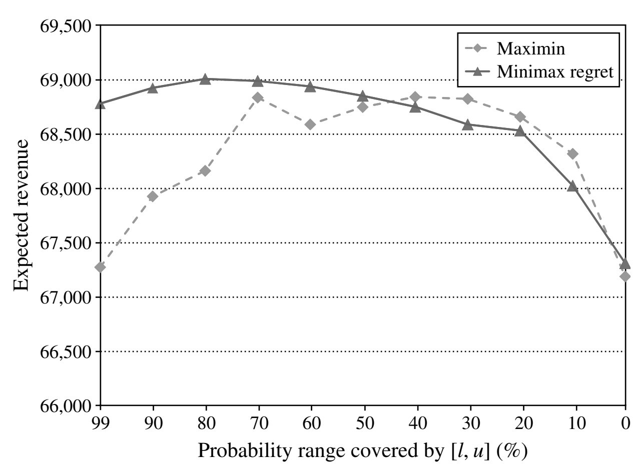
When the demand factor is smaller than 1, the minimax regret approach underperforms EMSR-b. Intuitively, the worst-case demand scenarios foreseen by the minimax regret are characterized by a high demand load factor (Lemma 3), which explains its superior performance under high demand factors.

5.1.3. Robustness to Choice of Uncertainty Set. Figure 2 displays the expected revenue as a function of the range of probability covered by the interval $[l_j, u_j]$. For instance, when l_j and u_j correspond to the tenth and ninetieth quantiles of the negative binomial demand distribution, the simulated revenues are equal to \$68,163 and \$69,009, respectively, under the maximin and the minimax regret nested booking limits. The concave shape of the functions illustrates

Table 2 Ninety Percent Confidence Intervals for the Mean Revenues for the Single-Resource Example from Talluri and van Ryzin (2004) with $\mathbf{r} = [1{,}050, 567, 534, 520]$

c	DF	Regret	Maximin	EMSR-b
80	1.70	49,479 ± 202	49,376 ± 120	49,426 ± 211
90	1.51	54,679 ± 202	54,568 ± 121	54,512 ± 223
100	1.36	59,853 ± 205	59,755 ± 131	59,697 ± 225
110	1.24	64,861 ± 223	64,702 ± 172	64,746 ± 238
120	1.13	69,253 ± 285	69,164 ± 254	69,252 ± 288
130	1.05	72,669 ± 375	72,790 ± 355	72,836 ± 371
140	0.97	74,902 ± 463	75,327 ± 453	75,343 ± 461
150	0.91	76,189 ± 527	76,861 ± 532	76,886 ± 537

Figure 2 Simulated Expected Revenue for Different Interval Sizes, Using the Minimax Regret and Maximin Nested Booking Limits in the Single-Leg Example by Talluri and van Ryzin (2004) when $c = 119$



[1] We choose the shape parameters of the beta distribution to ensure that high-fare demand almost certainly arrives after low-fare demand; in particular, we set the shape parameters to $\mathbf{a} = [10, 5, 2, 1]$ and $\mathbf{b} = [1, 2, 5, 10]$. For the gamma distribution, we set the shape parameter to $\mu_j^2/(\sigma_j^2 - \mu_j)$ and the scale parameter to $\sigma_j^2/\mu_j - 1$, consistently with the data presented in Table 1.

Table 3 Robustness of the Minimax Regret Controls with Respect to the Size of the Uncertainty Set for the Single-Resource Example from Talluri and van Ryzin (2004)

	[0, 1], [0, 1]		[1, 3], [0, 1]		[0, 1], [1, 3]		[1, 3], [1, 3]	
Range	Original	Adapted	Original	Adapted	Original	Adapted	Original	Adapted
0.5–99.5	63,767	63,812	72,441	72,445	47,487	47,455	69,107	69,131
10–90	66,778	66,813	72,052	72,069	57,590	57,545	69,281	69,295
20–80	68,070	68,127	71,795	71,795	61,848	61,885	69,490	69,520
30–70	69,234	69,269	71,533	71,526	65,663	65,741	69,921	69,948
40–60	69,613	69,736	70,919	70,947	67,896	68,050	69,848	69,952

the fundamental trade-off between flexibility, or realism, and conservativeness. Interestingly, the worse performance is attained when $l_j = u_j$, i.e., when the robust problems reduce to the classic DLP. Nevertheless, Figure 2 illustrates that, as long as the model captures at least 20% of uncertainty, the minimax regret approach is robust to the size of the range.

As a dual experiment, we consider different coefficients of variation of the demand and compare the performance of the original minimax regret booking limits, which appear in Table 1, to the performance of the minimax regret booking limits that are based on the twenty-fifth–seventy-fifth quantile range of the new demand distribution. The differences in simulated revenues ranged from -0.09% to 0.99% as the multiplying factor of the coefficient of variation increased from one-half to six.

We next test the robustness of the minimax regret control to the size of the uncertainty set under different demand distributions. Specifically, we assume that the rate of the demand for product j, denoted by G_j in (16), follows a beta distribution with range $[l_j, u_j]$. The parameters of this beta distribution a_j and b_j are uniformly distributed over either $[0, 1]$ or $[1, 3]$. The probability density function is increasing-decreasing when $a_j < 1$ and $b_j < 1$, increasing when $a_j > 1$ and $b_j < 1$, decreasing when $a_j < 1$ and $b_j > 1$, and decreasing-increasing when $a_j > 1$ and $b_j > 1$. We compare different interval sizes for $[l_j, u_j]$, corresponding to the 0.5–99.5, 10–90, 20–80, 30–70, and 40–60 quantile ranges of the original negative binomial demand distribution. Table 3 compares the simulated revenues of the original minimax regret booking limits (referred to as "original"), which appear in Table 1, with the revenues of the minimax regret booking limits that assume that the demand is distributed over $[l_j, u_j]$ when the demand rate G_j is distributed over the range $[l_j, u_j]$ (referred to as "adapted"). Although the simulated revenues are sensitive to the shape and the support of the demand distribution, the difference in revenues between the original and the adapted minimax regret controls is negligible. As a result, the performance of the minimax regret approach seems to be fairly insensitive to the size of the uncertainty set, for any shape of distribution.

5.1.4. Evenly Spaced Fares. With more evenly spaced fares, the performance of the robust approaches is still very good, as reported in Table 4, although the dominance of the EMSR-b heuristic is more pronounced.

5.1.5. Demand Correlation. We now investigate the impact of correlation among demands by assuming perfectly correlated demand rate distributions G_j in (16). Not surprisingly, the minimax regret approach tends to outperform EMSR-b, as shown in Table 5, because the worst-case demand scenarios against which the booking limits are protected are perfectly correlated (Lemma 3).

5.1.6. Censored Sales Data. The robust approaches clearly have the largest appeal when only limited demand information is available. Because

Table 4 Ninety Percent Confidence Intervals for the Mean Revenues for the Single-Resource Example from Talluri and van Ryzin (2004) with $\mathbf{r} = [1{,}050, 950, 699, 520]$

c	DF	Regret	Maximin	EMSR-b
80	1.70	67,587 ± 285	66,927 ± 177	67,804 ± 291
90	1.51	73,982 ± 331	72,152 ± 179	74,279 ± 362
100	1.36	79,211 ± 363	77,784 ± 200	79,597 ± 444
110	1.24	84,382 ± 401	83,839 ± 265	84,792 ± 445
120	1.13	89,382 ± 468	89,687 ± 377	89,914 ± 464
130	1.05	93,685 ± 574	94,429 ± 506	94,447 ± 533
140	0.97	96,774 ± 661	97,701 ± 630	97,683 ± 637
150	0.91	98,731 ± 730	99,651 ± 730	99,625 ± 730

Table 5 Ninety Percent Confidence Intervals for the Mean Revenues for the Single-Resource Example from Talluri and van Ryzin (2004) with $\mathbf{r} = [1{,}050, 950, 699, 520]$ with Perfectly Correlated Demand Rates

c	DF	Regret	Maximin	EMSR-b
80	1.70	65,979 ± 427	65,263 ± 327	65,994 ± 430
90	1.51	71,554 ± 517	70,397 ± 337	71,388 ± 555
100	1.36	76,745 ± 550	75,230 ± 403	76,066 ± 657
110	1.24	81,649 ± 603	80,001 ± 513	81,240 ± 659
120	1.13	86,064 ± 710	84,694 ± 646	85,899 ± 705
130	1.05	89,848 ± 845	88,783 ± 787	89,386 ± 806
140	0.97	92,790 ± 963	92,080 ± 921	92,233 ± 925
150	0.91	95,036 ± 1,072	94,585 ± 1,042	94,571 ± 1,041

Table 6 Ninety Percent Confidence Intervals for the Mean Revenues for the Single-Resource Example from Talluri and van Ryzin (2004) with $\mathbf{r} = [1{,}050, 950, 699, 520]$ with Censored Sales Observations

c	DF	Regret	Maximin	EMSR-b
80	1.70	62,678 ± 99	60,131 ± 85	61,321 ± 85
90	1.51	70,316 ± 147	67,664 ± 129	68,890 ± 124
100	1.36	78,107 ± 219	74,742 ± 188	76,357 ± 167
110	1.24	84,483 ± 312	83,070 ± 246	83,509 ± 248
120	1.13	90,025 ± 452	89,406 ± 366	89,757 ± 372
130	1.05	94,247 ± 559	94,486 ± 507	94,547 ± 513
140	0.97	97,226 ± 660	97,841 ± 631	97,812 ± 634
150	0.91	98,904 ± 732	99,760 ± 728	99,727 ± 728

only sales data are measured in practice, we now investigate the performance of the robust approaches with demand censoring.

We first compute the EMSR-b booking limits with the original demand data and simulate 100 booking processes, each covering 150 periods of demand, with these EMSR-b booking limits. We then estimate the means $\hat{\boldsymbol{\mu}}$, standard deviations $\hat{\boldsymbol{\sigma}}$, twenty-fifth quantiles $\hat{\mathbf{l}}$, and seventy-fifth quantiles $\hat{\mathbf{u}}$ from these sales data. These estimates are naive in the sense that they are not adjusted for the missing demand data. If the mean estimates $\hat{\boldsymbol{\mu}}$ are similar to the original mean $\boldsymbol{\mu}$, we use these estimates in the sequel. Otherwise, we recompute new EMSR-b booking limits, now based on the estimated demand parameters $\hat{\boldsymbol{\mu}}$ and $\hat{\boldsymbol{\sigma}}$, and simulate another 100 booking processes with these new booking limits. The procedure is repeated until the mean estimated from the sales data converges (within 5% in Euclidean norm) to the mean used to compute the EMSR booking limits. After the procedure has converged, we compute the maximin revenue, the minimax regret, and the EMSR booking limits using the most recent estimates $\hat{\boldsymbol{\mu}}$, $\hat{\boldsymbol{\sigma}}$, $\hat{\mathbf{l}}$, and $\hat{\mathbf{u}}$, and simulate their respective performance over 1,000 runs. In our experiments, the procedure had to be iterated only two or three times for the estimates to converge.

As shown in Table 6, the minimax regret approach tends to outperform EMSR when demand data are censored. In fact, the quantiles $\hat{\mathbf{l}}$ and $\hat{\mathbf{u}}$ are more robust to missing sales data than $\hat{\boldsymbol{\mu}}$ and $\hat{\boldsymbol{\sigma}}$, making the minimax regret approach perform better than EMSR-b. This dominance is stronger when the demand factor is high, i.e., when more sales are lost, making the estimates $\hat{\boldsymbol{\mu}}$ and $\hat{\boldsymbol{\sigma}}$ more distorted. The performance of the maximin booking limits, is in contrast, more dependent on accurate range estimation, because they focus on only one worst-case scenario (i.e., all demands equal to their lower bounds).

5.2. Small Network Example

We next consider the network example of de Boer et al. (2002), consisting of four cities in series. Flights are assumed to go only in one direction; accordingly, there are six possible OD pairs. Each OD pair has three fare classes, giving rise to 18 different products. There are 150 booking periods, and demand is assumed to follow a nonhomogeneous Poisson process. The fares, means, standard deviations, and shape parameters of the booking arrival process of all products appear in Tables 8 and 9 of de Boer et al. (2002). In addition to the base case, we consider a situation with larger variances (Table 10 in de Boer et al. 2002) and a smaller fare spread (Table 11 in de Boer et al. 2002). Each aircraft has 200 seats.

We benchmark the performance of the robust approaches against the the following static policies, which are commonly used in practice:

- DAVN EMSR/DLP: DAVN booking limits where products are clustered in at most five buckets, according to the algorithm described in the appendix to Bertsimas and de Boer (2005), and where the booking limits on each resource are respectively computed with EMSR-b (Belobaba 1992) and the DLP.
- Nested booking limit (BL) DLP: booking limits where products with the same OD pair are nested together (Curry 1990), and the booking limits are obtained from the DLP.
- Bid RLP: bid prices, obtained by averaging the shadow prices of 100 randomized DLPs, where the

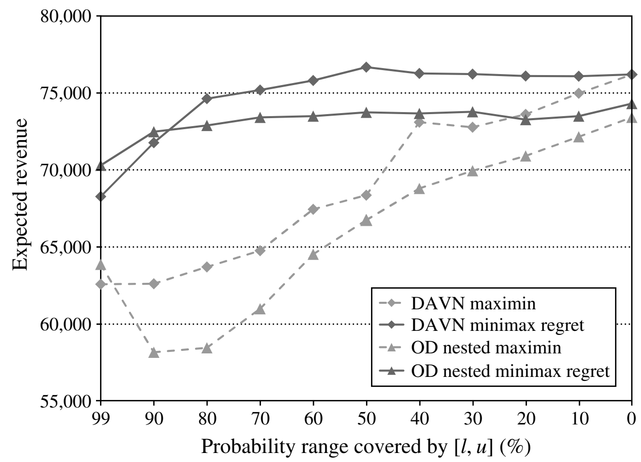
demand samples are independently generated from a negative binomial distribution.

The robust controls (DAVN-ρ/φ, Nested BL-ρ/φ, Bid-ρ/φ) are defined similarly, but are computed by solving (10). The MIPs were solved with the branch-and-bound algorithm of CPLEX 10, and the outer optimization problem was solved with a local gradient algorithm coded in C++. Note that, because of lack of convexity (Proposition 2), the gradient method is only guaranteed to converge to a local optimum.

As before, we assume that the only information available about the demand distributions is the range $[\mathbf{l}, \mathbf{u}]$, where $\mathbf{l}$ and $\mathbf{u}$ respectively correspond to the twenty-fifth and seventy-fifth quantiles. Similar to Figure 2, Figure 3 displays the simulated revenues as a function of the range of probability covered by the interval $[\mathbf{l}, \mathbf{u}]$ for the maximin and the minimax regret booking limits, either following a DAVN approach or nested by OD pairs. Figure 3 confirms that the performance of the minimax regret approach is very robust to the size of the uncertainty set, as long as $[\mathbf{l}, \mathbf{u}]$ does not cover more than 80% probability. In fact, when the range of uncertainty covers more than 80% probability, the maximum regret criterion is distorted because the interval midpoint $(\mathbf{l}+\mathbf{u})/2$ is very different from the median of the demand distribution, due to the large coefficients of variation (sometimes as large as 3/4) characterizing the demand in this problem.

We also noticed in our numerical tests that incorporating demand independence through (6) or imposing that, for each product, low-fare customers book first through (11) had no significant impact on the average

Figure 3 **Simulated Expected Revenue for Different Interval Sizes Using the Minimax Regret and Maximin Nested Booking Limits in the Network Example by de Boer et al. (2002)**



performance of the minimax regret approach. Therefore, the performance of the minimax regret is also robust to the shape of the uncertainty set.

Interestingly, in contrast to Figure 2, the performance of the robust approaches does not drop when the range of uncertainty tends to zero. In particular, the booking limits based on the DLP are almost performing the best. In fact, with a large number of products, the stochasticities of demand cancel each other out, and booking limits based on the mean demand are asymptotically optimal (Cooper 2002).

Table 7 presents the maximum regret $\rho(\mathbf{y})$ and the minimum revenue $\varphi(\mathbf{y})$ for the different methods, for the base case investigated by de Boer et al. (2002). We also report the 90% confidence intervals for the

Table 7 **Maximum Regret, Minimum Revenue, and 90% Confidence Intervals for the Mean Simulated Revenues for the Network Example by de Boer et al. (2002)**

	Base case			Lower fare spread	Inflated variance
Control	$\rho(\mathbf{y})$	$\varphi(\mathbf{y})$	$E[R(\mathbf{y})]$	$E[R(\mathbf{y})]$	$E[R(\mathbf{y})]$
DAVN-ρ	15,710	52,085	$76{,}667 \pm 382$	$63{,}853 \pm 213$	$75{,}863 \pm 371$
DAVN-φ	32,900	60,090	$68{,}358 \pm 134$	$59{,}476 \pm 101$	$69{,}652 \pm 168$
DAVN EMSR	16,320	51,370	$76{,}568 \pm 398$	$62{,}322 \pm 295$	$76{,}058 \pm 411$
DAVN DLP	16,510	53,730	$76{,}278 \pm 343$	$62{,}584 \pm 280$	$75{,}900 \pm 358$
Nested BL-ρ	10,595	56,890	$73{,}734 \pm 338$	$62{,}249 \pm 218$	$72{,}946 \pm 352$
Nested BL-φ	28,150	52,085	$66{,}744 \pm 134$	$58{,}994 \pm 91$	$64{,}346 \pm 163$
Nested BL DLP	14,040	54,820	$73{,}248 \pm 343$	$61{,}344 \pm 251$	$72{,}470 \pm 366$
Bid-ρ	19,130	48,730	$73{,}925 \pm 465$	$64{,}573 \pm 240$	$73{,}159 \pm 464$
Bid-φ	34,180	56,405	$72{,}911 \pm 225$	$62{,}849 \pm 145$	$72{,}391 \pm 265$
Bid RLP	23,715	54,810	$76{,}990 \pm 329$	$64{,}573 \pm 240$	$76{,}113 \pm 345$

mean revenues over 1,000 simulations of the booking process. In our simulation, because the maximin booking limits are, in general, smaller than the total capacity, we assumed that, whenever there was some leftover capacity after all booking limits had been reached, incoming requests were accepted on a first-come, first-served basis.

5.2.1. Performance of the Minimax Regret. Despite the asymptotic optimality of the booking limits based on the DLP, capturing demand uncertainty is valuable, as demonstrated by the performance gap between DAVN EMSR and DAVN DLP. Remarkably, the minimax regret is not only able to reap the additional benefits of capturing the stochastic nature of demand, it also often dominates the other approaches, within each class of policy.

Although capturing the stochastic nature of demand when choosing the *values* of the booking limits is valuable, it is even more important to choose the right *structure* for the static policy, as illustrated by the performance gap between the DAVN policy and the other policies. Dynamic policies could furthermore offer significant revenue gains.

Interestingly, the class of booking limit policy that performs the best on average (i.e., DAVN booking limits) differs from the policy that attains the lowest maximum regret (i.e., the Nested BL), at least among the policies we have considered. Although we have assumed the same booking curves for all products, like de Boer et al. (2002), it is important to keep in mind that there exists a worst-case demand pattern for which the DAVN is less appropriate than Nested BL-ρ. It is therefore paramount in practice to accurately characterize the demand arrival process and assess the validity of these assumptions before implementing a particular policy.

5.2.2. Performance of the Maximin. Not surprisingly, the maximin booking limits have lower performance than the other methods due to their conservative nature. The main benefit of the maximin approach is, however, that it guarantees a minimum revenue over all demand scenarios such that $\mathbf{d} \in \mathcal{P}(\eta)$, and as such, fits better with risk-averse decision makers.

5.2.3. Correlation. We now investigate the impact of correlation by assuming perfect correlation among the demands rates G_j in (16). With positive correlation, the demand stochasticities do not cancel out, and the relative performance of the DLP booking limits is expected to be lower. Moreover, as shown in Table 8, the relative performance of the EMSR booking limits also deteriorates as it assumes independent demands. In contrast, the minimax regret booking limits are adequate to deal with demand correlation, because most worst-case scenarios evaluated by the minimax regret objective assume correlation. As a result, when demands are not known to be independent, the minimax regret approach should be preferred because it is more amenable to dealing with correlation.

Table 8 Ninety Percent Confidence Intervals for the Mean Simulated Revenues for the Network Example by de Boer et al. (2002) with Perfectly Correlated Demand Rates

Control	Base case	Lower fare spread	Inflated variance
DAVN-ρ	$72{,}578 \pm 823$	$61{,}213 \pm 495$	$71{,}159 \pm 866$
DAVN-φ	$66{,}719 \pm 280$	$58{,}766 \pm 201$	$65{,}947 \pm 453$
DAVN EMSR	$72{,}396 \pm 851$	$59{,}667 \pm 621$	$70{,}900 \pm 929$
DAVN DLP	$72{,}511 \pm 760$	$60{,}209 \pm 592$	$70{,}972 \pm 851$
Nested BL-ρ	$72{,}353 \pm 755$	$61{,}072 \pm 480$	$70{,}814 \pm 842$
Nested BL-φ	$65{,}691 \pm 280$	$58{,}271 \pm 209$	$64{,}073 \pm 412$
Nested BL DLP	$71{,}935 \pm 793$	$60{,}444 \pm 587$	$69{,}913 \pm 892$
Bid-ρ	$69{,}262 \pm 956$	$60{,}745 \pm 565$	$66{,}532 \pm 1{,}001$
Bid-φ	$68{,}837 \pm 480$	$59{,}924 \pm 366$	$65{,}600 \pm 588$
Bid RLP	$71{,}733 \pm 749$	$60{,}745 \pm 565$	$68{,}519 \pm 826$

5.2.4. Censored Data. We now evaluate the performance of the robust approaches in the presence of censored sales observations. In particular, we estimate the mean, standard deviation, and quantiles from past sales data obtained with DAVN EMSR booking limits, with no adjustment for the missing data. The procedure for obtaining sales data is similar to that described in §5.1. Table 9 shows that, similar to the single-resource example, the minimax regret significantly dominates all traditional approaches in the presence of censored observations, sometimes by several percent. Because the minimax regret approach considers all scenarios as equally likely, a change in the shape of the demand distribution (through truncation, shift in mean, etc.) does not affect its performance, in contrast to the DLP, EMSR, and RLP approaches. Therefore, when the sales data are inaccurate or censored, the minimax regret booking limits should be preferred, because they are the most suitable to dealing with inaccurate demand estimates.

Table 9 Ninety Percent Confidence Intervals for the Mean Simulated Revenues for the Network Example by de Boer et al. (2002) with Censored Sales Observations

Control	Base case	Lower fare spread	Inflated variance
DAVN-ρ	$70{,}904 \pm 223$	$58{,}620 \pm 104$	$71{,}737 \pm 217$
DAVN-φ	$64{,}066 \pm 707$	$57{,}378 \pm 98$	$64{,}372 \pm 153$
DAVN EMSR	$69{,}402 \pm 163$	$57{,}906 \pm 104$	$69{,}391 \pm 180$
DAVN DLP	$68{,}371 \pm 164$	$57{,}896 \pm 109$	$68{,}601 \pm 177$
Nested BL-ρ	$70{,}758 \pm 241$	$58{,}849 \pm 104$	$69{,}994 \pm 221$
Nested BL-φ	$55{,}935 \pm 407$	$50{,}705 \pm 273$	$54{,}260 \pm 385$
Nested BL DLP	$66{,}163 \pm 192$	$57{,}555 \pm 113$	$66{,}511 \pm 192$
Bid-ρ	$68{,}771 \pm 210$	$60{,}737 \pm 132$	$70{,}647 \pm 252$
Bid-φ	$62{,}652 \pm 173$	$57{,}156 \pm 101$	$62{,}479 \pm 202$
Bid RLP	$62{,}652 \pm 173$	$57{,}156 \pm 101$	$62{,}479 \pm 202$

5.2.5. Minimax Randomized Regret. For large networks, finding the minimax regret booking limits is not practical, due to the complexity of (10). It is therefore important to develop good approximation schemes to compute the minimax regret booking limits efficiently. In Proposition 7, we showed that the minimax regret partitioned booking limits could be obtained by solving an LP when the regret is randomized. Because of the randomization, the optimal value of (15) is an upper bound on the maximum regret obtained with the same partitioned booking limits. The upper bound is in fact very tight. It is in fact no larger than 0.03% of the maximum regret associated with partitioned booking limits in the three scenarios considered.

The booking limits obtained from problem (15) can moreover be used in a nested fashion, either using the DAVN method or by nesting fare classes with the same OD pair, and the dual values obtained from problem (15) can be used as bid price controls. Table 10 displays the simulated mean revenues obtained with these controls (where the tilde refers to the fact that these controls are derived heuristically). Comparing these revenues with those obtained with the minimax (deterministic) regret (see Table 7) reveals that nearly the same level of performance can be attained with the randomized regret, at a much lower computational cost.

Table 10 Ninety Percent Confidence Intervals for the Mean Simulated Revenues, with the Booking Limits Obtained from the Minimax Randomized Regret Problem (15), for the Network Example by de Boer et al. (2002)

Control	Base case	Inflated variance	Lower fare spread
DAVN-$\tilde{\rho}$	$75{,}949 \pm 345$	$62{,}393 \pm 224$	$75{,}586 \pm 367$
Nested BL-$\tilde{\rho}$	$73{,}100 \pm 331$	$61{,}479 \pm 212$	$72{,}253 \pm 348$
Bid-$\tilde{\rho}$	$74{,}541 \pm 250$	$63{,}821 \pm 172$	$73{,}851 \pm 276$

5.3. Large Network Examples

We now investigate the viability of the robust approach for solving large-scale network RM problems. The first problem is the hub-and-spoke problem reported in Table 5.3. in Williamson (1988). Four cities are connected with a hub. Considering all possible origin–destination pairs, there is a total of 20 itineraries on 8 legs. In addition, there are four classes per itinerary. Each class on a given itinerary is associated with a mean demand, a standard deviation, and a fare. We use the same nonhomogenous Poisson model as before, assuming the same beta-distributed arrival pattern for all itineraries within a fare class and selecting the shape and scale parameters of the gamma distribution of the demand rate to match the given means and standard deviations.[2] We consider five variants of this problem, by taking the aircraft capacity equal to 100, 125, 150, 175, and 200 seats, corresponding to demand factors varying from 125% to 55.4%.

Table 11 reports the mean simulated revenues with the proposed policies. As before, the minimax regret approach is very comparable to more traditional approaches (except the bid price controls, which behave unevenly). As in the previous examples, the performance gap between the minimax regret approach and traditional approaches increases in the presence of demand correlation or censorship, but the results are not reported here for the sake of brevity.

The second problem is a real airline network problem with 517 different itineraries, each with 11 fare classes, and 67 segments. Arrivals follow a nonhomogenous Poisson process, with 16 changes of rates. The aggregate DF is 64.5%. We also analyze the TNG case study (van Ryzin 1998) consisting of 50 resources (weeks) with 200 products (number of lease durations offered times the number of weeks). In contrast to the previous examples, all demand occurs in the same period because there is no advance reservation.

[2] Because the binomial distribution requires $\sigma \geq \sqrt{\mu}$, we modify the problem data by taking $\sigma \doteq \max\{\sigma, \sqrt{\mu}\}$.

Table 11 **Mean Simulated Revenues for the Network Example by Williamson (1988)**

	Capacity				
Control	100	125	150	175	200
DAVN-ρ	129,634 ± 288	141,588 ± 334	152,906 ± 307	163,124 ± 313	171,625 ± 349
DAVN-φ	116,397 ± 104	124,877 ± 152	137,271 ± 175	150,659 ± 227	171,392 ± 316
DAVN EMSR	129,600 ± 288	141,882 ± 329	152,229 ± 321	163,176 ± 344	168,883 ± 363
DAVN DLP	129,396 ± 247	141,339 ± 272	151,131 ± 270	163,204 ± 284	173,274 ± 319
Nested BL-ρ	124,187 ± 244	137,640 ± 264	148,537 ± 275	158,620 ± 287	166,787 ± 345
Nested BL-φ	107,729 ± 103	118,567 ± 110	125,081 ± 119	125,081 ± 119	125,081 ± 119
Nested BL DLP	124,366 ± 239	137,306 ± 256	148,488 ± 271	158,516 ± 280	167,421 ± 299
Bid-ρ	126,328 ± 332	141,588 ± 334	147,228 ± 363	149,758 ± 369	171,042 ± 332
Bid-φ	124,010 ± 250	138,236 ± 152	149,806 ± 308	144,082 ± 326	171,042 ± 332
Bid RLP	124,010 ± 250	133,764 ± 307	149,806 ± 308	144,082 ± 381	171,042 ± 332

Because of the large computational cost involved with (10), we derive the controls from the maximin partitioned booking limits, which solve (13), and the randomized minimax regret partitioned booking limits, which solve (15). The partitioned booking limits are then transformed into nested booking limits to obtain the DAVN-$\tilde{\rho}/\tilde{\phi}$ and the Nested BL-$\tilde{\rho}/\tilde{\phi}$ controls, and the shadow prices of (13) and (15) are used to derive the Bid-$\tilde{\rho}/\tilde{\phi}$ controls.

Accordingly, the robust controls are based only on interval uncertainty. However, the TNG case is characterized by some degree of correlation between the demands for the same lease duration across successive periods and between the demands for different lease durations in the same period. Moreover, the TNG case assumes a strict order in the sequence of arrivals among the demands of different periods, because there is no advance reservation system. Although this additional demand information can easily be incorporated into (10) through (6) and (11), it cannot be incorporated into the LPs (13) and (15) without losing computational tractability. To quantify the value of this additional information, we evaluate the maximum regret associated with the robust controls DAVN-$\tilde{\rho}$, Nested BL-$\tilde{\rho}$, and Bid-$\tilde{\rho}$ using (10) with and without the constraints (6) and (11). Without constraints, the maximum regret of Nested BL-$\tilde{\rho}$ equals \$278,377, which represents about 10% of the simulated revenue. With the constraints, however, the maximum regret associated with Nested BL-$\tilde{\rho}$ drops to \$174,489. Therefore, although the actual loss from not incorporating the additional demand information may seem to be as high as \$278,377, it is in reality no larger than \$174,489. Although the maximum regret could be decreased by incorporating this additional demand information, there is only limited room for improvement.

Comparing Table 12 with Tables 7 and 11 confirms that, although the robust controls are derived only heuristically, without accounting for demand correlation and the strict sequence of arrivals, their performance is comparable to the best robust controls. In particular, the minimax regret typically outperforms the traditional controls by 0%–2%, and the maximin is very conservative, but has the merit of being associated with a minimum revenue guarantee. Because the minimax randomized regret and the modified DLP are simple LPs, robust methods can then be efficiently applied to large network RM problems.

6. Conclusions

In this paper, we develop robust formulations for the capacity allocation problem in RM under polyhedral

Table 12 **Ninety Percent Confidence Intervals for the Mean Simulated Revenues for the Real-World Hub-and-Spoke Network Problem and the TNG Case Study**

Control	Revenue hub and spoke	Revenue TNG
DAVN-$\tilde{\rho}$	3,421,810 ± 1,275	4,259,770 ± 4,119
DAVN-$\tilde{\varphi}$	3,204,490 ± 1,649	4,260,560 ± 3,557
DAVN EMSR	3,386,050 ± 1,471	4,213,610 ± 5,144
DAVN DLP	3,368,030 ± 1,461	4,270,230 ± 4,428
Nested BL-$\tilde{\rho}$	3,047,670 ± 815	4,000,828 ± 3,685
Nested BL-$\tilde{\varphi}$	1,506,220 ± 628	3,678,170 ± 2,292
Nested BL DLP	2,708,580 ± 822	3,904,470 ± 3,599
Bid-$\tilde{\rho}$	3,351,350 ± 1,734	4,289,990 ± 3,560
Bid DLP-$\tilde{\varphi}$	3,351,350 ± 1,734	4,289,990 ± 3,560
Bid RLP	3,351,350 ± 1,734	4,289,990 ± 3,560

uncertainty, using the maximin and the minimax regret criteria. We consider generic booking limit controls, including partitioned booking limits, nested booking limits, DAVN, and fixed bid prices. Under interval uncertainty, we provide closed-form solutions or LP formulations for the maximin and minimax regret controls for two problems that lie at the foundation of airline RM and hotel RM, namely, a single-resource problem and a a serial network with promotional packages. We also develop an efficient heuristic to compute minimax regret partitioned booking limits in a general network.

Our numerical study reveals that the minimax regret policy generally performs as well as traditional approaches and tends to outperform them by several percent on real problem instances, when demands are correlated or when the demand parameter estimates are inaccurate. The maximin policy is, in contrast, more conservative, but its ability to guarantee a minimum revenue over all demand realizations is definitely an attractive feature to risk-averse decision makers.

Robust approaches are pragmatic. They are scalable to solve practical network RM problems because they often combine efficient (exact or heuristic) solution methods with modest data requirements. In contrast to most traditional approaches, which assume independent demands, strict low-before-high fare arrivals, and constant product mix over time, robust approaches are also flexible because they require only the specification of a polyhedral uncertainty set, without being too sensitive to its specific characterization. Because of its unique combination of flexibility and scalability, the robust optimization paradigm has a tremendous potential for improving operations and increasing revenues, especially in fast-changing business environments.

Electronic Companion

An electronic companion to this paper is available on the *Manufacturing & Service Operations Management* website (http://msom.pubs.informs.org/ecompanion.html).

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