Pre-announced Pricing Strategies with Reservations

Wedad Elmaghraby, Steven A. Lippman, Christopher S. Tang, and Rui Yin[1]

First version: December 2, 2005; Second version: June 1, 2006

Abstract

At the beginning of the selling season, the retailer announces both the price p_h at which the product will be sold during the selling season and the post-season clearance price $p_\ell < p_h$ for unsold items. Customers arrive in accord with a Poisson process, and each of the $n + 1$ customer classes has a class-specific valuation. We analyze two operating regimes. Under the reservation regime, a buyer either can purchase the product (if available) at price p_h or reserve the product for purchase at the clearance price p_ℓ. If the buyer reserves the product and if it remains unsold at the end of the selling season, then he is obligated to purchase it at price p_ℓ. Under the no reservation regime, a buyer either purchases the product when he arrives at price p_h or he enters a lottery to purchase at price p_ℓ if the product remains unsold. In the presence of stochastic arrivals, we characterize rational purchasing behavior wherein each buyer takes other buyers' purchasing behavior into consideration, and we analyze the retailer's expected payoff and the customer's expected surplus under the two operating regimes.

Keywords: Pre-announced Pricing, Progressive markdown, Reservation.

[1]Direct correspondence to any of the authors. Elmaghraby: Smith School of Business, University of Maryland, College Park, MD 20742, email address: Wedad_Elmaghraby@rhsmith.umd.edu; Lippman, Tang, and Yin: UCLA Anderson School, UCLA, 110 Westwood Plaza, CA 90095, email addresses: slippman@anderson.ucla.edu, ctang@anderson.ucla.edu, ryin@anderson.ucla.edu. The authors are grateful to Professors Sushil Bikhchandani and Kevin McCardle for sharing their ideas throughout this research project.

1 Introduction

At the end of the selling season, many retailers dispose of unsold seasonal goods at a "clearance price." One way to obtain a higher clearance price is to announce the price schedule in advance so as to entice customers to return to purchase unsold items. A pre-announced pricing schedule, known as 'progressive markdown' or 'automatic markdown' in retailing, has been adopted by many companies including FairMarket, Comp USA, Land's End, J.C. Penny, etc.

We consider the situation in which the retailer has 1 unit to sell during the selling season. At the beginning of the season, the retailer announces that the product will be sold at the reduced price p_ℓ if it is not sold at the regular price p_h ($> p_\ell$) by the end of the season. We consider two operating regimes: 'No Reservation' and 'With Reservation'. Under either regime, a buyer either can purchase the product (if available) during the season at price p_h or attempt to purchase the product at the reduced price p_ℓ after the selling season ends. Suppose the product was not sold during the season. Under the NR (No Reservation) regime, all interested buyers enter a lottery, and the lottery winner purchases the product at the reduced price p_ℓ.[1] A more general version of the NR regime has been implemented since 1908 at the Filene's Basement store in Boston. At the Filene's Basement store, the markdown price schedule is pre-announced: each unsold unit after 14, 21, or 28 days will be sold at 25%, 50%, or 75% off the regular price, respectively. Items are sold on a first-come-first-serve basis. (Filene's Basement donates all unsold items to charitable organizations after 35 days: see www.filenesbasement.com and Bell and Starr (1998) for more details.)

From the customer's perspective, the NR regime has two drawbacks. First, to buy the product at the reduced price, each customer must return to the store (or the web site) at the end of the season to check upon the item's availability. Second, because the item is sold on a first-come-first-serve basis, purchase at the reduced price is not guaranteed. Thus, the NR regime creates additional inconveniences for the buyers, thereby imposing additional

[1]In practice, at the end of the season, all markdown items are sold on a first-come-first-serve basis. To capture the fact that the first customer to return to the store after the season ends purchases at price p_ℓ, we assume that each buyer has equal probability of winning the lottery. Clearly, the retailer can use other mechanisms, such as an auction, to dispose of unsold items. But an auction may not be practical in the context of retailing because it requires all customers to be present simultaneously for the auction. Practicality aside, in Section 5 we investigate use of an auction as it is a reasonable approximation of a mark down schedule with several price reductions.

‘imputed’ costs.[2]

In order to reduce these imputed costs, some retailers have introduced the WR (With Reservation) regime which allows a buyer to attempt to purchase at the reduced price p_ℓ by placing a reservation (or hold). If the product remains unsold at the end of the season, his reservation obligates him to purchase at price p_ℓ. Because only 1 unit is for sale, only one buyer can place a reservation.[3] Under the WR regime, a buyer who has placed a reservation does not need to return to the store (or web site) and his purchase is guaranteed (provided that the product was not sold during the season): the WR regime simplifies the purchasing experience for the buyers. A more general version of the WR regime has been implemented by Sam’s club (http://auctions.samsclub.com) in which the “buy later” option obligates the buyer to purchase at price p_ℓ if the item remains available.

In light of these two operating regimes three natural questions arise. Fix p_h and p_l:
Given rational buyers, what is the optimal purchasing behavior under each regime?
Which regime (NR or WR) generates a higher expected payoff for the retailer?
Which regime (NR or WR) generates a higher expected surplus for the customer?

To answer these questions, our model incorporates stochastic customer arrivals, rational purchasing behavior, and Poisson arrivals of multiple classes of customers with class-specific valuations. It captures rational purchasing behavior in the sense that each customer’s purchasing decision accounts for the rational purchasing behavior of other customers. As an initial attempt to examine the optimal purchasing behavior, the retailer’s expected payoff, and the customer’s expected surplus under each regime, we shall consider the case when the retailer charges the same prices p_h and p_l under both regimes. This case occurs when the products have well established price points so that it is not practical for the retailer to charge different prices under different regimes (c.f., Elmaghraby and Keskinocak (2003)).

With two exceptions the entirety of the extant literature, beginning with Stokey (1979), is predicated upon having all potential buyers present at the very beginning of the season: stochastic arrival of customers has not been considered until recently by Aviv and Pazgal (2005). Stokey (1979) was the first to address the issue of rational purchasing behavior when the price schedule is pre-announced. She analyzed the optimal price schedule when

[2]Based on an empirical study of the shopping trips made by 12,000 shoppers to 5 different supermarket over a 2-year period, Bell et al. (1998) confirm that there is an imputed cost associated with travel distance.

[3]Notice that the WR regime is equivalent to an English auction with 2 pre-specified permissable bid prices p_l and p_h in which bidding p_l is equivalent to placing a reservation under the WR regime.

the number of buyers is fixed and known. Besanko and Winston (1990) modifies Stokey's model by assuming that the seller/retailer can make only a finite number of price adjustments in contrast to Stokey's infinite number of adjustments. With a known number of strategic customers, Harris and Raviv (1981) showed that the optimal price schedule depends on the number of units available for sales. More recently, Elmaghraby, Gulcu and Keskinocak (2004) developed a model analyzing the optimal price schedule when each buyer demands multiple units. All of these models assume that all customers are present at the beginning of the season.

A second major difference between the literature (especially Aviv and Pazgal (2005)) and our model is the buyer's option to reserve the product. In one of the first papers to consider granting the buyer the right (*i.e.,* reservation), but not the obligation, to purchase the product at a later date, Png (1989) considers an airline which, in order to increase utilization, takes customer reservations. After booking the reservation, the customer discerns his own valuation/type: prices are set so that only high valuation customers exercise the purchase option. However, overbooking allows the airline to renege on the offered reservation: the customer has no guarantee that his reservation will be honored. Thus, Png's model is akin to our WR model but differs from ours wherein the customer is afforded the opportunity to make a guaranteed purchase.

Biyalogorsky and Gerstner's 2004 paper is the aforementioned exception; they examine a two period model in which one low valuation customer arrives in period 1 and one high valuation customer arrives in period 2 with known probability $q < 1$. As in our paper, the seller has one unit available for sale. They show that the seller does best by (1) selling to the high valuation customer at price p_h and allowing the low valuation customer to purchase at a price less than p_h if there is no high valuation customer. While the arrival process is considerably simpler than ours and there are but two points in time, their model shares the reservation aspect of our model.

A third major difference is that buyer behavior is myopic in the literature: a buyer makes a purchase only if the price is below his own reservation price, the behavior of other buyers is not considered. See Elmaghraby and Keskinocak (2003) for a comprehensive review of dynamic pricing models.

In Section 2 we explicate the underlying mathematical structure of the model with a single class of customers. We determine rational purchasing behavior under both the NR and WR regimes and demonstrate that the retailer always enjoys a higher expected payoff under the WR regime. The customers, however, always obtain a higher expected payoff under the NR regime. Extention to the case of two classes of customers with different valuations for the item is treated in Section 3 where we establish conditions under which the retailer enjoys a higher expected payoff for the WR regime as well as conditions under which the customers obtain a higher expected payoff for the NR regime. The general case of multiple classes of customers is presented in Section 4 where numerical examples are used to compare the performance between the NR and WR regimes. A variant of the NR regime under which the retailer utilizes a second-price sealed-bid auction to allocate the product in the post-season is considered in Section 5 while Section 6 concludes with suggestions for related future research.

2 The Base Model

Consider a retailer selling a single unit of a product over the selling season $[0, T]$. At the beginning of the season, the retailer pre-announces that if the product is not sold at the regular price p_h during the season, then it will be sold at the post-season clearance (reduced) price p_l (and eventually at the salvage value $s < p_l$ if not sold at clearance). During the season, customers arrive in accord with a Poisson process with rate λ.[4] In the base model, the market is comprised of a single class of customers (class 1), and all class 1 customers have identical valuation v_1. To ensure that a class 1 customer might purchase the product during the season, we assume $v_1 \geq p_h > p_l$ throughout this paper. The parameter values v_1, p_h, p_l, and λ are common knowledge. In addition, each customer knows his own arrival time t and whether the product has been reserved by another customer. This base model with a single class of customers enables us to understand the underlying structure of the model and to generate specific insights.

[4]Because the imputed cost for the customers associated with the WR regime is lower, it is plausible that the arrival rate λ is higher under the WR regime. For ease of exposition, we assume that the arrival rate λ remains the same under the NR and WR regimes.

2.1 No Reservation Regime

We start by exhibiting a purchasing rule which is a Nash equilibrium: it is used by all arriving customers ensures rational purchasing behavior. This purchasing rule is characterized by a single threshold. A customer is said to follow the "no reservation threshold" (NR-threshold) if, given his arrival time t, he purchases the item when $t < t^*$ and he joins the lottery when $t \geq t^*$. The threshold value t^* is defined in (2.1).

To begin, let $B_1(t)$ and $A_1(t)$ be the number of class 1 customers to arrive, respectively 'before' and 'after' time t, and notice that $B_1(t)$ and $A_1(t)$ are independent Poisson random variables with parameters λt and $\lambda(T-t)$, respectively. Define P_t as the probability that a customer who arrives at time t wins the lottery when all customers who arrive after time t join the lottery and there are no customers who arrive before time t. Clearly, $P_t = E\{1/[1+A_1(t)]\}$. Under the NR regime, a customer who arrives at time t enjoys a surplus of $(v_1 - p_h)$ if he purchases the product at time t. Alternatively, he can elect not to purchase the product and enter the lottery with $A_1(t)$ other class 1 customers who arrive after him. If none of these $A_1(t)$ customers purchase the product and there were no arrivals prior to time t, then P_t is the probability that he wins the lottery, and he obtains expected surplus $(v_1 - p_l)P_t$. Because the number $A_1(t)$ of arrivals on the time interval $(t, T]$ is (stochastically) non-increasing in t, P_t is strictly increasing in t. Moreover, P_t is continuous in t, $P_T = 1$, and $\lim_{t\to-\infty} P_t = 0$. Consequently, there is a unique value t^* of t which equalizes these two surpluses:

$$v_1 - p_h \quad = \quad (v_1 - p_l)P_{t^*}. \tag{2.1}$$

If $t^* < 0$, then (2.1) tells us that the customer is always better off joining the lottery, and hence, we can set $t^* = 0$.[5] Because $P_T = 1$ and $v_1 - v_h < v_1 - p_l$, we must have $t^* < T$: joining the lottery is optimal if the season is nearly over. In Proposition 1 we prove that rational purchasing behavior requires that all customers act in accord with the threshold t^*.

Proposition 1 *There is a Nash equilibrium in which all arriving customers follow the NR-threshold. The threshold t^* is increasing in λ, v_1, and p_l, and decreasing in p_h.*

[5] As we shall see later, $P_{t^*} = \frac{1-e^{-\lambda(T-t^*)}}{\lambda(T-t^*)}$. Substitute P_{t^*} into (2.1), we can show that $t^* \geq 0$ when $\frac{1-e^{-\lambda T}}{\lambda T} \leq \frac{v_1-p_h}{v_1-p_l}$ (which holds when T is sufficiently large). To simplify our exposition, we shall assume that T is sufficiently large so that $\frac{1-e^{-\lambda T}}{\lambda T} \leq \frac{v_1-p_h}{v_1-p_l}$ throughout this paper.

Proof: Assume all customers follow NR-threshold. Suppose a customer arrives at time $t < t^*$ and decides to join the lottery. Because any other customer who arrives in (t, t^*) purchases the item and those who arrive in $[t^*, T]$ join the lottery, his expected surplus is

$$(v_1 - p_l)P_{t^*}e^{-\lambda(t-t^*)} < (v_1 - p_l)P_{t^*} = v_1 - p_h.$$

Hence, purchasing at time $t < t^*$ strictly dominates joining the latter. Now suppose a customer arrives at time $t > t^*$ and decides to purchase at p_h. Then his surplus is

$$v_1 - p_h = (v_1 - p_l)P_{t^*}.$$

Hence, purchasing at time $t > t^*$ does not yield a higher surplus.

The comparative statics results for v_1, p_l, and p_h are immediate from (2.1) and P_t strictly increasing. Similarly, for each t, $A_1(t)$ is (stochastically) non-decreasing in λ whence P_t is strictly decreasing in λ which, in turn, requires that t^* increases if λ increases. More formally, fix $\lambda^+ > \lambda$, consider two independent Poisson processes with parameters λ and $\lambda^+ - \lambda$, and let $A_1(t)$ and $A_2(t)$ be the number of arrivals on $(t, T]$, respectively. Because the sum of these two independent Poisson processes is itself a Poisson process with parameter λ^+, P_t^+ satisfies $P_t^+ = E\{1/[1 + A_1(t) + A_2(t)]\} < E\{1/[1 + A_1(t)]\} = P_t$. ■

2.1.1 Payoffs under the NR Regime

Because all customers follow the threshold rule in equilibrium, the retailer's payoff depends on $B_1(t^*)$ and $A_1(t^*)$. Consider the following three mutually exclusive and exhaustive events and the retailer's surplus when each occurs. The retailer's surplus is s when $B_1(t^*) + A_1(t^*) = 0$, p_l when $B_1(t^*) = 0$ and $A_1(t^*) > 0$, and p_h when $B_1(t^*) > 0$ and $A_1(t^*) \geq 0$. Coupling these observations with the facts that $B_1(t)$ and $A_1(t)$ are independent Poisson random variables with parameters λt and $\lambda(T - t)$, respectively, enables us to compute π_r, the retailer's expected payoff.

Lemma 1 *Under the NR regime, the retailer's expected payoff π_r can be written as:*

$$\begin{aligned} \pi_r &= (1 - e^{-\lambda t^*})p_h + e^{-\lambda t^*}(1 - e^{-\lambda(T-t^*)})p_l + e^{-\lambda t^*}e^{-\lambda(T-t^*)}s \\ &= p_h - e^{-\lambda t^*}(p_h - p_l) - e^{-\lambda T}(p_l - s) \end{aligned} \tag{2.2}$$

Lemma 1 suggests that the retailer's expected payoff is equal to p_h minus the "opportunity loss" for the two cases in which no customers arrive before time t^*.

In equilibrium, the customer earns a surplus of $(v_1 - p_h)$ when $B_1(t^*) > 0$ and $A_1(t^*) \geq 0$, and he earns a surplus of $(v_1 - p_l)$ when $B_1(t^*) = 0$ and $A_1(t^*) > 0$. Thus, Π_c, the expected surplus earned by the set of customers, satisfies

$$\begin{aligned} \Pi_c &= (v_1 - p_h)(1 - e^{-\lambda t^*}) + (v_1 - p_l)e^{-\lambda t^*}(1 - e^{-\lambda(T-t^*)}) \\ &= (v_1 - p_l)(1 - e^{-\lambda T}) - (p_h - p_l)(1 - e^{-\lambda t^*}) \end{aligned} \tag{2.3}$$

2.2 With Reservation Regime

We now consider rational purchasing behavior under the With Reservation (WR) regime. A customer arriving at time t is said to follow the "with reservation threshold" (WR-threshold) if he (a) purchases the product at price p_h whenever it has already been reserved; (b) purchases the product at price p_h if $t < \hat{t}$; and (c) reserves the product if $t \geq \hat{t}$, where $\hat{t}$ is defined below in (2.4). Consider a customer arriving at time t. First, if the product was reserved earlier by another customer, then the customer arriving at time t should purchase the product at price p_h and earn a surplus of $v_1 - p_h$; otherwise, he will earn a surplus of 0. Second, if the product is available for purchase or reservation at time t, then he earns a surplus of $(v_1 - p_h)$ if he immediately purchases the product at price p_h. If instead he reserves the product, then he obtains a surplus of $v_1 - p_l$ if no customer arrives after time t or he obtains a surplus of 0 if one or more customers arrive after time t because the next arriving customer will in fact purchase the reserved product. Combining these observations with the fact that $e^{-\lambda(T-t)}$ is the probability no customer arrives after t, we see that $\hat{t}$ satisfies

$$v_1 - p_h = e^{-\lambda(T-\hat{t})}(v_1 - p_l). \tag{2.4}$$

It is useful to rewrite (2.4) as

$$\hat{t} = T - \frac{1}{\lambda} ln(\frac{v_1 - p_l}{v_1 - p_h}). \tag{2.5}$$

As in the NR regime, we must have $\hat{t} < T$. If $\hat{t} < 0$, then (2.4) is telling us that the customer never purchases the product at price p_h: in essence, $\hat{t} = 0$.[6] By examining (2.5), we have:

[6] Notice from (2.5) that $\hat{t} \geq 0$ when $T \geq \frac{1}{\lambda} ln(\frac{v_1 - p_l}{v_1 - p_h})$. To simplify our exposition, we shall assume T is sufficiently large so that $T \geq \frac{1}{\lambda} ln(\frac{v_1 - p_l}{v_1 - p_h})$ throughout this paper.

Proposition 2 *There is a Nash equilibrium in which all arriving customers follow the WR-threshold. The threshold* $\hat{t}$ *is increasing in* λ, v_1, *and* p_l, *and decreasing in* p_h. *Moreover,* $\hat{t} > t^*$.

Proof: Proving that all customers following the WR-threshold is a Nash equilibrium is nearly identical to the proof given for the NR regime. The comparative statics on $\hat{t}$ is straightforward. To compare $\hat{t}$ and t^* observe

$$P_t = E[\frac{1}{1 + A_1(t)}] = \sum_{n=0}^{\infty} \frac{1}{n+1} \frac{g(t)^n}{n!} e^{-g(t)} = (1 - e^{-g(t)})/g(t), \tag{2.6}$$

where $g(t) = \lambda(T - t)$. Because $e^x > 1 + x$ for $x > 0$, it follows that

$$P_{t^*} = (1 - e^{-g(t^*)})/g(t^*) = e^{-g(t^*)}[e^{g(t^*)} - 1]/g(t^*) > e^{-g(t^*)} = e^{-\lambda(T-t^*)}.$$

Employing (2.1), (2.6), and (2.5) yields $t^* < T - \frac{1}{\lambda} \cdot ln(\frac{v_1 - p_l}{v_1 - p_h}) = \hat{t}$. ■

2.2.1 Payoffs under the WR regime

Lemma 2 *In equilibrium,* $\hat{\pi}_r$, *the retailer's expected payoff under the WR regime, satisfies*

$$\begin{aligned} \hat{\pi}_r &= (1 - e^{-\lambda\hat{t}})p_h + e^{-\lambda\hat{t}}\{(\lambda(T - \hat{t})e^{-\lambda(T-\hat{t})})p_l + (1 - e^{-\lambda(T-\hat{t})} - \lambda(T - \hat{t})e^{-\lambda(T-\hat{t})})p_h\} + e^{-\lambda T}s \\ &= p_h - (p_l - s)e^{-\lambda T} - (p_h - p_l)e^{-\lambda T}(1 + \lambda(T - \hat{t})) \end{aligned} \tag{2.7}$$

Proof: We utilize the argument used to establish Lemma 1: consider four mutually exclusive and exhaustive events and compute the retailer's surplus in each case. In particular, the retailer's surplus is (a) p_h when $B_1(\hat{t}) > 0$ and also when $B_1(\hat{t}) = 0$ and $A_1(\hat{t}) \geq 2$; (b) p_l when $B_1(\hat{t}) = 0$ and $A_1(\hat{t}) = 1$; and (c) s when $B_1(\hat{t}) + A_1(\hat{t}) = 0$. Combining these observations with the fact that $B_1(\hat{t})$ and $A_1(\hat{t})$ are independent Poisson random variables suffices to establish (2.7). ■

The customers' equilibrium expected payoff is determined using the same four set partition. In particular, $\hat{\Pi}_c$, the customers' expected payoff, is easily seen to satisfy

$$\begin{aligned} \hat{\Pi}_c &= (1 - e^{-\lambda\hat{t}})(v_1 - p_h) + e^{-\lambda\hat{t}}\{(\lambda(T - \hat{t})e^{-\lambda(T-\hat{t})})(v_1 - p_l) + \\ &\quad (1 - e^{-\lambda(T-\hat{t})} - \lambda(T - \hat{t})e^{-\lambda(T-\hat{t})})(v_1 - p_h)\} \\ &= (v_1 - p_h)(1 - e^{-\lambda T}) + (p_h - p_l)e^{-\lambda T}\lambda(T - \hat{t}). \end{aligned} \tag{2.8}$$

2.3 Comparison of Payoffs

Proposition 3 *Given prices $p_h > p_l$, the retailer's expected payoff is higher under the WR regime whereas the customers' expected payoff is lower: $\hat{\pi}_r > \pi_r$ and $\hat{\Pi}_c < \Pi_c$.*

Proof: Using $\hat{t} > t^*$ and then $e^x > 1 + x$ for $x > 0$, we see that

$$e^{-\lambda T}[1 + \lambda(T - \hat{t})] < e^{-\lambda T}[1 + \lambda(T - t^*)] < e^{-\lambda T} e^{\lambda(T - t^*)} = e^{-\lambda t^*}.$$

Consequently, $\hat{\pi}_r > \pi_r$ follows from (2.2) and (2.7). Next, observe that

$$\begin{aligned} [\Pi_c - \hat{\Pi}_c]/[(p_h - p_l)e^{-\lambda T}] &= e^{\lambda(T - t^*)} - 1 - \lambda(T - \hat{t}) \\ &> 1 + \lambda(T - t^*) - 1 - \lambda(T - \hat{t}) = \lambda(\hat{t} - t^*) > 0, \end{aligned}$$

where the inequalities follow from $e^x > 1 + x$ for $x > 0$ and Proposition 2, respectively. ■

We anticipate that the arrival rate λ will be higher for the WR regime due to its aforementioned customer convenience. Because an increase in λ increases the retailer's expected payoff $\hat{\pi}_r$, this further augments the retailer's incentive to use the WR regime.

3 Extension 1: Two Classes of Customers

Section 2 established the existence of threshold rules t^* and $\hat{t}$ which are Nash equilibria for the NR and the WR regime, respectively. In addition to the comparative statics results of Propositions 1 and 2, we also demonstrated that $\hat{t} > t^*$ and that the retailer prefers the WR regime whereas customers prefer the NR regime. Our analysis in this section demonstrates that some, but not all, of these results extend to the case of two classes of customers.

In this section we assume that for $i = 0$ and $i = 1$ each class i customer has valuation v_i for the item, and, moreover, $v_1 \geq p_h > v_0 \geq p_l$: class 0 customers never purchase during the selling season. Define $\alpha > 0$ to be the fraction of class 1 customers. Customers arrive in accord with independent Poisson processes with parameter $\alpha\lambda$ for class 1 customers and $(1-\alpha)\lambda$ for class 0 customers. As before, $A_i(t)$ and $B_i(t)$ are the number of class i customers who arrive before time t and after time t, respectively.

3.1 No Reservation Regime

We now extend the NR-threshold to two customer classes under the NR regime. A class i customer is said to follow the NR threshold if, given his arrival time t, he purchases the item at price p_h when $t < t_i^*$ and he joins the lottery when $t \geq t_i^*,\ i = 1, 0$. The assumption $v_0 < p_h$ implies that all class 0 customers join the lottery: $t_0^* = 0$. As before, define P_t to be the probability that a class 1 customer who arrives at time t wins the lottery when all customers who arrive after him join the lottery and there are no class 1 customers who arrive before him. Clearly, $P_t = E\{1/[1 + A_0(0) + A_1(t)]\}$. As in Section 2, t_1^* equilibrates the surplus for a class 1 customer who purchases the product and one who enters the lottery:

$$v_1 - p_h \quad = \quad (v_1 - p_l)P_{t_1^*}. \tag{3.1}$$

Proposition 4 *There is a Nash equilibrium in which all arriving customers follow the NR-threshold. The threshold t_1^* is increasing in λ, v_1 and p_l and decreasing in p_h and α. Moreover, $t_1^* \geq 0$. Furthermore, there exists a critical value c^* with $0 \leq c^* \leq 1$ such that $T \geq t_1^* = t^*/\alpha > t^*$ when $\alpha \geq c^*$.*

Proof: The proof of the Nash equilibrium follows as per the proof of Proposition 1. Most of the comparative statics proof follows as per Proposition 1. From (2.6) we see that $P_t = [1 - e^{-g(t)}]/g(t)$ with $g(t) = \lambda T(1-\alpha) + \alpha\lambda(T - t) = \lambda T - \alpha\lambda t$. Substitute $P_{t_1^*}$ into (3.1) and use the fact that $\frac{1-e^{-\lambda(T-\alpha t_1^*)}}{\lambda(T-\alpha t_1^*)}$ is increasing in αt_1^*, $t_1^* \geq 0$ if $\frac{1-e^{-\lambda T}}{\lambda T} \leq \frac{v_1-p_h}{v_1-p_l}$ (which we assume hold in Section 2.1). In addition, we can show that $t_1^* \leq T$ when $\frac{v_1-p_h}{v_1-p_l} \leq \frac{1-e^{-\lambda(1-\alpha)T}}{\lambda(1-\alpha)T}$, which holds when α exceeds a certain critical value c^*. Therefore, when $\alpha \geq c^*$, the desired result now follows as $\alpha\lambda t_1^* = \lambda t^*$. ■

The explicit formula $t_1^* = t^*/\alpha$ asserts that t_1^* increases as the proportion $1 - \alpha$ of class 0 customers increases. This is intuitive because more customers join the lottery as $1 - \alpha$ increases. Consequently, the incentive for a class 1 customer to purchase rather than join the lottery (with more customers) increases: t_1^* increases as α decreases. If the fraction $1-\alpha$ of class 0 customers is sufficiently large (and λ is not small), then $t_1^* = T$: a class 1 customer will purchase the item if it is available, without regard to when he arrives. Henceforth, we assume α is sufficiently large to ensure $t_1^* < T$.

3.1.1 Payoffs under the NR Regime

Because $t_1^* < T$, we can use the same approach as in Section 2.1.1 and use the fact that $\alpha t_1^* = t^*$ when $\alpha \geq c^*$ (by Proposition 4) to show that the retailer's expected payoff in the presence of two customer classes is the same as when there is only one class.

Lemma 3 *Under the NR regime, π_r', the retailer's expected payoff with two customer classes case, is given by*

$$\begin{aligned}\pi_r' &= (1 - e^{-\alpha\lambda t_1^*})p_h + e^{-\alpha\lambda t_1^*}(1 - e^{-\alpha\lambda(T-t_1^*)}e^{-(1-\alpha)\lambda T})p_l + e^{-\lambda T}s \\ &= p_h - e^{-\alpha\lambda t_1^*}(p_h - p_l) - e^{-\lambda T}(p_l - s).\end{aligned} \tag{3.2}$$

Furthermore, when $\alpha \geq c^*$, $\pi_r' = \pi_r$.

Becuase π_r is independent of α, Lemma 3 implies that π_r' is independent of α when $\alpha \geq c^*$.

Computing the customers' expected surplus, denoted by Π_c', is more difficult than computing the customers' expected surplus Π_c when there is only one class because we must account for which customer class purchases the item when it is sold in the lottery. Fortunately, independent Poisson arrivals enable us to assert that if the item is sold in the lottery, then a class 1 customer purchases the item with probability $\alpha\lambda(T - t_1^*)/[(1-\alpha)\lambda T + \alpha\lambda(T - t_1^*)]$. Coupling this fact with the method used in Section 2.1.1 to compute Π_c, we obtain

$$\begin{aligned}\Pi_c' &= (v_1 - p_h)(1 - e^{-\lambda\alpha t_1^*}) + \\ &\quad (v_1 \frac{\alpha\lambda(T - t_1^*)}{\lambda(T - \alpha t_1^*)} + v_0(1 - \frac{\alpha\lambda(T - t_1^*)}{\lambda(T - \alpha t_1^*)}) - p_l)(e^{-\lambda\alpha t_1^*} - e^{-\lambda T}).\end{aligned} \tag{3.3}$$

Because $\alpha t_1^* = t^*$, the probability that some customer purchases the item at price p_h is the same as when there was just one customer class. Furthermore, there is a positive probability that some customer with value $v_0 < v_1$ purchases the item in the lottery. Consequently, both intuition and comparing (2.3) with (3.3) reveal that $\Pi_c' \leq \Pi_c$.

3.2 With Reservation Regime

A class i customer is said to follow the WR-threshold if, given his arrival time t, he (a) purchases the item at price p_h when $t < \hat{t}_i$, (b) reserves the item if $t \geq \hat{t}_i$, and (c) purchases the item at price p_h if it has been reserved by another customer and if $v_i \geq p_h$. Because $v_0 < p_h$, it is clear that $\hat{t}_0 = 0$: any arriving class 0 customer will reserve the item if it is available. We employ the argument of Section 2.2 to show that $\hat{t}_1$ satisfies

$$v_1 - p_h \;=\; e^{-\alpha\lambda(T-\hat{t}_1)}(v_1 - p_l). \tag{3.4}$$

Equivalently,

$$\hat{t}_1 \;=\; T - \frac{1}{\alpha\lambda} ln(\frac{v_1 - p_l}{v_1 - p_h}). \tag{3.5}$$

Recall from Section 2.2 that we assume $T > \frac{1}{\lambda} ln(\frac{v_1 - p_l}{v_1 - p_h})$ so as to ensure that $\hat{t} > 0$. This assumption and (3.5) imply that there exists a critical value $\hat{c}$ with $0 \leq \hat{c} \leq 1$ such that $\hat{t}_1 \geq 0$ if and only if $\alpha \geq \hat{c}$.

Proposition 5 *There is a Nash equilibrium in which all arriving customers follow the WR-threshold: $\hat{t}_1$ is increasing in λ, v_1, p_l, and α and is decreasing in p_h. Also, $\hat{t}_1 < \hat{t}$ for $0 < \alpha < 1$. Furthermore, there exists a critical value c with $0 < c < 1$ such that $\hat{t}_1 < t_1^*$ when $\alpha < c$ and $\hat{t}_1 \geq t_1^*$ when $\alpha \geq c$.*

Proof: Existence is verified as before, and the comparative statics are immediate from (3.5). Because $\hat{t}_1$ is increasing in α and $\hat{t}_1 = \hat{t}$ when $\alpha = 1$ (where $\hat{t}$ is given in (2.5)), it follows that $\hat{t}_1 < \hat{t}$ for $0 < \alpha < 1$.

The existence of the critical value c follows from noting that t_1^* is a decreasing function of α (by Proposition 4); $\hat{t}_1$ is an increasing function of α (by Proposition 5); both t_1^* and $\hat{t}_1$ are continuous functions of α; when $\alpha = 1$, $t_1^* = t^* < \hat{t} = \hat{t}_1$ (where the inequality follows from Proposition 2); and when $\alpha = \hat{c}$, $\hat{t}_1 = 0$. From Proposition 4, $t_1^* \geq 0 = \hat{t}_1$ when $\alpha = \hat{c}$. Hence, the critical value c satisfies $1 > c > \hat{c}$. ■

In accord with intuition, Proposition 5 asserts that $\hat{t}_1$ decreases as the proportion $1 - \alpha$ of class 0 customers increases. This is so because a higher proportion of class 0 customers (who never purchase at price p_h) means fewer class 1 customers with whom to compete; consequently, an arriving class 1 customer can afford to delay purchase at price p_h longer than before.

3.2.1 Payoffs under the WR Regime

Computing the retailer's expected payoff, denoted by $\hat{\pi}'_r$, under the WR regime is more difficult than in the NR regime because of one additional mutually exclusive and exhaustive event which we label E: $E \equiv B_1(\hat{t}_1) = 0$, $A_1(\hat{t}_1) = 1$, and $A_0(\hat{t}_0) \geq 0$. When the customers follow the WR-threshold and event E occurs, the retailer will earn p_h if at least one customer of class 0 arrives before the one arriving customer of class 1, and the retailer will earn p_l if no customers of class 0 arrive before the one arriving customer of class 1. Define u to be the conditional probability that no customers of class 0 arrive before the class 1 customer arrives given that E occurs. Because there is exactly one class 1 customer arriving between $\hat{t}_1$ and T, the distribution of the arrival time t for this class 1 customer is uniformly distributed between $\hat{t}_1$ and T. Combining this observation with the fact that the number of class 0 customers arriving before t is a Poisson random variable with parameter $(1-\alpha)\lambda t$, it follows that the retailer's expected payoff for this event equals $p_h(1-u) + p_l u$, where

$$u \equiv \int_{\hat{t}_1}^{T} \frac{e^{-(1-\alpha)\lambda t}}{T - \hat{t}_1} dt = \frac{e^{-(1-\alpha)\lambda \hat{t}_1} - e^{-(1-\alpha)\lambda T}}{(1-\alpha)\lambda(T - \hat{t}_1)}. \tag{3.6}$$

Lemma 4 *Under the WR regime with two customer classes, the retailer's expected payoff $\hat{\pi}'_r$ satisfies*

$$\hat{\pi}'_r \;=\; p_h - (p_l - s)e^{-\lambda T} - (p_h - p_l)e^{-\alpha\lambda T}(1 + u \cdot \alpha\lambda(T - \hat{t}_1)). \tag{3.7}$$

Moreover, $\hat{\pi}_r \geq \hat{\pi}'_r$.

The ordering $\hat{\pi}_r \geq \hat{\pi}'_r$ is intuitive: the presence of class 0 customers whose valuation is less than the valuation of the class 1 customers is deleterious to the retailer. Substantiating this intuition entails considerable computation and can be found in Supplementary Note 1.

Employing the same set of mutually exclusive and exhaustive events used in computing $\hat{\pi}'_r$, we can show that the expected surplus for the totality of customers, denoted $\hat{\Pi}'_c$, is

$$\begin{aligned} \hat{\Pi}'_c \;=\; & (1 - e^{-\alpha\lambda T} - u\alpha\lambda(T - \hat{t}_1)e^{-\alpha\lambda T})(v_1 - p_h) \\ & + u\alpha\lambda(T - \hat{t}_1)e^{-\alpha\lambda T}(v_1 - p_l) + (e^{-\alpha\lambda T} - e^{-\lambda T})(v_0 - p_l) \end{aligned} \tag{3.8}$$

3.3 Comparison of Payoffs

In Proposition 3 we showed that the retailer's expected payoff is larger under the WR regime than under the NR regime whereas the customers expected payoff is larger under the NR regime. With two customer classes these results no longer hold. Instead, the ordering depends upon the proportion of class 1 customers.

Proposition 6 *Suppose there are two customer classes. There exists a critical value c' such that the retailer's expected payoff under the WR regime is greater than the expected payoff under the NR regime when $\alpha \geq c'$: $\hat{\pi}'_r \geq \pi'_r$ when $\alpha \geq c'$. Conversely, there is a critical number c'' such that the customers' expected payoff is greater under the NR regime when $\alpha \geq c''$: $\hat{\Pi}'_c \leq \Pi'_c$ when $\alpha \geq c''$.*

The proof of Proposition 6 is similar to the proof of Proposition 5 and is given in Supplementary Note 2. The numerical results represented in Figures 1 and 2 (below) are consistent with our conjectures that $\hat{\pi}'_r < \pi'_r$ when $\alpha < c'$ and $\hat{\Pi}'_c > \Pi'_c$ when $\alpha < c''$. In our example, we set $\lambda = 1, T = 6, p_h = 32, p_l = 10, s = 4, v_0 = 15$, and $v_1 = 40$ so that $p_l < v_0 < p_h < v_1$. In addition, we vary α from 0.01 to 0.9 according to an increment of 0.01. Observe from Figures 1 and 2 that $\hat{\pi}'_r \geq \pi'_r$ when $\alpha \geq 0.41 = c'$ and $\hat{\Pi}'_c \leq \Pi'_c$ when $\alpha \geq 0.47 = c''$, respectively. Hence, our numerical results comport with Proposition 6. Notice the odd shape of the customer's surplus under the WR regime (i.e., $\hat{\Pi}'_c$) reported in Figure 2 is due to the fact that $\hat{t}_1 = 0$ when $\alpha \leq 0.22 = \hat{c}$.

Insert Figures 1 and 2 about here.

4 Extension 2: Multiple Customer Classes

In this section there are $n + 1$ customer classes, and v_i is the item's valuation for each class i customer, $i = 0, 1, \ldots, n$. Without loss of generality, we assume $p_l < v_0 < p_h < v_1 < v_2 < \cdots < v_n$. We do this because any customer class with valuation below p_l is irrelevant, and all customers with valuation between p_l and p_h have identical behavior in terms of purchasing the item, joining the lottery, and reserving the item. Let α_i denote the probability that an arriving customer is from class i, $i = 0, 1, 2, \cdots, n$, where $\sum_{i=0}^{n} \alpha_i = 1$.

4.1 No Reservation Regime

We extend the NR-threshold to $n+1$ customer classes. A class i customer is said to follow the NR threshold if, given his arrival time t, he (a) purchases the product at price p_h when $t < t_i^*$; and (b) joins the lottery when $t \geq t_i^*$, where $t_0^* < t_1^* < \cdots < t_n^*$. Of course, $t_0^* = 0$ because $v_0 < p_h$.

As defined earlier, let the Poisson random variables $A_i(t)$ and $B_i(t)$ denote the number of arrivals of class i customers before time t and after time t, $i = 0, 1, \ldots, n$. To compute the thresholds, we consider the expected surplus of a class j customer who arrives at time t_j^*. Because t_j^* is defined so as to render a class j customer indifferent between purchasing at time t_j^* and joining the lottery, his expected surplus from joining the lottery equals $v_j - p_h$. We can assume $\sum_{i=1}^{j} B_i(t_i^*) = 0$; otherwise, the product already would have been sold. Similarly, to ensure that the product is not sold between times t_j^* and T, we require that for each $i > j$, no class i customer arrives in the time interval $[t_j^*, t_i^*]$. The probability of this event equals $\Pi_{i=j+1}^{n} e^{-\alpha_i \lambda (t_i^* - t_j^*)}$. The number of customers who join the lottery equals $1 + \sum_{i=0}^{n} A_i(t_i^*)$. Using (2.6) to compute $E\{1/[1 + \sum_{i=0}^{n} A_i(t_i^*)]\}$ and coupling the resulting expression with the above probability and $\sum_{i=0}^{n} \alpha_i = 1$, for $j = 1, 2, \ldots, n$, we obtain

$$v_j - p_h \;=\; (v_j - p_l) \cdot \Pi_{i=j+1}^{n} e^{-\alpha_i \lambda (t_i^* - t_j^*)} \cdot \frac{1 - e^{-\lambda \sum_{i=0}^{n} \alpha_i (T - t_i^*)}}{\lambda T - \lambda \sum_{i=0}^{n} \alpha_i t_i^*}, \tag{4.1}$$

where $\Pi_{i=n+1}^{n} z_i \equiv 1$ for any set $\{z_i\}$ of constants. From (4.1) we obtain a relationship between t_{j+1}^* and t_j^*:

$$t_{j+1}^* - t_j^* \;=\; \frac{1}{\lambda \sum_{k=j+1}^{n} \alpha_k} \cdot ln\{\frac{(v_{j+1} - p_h)(v_j - p_l)}{(v_j - p_h)(v_{j+1} - p_l)}\} > 0 \quad \text{for } j = 1, 2, \cdots, n-1. \tag{4.2}$$

The inequality in (4.2) follows from $v_j < v_{j+1}$. Hence, $t_1^* < t_2^* < \cdots < t_n^*$.

From (4.2), we have $t_{j+1}^* - t_j^* \equiv x_j$ where x_j is not a function of $\{\, t_i^* : i = 0, \ldots, n\}$. Thus, $t_{n-1}^* = t_n^* - x_{n-1}$. Iterating, we obtain $t_{n-j}^* = t_n^* - \sum_{i=i}^{j} x_{n-i}$. Now substitute these values for t_i^* into (4.1) to obtain an expression for t_n^*. Knowing the value of t_n^* we then compute $t_{n-1}^*, t_{n-2}^*, \ldots, t_1^*$ in order.

Proposition 7 *There is a Nash equilibrium in which all arriving customers follow the NR-threshold.*

4.1.1 Payoffs under the NR Regime

Employing the approach used in Section 2.1.1, we find that the retailer's expected payoff π_r satisfies

$$\pi_r = (1 - \Pi_{i=1}^{n} e^{-\lambda\alpha_i t_i^*})p_h + (\Pi_{i=1}^{n} e^{-\lambda\alpha_i t_i^*} - e^{-\lambda T})p_l + e^{-\lambda T} s. \tag{4.3}$$

See Supplementary Note 3 for the details.[7] When $n = 1$, (4.3) reduces to (3.2) given in Section 3.1.1 for the 2-customer class case.

As shown in Supplementary Note 4, the customer's expected payoff Π_c satisfies

$$\begin{aligned}\Pi_c &= (1 - \Pi_{i=1}^{n} e^{-\lambda\alpha_i t_i^*}) \cdot \sum_{i=1}^{n} \frac{\alpha_i t_i^*}{\sum_{j=1}^{n} \alpha_j t_j^*}(v_i - p_h) + \\ &\quad (\Pi_{i=1}^{n} e^{-\lambda\alpha_i t_i^*} - e^{-\lambda T}) \cdot \sum_{i=0}^{n} \frac{\alpha_i (T - t_i^*)}{\sum_{j=0}^{n} \alpha_j (T - t_j^*)}(v_i - p_l).\end{aligned} \tag{4.4}$$

4.2 With Reservation Regime

A class i customer is said to follow the WR threshold if, given his arrival at time t, he (a) purchases the product at price p_h if $t < \hat{t}_i$, (b) reserves the product if $t \geq \hat{t}_i$, and (c) purchases the product if the product is reserved by another customer. Again, $v_0 < p_h$ implies $\hat{t}_0 = 0$.

Computing $\hat{t}_i$ is easy because reserving the product produces a positive surplus if and only if no customer of class j, with $j > 0$, arrives after time $\hat{t}_i$, and the probability of this event is $\Pi_{j=1}^{n} e^{-\alpha_j \lambda (T - \hat{t}_i)} = e^{-(1-\alpha_0)\lambda(T-\hat{t}_i)}$. Consequently, $\hat{t}_i$ satisfies

$$v_i - p_h = (v_i - p_l) e^{-(1-\alpha_0)\lambda(T-\hat{t}_i)}, \qquad \text{for } i = 1, 2, \cdots, n. \tag{4.5}$$

A simple rearrangement of (4.5) produces the useful expression

$$\hat{t}_i = T - \frac{1}{(1-\alpha_0)\lambda} ln(\frac{v_i - p_l}{v_i - p_h}), \qquad \text{for } i = 1, 2, \cdots, n. \tag{4.6}$$

Because $v_1 < v_2 < \cdots < v_n$, (4.6) reveals that $\hat{t}_1 < \hat{t}_2 < \cdots < \hat{t}_n \leq T$. The existence of a Nash equilibrium and the comparative statics follow as per Proposition 2.

[7] Because $t_0^* = 0$, (4.1) implies that $\frac{1 - e^{-\lambda(T - \sum_{i=1}^{n} \alpha_i t_i^*)}}{\lambda(T - \sum_{i=1}^{n} \alpha_i t_i^*)} = \frac{v_n - p_h}{v_n - p_l}$ when $j = n$. Hence, the term $\sum_{i=1}^{n} \alpha_i t_i^*$ can be expressed as an implicit function of v_n, p_h, p_l, λ and T, which is independent of the individual value of α_i, for $i = 1, \cdots, n$. This observation and (4.3) imply that the retailer's expected payoff π_r is also independent of the individual value of α_i.

Proposition 8 *There is a Nash equilibrium in which all arriving customers follow the WR-threshold. Moreover, $\hat{t}_i$ is increasing in λ, v_i, p_l and decreasing in α_0 and p_h.*

4.2.1 Payoffs under the WR Regime

Computing the retailer's expected payoff $\hat{\pi}_r$ and the totality of the customers' payoff $\hat{\Pi}_c$ with more than two customer classes entails a tedious extension the arguments used in Section 3.2.1: the details are given in Supplementary Note 5. We have

$$\hat{\pi}_r = p_h - (p_l - s)e^{-\lambda T} - e^{-(1-\alpha_0)\lambda T}[(p_h - p_l) + (p_h - Z_r)((1-\alpha_0)\lambda(T - \hat{t}_1)], \qquad (4.7)$$

where Z_r is the natural extension of u given in Section 3.2.1 to $n+1$ customer classes and

$$Z_r = p_h \int_{\hat{t}_1}^{T} \frac{1 - e^{-\alpha_0 \lambda t}}{T - \hat{t}_1} dt + \sum_{j=1}^{n} (\frac{\alpha_j}{1-\alpha_0})\{p_l \int_{\hat{t}_j}^{T} \frac{e^{-\alpha_0 \lambda t}}{T - \hat{t}_1} dt + p_h \int_{\hat{t}_1}^{\hat{t}_j} \frac{e^{-\alpha_0 \lambda t}}{T - \hat{t}_1} dt\}. \qquad (4.8)$$

Due to its complexity, we omit the expression for $\hat{\Pi}_c$ which is given in Supplementary Note 6.

4.3 Numerical Examples

Because the expression for t_i^* given in (4.2) is complex, we are forced to compare the thresholds t_i^* and $\hat{t}_i$ given in (4.6) numerically rather than analytically. For the same reason, we shall compare the retailer's payoffs π_r given in (4.3) and $\hat{\pi}_r$ given in (4.7) numerically. In our example, we set $n = 2$, $\lambda = 1$, $T = 6$, $p_h = 32$, $p_l = 15$, and $s = 2$. Also, we set $v_0 = 20, v_1 = 38, v_2 = 40$ so that $p_l < v_0 < p_h < v_1 < v_2$. In addition, we vary α_1 from 0.02 to 0.88 according to an increment of 0.02, while we set $\alpha_0 = \alpha_2 = \frac{1-\alpha_1}{2}$.

Observe from Figure 3 that t_1^* is decreasing in α_1, that $\hat{t}_1$ is increasing in α_1, and that $\hat{t}_1 \geq t_1^*$ if and only if $\alpha_1 \geq c = 0.39$. Hence, it is consistent with Proposition 5 for class 1 customers. As $\alpha_2 = \frac{1-\alpha_1}{2}$ is a linear decreasing function of α_1, one can prove that t_2^* is convex in α_1 by examining (4.2) and that $\hat{t}_2$ is increasing in α_1 by examining (4.6). This analytical result is illustrated numerically in Figure 3. However, the critical number c of Proposition 5 no longer exists for class 2 customers. Next, observe from Figures 4 and 5 that $\hat{\pi}_r \geq \pi_r$ when $\alpha_1 \geq 0.06 = c'$ and $\hat{\Pi}_c \leq \Pi_c$ when $\alpha_1 \geq 0.06 = c''$, respectively. Hence, our numerical results comport with Proposition 6.

Insert Figures 3, 4, and 5 about here.

5 Auction Regime

Until this point, the no reservation regime entailed disposal of the unsold item at the reduced price p_l on a first-come-first-serve basis. In this section, the retailer employs an auction in the after season to sell the item if it was not sold during the season: if the item was not sold during the season, all of the customers who arrived between $[0, T]$ enter a second-price auction. As is well known for independent private values (which we assume to hold for the customers), the second-price (or Vickrey (1961)) auction produces the same expected revenue as a sealed-bid first-price auction and has the added advantage that a (weakly) dominant strategy for each bidder is to bid his true valuation.

5.1 One Customer Class

As per the base model of Section 2, suppose there is a single class of customers, each with valuation v_1, where $v_1 \geq p_h > p_l$. Consider a customer who arrives at time t. He earns a surplus $(v_1 - p_h)$ if he purchases the product; alternatively, he can enter the second-price auction at the end of the selling season. The retailer's "reserve price" is taken to be p_l: if there is only one bidder in the auction, the product will be sold at p_l provided the bid is at least p_l; otherwise, it is not sold. If no other customer arrives after t, then he wins the auction and pays the reduced price p_l, thereby earning a surplus $(v_1 - p_l)$. If at least one other customer arrives after t, then he earns a surplus of $(v_1 - v_1) = 0$ because all bids equal v_1. A customer is said to follow the "auction threshold" if, given his arrival time t, he purchases the item at price p_h when $t < \tilde{t}$ and he enters the auction and bids v_1 when $t \geq \tilde{t}$, where $\tilde{t}$ satisfies

$$v_1 - p_h \quad = \quad e^{-\lambda(T-\tilde{t})}(v_1 - p_l). \tag{5.1}$$

Rearranging terms in (5.1) yields

$$\tilde{t} \quad = \quad T - \frac{1}{\lambda} ln(\frac{v_1 - p_l}{v_1 - p_h}). \tag{5.2}$$

Observing that (5.4) is identical to (2.5), it follows that $\tilde{t} = \hat{t} > t^*$. Hence, the existence of a Nash equilibrium and the comparative statics results of Proposition 2 hold for the auction with one customer class.

The threshold in the auction regime is the same as the threshold in the WR regime; hence, the product is sold at price p_h if at least one customer arrives before time $\hat{t} = \tilde{t}$. If no customer arrives before time $\tilde{t}$ and exactly one customer arrives after time $\tilde{t}$, then it is sold at price p_l. However, if no customer arrives before time $\tilde{t}$ and more than one customer arrives after time $\tilde{t}$, then the item is sold at price v_1 in the auction regime and at price p_h in the WR regime. Consequently, as $v_1 \geq p_h$, we have shown that $\tilde{\pi}_r$, the retailer's expected surplus in the auction regime, exceeds $\hat{\pi}_r$, his expected surplus in the WR regime. The simple argument above also serves to establish that $\tilde{\Pi}_c$, the customers' expected surplus in the auction regime, is smaller than $\hat{\Pi}_c$, the customers' expected surplus in the WR regime. Consequently, invoking Proposition 3, we obtain

Proposition 9 *The retailer's and the customers' expected surplus in the auction regime, the WR regime, and the NR regime satisfy* $\tilde{\pi}_r > \hat{\pi}_r > \pi_r$ *and* $\tilde{\Pi}_c < \hat{\Pi}_c < \Pi_c$.

5.2 Two Customer Classes

In this subsection, we suppose there are two customer classes. Class 1 customers have valuation v_1 and arrive in accord with a Poisson process with parameter $\alpha\lambda$ whereas class 0 customers have valuation v_0 and arrive in accord with a Poisson process, independent of the class 1 process, at rate $(1-\alpha)\lambda$. As in Section 3, we assume $v_1 \geq p_h > v_0 \geq p_l$ and a class i customer is said to follow the "auction threshold" if, given his arrival time t, he purchases the item at price p_h when $t < \tilde{t}_i$ and he enters the auction and bids v_i when $t \geq \tilde{t}_i$. Of course, $\tilde{t}_0 = 0$ because $v_0 < p_h$.

To compute $\tilde{t}_1$, note that $y = e^{-(1-\alpha)\lambda T}$ is the probability that no class 0 customer arrives during the season. Noting that an arrival of a class 1 customer after time $\tilde{t}_1$ implies a surplus of 0 for the class 1 customer arriving at time $\tilde{t}_1$, computation nearly identical to that which produced (5.1) yields

$$v_1 - p_h \quad = \quad e^{-\alpha\lambda(T-\tilde{t}_1)}[y(v_1 - p_l) + (1-y)(v_1 - v_0)]. \tag{5.3}$$

As before, each customer following the auction-threshold constitutes a Nash equilibrium. The impact of changes in the problem parameters is easily analyzed via (5.4) which is obtained by rearranging terms in (5.3):

$$\tilde{t}_1 \quad = \quad T - \frac{1}{\alpha\lambda} ln(\frac{y(v_1 - p_l) + (1-y)(v_1 - v_0)}{v_1 - p_h}). \tag{5.4}$$

Comparing the retailer's expected payoff $\tilde{\pi}_r'$ in the auction regime to his expected payoff π_r' in the NR (lottery) regime is complex and requires a sample path by sample path comparison (*i.e.* a coupling argument). Doing so yields (see Supplementary Note 7):

Proposition 10 *If* $\tilde{t}_1 > t_1^*$, *then* $\tilde{\pi}_r' \geq \pi_r'$.

When we compare the customer's expected payoff under the NR-threshold (with auction); i.e., $\tilde{\Pi}_c$, and the customer's expected payoff under the NR-threshold (with lottery); i.e., Π_c, the comparison is inconclusive.

6 Concluding Remarks and Future Research Directions

Our model in which the retailer pre-announces the price markdown mechanism at the beginning of the selling season captures the essence of two operating regimes (NR and WR) that have been implemented in practice. To capture the inherent uncertainty of customer shopping behavior, we use a Poisson process to model customer arrivals. The end goal of our analysis was to obtain equilibrium customer behavior in which all customer decisions are made in full cognizance that all other customers not only will act rationally but also will act with full awareness of how other customers will act. In all cases, equilibrium behavior reduced to following a threshold policy which, in essence, can be stated as: purchase if arrival is before the threshold and enter the lottery if arrival is after the threshold. (In the WR regime [auction regime], place a reservation [enter the auction] if arrival is after the threshold.) When there is a single class of customers, our analysis reveals that the retailer's expected surplus is highest with an auction and lowest with a lottery. The opposite is true for the customers: customer expected surplus is highest in the lottery and lowest in the auction. However, these results are not always true when there are multiple customer classes.

While not without its limitations, the model presented in this paper can serve as a building block for future research in this area. Extensions worthy of consideration include investigations of: the optimal regular price p_h and the optimal post-season clearance price p_l for each regime; the impact of changes in T and the price schedule; customer valuations which decrease over the course of the selling season; time dependent customer arrival rates; the retailer having several of the same item for sale; the impact upon customer arrival rates of regime changes (NR and WR) and changes in the price schedule; multiple price changes during the selling season; and retailers competing via different pricing regimes.

As evident from this list of potential future research directions, a considerable amount of research work remains to be done in this area.

References

[1] Aviv, Y., and A. Pazgal, "Optimal Pricing of Seasonal Products in the Presence of Forward-Looking Consumers," *Working Paper, Olin School of Business, Washington University,* 2005.

[2] Bell, D., T. Ho, and C.S. Tang, "Determining Where to Shop: Fixed and Variable Costs of Shopping" *Journal of Marketing Research*, Vol. 35, pp. 352-369, 1998.

[3] Bell, D., and D. Starr, "Filene's Basement" *Harvard Business School Case*, case number 594018, 1998.

[4] Besanko, D., and W. Winston, "Optimal Price Skimming by a Monopolist Facing Rational Consumers," *Management Science*, Vol. 36, No. 5, pp. 555-567, 1990.

[5] Biyalogorsky, E., and E. Gerstner, "Note: Contingent Pricing to Reduce Price Risks," *Marketing Science*, Vol. 23, No. 1, pp. 146-155, Winter, 2004.

[6] Elmaghraby, W., and P. Keskinocak, "Dynamic Pricing in the Presence of Inventory Considerations: Research Overview, Current Practices, and Future Directions," *Management Science*, Vol. 49, No. 10, pp. 1287-1309, October 2003.

[7] Elmaghraby, W., A. Gulcu, and P. Keskinocak, "Optimal Markdown Mechanisms in the Presence of Rational Customers with Multi-unit Demands," Working Paper, School of Industrial and Systems Engineering, Georgia Institute of Technology, Atlanta, GA 30332, 2004.

[8] Harris, M., and A. Raviv, "A Theory of Monopoly Pricing Schemes with Demand Uncertainties," *The American Economic Review*, Vol. 71, No. 3, 347-365, 1981.

[9] Kagel, J., "Auctions," Chapter 7 in *The Handbook of Experimental Economics*, eds. J. Kagel and A.E. Roth, Princeton University Press, Princeton, NJ, 1995.

[10] Png, I.P.L., "Reservations: Customer Insurance in the Marketing of Capacity," *Marketing Science*, Vol. 8, No. 3, pp. 248-264, Summer, 1989.

[11] Ross, S., *Introduction to Probability Models,* Academic Press, 1980.

[12] Vickrey, W., "Counterspeculation, Auctions, and Competitive Sealed Tenders," *Journal of Finance*, Vol. 16, pp. 8-37, 1961.

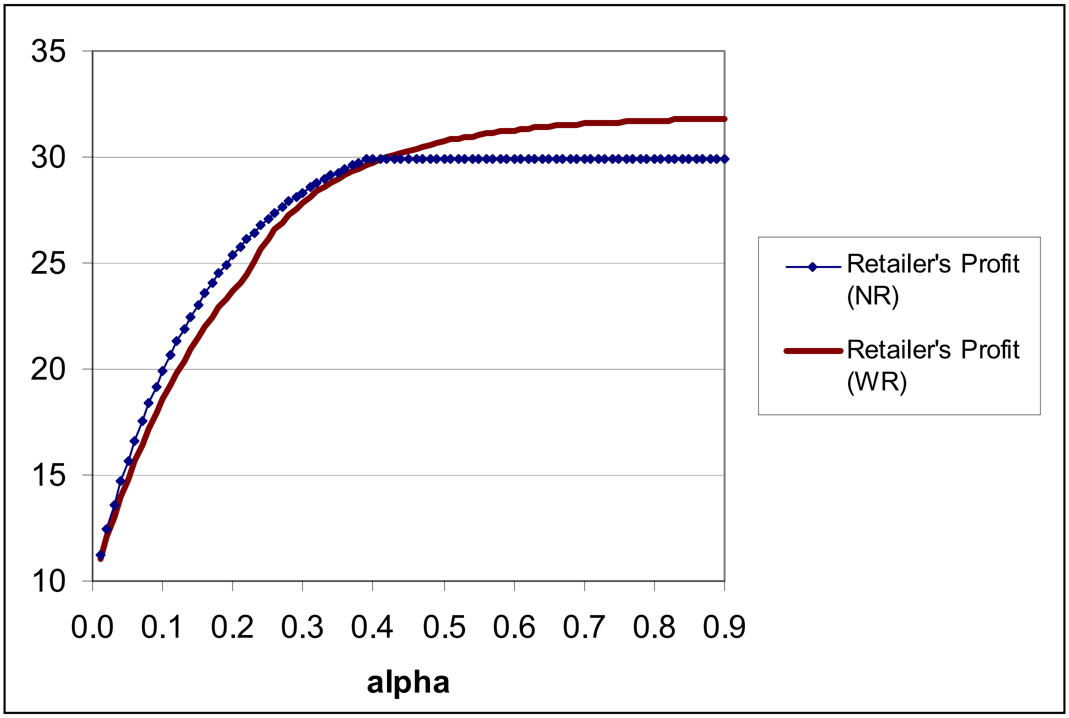

Figure 1. Impact of alpha on the retailer's expected payoff

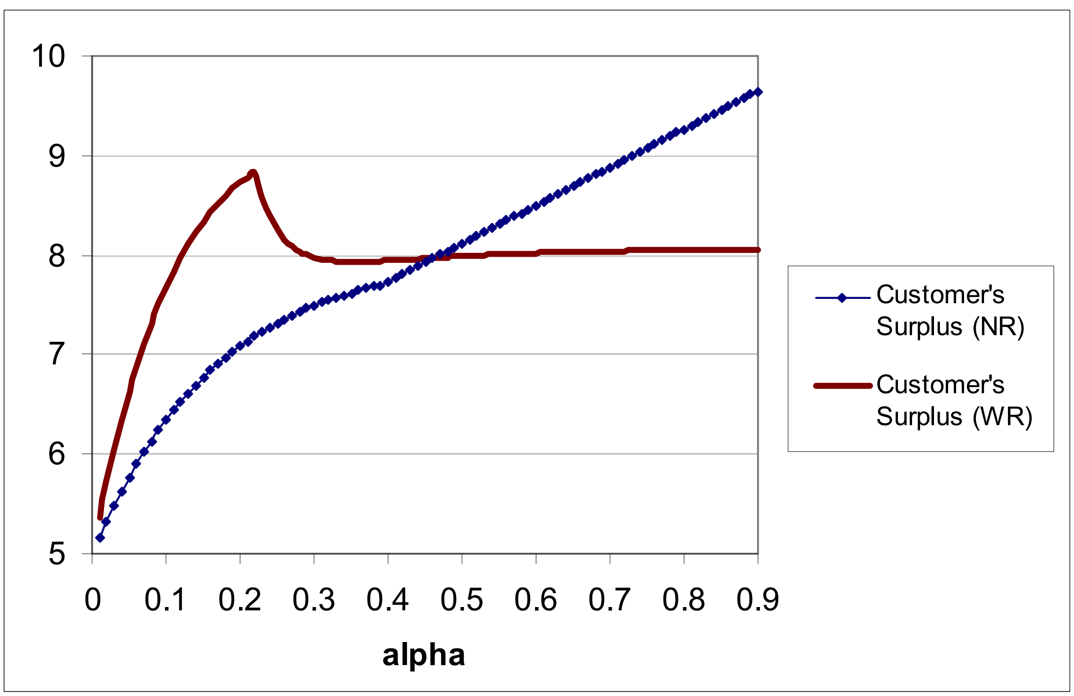

Figure 2. Impact of alpha on the customer's expected surplus

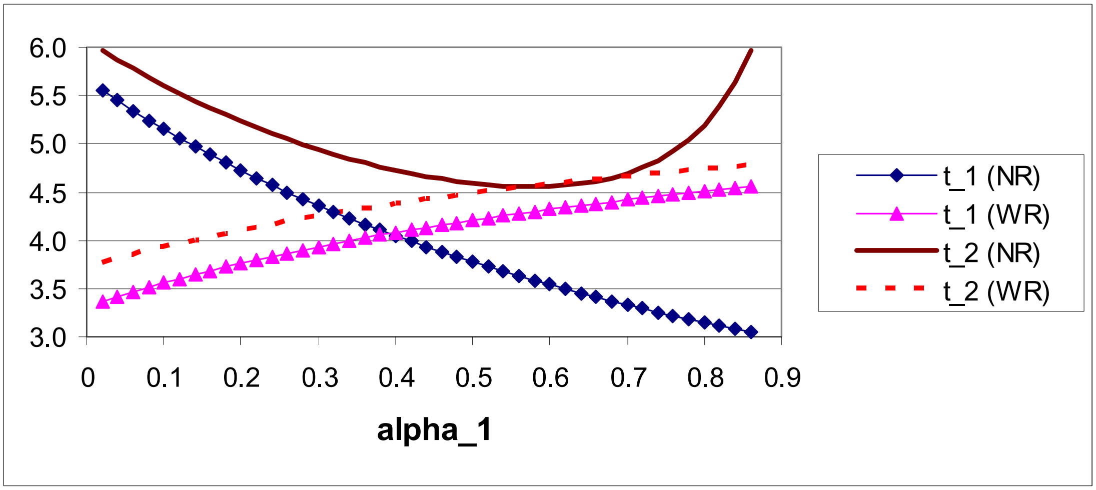

Figure 3. Impact of alpha_1 on the thresholds under the NR and WR regimes

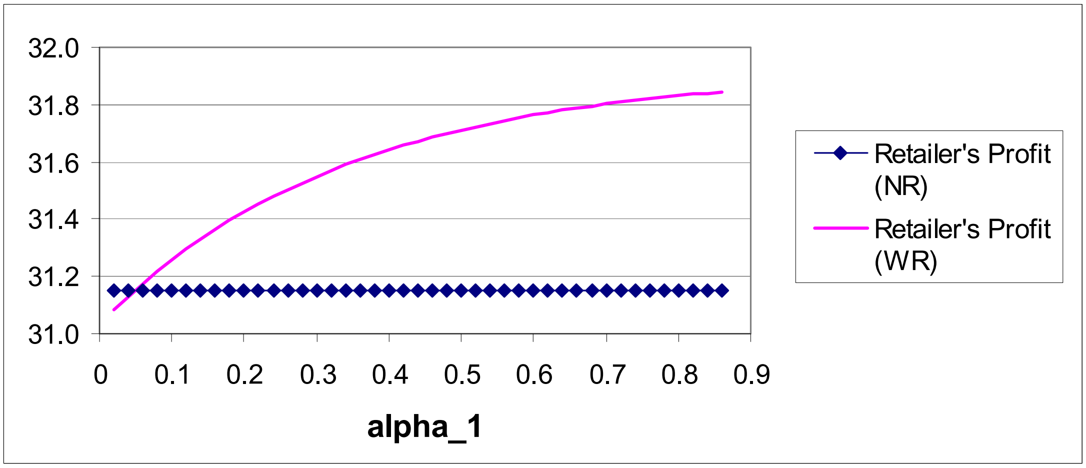

Figure 4. Impact of alpha_1 on the retailer's expected profit

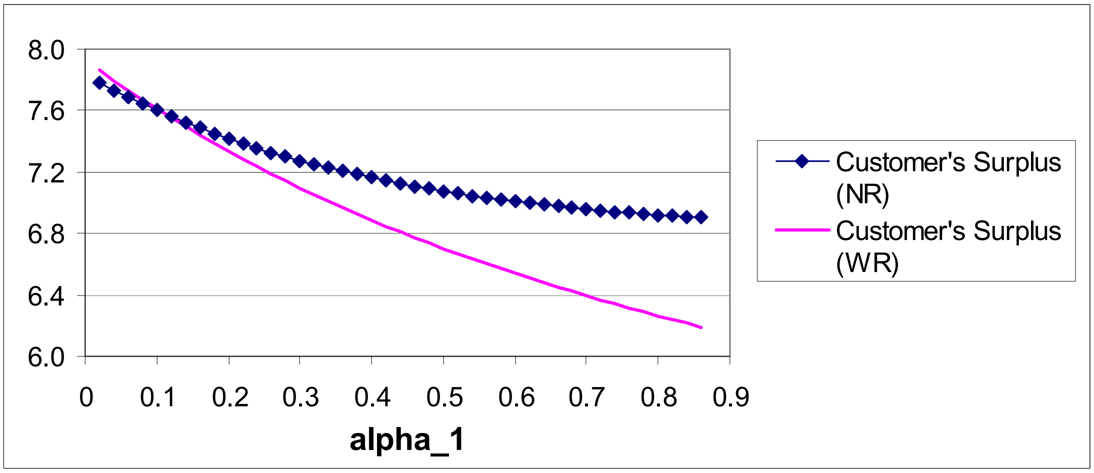

Figure 5. Impact of alpha_1 on the customer's expected surplus