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# Gray markets, Contracts and Supply Chain Coordination

Reza Ahmadi

Anderson School of Management, University of California, Los Angeles, 90095, rahmadi@anderson.ucla.edu

Foad Iravani, Hamed Mamani

Foster School of Business, University of Washington, Seattle, 98195,  
firavani@uw.edu, hmamani@uw.edu

The practice of diverting genuine products to unauthorized gray markets continues to challenge many companies in various industries and creates an intense competition for authorized channels. Recent industry surveys report that the abuse of channel incentives is a primary reason for the growth of gray market activities. Therefore, it is crucial that companies take the potential presence of gray markets into consideration when they design contracts to distribute their products through authorized retailers. This issue has received little attention in the extensive literature on contracting and supply chain coordination. In this paper, we analyze the impacts of gray markets on two classic contracts, wholesale price and quantity discount, in a supply chain with one manufacturer and one retailer when the retailer has the opportunity to sell the product to a domestic gray market. Consumers are forward-looking and anticipate the diversion of the product to the gray market. Our analysis provides interesting and counterintuitive results. First, a classic quantity-discount contract that normally coordinates the supply chain can perform so poorly in the presence of a gray market that the supply chain would be better off using a wholesale price contract instead. Second, the presence of a gray market can also degrade the performance of the wholesale price contract; thereby, necessitating a more sophisticated contract for coordinating the supply chain in the presence of the gray market. We show that any contract that solely depends on retailer order quantity cannot coordinate the supply chain and provide the conditions for coordinating the supply chain with price-dependent quantity discount contracts. We also provide comparative statics and show that when there is a gray market, coordinating the supply chain also enhances total consumer welfare.

*Key words:* gray markets, parallel markets, contracting, supply chain coordination, strategic customers, price of anarchy, wholesale price, quantity discount, consumer welfare

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## 1. Introduction

The diversion of branded goods from authorized distribution channels to unauthorized resale channels, also known as gray markets, continues to be of a major concern to many companies around the world. Although the existence of gray markets was documented and discussed in the 80's, their impact on various industries has grown significantly due to diminishing trade barriers and the rapid growth of online sales. Cespedes et al. (1988) estimated that the value of products sold outside manufacturers' authorized distribution channels in the United States reached \$7 billion to \$10 billion each year. Twenty years later, a study by KPMG and AGMA reported that the IT industry alone lost \$58 billion worth of products to gray markets (KPMG 2008).

Gray markets operate in different forms; some gray marketers import products from low price markets to high price markets for resale (also known as parallel imports) while others operate in the same region or market as authorized channels (also known as domestic gray markets). Gray market trades span a large variety of goods such as watches, luxury goods, food items, textbooks, apparel, electronics, and pharmaceuticals (Schonfeld 2010).

Gray markets pose several challenges on manufacturers. They undercut the authorized channel price and pressure retailers and manufacturers into cutting their prices and compromising on profit margins. Gray markets also freeride on the authorized channel's investment in promotional and sales efforts such as advertising and in-store assistance. They can also damage the brand value by making products available to segments that manufacturers deliberately avoid. Besides, consumers who buy products from gray markets tend to blame manufacturers for loss of warranty and after-sales support (Cespedes et al. 1988, Deloitte 2011). Gray markets can, however, be beneficial to manufacturers. They generate an additional source of demand for products and can help manufacturers to steal market share from their competitors.

Gray markets are generally blessed with the first-sale doctrine, also referred to as the right of first sale or the exhaustion rule. The doctrine recognizes the right of an owner of a legally obtained product to dispose of or resell the product without permission from the copyright owner. Therefore, when companies attempt to cease gray markets through legal actions, their hands are tight. As a result, companies are left with business solutions to manage the competition from gray markets.

There are different drivers of gray markets. One factor is price differentials. Ideally, a global company would want to price its products locally based on consumers purchase power in each market and exchange rates to maximize profit. The downside of a price discrimination strategy is providing the opportunity for parallel importers to import the product to the high price market and undercut the authorized channel price (Ahmadi and Yang 2000, Xiao et al. 2011). Another driver of gray markets is demand uncertainty; a retailer who is left with excess inventory may sell unsold items to a gray market instead of salvaging them (Dasu et al. 2012).

Although price differentials and demand uncertainty have been extensively studied in the literature, inefficient channel incentives which are another important driver of gray markets, especially domestic gray markets, have received very limited attention despite the documented evidence in practice. Recently, Deloitte conducted a survey of leading high tech companies with revenues exceeding \$25 billion that received a significant percent of their revenue through channel partners (Deloitte 2011). The survey estimates that the abuse of channel incentives costs high tech companies as much as \$1.4 billion in lost profits each year. Moreover, there appears to be a direct link between channel incentive abuse and the gray market for high tech products. In fact, 84% of the respondents agreed that incentives are responsible for gray market activities. According to one of the respondents, the VP of global channel development at APC by Schneider Electric, “the most common abuses are through special pricing requests and other channel promotions, in which brokers buy into the channel partners deal and purchase products through the channel for sale on the gray market.” The 2008 study by KPMG and AGMA also reflects that effective channel management plays an important role in controlling gray market activities.

Motivated by these observations, in this paper we analyze the performance of classic contracts and supply chain coordination in the presence of gray market activities. We study a supply chain consisting of one manufacturer selling a single product through an opportunistic retailer who may divert the product to a domestic gray market. Our model also incorporates strategic consumer behavior. Retailers recognize that consumers are increasingly more sophisticated and aware of market dynamics. Some consumers are willing to postpone their purchase to obtain products at lower prices (Silversetin and Butman 2006). Therefore, demand depends on the prices set by the retailer and the gray market.

**Summary of key results.** Our analysis yields interesting and somewhat counterintuitive results:

- (i) We find that gray market activities have a significant impact on the performance of classic contracts. For example, it is known that the wholesale price contract suffers from double marginalization and generally fails to coordinate the supply chain whereas a classic quantity discount contract whose parameters are set appropriately can coordinate the supply chain. We show, however, that if such an (otherwise coordinating) classic quantity discount contract is implemented in the presence of a gray market, the retailer may abuse the discount and divert the product to the gray market. This abuse of the contract can hurt total supply chain profit so much that the supply chain would be better off using a wholesale price contract.
- (ii) We show that the retailer always benefits from the gray market, whereas the impact of the gray market on the manufacturer depends on the contract. Specifically, under the wholesale price contract the manufacturer may be better off if the penalty for the loss of brand image

is not too high, while the manufacturer is always worse off with the gray market under the classic quantity discount contract.

- (iii) Based on our first finding, we explore the performance of the wholesale price contract further. Prior research has shown that the double marginalization effect of wholesale price contracts can help supply chains reach coordination in a network of competing retailers. Since in our model the gray market creates competition in the supply chain, one would expect that the presence of the gray market will help to coordinate the supply chain. Interestingly, our analysis reveals that the gray market can actually increase the efficiency loss of the wholesale price contract, measured with the notion of *Price of Anarchy*, significantly. Therefore, it is imperative to design a more sophisticated contract to achieve supply chain coordination in the presence of a gray market.
- (iv) We find that any contract that solely depends on the order quantity of the retailer fails to coordinate the supply chain. As a result, we provide the conditions for coordinating the supply chain with a general class of price-dependent quantity discount contracts.
- (v) Although consumer welfare is generally not considered in the design of optimal contracts, we find that our proposed coordinating contract greatly enhances total consumer welfare. In other words, implementing a coordinating contract results in a win-win situation in which not only the supply chain but also the consumers are better off.

This paper is organized as follows. Section 2 reviews the literature. Section 3 describes the structure and assumptions of our model. Section 4 examines the performance of the wholesale price and classic quantity discount contracts in the presence of the gray market. Section 5 uses the so-called Price of Anarchy to study the effect of the gray market on the inefficiency resulting from implementing the wholesale price contract. Section 6 proposes a class of contracts for coordinating the supply chain, accounting for the impact of the gray market. Section 7 studies the effect of coordination on consumer welfare and Section 8 concludes the paper with a discussion.

## 2. Literature

Our research relates to the literature on gray markets and the literature on contracting and supply chain coordination. Empirical evidence of gray market activities and qualitative discussions of factors leading to the emergence of gray markets can be found in Banerji (1990), Myers (1999), Duttal et al. (1999), Maskus (2000), and Ganslandt and Maskus (2004) among others. Antia et al. (2004) discuss some policies (e.g. uniform pricing) that companies can use to cope with gray markets. Dutta et al. (1999) highlight the importance of business efficiency in territorial restriction policies.

Since the primary driver of gray markets in the form of parallel importation is price differentials, pricing decisions in the presence of gray markets and whether or not gray markets should be

tolerated have been explored in studies such as Dutta et al. (1994), Bucklin (1993), Ahmadi and Yang (2000), and Richardson (2002) under various modeling settings. Results from these studies mainly indicate that manufacturers should tolerate some level of parallel importation. Ahmadi and Yang (2000) find that gray markets may increase manufacturer profit under some circumstances. Xiao et al. (2011) show that whether a manufacturer sells directly or through a retailer is critical to determining the increase or reduction in manufacturer profit due to parallel importation. Iravani et al. (2014) consider both price and quantity decisions of a manufacturer facing parallel importation and uncertain demand and investigate the impact of market parameters and product characteristics on the manufacturer policy towards parallel importation. They also show that strategic price discrimination is more valuable when market conditions are moderately different and the product has not turned to a commodity. Autrey et al. (2013) consider two firms that engage in a Cournot competition in a domestic market and face parallel importation when they enter a foreign market. They find that when the products are close substitutes, it is better to decentralize the management structure in the foreign market.

Designing contracts to coordinate decisions in decentralized supply chains has been studied extensively. Cachon (2003) provides a comprehensive review of supply chain coordination under different settings. This rich stream of literature, however, pays very limited attention to the possibility of product diversion to gray markets. Recently, a few papers have analyzed the effect of gray markets on decentralized supply chains that face competition from gray markets. Dasu et al. (2012) consider a supply chain with exogenous pricing in which a retailer could salvage leftover inventory or sell it to the gray market. They show that the performance of a classic buyback contract may be unsatisfactory. Su and Mukhopadhyay (2012) show that a two-part tariff revenue-sharing contract can coordinate the supply chain when there is a gray market. In addition, a dynamic quantity discount contract fails to achieve coordination but may be near optimal profit. Su and Mukhopadhyay consider a specific type of gray market trade in which multiple "fringe" retailers get a predetermined proportion of the market size and their profit is determined by the price and service decisions of a dominant retailer. Their proposed contract assumes that the manufacturer offers a revenue-sharing contract both to the dominant retailer and the fringe retailers, which will likely affect the implementation of their contract.

The closest papers to our work are Hu et al. (2013) and Altug and van Ryzin (2014). Hu et al. (2013) study an Economic Order Quantity (EOQ) setting in which a supplier offers all-unit quantity discounts to a reseller who can divert the product to a gray market. They show that when the reseller's batch inventory holding cost is high, the gray market improves channel performance. Altug and van Ryzin (2013) consider a manufacturer selling a product through a large number of retailers that sell their leftover inventory to a domestic gray market. They show that the gray

market alleviates the negative effect of double marginalization in a wholesale price contract and propose a partial buy–back contract to coordinate the supply chain.

Our work is different from Hu et al. (2013) in that they consider an all-unit quantity discount contract and focus on the role of inventory holding cost, whereas we study both quantity discount and wholesale price contracts and provide insights into the efficiency of a wholesale price contract in the presence of a gray market. Furthermore, our model proposes a class of coordinating contracts. Another distinction between our setting and that of Hu et al. is that we incorporate strategic behavior of consumers in the model. Altug and van Ryzin (2013) assume that retail price is exogenous and demand is uncertain, whereas we model an environment in which the retail price is treated as a decision but demand is deterministic. Furthermore, a key feature of both of the above-mentioned papers is that they assume an exogenous or a market-clearing gray market price, which leads to little to no profit for the gray market. The legality of gray trade and the advancement of web technology has led to the emergence of many gray marketers that have established sophisticated enterprises. For example, Amazon, eBay, Kmart, and Costco are among the famous retailers known to have sold or facilitated the sale of gray goods (Schonfeld 2010). Other examples are online gray marketers Authenticwatches.com, Jomashop.com and Prestigetime.com that sell genuine luxury watches and directly provide services to customers. Therefore, in contrast to these papers, we consider a domestic gray market that strategically sets its price to maximize profit. Finally, we also analyze consumer welfare implications of supply chain coordination in the presence of a gray market.

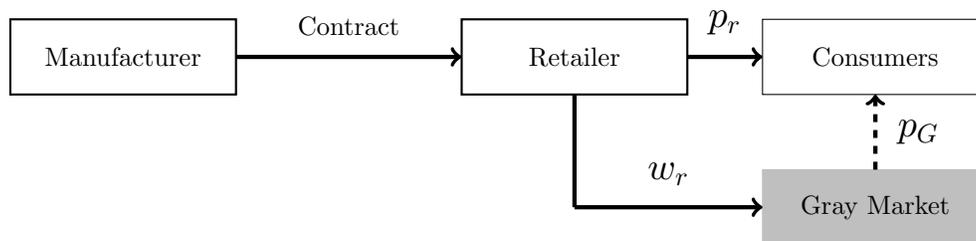
Our paper also relates to the literature on strategic consumer behavior. Su and Zhang (2009) study the impact of strategic consumers on supply chain performance when a newsvendor salvages excess inventory at the end of selling season. Cachon and Swinney (2009) study inventory, pricing and quick response decisions when some consumers in the market are strategic. They find that quick response is generally more valuable when there is strategic consumer behavior. Kim and Swinney (2013) find that strategic consumers lead firms to invest in a higher quality product but with less inventory. Other papers that consider forward–looking consumers are Dasu and Tong (2010), Aviv and Pazgal (2008), Parlakturk (2012), Liu and van Ryzin (2008), and Kabul and Parlakturk (2013). None of these papers considers product diversion to a gray market which is different from salvaging products; diversion to gray markets entails negative consequences for manufacturers such as loss of brand image and consumer confusion, and gray marketers are not affiliated with manufacturers and seek to maximize their profits.

Finally, our paper is the first to study the *Price of Anarchy* (PoA) in the context of gray markets. The PoA has been used in the literature to quantify the efficiency of decentralized systems relative to the performance of centralized ones. The concept of PoA was introduced by Koutsoupias and

Papadimitriou (1999) and Papadimitriou (2001) and has been used in transportation networks (Roughgarden and Tardos 2000, 2002; Perakis 2007) and network resource allocation games (Johari and Tsitsiklis 2004). Perakis and Roels (2007) measure the PoA for various decentralized supply chains that use price-only contracts. They find that the loss of efficiency from double marginalization is a major concern and increases with the number of stages in the supply chain. None of the papers that use the PoA incorporate gray market activities.

### 3. Assumptions and Modeling Framework

Consider the supply chain in Figure 1 with one manufacturer selling a single product through a retailer. For simplicity, we refer to the manufacturer as a female and to the retailer as a male. The manufacturer produces the product at unit cost  $c$  and sells the product to the retailer through a contract. In addition to the manufacturer and the retailer, a domestic gray market is also present.



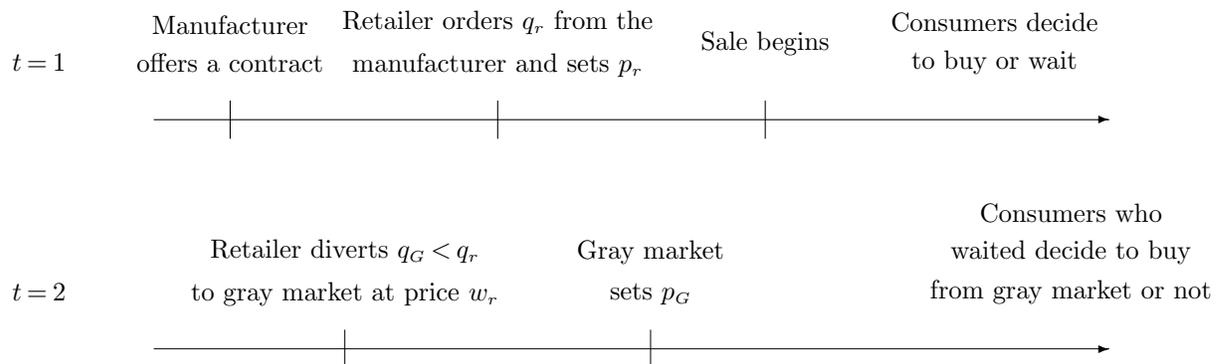
**Figure 1** Model setting

Our model has two periods and the sequence of events in each period is described in Figure 2. In period 1, the manufacturer offers a contract to the retailer. Then, the retailer offers the product at price  $p_r$  to consumers. The retailer has access to the domestic gray market and may divert the product to the gray market at the beginning of period 2. If the retailer decides to divert to the gray market, he will sell to the gray market at price  $w_r$ . In period 2, the gray market offers the product at price  $p_G$  to maximize its profit. Each unit of the product that is diverted to the gray market by the retailer incurs a loss of brand image penalty  $\rho$  for the manufacturer.

We assume that the retailer does not change his price in period 2 in order to keep the model tractable and focus on the role of contracts rather than dynamic pricing. We also assume that the retailer retains a positive market share when he diverts the product to the gray market. In other words, we do not consider the situation where the retailer diverts his entire inventory to the gray market and exits the market completely. Although this assumption is for tractability, we believe it is a realistic assumption because in reality we rarely observe authorized channels that divert all of their inventory to gray markets and abandon their primary business of direct sales to consumers.

We assume the size of the market is normalized to 1, all consumers are present at the beginning of period 1, and each consumer buys at most one unit of the product. We assume that consumers are strategic (forward-looking) and anticipate the possibility of the retailer diverting the product to the gray market. Therefore, they may postpone their purchase decision to period 2 if buying from the gray market yields a higher surplus. Note that these assumptions imply that consumers who do not purchase the product from the retailer in period 1 either wait to purchase the product from the gray market in period 2 or do not purchase the product at all.

Similar to the works of Su and Zhang (2008) and Cachon and Swinney (2009), in this paper we seek to find the Rational Expectations equilibrium as follows: (i) the (strategic) consumers form an expectation of the authorized channel and gray market prices of the product and make their decisions (purchase from authorized channel, wait and buy from the gray market, or make no purchase) based on their expectations; (ii) the retailer sets his authorized channel retail price at the beginning of period 1 and sets his wholesale price for selling to the gray market at the beginning of period 2, given his expectations of the consumer market segmentation and the gray market price; (iii) the gray market sets the gray market price in period 2; and (iv) everyone's expectations are consistent with the actual outcomes.



**Figure 2** Sequence of events

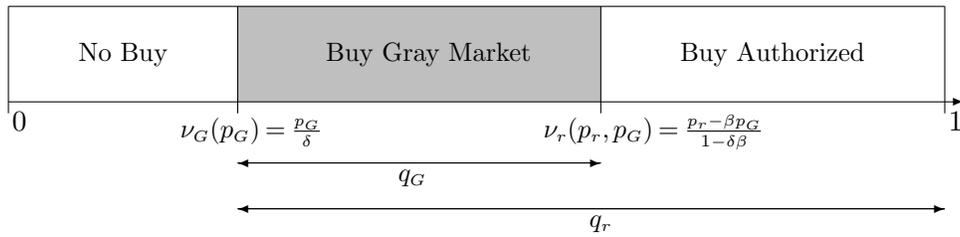
Since we assume demand to be deterministic, we focus our analysis on two contracts; wholesale price and quantity discount contracts. The wholesale price contract has a simple form and is considered a benchmark in the literature, even though it generally fails to coordinate the supply chain. A quantity discount contract, on the other hand, can coordinate the supply chain if the parameters of the contract are chosen appropriately (e.g. Jeuland and Shugan 1983). Other contracts such as the buy-back contract are more suitable for coordinating supply chains that face uncertain demand.

We denote consumer utility of the product with  $\nu$ , which is uniformly distributed between 0 and 1. If consumers buy the product from the retailer in period 1, they earn a surplus of  $\nu - p_r$ .

On the other hand, if consumers buy the product from the gray market in period 2, their surplus will be  $(\delta\nu - p_G)\beta$ . Parameter  $0 < \delta < 1$  represents consumers' lower valuation of the gray market relative to the authorized channel due to factors such as loss of warranty and after-sales services. Parameter  $0 < \beta < 1$  is a time discount factor that captures the disutility of having to wait for one period to obtain the product. Note that buying from the retailer in period 2 is a suboptimal strategy for consumers with a positive surplus since  $\beta(\nu - p_r) < \nu - p_r$ , for  $\nu > p_r$ . Let  $\nu_r$  be the utility of a consumer who is indifferent between buying from the retailer in period 1 and buying from the gray market in period 2 and let  $\nu_G$  be the utility of a consumer who is indifferent between buying from the gray market in period 2 and not buying the product at all. Then

$$\begin{aligned} \nu_r - p_r &= (\delta\nu_r - p_G)\beta \implies \nu_r(p_r, p_G) = \frac{p_r - \beta p_G}{1 - \delta\beta}, \\ (\delta\nu_G - p_G)\beta &= 0 \implies \nu_G(p_G) = \frac{p_G}{\delta}. \end{aligned} \tag{1}$$

Given our assumption that the retailer retains a positive market share when he diverts to the gray market,  $\nu_r(p_r, p_G) < 1$ . In fact, we can show that the equilibrium solutions for the centralized supply chain, wholesale price, and quantity discount contracts in Lemmas 1–3 indeed result in a positive market share for the authorized channel. Thus, the expected total demand for the retailer in period 1 and for the gray market in period 2 will be  $q_r(p_r, p_G) - q_G(p_r, p_G) = 1 - \nu_r(p_r, p_G)$  and  $q_G(p_r, p_G) = \nu_r(p_r, p_G) - \nu_G(p_G)$ , respectively, as depicted in Figure 3. For brevity, we suppress the arguments and use  $q_r$  and  $q_G$  for the rest of the analysis.



**Figure 3** Market segmentation when the retailer diverts to the gray market

Table 1 summarizes the basic notations used throughout the paper. We denote optimal decisions with superscripts  $c$  for the centralized supply chain,  $w$  for the wholesale price contract, and  $QD$  for the quantity discount contract. We use subscripts  $m$ ,  $r$ ,  $sc$ , and  $G$  for manufacturer, retailer, supply chain, and gray market. Furthermore, we add  $N$  to the superscripts when referring to a benchmark supply chain in which no gray market exists.

Before starting our analysis, we introduce an assumption that imposes an upper bound on the manufacturer's production cost. The analyses in subsequent sections will reveal that this assumption is a necessary (but not sufficient) condition for the diversion of the product to the gray market

**Table 1** Notations

$w_m$	manufacturer's wholesale price for selling to the retailer
$w_r$	retailer's wholesale price for selling to the gray market
$c$	manufacturer's unit production cost
$\delta$	consumer relative valuation of the gray market
$\rho$	penalty to the manufacturer for each unit diverted to the gray market
$\beta$	time discount parameter
$b$	slope of the classic quantity discount contract
$p_r, q_r$	retailer's retail price and order quantity
$\pi_m, \pi_r, \pi_{sc}$	total profit of the manufacturer, retailer, and the supply chain
$q_G, p_G, \pi_G$	quantity, price, and total profit of the gray market

in the centralized supply chain as well as in the wholesale price and quantity discount contracts. Therefore, the following assumption allows us to analyze the impact of the gray market.

$$\text{ASSUMPTION 1. } c < \frac{\delta(1-\beta)}{2-\delta-\delta\beta}$$

### 3.1. No Gray Market

We start our analysis assuming that no gray market exists. In this situation, consumers who wish to obtain the product can only buy from the retailer in period 1. If the manufacturer and the retailer are integrated, the profit of the supply chain will be  $(p_r - c)(1 - p_r)$  which is maximized at  $p_r^{cN} = \frac{1+c}{2}$ . If the supply chain is decentralized and the manufacturer charges the retailer a wholesale price  $w_m$ , then the retailer's optimal price will be  $p_r^{wN} = \frac{1+w_m}{2}$ .

When there is no gray market, the wholesale price contract fails to coordinate the supply chain unless  $w_m = c$  and the manufacturer earns zero profit. However, a quantity discount contract can achieve coordination. Suppose the manufacturer offers the retailer an all-unit quantity discount contract where the retailer has to pay the manufacturer  $h(q_r) = a - bq_r$  per unit when he orders  $q_r$  units. Under this contract, the retailer determines his retail price by maximizing  $[p_r - (a - b(1 - p_r))](1 - p_r)$ . As a result, the retailer sets  $p_r^{QDN} = \frac{1+a-2b}{2(1-b)}$ . If  $a = b(1 - c) + c$ , then  $p_r^{QDN} = p_r^{cN}$  and the quantity discount contract coordinates the supply chain. The profit margin of the retailer in this case will be  $p_r^{QDN} - [b(1 - c) + c - b(1 - p_r^{QDN})] = \frac{(1-b)(1-c)}{2}$ . Therefore,  $b$  must be strictly smaller than 1.

### 3.2. Centralized Supply Chain with Gray Market

We now analyze the optimal decisions of the supply chain assuming that the gray market exists. As a benchmark, we first assume that the supply chain is centralized. If the product is diverted to the gray market, the gray market will solve the following profit-maximization problem in period 2

$$\max_{p_G} \pi_G = (p_G - w_r) \underbrace{[\nu_r(p_r, p_G) - \nu_G(p_G)]}_{q_G}.$$

The gray market's best response to the authorized channel is thus  $p_G^*(p_r, w_r) = \frac{1}{2}(\delta p_r + w_r)$ . In the interest of brevity, we suppress  $(p_r, w_r)$  and denote the gray market's best response with

$p_G^*$  throughout the analysis. Taking the gray market's response into consideration, the centralized supply chain finds its optimal decision by solving

$$\max_{p_r, w_r} \pi_{sc} = (p_r - c) \underbrace{[1 - \nu_r(p_r, p_G^*)]}_{q_r - q_G} + (w_r - c - \rho) \underbrace{[\nu_r(p_r, p_G^*) - \nu_G(p_G^*)]}_{q_G}. \quad (2)$$

The first term is the profit from selling to consumers who buy in the first period. The second term is the profit from diverting the product to the gray market. The quantity sold to the gray market is  $\nu_r(p_r, p_G^*) - \nu_G(p_G^*)$  and the profit margin is reduced by  $\rho$ . In order to characterize the product diversion to the gray market, we define the following threshold for  $\rho$

$$\rho^c = \frac{\delta}{2} [(1 + \beta)c + 1 - \beta] - c. \quad (3)$$

We note that  $\rho^c > 0$  due to Assumption 1. The next results when the centralized supply chain will divert to the gray market.

LEMMA 1. *Let  $\rho^c$  be defined as (3). Then,*

(a) *If  $\rho < \rho^c$ , then the centralized supply chain will divert to the gray market and*

$$p_r^c = \frac{(1 - \delta\beta)[4 + (3 + \beta)c] - (1 - \beta)\rho}{8 - \delta(1 + \beta^2 + 6\beta)}, \quad (4)$$

$$w_r^c = \frac{(1 - \delta\beta)[4 + (1 - \beta)\delta]c + 2\delta(1 + \beta)(1 - \delta\beta) + [4(1 - \delta\beta) - (1 - \beta)\delta]\rho}{8 - \delta(1 + \beta^2 + 6\beta)}. \quad (5)$$

(b) *If  $\rho \geq \rho^c$ , then the centralized supply chain will not divert to the gray market, i.e.,  $p_r^c = p_r^{cN}$  and  $w_r^c = \delta p_r^{cN}$ .*

If the loss of brand image penalty imposed by the gray market is not excessively high, the additional revenue from selling to the gray market will offset the penalty and the centralized supply chain will divert to the gray market at the end of period 1. When diversion occurs, the supply chain increases its retail price to extract more profit from customers who prefer to buy from the authorized channel (i.e.,  $p_r^c > p_r^{cN}$ ). Also, the equilibrium retail price,  $p_r^c$ , decreases with  $\rho$  while the equilibrium wholesale price for selling to the gray market,  $w_r^c$ , increases with  $\rho$ .

#### 4. Wholesale Price vs. Classic Quantity Discount

We now derive the equilibrium solutions for the wholesale price contract and the classic quantity–discount contract,  $h(q_r) = b(1 - c) + c - cq_r$ , and investigate their performance in the presence of the gray market. By comparing the solutions of the contracts with the centralized supply chain and with each other, we explore the impacts of the gray market on the contracts and the impacts of the contracts on the supply chain.

#### 4.1. Wholesale Price Contract

Suppose the manufacturer sells the product to the retailer at wholesale price  $w_m$ . Then, the retailer's problem will be

$$\max_{p_r, w_r} \pi_r = p_r \underbrace{[1 - \nu_r(p_r, p_G^*)]}_{q_r - q_G} + w_r \underbrace{[\nu_r(p_r, p_G^*) - \nu_G(p_G^*)]}_{q_G} - w_m \underbrace{[1 - \nu_G(p_G^*)]}_{q_r},$$

and the manufacturer's profit will be

$$\pi_m = (w_m - c)[1 - \nu_G(p_G^*)] - \rho[\nu_r(p_r, p_G^*) - \nu_G(p_G^*)].$$

The next lemma describes the equilibrium decisions of the retailer in the wholesale price contract.

LEMMA 2. *In the wholesale price contract, the retailer will sell to the gray market and set the following prices*

$$p_r^w = \frac{(1 - \delta\beta)[4 + (3 + \beta)w_m]}{8 - \delta(1 + \beta^2 + 6\beta)}, \quad w_r^w = \frac{(1 - \delta\beta)[(4 + (1 - \beta)\delta)w_m + 2\delta(1 + \beta)]}{8 - \delta(1 + \beta^2 + 6\beta)},$$

if and only if  $w_m < \frac{\delta(1-\beta)}{2-\delta-\delta\beta}$ . Otherwise,  $p_r^w = p_r^{w^N}$ ,  $w_r^w = \delta p_r^{w^N}$ , and the gray market will not emerge.

This lemma provides a necessary and sufficient condition for the emergence of the gray market in the wholesale price contract. If the manufacturer's wholesale price is sufficiently low, the retailer can afford to sell a quantity of the product to the gray market. Otherwise, double marginalization precludes the retailer from diverting to the gray market. Note that if Assumption 1 does not hold, then  $w_m \geq \frac{\delta(1-\beta)}{2-\delta-\delta\beta}$  and the gray market will never exist in the wholesale price contract.

Now we look at how the equilibrium decisions for a wholesale price contract differ from those for the centralized supply chain in the presence of the gray market.

PROPOSITION 1. *The effects of using a wholesale price contract when there is a gray market are as follows:*

(a) *The optimal retail price in the wholesale price contract will be greater than or equal to the optimal retail price in the centralized supply chain, i.e.,  $p_r^w \geq p_r^c$ , with equality obtained when  $w_m = c$  and  $\rho = 0$ .*

(b) *The price for selling to the gray market will be smaller in the wholesale price contract than in the centralized supply chain (i.e.  $w_r^w < w_r^c$ ) if and only if  $\rho > \frac{(w_m - c)(1 - \delta\beta)[4 + \delta(1 - \beta)]}{4 - \delta(1 + 3\beta)}$ .*

Double marginalization once again increases the retail price. The relationship between equilibrium prices for selling to the gray market depends on  $\rho$ , because in the centralized supply chain this penalty is borne by the entire supply chain whereas it is only borne by the manufacturer when the supply chain is decentralized.

Next we investigate how the emergence of the gray market impacts the retail price and the profits of the manufacturer, the retailer, and the supply chain when a wholesale price contract is in place. For this purpose, we define two thresholds for  $\rho$

$$\hat{\rho}_w = \frac{1}{2}(2 - \delta - \delta\beta)(w_m - c), \quad \tilde{\rho}_w = \frac{1}{4}(w_m - 2c)(2 - \delta - \delta\beta) + \frac{\delta}{4}(1 - \beta), \quad (6)$$

where  $0 \leq \hat{\rho}_w < \tilde{\rho}_w$  due to Assumption 1.

**PROPOSITION 2.** *The effects of the gray market on supply chain profits in the wholesale price contract are as follows:*

- (a) *The retailer sets a higher retail price and his profit is always higher with the gray market, i.e.,  $p_r^w > p_r^{w^N}$  and  $\pi_r^w > \pi_r^{w^N}$ .*
- (b) *The manufacturer will be better off with the gray market if and only if  $\rho < \hat{\rho}_w$ , and the supply chain will be better off with the gray market if and only if  $\rho < \tilde{\rho}_w$ , where  $\hat{\rho}_w, \tilde{\rho}_w$  are defined in (6).*

This result explains that selling to the gray market always benefits the retailer. If the gray market does not damage the brand image severely, the manufacturer, and as a result the supply chain, will benefit from the gray market, because the loss of brand image from diversion to the gray market will be more than compensated by the additional profit from selling to the gray market. If the penalty is moderately high, the manufacturer no longer benefits from the gray market but the retailer's extra profit continues to benefit the entire supply chain. Eventually, when the negative consequences of the gray market for the manufacturer are significantly high, the manufacturer and the supply chain lose profit.

Next we describe the impact of strategic consumer behavior and their valuation of the gray market on decisions and profits under the wholesale price contract. Our model uses two "discount" parameters for the surplus of the consumers who wait to buy the product from the gray market. The first parameter,  $\delta$ , represents consumer valuation of the gray market relative to the authorized channel (retailer). The next result summarizes the effects of  $\delta$ .

**PROPOSITION 3.**

- (a) *The profit of the gray market increases with  $\delta$ .*
- (b) *The retail price and profit of the retailer increase with  $\delta$ .*

When  $\delta$  grows, consumers' relative valuation of the gray market increases and they are more willing to wait and buy from the gray market in period 2. This is obviously beneficial to the gray market. It is also beneficial to the retailer because it enables him to offer the product at a higher retail price in period 1 and make the product available to consumers who have a lower willingness to pay via the gray market channel in period 2. In other words, the gray market allows the retailer to indirectly segment the market.

The second parameter that influences consumer surplus for buying from the gray market is the time discount parameter  $\beta$  which captures consumers' *patience* for waiting to obtain the product from the gray market. Higher  $\beta$  means consumers are more willing to wait for the gray market. The next lemma describes the effects of  $\beta$  under the wholesale price contract.

PROPOSITION 4.

- (a) *The profit of the gray market decreases with  $\beta$ .*
- (b) *The retail price and profit of the retailer decrease with  $\beta$ .*

The contrast between the statements of propositions 3 and 4 is interesting. Even though  $\delta$  and  $\beta$  are defined to represent a similar notion about consumer surplus for buying from the gray market, they influence the outcomes very differently; in fact, in opposite directions. One would expect that the gray market should benefit from higher values of  $\beta$  because the consumers are more willing to wait for the gray market. However, the profit of the gray market and the retailer decrease with  $\beta$ . This difference can be explained by examining how  $\delta$  and  $\beta$  influence market segments. According to equations in (1), given  $p_r$  and  $p_G$ , the market share of the gray market is the difference between the consumer who is indifferent between buying from the retailer and buying from the gray market,  $\nu_r(p_r, p_G)$ , and the consumer who is indifferent between buying from the gray market and not buying at all,  $\nu_G(p_G)$ . As  $\delta$  increases,  $\nu_r(p_r, p_G)$  increases and the gray market cannibalizes more of the retailer's demand. At the same time,  $\nu_G(p_G)$  decreases and more consumers who would not buy the product if there were no gray markets are induced to buy from the gray market. Since the gray market buys from the retailer, the retailer's total demand will be  $1 - \nu_G(p_G) = 1 - \frac{p_G}{\delta}$ , which means higher  $\delta$  expands his total demand. As a result, the profits of the retailer and the gray market go up with  $\delta$ . In contrast, when  $\beta$  increases,  $\nu_r(p_r, p_G)$  increases whereas  $\nu_G(p_G)$  does not change. Therefore, higher consumer patience only cannibalizes the retailer's demand, but does not expand total demand for the product. As a result, the retailer limits the gray market by reducing his retail price to encourage more consumers to buy in the first period. This reaction by the retailer reduces his profit and the profit of the gray market.

#### 4.2. Classic Quantity Discount Contract

Under the classic quantity discount contract the retailer's profit maximization problem will be

$$\max_{p_r, w_r} \pi_r = p_r \underbrace{[1 - \nu_r(p_r, p_G^*)]}_{q_r - q_G} + w_r \underbrace{[\nu_r(p_r, p_G^*) - \nu_G(p_G^*)]}_{q_G} - \underbrace{h(1 - \nu_G(p_G^*))}_{h(q_r)} \underbrace{[1 - \nu_G(p_G^*)]}_{q_r},$$

The manufacturer's profit under this contract will be

$$\pi_m = b[(1 - c) - (1 - \nu_r(p_r, p_G^*))][1 - \nu_r(p_r, p_G^*)] - \rho[\nu_r(p_r, p_G^*) - \nu_G(p_G^*)].$$

The next lemma provides the conditions under which the classic quantity discount contract encourages the retailer to divert to the gray market.

LEMMA 3. *When the manufacturer offers the classic quantity discount contract, the retailer diverts the product to the gray market and the gray market will make positive profit if Assumption 1 and the following condition hold:*

$$b \leq \bar{b} = \frac{4c + [2(1 + \beta) + c(1 - \beta)]\delta}{4c + 2 + [1 + \beta + c(1 - \beta)]\delta}, \quad (7)$$

where  $\bar{b} < 1$ . The equilibrium decisions of the retailer will be

$$p_r^{QD} = \frac{2(1 - \delta\beta) + w_r^{QD}(3 + \beta)}{4 + \delta(1 - \beta)},$$

$$w_r^{QD} = \frac{\delta(1 - \delta\beta) [(2(1 + \beta) + (c(1 - \beta)(1 - b) - b(1 + \beta)))\delta + 2(2c(1 - b) - b)]}{\mu},$$

where  $\mu = -[1 + \beta^2 + 2(3 - 2b)\beta]\delta^2 + 4[2 - (1 - \beta)b]\delta - 4b$ .

REMARK 1. The inequality in (7) and Assumption 1 guarantee that  $w_r^{QD}$  is an interior solution between 0 and  $\delta p_r^{QD}$ . If one or both of these conditions are violated, then the optimal value of  $w_r^{QD}$  will be a boundary value. Specifically, if  $b < \bar{b}$  and  $c \geq \frac{\delta(1-\beta)}{2-\delta-\delta\beta}$ , then  $p_r^{QD} = p_r^{QD^N}$  and  $w_r^{QD} = \delta p_r^{QD^N}$  which means the gray market will not operate. If  $b \geq \bar{b}$  and  $c < \frac{\delta(1-\beta)}{2-\delta-\delta\beta}$ , then  $w_r^{QD} = 0$  which means that the quantity discount is so steep that the retailer sells the product to the gray market for free. This is not a realistic case. Finally if both  $b \geq \bar{b}$  and  $c \geq \frac{\delta(1-\beta)}{2-\delta-\delta\beta}$ , then either  $w_r^{QD} = \delta p_r^{QD^N}$  or  $w_r^{QD} = 0$ .

First we compare the solution of the classic quantity discount contract with the solution of the centralized supply chain to understand the effect of using this contract when there is a gray market.

PROPOSITION 5. *Let  $\rho^c$  be defined as (3). Then, the effects of using the classic quantity discount contract when the gray market exists are as follows:*

- (a) *The retail price under the classic quantity discount contract can be higher or lower than the retail price in the centralized supply chain. Specifically, if  $b > \frac{\delta(1-\beta)}{2(1-\delta\beta)}$ , then  $p_r^{QD} < p_r^c$  for all  $\rho < \rho^c$ . Otherwise, there exist  $\rho^* < \rho^c$  such that  $p_r^{QD} < p_r^c$  for  $\rho < \rho^*$  and  $p_r^{QD} \geq p_r^c$  for  $\rho^* \leq \rho < \rho^c$ .*
- (b) *Retailer's wholesale price for selling to the gray market is always smaller in the classic quantity discount contract than in the centralized supply chain, i.e.,  $w_r^{QD} < w_r^c$  for all  $\rho < \rho^c$ .*

The first finding in Proposition 5 is that the double marginalization phenomenon does not necessarily hold for the quantity discount contract. The wholesale price that the retailer charges for selling to the gray market is always smaller in the quantity discount contract, because in the decentralized supply chain the retailer is not exposed to the loss of brand image penalty. As a result, he can afford to charge the gray market a lower price.

Next we look at the effect of gray market on the retail price and the profit of the manufacturer, the retailer, and the supply chain when the classic quantity discount contract is in place.

PROPOSITION 6. *The effects of gray market on supply chain profits in the classic quantity discount contract are as follows:*

- (a) *The retailer is always better off with gray market. The retail price will increase (i.e.,  $p_r^{QD} > p_r^{QD^N}$ ) if and only if  $b < \frac{\delta(1-\beta)}{2(1-\delta\beta)}$ .*
- (b) *The manufacturer is always worse off with gray market.*

Similar to the wholesale price contract, the retailer always benefits from selling to the gray market because the gray market is an extra source of demand. Contrary to the wholesale price contract, however, the manufacturer is always hurt when the retailer sells to the gray market *even* if  $\rho = 0$  and the gray market does not damage the manufacturer's brand image. The reason is that the classic quantity discount contract is designed based on the premise that the retailer only sells the product in the authorized channel. When the retailer diverts a portion of his order to the gray market, his order quantity differs from what the manufacturer anticipates and leads to profit loss for the manufacturer. The effect of the gray market on the total profit of the supply chain is undetermined. It depends on whether or not the retailer's benefit from the gray market outweighs the manufacturer's loss to the gray market.

### 4.3. Comparing Contracts

So far, we have analyzed the effects of the gray market on the wholesale price and classic quantity discount contracts and compared their performance with the centralized system. Now we compare these contracts with each other to understand which contract serves the supply chain better. The next proposition compares the equilibrium prices of the two contracts.

- PROPOSITION 7. (a) *The retailer's retail price and wholesale price for selling to the gray market are higher in the wholesale price contract, i.e.,  $p_r^w > p_r^{QD}$  and  $w_r^w > w_r^{QD}$ .*
- (b) *The gray market's price is higher in the wholesale price contract, i.e.,  $p_G^w > p_G^{QD}$ .*

Under the wholesale price contract, both the retailer and the gray market charge higher prices. This is mainly due to the double marginalization effect in the wholesale price contract. Because the retailer charges a higher price both to consumers and to the gray market when the manufacturer offers a wholesale price contract, the gray market price is higher in the wholesale price contract than in the quantity discount contract. Note that the quantity discount contract encourages the retailer to purchase a larger quantity and divert more to the gray market; therefore all optimal prices are smaller than the optimal prices in the wholesale price contract. This means that if diversion of products to the gray market is costly to the manufacturer, then she, and subsequently the supply chain, can be worse off under the quantity discount contract. The next result, which presents one of the key findings of this paper, formalizes this argument by comparing total profits of the authorized supply chain under each contract.

PROPOSITION 8. *There exist a wholesale price  $\tilde{w}_m > c$ , such that if  $c \leq w_m < \tilde{w}_m$ , then the supply chain will be better off under the wholesale price contract than the classic quantity discount contract. In addition,  $\tilde{w}_m$  is increasing in  $\rho$ .*

This proposition shows that gray markets significantly impact the performance of classic coordinating contracts so much so that a range of simple wholesale price contracts, which suffer from double marginalization, outperform the classic quantity discount contract that coordinates the supply chain in the absence of gray markets. The classic quantity discount contract coordinates the supply chain by reducing the per unit order cost of the retailer and inducing him to raise his order quantity. However, when there is a gray market, the retailer exploits the contract to increase his profit by diverting to the gray market.

Proposition 8 has important implications for companies that operate in environments where their authorized distributors can potentially divert products to gray markets. Such companies need to design and offer channel incentives that carefully take into consideration the extent of gray market activities in the market. While a classic quantity discount contract alleviates double marginalization and induces retailers to place larger orders, the abuse of the contract by the retailers and the diversion of a large volume of products to gray markets may erode the manufacturer's brand image severely and ultimately hurt the total supply chain profit.

## 5. Wholesale Price Contract and the Price of Anarchy

Based on our observation that a wholesale price contract can serve the supply chain better than a classic quantity discount contract, in this section we investigate the performance of wholesale price contracts further. Although wholesale price contracts are considered to be inefficient in general, previous research has shown the effectiveness of such contracts in certain situations. For instance, Bernstein and Federgruen (2005) show that in a decentralized supply chain with one supplier servicing a network of competing retailers facing stochastic demand, double marginalization is beneficial and wholesale price contracts can induce perfect coordination under some conditions. In our model, the gray market creates competition for the manufacturer's authorized channel; therefore, one would expect that the inefficiency resulting from implementing a wholesale price contract would be less when a gray market exists. In fact, Altug and van Ryzin (2014) find that when the authorized channel price is fixed but demand is random, the efficiency loss of wholesale price contracts decreases when there is a gray market. Surprisingly, our analysis in this section shows that in our model (price-setting authorized channel and gray market) the presence of the gray market is likely to cause wholesale price contracts to be even more inefficient.

In order to investigate the performance of wholesale price contracts in our model, we measure the Price of Anarchy (PoA) for the wholesale price contract. PoA is defined as the ratio of the

optimal profit of the centralized supply chain over the profit of the decentralized supply chain when the manufacturer optimizes her wholesale price value. Mathematically, the PoA is defined as

$$\text{PoA} = \frac{\pi_{sc}^c}{\pi_{sc}(w_m^*)} \quad (8)$$

where  $\pi_{sc}^c$  is the optimal profit of the centralized supply chain and  $\pi_{sc}(w_m^*)$  is the profit of the decentralized supply chain given the manufacturer's *optimal* wholesale price  $w_m^*$ . According to Lemma 2, if  $w_m < \frac{\delta(1-\beta)}{2-\delta-\delta\beta}$ , then the retailer will sell to the gray market. Otherwise,  $w_m$  will prevent the retailer from diverting to the gray market. When the retailer does not sell to the gray market, he sets  $p_r = \frac{1+w_m}{2}$  which means the manufacturer's profit will be  $(w_m - c)(1 - p_r)$ . Thus, the manufacturer has to solve two optimization problems and compare the profits to find  $w_m^*$ . The first problem is for allowing the retailer to sell to the gray market and the second problem is for dissuading the retailer from selling to the gray market:

$$\max_{w_m} \quad \pi_m = (w_m - c) [1 - \nu_G(p_G^*)] - \rho [\nu_r(p_r^w, p_G^*) - \nu_G(p_G^*)], \quad (9)$$

*s.t.*

$$w_m < \frac{\delta(1-\beta)}{2-\delta-\delta\beta},$$

$$\max_{w_m} \quad \pi_m = (w_m - c) \left( \frac{1-w_m}{2} \right), \quad (10)$$

*s.t.*

$$w_m \geq \frac{\delta(1-\beta)}{2-\delta-\delta\beta}.$$

Note that (10) is concave in  $w_m$ . Thus, when the manufacturer's decision is to dissuade the retailer from diverting to the gray market,  $w_m^* = \max\left(\frac{1+c}{2}, \frac{\delta(1-\beta)}{2-\delta-\delta\beta}\right)$ . To understand the effects of the gray market on the PoA, we first look at the benchmark case when  $\delta \rightarrow 0$  and there will be no diversion to the gray market under both centralized and wholesale price scenarios (as if the gray market did not exist). For this case,  $\pi_{sc}^c = \frac{(1-c)^2}{4}$ ,  $w_m^* = \frac{1+c}{2}$ ,  $p_r^{w^*} = \frac{3+c}{4}$ , and  $\pi_{sc}(w^*) = \frac{3(1-c)^2}{16}$  which result in  $\text{PoA} = \frac{4}{3}$ . It is interesting to note that the same value of the PoA is observed in other contexts such as traffic routing (Roughgarden and Tardos 2000).

Table 2 summarizes the analytical results for the PoA based on whether or not the product is diverted to the gray market in the centralized and wholesale price settings. Interestingly, the efficiency loss of the wholesale price contract can increase when there is a gray market. When the manufacturer's wholesale price prevents the retailer from diverting to the gray market (i.e. (No, No) and (Yes, No) cases in Table 2), the PoA increases from  $\frac{4}{3}$  and the gray market worsens the inefficiency of the wholesale price contract. The reason for (No, No) is the concavity of (10) and

	Centralized supply chain	Wholesale price contract	PoA
Diversion to gray market	No	No ( $w_m^* = \frac{1+c}{2}$ )	$= \frac{4}{3}$
	No	No ( $w_m^* = \frac{\delta(1-\beta)}{2-\delta-\delta\beta}$ )	$> \frac{4}{3}$
	Yes	No	$> \frac{4}{3}$
	No	Yes	$> \text{ or } < \frac{4}{3}$
	Yes	Yes	$> \text{ or } < \frac{4}{3}$

**Table 2** The Price of Anarchy for different diversion scenarios

the reason for (Yes, No) is that the centralized supply chain will sell to the gray market only when doing so increases total supply chain profit.

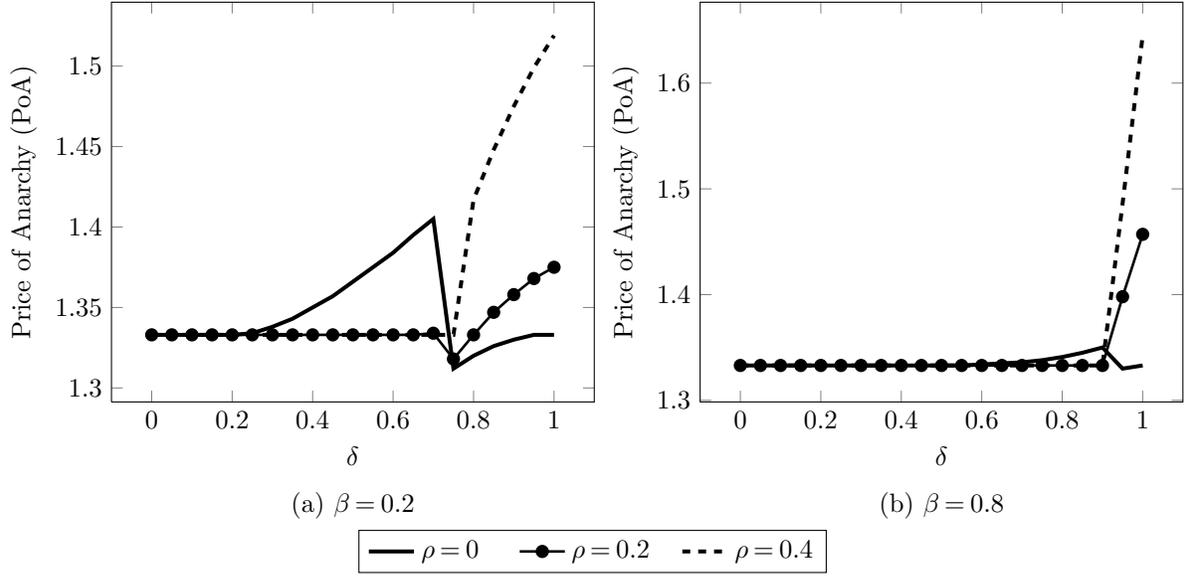
There are two scenarios, (No, Yes) and (Yes, Yes), in which the PoA can potentially be higher or lower than  $\frac{4}{3}$ . It is difficult to derive lower or upper bounds on the PoA when  $\delta$  is strictly between 0 and 1. The next result characterizes the PoA for a special case of the (Yes, Yes) scenario.

**PROPOSITION 9.** *When  $\delta \rightarrow 1$ , PoA increases with  $\rho$  and varries between  $\frac{4}{3}$  and  $\frac{32}{3(\beta+7)}$ .*

This proposition indicates that when  $\delta$  approaches 1 and consumers almost value the gray market same as the authorized channel, the efficiency loss of the wholesale price contract increases. As the loss of brand image penalty gets close to its upper bound for selling to the gray market in the centralized setting, i.e.,  $\rho \rightarrow \lim_{\delta \rightarrow 1} \rho^c = \frac{(1-\beta)(1-c)}{2}$ , the PoA approaches  $\frac{32}{3(\beta+7)}$  which reaches  $\frac{32}{21} = 1.52$  as  $\beta \rightarrow 0$ . Therefore, the efficiency loss of the wholesale price contract can be as high as 52% (a 14.28% increase from the case of no gray market with  $\text{PoA} = \frac{4}{3}$ ).

We conducted experiments for different parameter values to fully examine the PoA when analytical results are not available. Figure 4 provides two examples of our experiments and represents the pattern we observed for the PoA. Although the PoA can take values less than  $\frac{4}{3}$ , the efficiency loss did not drop below  $\frac{4}{3}$  by more than 2.5% in our experiments. As a matter of fact, the vast majority of the experiments generated PoAs that were significantly higher than  $\frac{4}{3}$ . Figure 4(a) shows that the PoA can be as high as 1.64.

As one can see from Figure 4, the PoA increases the most when  $\delta$  and  $\rho$  are sufficiently high. Recall Proposition 3 that the retailer will sell more to and benefit from the gray market when  $\delta$  increases and consumers perceive the gray market more valuable. Also recall Proposition 2 that the manufacturer benefits from diversion to the gray market when  $\rho$  is below a threshold,  $\hat{\rho}$ . When  $\delta$  grows large and  $\rho$  is also high, diversion to the gray market hurts the manufacturer's profit. The only weapon the manufacturer has in her arsenal to dissuade the retailer from selling to the gray market is her wholesale price. Therefore, the manufacturer charges a higher wholeslae price to



**Figure 4** Effects of the gray market on the Price of Anarchy ( $c=0.1$ )

increase the purchase cost for the retailer and make it difficult for him to sell to the gray market. In fact, we can show that when  $w_m^*$  solves (9),  $w_m^* = \frac{1+c}{2} + \frac{(2-\delta-\delta\beta)(\delta+\rho-1)}{4(1+\delta)(1-\delta\beta)}$ , which shows that  $w_m^* > \frac{1+c}{2}$  when  $\delta$  and  $\rho$  are high. This reaction by the manufacturer intensifies double marginalization and the efficiency loss of the wholesale price contract worsens. Therefore, even though wholesale price contracts dominate classic quantity classic discount contracts when there is a gray market, they still perform poorly and are far from achieving supply chain coordination.

## 6. Coordination

Having established the inefficiency of wholesale price and classic quantity discount contracts, in this section we turn our attention to designing optimal coordinating mechanisms. In the context of deterministic demand, the primary contract of interest is a quantity discount contract. Section 4 demonstrated the inefficiency of classic quantity discount contracts when gray market externalities are ignored. Our next result shows that any one-time payment contract, including quantity discount, that only depends on the quantity ordered from the manufacturer fails to coordinate the supply chain in the presence of a gray market.

**PROPOSITION 10.** *Any retailer-to-manufacturer payment that only depends on the retailer's order quantity will fail to coordinate the supply chain, unless  $\rho = 0$ .*

This result shows that any payment scheme offered by the manufacturer that ignores the retail price of the product and is solely based on the quantity purchased is sub-optimal. This is because quantity-only contracts while promote larger orders by the retailer, they offer incentives for diversion of products to gray markets beyond the level that is optimal for the supply chain. There

are two retailer decisions that the manufacturer is interested in: total quantity purchased by the retailer and the quantity diverted to the gray market. If the payment scheme is dependent on the quantity purchased as well as the retail price of the product, then it will provide two levers for the manufacturer to control both the quantity purchased by the retailer and the quantity that he diverts to the gray market. Price-dependent contracts have been used in prior research such as Bernstein and Federgruen (2005).

Consequently, in this section we rely on the following payment schemes. We define  $C(p_r, q_r)$  to be the total payment from the retailer to the manufacturer if his total order quantity is  $q_r$  and he sets the retail price to  $p_r$ . Using the market segmentation equations in (1) and the gray market's best response price, we can find a relationship between the retailer's order quantity  $q_r$  and his prices  $p_r$  and  $w_r$ . More specifically,  $q_r = 1 - \nu_G(p_G^*) = 1 - \frac{p_G^*}{\delta} = 1 - \frac{\delta p_r + w_r}{2\delta}$ . Therefore, for a given value of  $p_r$ , there is a one-to-one relationship between  $q_r$  and  $w_r$ . For our analysis of coordination, we replace  $w_r$  in the profit function of the retailer with  $w_r = 2\delta(1 - q_r) - \delta p_r$ . We make this change of variables for ease of exposition and the fact that in most practical scenarios, the retailer's price for selling to the gray market,  $w_r$ , is not contractable (unlike the order quantity  $q_r$ , which in most cases is contractable).

Under the proposed contract, the retailer's profit,  $\bar{\pi}_r$ , can be stated as a function of  $p_r$  and  $q_r$

$$\bar{\pi}_r(p_r, q_r) = p_r \underbrace{[1 - \nu_r(p_r, p_G^*)]}_{q_r - q_G} + \underbrace{[2\delta(1 - q_r) - \delta p_r]}_{w_r} \underbrace{[\nu_r(p_r, p_G^*) - \nu_G(p_G^*)]}_{q_G} - C(p_r, q_r),$$

where  $p_G^* = \delta(1 - q_r)$ . The next proposition, which is the main result of this section, characterizes a family of payment mechanisms that are able to coordinate the decisions of the retailer with those of the centralized supply chain.

**PROPOSITION 11.** *Let  $\rho^c$  be defined as (3), and let  $C(p_r, q_r)$  denote the retailer's total payment to the manufacturer when his order quantity and retail price are  $q_r$  and  $p_r$ , respectively. Then payment scheme  $C(p_r, q_r)$  can coordinate the supply chain if it satisfies the following conditions:*

(a) when  $\rho \leq \rho^c$ ,

$$\begin{aligned} \left. \frac{\partial C(p_r, q_r)}{\partial p_r} \right|_{p_r=p_r^c} &= \frac{\rho}{1 - \delta\beta}, & \frac{-2(1 + \delta)}{1 - \delta\beta} &< \frac{\partial^2 C(p_r, q_r)}{\partial p_r^2}, \\ \left. \frac{\partial C(p_r, q_r)}{\partial q_r} \right|_{q_r=q_r^c} &= \frac{\rho}{1 - \delta\beta} + c, & \frac{-4\delta}{1 - \delta\beta} &< \frac{\partial^2 C(p_r, q_r)}{\partial q_r^2} < 0, \end{aligned}$$

where  $q_r^c$  is determined by  $p_r^c$  and  $w_r^c$  in (4)–(5).

(b) when  $\rho > \rho^c$ ,

$$C(p_r, q_r) = \begin{cases} \widehat{C}(p_r, q_r) & \text{if } p_r + q_r \geq 1, \\ \widetilde{C}(q_r) & \text{otherwise,} \end{cases}$$

where

$$\begin{aligned} \left. \frac{\partial \widehat{C}(p_r, q_r)}{\partial p_r} \right|_{p_r=p_r^c} &= \frac{\delta(1-\beta) - (2-\delta-\delta\beta)c}{2(1-\delta\beta)}, & \frac{-2(1+\delta)}{1-\delta\beta} &< \frac{\partial^2 \widehat{C}(p_r, q_r)}{\partial p_r^2}, \\ \left. \frac{\partial \widehat{C}(p_r, q_r)}{\partial q_r} \right|_{q_r=q_r^c} &= \frac{\delta(1-\beta)(1+c)}{2(1-\delta\beta)}, & \frac{-4\delta}{1-\delta\beta} &< \frac{\partial^2 \widehat{C}(p_r, q_r)}{\partial q_r^2} < 0, \end{aligned}$$

and

$$\left. \frac{d\widetilde{C}(q_r)}{dq_r} \right|_{q_r=q_r^c} = c, \quad -2 < \frac{d^2 \widetilde{C}(q_r)}{dq_r^2} < 0.$$

This result provides conditions for coordinating the supply chain both when the supply chain benefits from selling to the gray market and when it does not. When the gray market emerges under the centralized scenario (i.e.,  $\rho \leq \rho^c$ ), the first and second derivative conditions with respect to the order quantity ensure that  $C(p_r, q_r)$  is a concave and increasing function of  $q_r$ . That is, the payment scheme offers quantity discount to the retailer. In short, these conditions incentivize the retailer to purchase larger quantities from the manufacturer. On the other hand, the contract has to discourage “too much” diversion to the gray market. This can be achieved by controlling the retail price of the product. When the retailer diverts to the gray market, he has an incentive to increase the retail price because he can segment the market better with the gray market. The first and second order conditions for  $p_r$  precisely provide such a control on the retail price since the payment is increasing in  $p_r$ , at least for prices close to  $p_r^c$ .

When the centralized supply chain does not divert to the gray market (i.e.,  $\rho > \rho^c$ ), the contract should discourage the retailer from diverting the product to the gray market as well. For this case, the form of the payment depends on the relationship between  $p_r$  and  $q_r$ . The payment is more complex in this case as it needs to achieve two objectives: first to discourage diversion to the gray market (captured by  $\widehat{C}(p_r, q_r)$ ), second to align the retailer’s optimal order quantity to that of the centralized supply chain when  $\rho > \rho^c$  (handled by  $\widetilde{C}(q_r)$ ). The intuition behind the conditions for  $\widehat{C}(p_r, q_r)$  is similar to that of the conditions for the payment in part (a). The  $\widetilde{C}(q_r)$  payment only depends on the retailer’s order quantity as the optimal retail price is always  $p_r = 1 - q_r$  when the retailer does not intend to sell to the gray market.

We now provide one example of a price–dependent quantity discount contract that can coordinate the supply chain.

(a) For  $\rho < \rho^c$  where the gray market emerges in the centralized supply chain, consider the following payment function

$$C(p_r, q_r) = A_1 p_r + A_2 q_r + A_3 q_r^2,$$

in which

$$A_1 = \frac{\rho}{1-\delta\beta}, \quad A_2 = c - \frac{2\zeta K - \rho\lambda}{\lambda(1-\delta\beta)}, \quad A_3 = -\frac{2\delta\zeta}{1-\delta\beta},$$

where  $0 < \zeta < 1$ ,  $K = 2[(1 - \beta) + c\beta + 4\beta(1 - c)]\delta^2 - 2[(1 + \beta)(2c + \rho) + (1 - \beta) + 4(1 - c)]\delta + 4(c + \rho) < 0$ , and  $\lambda = 8 - \delta(1 + \beta^2 + 6\beta)$ .

(b) For  $\rho > \rho^c$  where the gray market does not emerge in the centralized supply chain, consider the following payment functions

$$\widehat{C}(p_r, q_r) = A_1 p_r + A_2 q_r + A_3 q_r^2, \quad \widetilde{C}(q_r) = B_1 + B_2 q_r + B_3 q_r^2,$$

in which

$$A_1 = \frac{\delta(1 - \beta)(1 + c)}{2(1 - \delta\beta)}, \quad A_2 = \frac{\delta[(1 - \beta)(1 + c) + 4\xi_1(1 - c)]}{2(1 - \delta\beta)}, \quad A_3 = -\frac{2\delta\xi_1}{1 - \delta\beta},$$

$$B_1 = \frac{\delta(\xi_1(1 - c)^2 + (1 - \beta)(1 + c)) - \xi_2(1 - c)^2(1 - \delta\beta)}{2(1 - \delta\beta)}, \quad B_2 = 2\xi_2(1 - c), \quad B_3 = -2\xi_2,$$

and  $0 < \xi_1, \xi_2 < 1$ . We note that the value of  $B_1$  is determined such that the payment scheme  $C(p_r, q_r)$  is a continuous function. Furthermore, if  $\xi_2 < \min\left\{1, \frac{\delta(\xi_1(1 - c)^2 + (1 - \beta)(1 + c))}{(1 - \delta\beta)(1 - c)^2}\right\}$ , then  $B_1$  and therefore  $C(p_r, q_r)$  will be positive for all  $p_r$  and  $q_r$ .

## 7. Consumer Welfare

Our analysis thus far has focused on supply chain profit and offering contracts to coordinate the supply chain. In this section, we examine the implications of the gray market for consumer welfare. From the consumer viewpoint, gray markets can be beneficial because they create competition that results in lower prices and provide more purchase options for consumers. One prime example is the case of prescription drugs. Many senior citizens in the United States whose prescriptions are not covered by insurance take a trip to Canada and buy drugs from Canadian pharmacies. Although the FDA prohibits importing medication from foreign pharmacies, the agency rarely enforces the ban on consumers. Recently, the state of Maine passed legislation that allows the direct purchase of mail-order drugs from some foreign pharmacies (*Wall Street Journal*, 2013). Although imported drugs create a fierce competition for U.S. drug manufacturers, legislators hope it will increase the welfare of U.S. patients.

To analyze consumer welfare, let  $\gamma^c$  and  $\gamma^{w*}$  represent total consumer welfare when the decentralized supply chain is coordinated through a contract and when the manufacturer's optimal wholesale price contract is used, respectively. Then

$$\gamma^c = \int_{\nu_r(p_r^c, p_G^*)}^1 (u - p_r^c) du + \int_{\nu_G(p_G^*)}^{\nu_r(p_r^c, p_G^*)} \beta(\delta u - p_G^*) du,$$

$$\gamma^{w*} = \int_{\nu_r(p_r^{w*}, p_G^*)}^1 (u - p_r^{w*}) du + \int_{\nu_G(p_G^*)}^{\nu_r(p_r^{w*}, p_G^*)} \beta(\delta u - p_G^*) du.$$

The next result compares the total consumer welfare functions when consumers highly value the gray market goods.

PROPOSITION 12. *When  $\delta \rightarrow 1$ , total consumer welfare is higher when the decentralized supply chain is coordinated than when the optimal wholesale price contract is implemented.*

This result provides insights about the impacts of a domestic gray market on total consumer welfare and highlights the value of coordinating contracts in the presence of gray markets. When  $\delta$  is sufficiently close to 1 and consumers value the gray market almost equally as they value the authorized channel, total consumer welfare will be higher when the supply chain is coordinated with our proposed contract. Although this analytical result is for  $\delta \rightarrow 1$ , our numerical experiments indicate that the statement continues to hold for all ranges of  $\delta$ . Figure 5 provides an example from our experiments. It is evident from the figure that coordinating the supply chain not only increases total supply chain profit, but also enhances consumer welfare remarkably.

We note that as  $\delta$  increases and the gray market becomes more competitive, total consumer welfare decreases in both centralized and decentralized settings. This observation is similar to the finding in Li and Maskus (2006) that the impact of parallel importation between two markets on total consumer welfare depends on the magnitude of trade costs, which are the costs of importing products from one market into the other market. If trades costs are in an intermediate range, then parallel imports reduce total consumer welfare because the manufacturer deters parallel imports with a high wholesale price in the country that sources parallel imports.

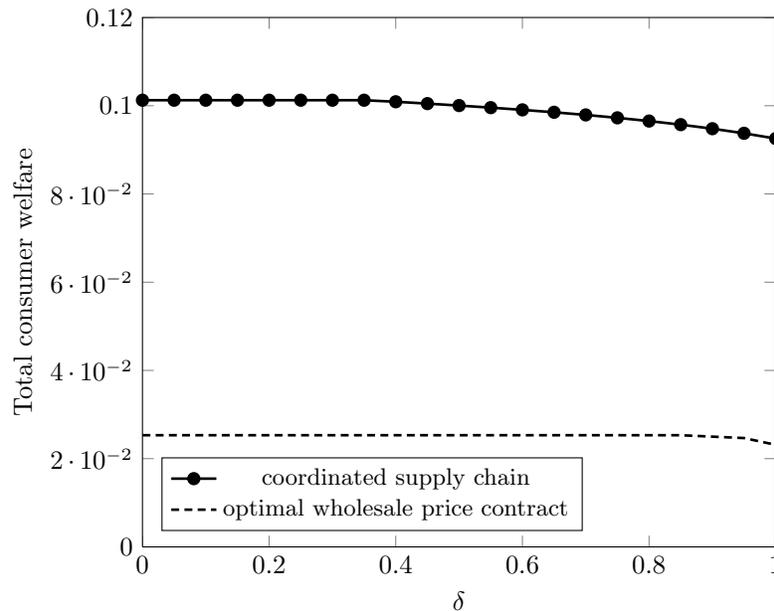


Figure 5 Total consumer welfare for the coordinated supply chain and the optimal wholesale price contract ( $\rho = 0$ ,  $c = 0.1$ ,  $\beta = 0.6$ )

## 8. Discussion and Conclusion

The practice of diverting genuine products from authorized channels for resale in gray markets is increasingly confronting numerous companies. The legality of the gray market business requires that companies account for gray market activities in their business decisions. The industry has documented evidence that inefficient channel incentives greatly encourage gray market activities. Therefore, understanding the effects of gray markets on channel incentives is crucial.

In this paper, we analyzed the impacts of a domestic gray market on the contractual relationship between a manufacturer and its authorized retailer. Our analysis of the wholesale price and quantity discount contracts shows the presence of a gray market can render a classic quantity discount contract so inefficient that the supply chain will be better off with a wholesale price contract that suffers from double marginalization. Nevertheless, the wholesale price contract is not ideal for coordinating the supply chain. In fact, the presence of the gray market can significantly worsen the efficiency loss of the manufacturer's optimal wholesale price contract. Therefore, achieving supply chain coordination in the presence of a gray market demands a more sophisticated contract. Our analysis shows that any one-time payment contract, including quantity discount, that only depends on the retailer's order quantity fails to coordinate the supply chain. As a result, we provide the conditions for coordinating the supply chain with price-dependent quantity discount contracts. We observe that, in addition to increasing total supply chain profit, coordinating the supply chain also benefits consumers and increases their total welfare.

Our model is developed based on some assumptions. We assume all consumers are strategic and anticipate the diversion of the product to the gray market. We can extend our analysis to the setting where proportion  $\alpha$  of consumers are myopic (not strategic) and  $1 - \alpha$  of consumers are strategic. Myopic consumers only buy the product in the first period if the retail price is below their valuation.

We assume demand is deterministic and there is no capacity limit so consumers who wait for the gray market will not face stockouts. Rather, their surplus is influenced by the lower valuation of the gray market and the disutility of obtaining the product one period later. Incorporating demand uncertainty into the model is an interesting extension of our model. Another possible extension is to analyze other supply chain structures such as competing manufacturers selling through the same retailer, or competing retailers selling the manufacturer's product. One can also analyze supply chain coordination when the manufacturer operates in two markets and faces parallel importation by a gray marketer. We leave these extensions for future research.

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## Appendix

We use  $\lambda = 8 - \delta(1 + \beta^2 + 6\beta)$  in the proofs.

**Proof of Lemma 1.** Equations (4)–(5) are the optimal solutions to (2). The equilibrium profit margin and market share of the gray market are

$$p_G^* - w_r^c = \frac{(1 - \delta\beta)[((c - 1)\beta + c + 1)\delta - 2(c + \rho)]}{\lambda}, \quad (11)$$

$$\nu_r(p_r, p_G^*) - \nu_G(p_G^*) = \frac{p_G^* - w_r^c}{\delta(1 - \delta\beta)}. \quad (12)$$

Since  $\lambda > 0$ , the gray market will earn a positive profit if  $p_G^* - w_r^c > 0$  or  $\rho < \rho^c$ . In addition,  $p_r^c - p_r^{cN} = \frac{(1-\beta)[((c-1)\beta+c+1)\delta-2(c+\rho)]}{16-2\delta(\beta^2+6\beta+1)} > 0$  when  $\rho < \rho^c$ . To show  $w_r^c - c - \rho > 0$ , we note that  $\frac{d}{d\rho}(w_r^c - c - \rho) = \frac{\delta(1+3\beta)-4}{8-\delta(1+\beta^2+6\beta)} < 0$ , and when  $\rho = \rho^c$ , we have  $w_r^c - c - \rho = \frac{\delta\beta(1-c)}{2} > 0$ . Therefore,  $w_r^c - c - \rho > 0$ .

**Proof of Lemma 2.** The proof follows from

$$p_G^w - w_G^w = \frac{(1 - \delta\beta)[\delta(1 - \beta) - (2 - \delta - \delta\beta)w_m]}{\lambda} > 0 \implies w_m < \frac{\delta(1 - \beta)}{2 - \delta - \delta\beta}.$$

**Proof of Proposition 1.** For part (a), we have  $p_r^w - p_r^c = \frac{(w_m - c)(1 - \delta\beta)(3 + \beta) + \rho(1 - \beta)}{\lambda} > 0$ . For part (b), we know that  $w_r^c$  is increasing in  $\rho$ . Solving  $w_r^c - w_r^w = 0$  for  $\rho$ , we get

$$w_r^c - w_r^w > 0 \iff \rho > \frac{(w_m - c)(1 - \delta\beta)[4 + \delta(1 - \beta)]}{4 - \delta(1 + 3\beta)}.$$

**Proof of Proposition 2.** First, we have  $p_r^w - p_r^{w^N} = \frac{(1-\beta)[((w_m-1)\beta+1+w_m)\delta-2w_m]}{2\lambda} > 0$ . Next, we have

$$\pi_r^w - \pi_r^{w^N} = \frac{[\delta(1-\beta) - (2-\delta-\delta\beta)w_m]^2}{4\delta\lambda} > 0.$$

Thus, part (a) follows. For part (b), we write

$$\begin{aligned} \pi_m^w - \pi_m^{w^N} &= \frac{[((w_m-1)\beta+1+w_m)\delta-2w_m][(2-\delta(1+\beta))(w_m-c)-2\rho]}{2\delta\lambda}, \\ \pi_{sc}^w - \pi_{sc}^{w^N} &= \frac{[((w_m-1)\beta+1+w_m)\delta-2w_m] \left[ \left(1 - \frac{\delta(1+\beta)}{2}\right)w_m - (2-\delta(1+\beta))c + \frac{\delta}{2}(1-\beta) - 2\rho \right]}{2\delta\lambda}. \end{aligned}$$

$\pi_m^w - \pi_m^{w^N} = 0$  when  $\rho = \hat{\rho}_w = \frac{1}{2}(2-\delta-\delta\beta)(w_m-c)$  and  $\pi_{sc}^w - \pi_{sc}^{w^N} = 0$  when  $\rho = \tilde{\rho}_w = \frac{1}{4}(w_m-2c)(2-\delta-\delta\beta) + \frac{\delta}{4}(1-\beta)$ . In addition,  $\hat{\rho}_w - \tilde{\rho}_w = \frac{(2-\delta-\delta\beta)w_m-\delta(1-\beta)}{4}$ . Since the retailer sells to the gray market when  $w_m < \frac{\delta(1-\beta)}{2-\delta-\delta\beta}$ , we get  $0 \leq \hat{\rho}_w < \tilde{\rho}_w$ . Therefore,  $\pi_m^w > \pi_m^{w^N}$  when  $\rho < \hat{\rho}_w$  and  $\pi_{sc}^w > \pi_{sc}^{w^N}$  when  $\rho < \tilde{\rho}_w$ .

**Proof of Proposition 3.** Let  $q_G^w = \nu_r(p_r^w, p_G^*) - \nu_G(p_G^*)$  denote the equilibrium profit of the gray market. We have

$$\begin{aligned} \frac{d}{dw_m} \left( \frac{d}{d\delta} q_G^w \right) &= \frac{[16 - \delta(4 - \delta - \delta\beta)(1 + \beta^2 + 6\beta)]}{\delta^2 \lambda^2} > 0, \\ \frac{d}{d\delta} q_G^w \Big|_{w_m=0} &= \frac{(1 + \beta^2 + 6\beta)(1 - \beta)}{\lambda^2} > 0. \end{aligned}$$

Therefore  $q_G^w$  is increasing in  $\delta$ . Next we note that

$$\frac{d}{d\delta} (p_G^* - w_r^w) = q_G^w \left[ \frac{\delta(1-\delta\beta)T}{\delta(1-\beta) - (2-\delta-\delta\beta)w_m} + 1 - 2\delta\beta \right],$$

where  $T = \frac{\delta(1+\beta^2+6\beta)(4w_m-\delta((w_m-1)\beta+w_m+1))-16w_m}{\delta\lambda}$ . The expression inside the brackets is an increasing function of  $w_m$  because its value for  $w_m = 0$  is  $\frac{\beta\delta^2(1-\beta)^2+8(1-\delta\beta)^2}{\lambda} > 0$  and its derivate with respect to  $w_m$  is  $\frac{2\delta(1-\delta\beta)}{[\delta(1-\beta)-(2-\delta-\delta\beta)w_m]^2}$ . Therefore,  $\frac{d}{d\delta} (p_G^* - w_r^w) > 0$ . Since both the market share and profit margin of the gray market are increasing in  $\delta$ ,  $\pi_G^w$  is increasing in  $\delta$ .

For the price and profit of the retailer, we have

$$\begin{aligned} \frac{d}{d\delta} p_r^w &= \frac{(1-\beta)^2[4+w_m(3+\beta)]}{\lambda^2} > 0, \\ \frac{d}{d\delta} \pi_r^w &= \frac{-[\delta(1-\beta) - (2-\delta-\delta\beta)w_m][((\beta^2+4\beta-1)\delta-4)w_m - 2\delta(1-\beta)]}{\delta^2 \lambda^2}. \end{aligned}$$

Because  $(\beta^2+4\beta-1)\delta-4 < 0$ , the second expression in the numerator of  $\frac{d}{d\delta} \pi_r^w$  is always negative and  $\frac{d}{d\delta} \pi_r^w > 0$ . Therefore,  $\pi_r^w$  increases with  $\delta$ .

**Proof of Proposition 4.** Recall that the market share of the gray market is  $q_G^w = \nu_r(p_r^w, p_G^*) - \nu_G(p_G^*)$  and the profit margin of the gray market is  $p_G^* - w_r^w = q_G^w \delta(1 - \delta\beta)$ . We have

$$\begin{aligned} \frac{d}{d\beta} q_G^w &= \frac{(\beta - 1)[((w_m - 1)\beta + 1 + 3w_m)\delta - 4w_m] - 8(1 - \delta)(1 + w_m)}{\lambda^2}, \\ \frac{d}{d\beta} q_G^w \Big|_{w_m=0} &= \frac{8(\delta - 1) - \delta(1 - \beta)^2}{\lambda^2} < 0, \\ \frac{d}{d\beta} q_G^w \Big|_{w_m=\frac{\delta(1-\beta)}{2-\delta-\delta\beta}} &= \frac{\delta - 1}{(2 - \delta - \delta\beta)\lambda} < 0. \end{aligned}$$

Thus  $\frac{d}{d\beta} q_G^w < 0$  which means  $p_G^* - w_r^w$  decreases with  $\beta$ . Because  $\pi_G^w = (p_G^* - w_r^w)q_G^w$ ,  $\pi_G^w$  decreases with  $\beta$ .

For the price of the retailer, we have

$$\frac{d}{d\beta} p_r^w = - \frac{[(-3\beta^2 - 2\beta - 3)\delta^2 + (-\beta^2 + 10\beta + 7)\delta - 8]w_m - 4\delta(1 - \beta)(2 - \delta - \delta\beta)}{\lambda^2}.$$

Now we evaluate  $\frac{d}{d\beta} p_r^w$  at boundary points  $w_m = 0$  and  $w_m = \frac{\delta(1-\beta)}{2-\delta-\delta\beta}$ .

$$\begin{aligned} \frac{d}{d\beta} p_r^w \Big|_{w_m=0} &= - \frac{4\delta(1 - \beta)(2 - \delta - \delta\beta)}{\lambda^2} < 0, \\ \frac{d}{d\beta} p_r^w \Big|_{w_m=\frac{\delta(1-\beta)}{2-\delta-\delta\beta}} &= - \frac{\delta(1 - \delta)(1 - \beta)}{(2 - \delta - \delta\beta)\lambda} < 0. \end{aligned}$$

Therefore,  $p_r^w$  is decreasing in  $\beta$ . For  $\pi_r^w$ , we use the Envelope Theorem. Because  $\delta p_r^w - w_r^w > 0$  when the gray market exists we get

$$\frac{d}{d\beta} \pi_r^w = - \frac{(\delta p_r - w_r)(p_r - w_r)}{2(1 - \delta\beta)^2} \Big|_{p_r=p_r^w, w_r=w_r^w} < 0.$$

**Proof of Lemma 3.** The optimal values of  $p_r^{QD}$  and  $w_r^{QD}$  can be obtained from first-order conditions, which we omit due to space limitation. The upper bounds on  $b$  and  $c$  ensure that  $w_r^{QD} > 0$  and  $\delta p_r^{QD} - w_r^{QD} > 0$ . Taking the derivatives of the retailer's profit function, we have  $\frac{\partial \pi_r^2}{\partial p_r^2} = -\frac{2-\delta\beta}{1-\delta\beta} + \frac{b}{2} < 0$  and  $\frac{\partial \pi_r^2}{\partial w_r^2} = \frac{(2+\beta b)\delta - b}{2\delta^2(\delta\beta - 1)} < 0$  because  $b < 1$ . We also get

$$\left( \frac{\partial \pi_r^2}{\partial p_r^2} \right) \left( \frac{\partial \pi_r^2}{\partial w_r^2} \right) - \left( \frac{\partial \pi_r^2}{\partial p_r \partial w_r} \right)^2 = \frac{\mu}{4\delta^2(1 - \delta\beta)^2}.$$

We note that  $\mu$  is linear in  $b$  and

$$\begin{aligned} \mu \Big|_{b=0} &= \delta\lambda, \\ \mu \Big|_{b=\frac{4c+[2(1+\beta)+c(1-\beta)]\delta}{4c+2+[1+\beta+c(1-\beta)]\delta}} &= - \frac{[\delta(1 - \beta) - (2 - \delta - \delta\beta)c][\delta(\beta + 1) - 2][ -4 - (1 - \beta)\delta]}{[\beta(c - 1) - c - 1]\delta - 4c - 2} > 0. \end{aligned}$$

Thus, inequalities (7) also ensure that  $\mu > 0$  and the second order optimality condition is satisfied.

**Proof of Proposition 5.** We showed that in the centralized supply chain  $p_r^c$  is decreasing in  $\rho$  and  $w_r^c$  is increasing in  $\rho$ . Now we write

$$p_r^c - p_r^{QD} \Big|_{\rho=0} = \frac{b(1-\delta\beta)(2-\delta-\delta\beta)[\delta(1-\beta) - (2-\delta-\delta\beta)c](3+\beta)}{\lambda\mu} > 0,$$

$$p_r^c - p_r^{QD} \Big|_{\rho=\rho^c} = -\frac{[\delta(1-\beta) - (2-\delta-\delta\beta)c][(\frac{1}{2} + (b-\frac{1}{2})\beta)\delta - b]}{\mu}.$$

Therefore, if  $b > \frac{\delta(1-\beta)}{2(1-\delta\beta)}$ , then  $p_r^c > p_r^{QD}$  for all  $\rho < \rho^c$ . Otherwise, there exist  $\rho^* < \rho^c$  such that  $p_r^c > p_r^{QD}$  if  $\rho < \rho^*$ . Second, we have

$$w_r^c - w_r^{QD} \Big|_{\rho=0} = \frac{b(2-\delta-\delta\beta)(1-\delta\beta)(4+(1-\beta)\delta)[\delta(1-\beta) - (2-\delta-\delta\beta)c]}{\lambda\mu} > 0.$$

Thus,  $w_r^c > w_r^{QD}$  for any  $\rho$ .

**Proof of Proposition 6.** When there is no gray market, the coordinating QD contract results in the following profits for the manufacturer, the retailer, and the supply chain

$$\pi_m^{QD^N} = \frac{b(1-c)^2}{4}, \quad \pi_r^{QD^N} = \frac{(1-b)(1-c)^2}{4}, \quad \pi_{sc}^{QD^N} = \frac{(1-c)^2}{4}.$$

To prove part (a), we write

$$p_r^{QD} - p_r^{QD^N} = \frac{[\delta(1-\beta) - (2-\delta-\delta\beta)c][\delta(1-\beta) - 2b(1-\delta\beta)]}{2\mu} > 0 \iff b < \frac{\delta(1-\beta)}{2(1-\delta\beta)},$$

$$\pi_r^{QD} - \pi_r^{QD^N} = \frac{(1-b)[\delta(1-\beta) - (2-\delta-\delta\beta)c]^2}{4\mu} > 0.$$

For part (b),  $\pi_m^{QD}$  is decreasing in  $\rho$  and

$$\pi_m^{QD} - \pi_m^{QD^N} \Big|_{\rho=0} = -\frac{(2-\delta-\delta\beta)^2[\delta(1-\beta) - (2-\delta-\delta\beta)c]^2 b}{4\mu^2} < 0.$$

Thus  $\pi_m^{QD} - \pi_m^{QD^N} < 0$  for any  $\rho > 0$  and the manufacturer is always worse off in the QD contract when there is a gray market.

**Proof of Proposition 7.** (a) Equilibrium retail price and diversion price in the wholesale setting ( $p_r^w$  and  $w_r^w$ ) are increasing in  $w_m > c$ . We have

$$p_r^w - p_r^{QD} \Big|_{w_m=c} = \frac{b(1-\delta\beta)(2-\delta(1+\beta))(3+\beta)[\delta(1-\beta) - (2-\delta-\delta\beta)c]}{\lambda\mu} > 0,$$

$$w_r^w - w_r^{QD} \Big|_{w_m=c} = \frac{b(1-\delta\beta)(2-\delta(1+\beta))(4+\delta(1-\beta))[\delta(1-\beta) - (2-\delta-\delta\beta)c]}{\lambda\mu} > 0.$$

Since  $p_G^* = \frac{\delta p_r + w_r}{2}$  and both  $p_r$  and  $w_r$  are higher in the wholesale price contract, the equilibrium gray market price will also be higher in the wholesale price contract.

**Proof of Proposition 8.** Supply chain profit under wholesale and quantity discount contracts are:

$$\begin{aligned}\pi_{sc}^w &= (p_r^w - c)[1 - \nu_r(p_r^w, w_r^w)] + (w_r^w - c - \rho)[\nu_r(p_r^w, w_r^w) - \nu_G(p_G^*)], \\ \pi_{sc}^{QD} &= (p_r^{QD} - c)[1 - \nu_r(p_r^{QD}, w_r^{QD})] + (w_r^{QD} - c - \rho)[\nu_r(p_r^{QD}, w_r^{QD}) - \nu_G(p_G^*)].\end{aligned}$$

Substituting the equilibrium solutions in the profit of the supply chain, we find that  $\frac{d^2}{dw_m^2}\pi_{sc}^w = \frac{2(1-\delta\beta)(-\delta-2)}{\delta\lambda} < 0$  so  $\pi_{sc}^w$ , and as a result  $\pi_{sc}^w - \pi_{sc}^{QD}$ , are concave in  $w_m$ . Solving  $\pi_{sc}^w - \pi_{sc}^{QD} = 0$ , we obtain two roots,  $\hat{w}_m$  and  $\tilde{w}_m$ . First,  $\hat{w}_m$  is an infeasible wholesale price because

$$\hat{w}_m - c = \frac{b(\delta(1+\beta) - 2)[\delta(1-\beta) - (2-\delta-\delta\beta)c]}{\mu} < 0 \implies \hat{w}_m < c,$$

Second,  $\tilde{w}_m$  is increasing in  $\rho$  since  $\frac{d}{d\rho}\tilde{w}_m = \frac{2-\delta(1+\beta)}{(1-\delta\beta)(1+\delta)} > 0$  and

$$\tilde{w}_m - c \Big|_{\rho=0} = \frac{b(2-\delta(1+\beta))[\delta(1-\beta) - (2-\delta-\delta\beta)c]}{\mu} > 0.$$

Therefore,  $\tilde{w}_m > c$  for all  $\rho$  and  $\pi_{sc}^w > \pi_{sc}^{QD}$  when  $c < w_m < \tilde{w}_m$ . Finally,  $\frac{d}{d\rho}\tilde{w}_m = \frac{(2-\delta(1+\beta))}{(1-\delta\beta)(1+\delta)} > 0$  and  $\tilde{w}_m$  is increasing in  $\rho$ .

**Proof of Proposition 9.** As  $\delta \rightarrow 1$ ,  $\frac{\delta(1-\beta)}{2-\delta-\delta\beta} \rightarrow 1$ , which means the manufacturer's optimal wholesale price  $w_m^*$  will make it possible for the retailer to sell to the gray market. Also, as  $\delta \rightarrow 1$ ,  $\rho^c \rightarrow \frac{(1-\beta)(1-c)}{2}$ . Define  $\widehat{\text{PoA}} = \lim_{\delta \rightarrow 1} \frac{\pi_{sc}^c}{\pi_{sc}(w_m^*)}$  to be the value of the PoA when  $\delta$  approaches 1. Then

$$\frac{d}{d\rho}\widehat{\text{PoA}} = \frac{32\rho(7+\beta)(1-c)}{3(1-\beta)(4(1-c)-\rho)}.$$

Since  $\rho < \frac{(1-\beta)(1-c)}{2}$ ,  $\widehat{\text{PoA}}$  is increasing in  $\rho$ . For  $\rho = 0$ , we get  $\widehat{\text{PoA}} = \frac{4}{3}$  and for  $\rho = \frac{(1-\beta)(1-c)}{2}$ , we get  $\widehat{\text{PoA}} = \frac{32}{3(\beta+7)}$ , which approaches  $\frac{32}{21}$  as  $\beta \rightarrow 0$ .

**Proof of Proposition 10.** Suppose the manufacturer offers total order cost  $C(q_r)$  to the retailer. We know that the retailer's order quantity will be

$$q_r(p_r, w_r) = 1 - \frac{p_G^*}{\delta} = 1 - \frac{\delta p_r + w_r}{2\delta}.$$

If  $\rho < \rho^c$ , the retailer maximizes

$$\pi_r = p_r[1 - \nu_r(p_r, p_G^*)] + w_r[\nu_r(p_r, p_G^*) - \nu_G(p_G^*)] - C(q_r(p_r, w_r)).$$

$C(q_r)$  will coordinate the supply chain if  $\frac{\partial \pi_r}{\partial p_r} = 0$  and  $\frac{\partial \pi_r}{\partial w_r} = 0$  at  $(p_r^c, w_r^c)$ . These conditions translate to

$$\begin{aligned}\frac{d}{dq}C(q_r(p_r, w_r)) \Big|_{p_r=p_r^c, w_r=w_r^c} &= c - \frac{\rho}{1-\delta\beta}, \\ \frac{d}{dq}C(q_r(p_r, w_r)) \Big|_{p_r=p_r^c, w_r=w_r^c} &= c + \frac{\rho}{1-\delta\beta}.\end{aligned}$$

Therefore, unless  $\rho = 0$ , no contract in the form of  $C(q_r)$  can coordinate the supply chain.

Now, suppose  $\rho > \rho^c$  and the manufacturer wants to prevent the retailer from selling to the gray market. Then  $C(q_r)$  will achieve this goal if  $\frac{\partial \pi_r}{\partial p_r} = 0$  and  $\frac{\partial \pi_r}{\partial w_r} = 0$  at  $(\frac{1+c}{2}, \frac{\delta(1+c)}{2})$ . These conditions then require that the following equations hold simultaneously which is not possible

$$\begin{aligned} \frac{d}{dq} C(q_r(p_r, w_r)) \Big|_{p_r = \frac{1+c}{2}, w_r = \frac{\delta(1+c)}{2}} &= \frac{4c - \delta[(3\beta + 1)c\beta + (1 - \beta)]}{2(1 - \delta\beta)}, \\ \frac{d}{dq} C(q_r(p_r, w_r)) \Big|_{p_r = \frac{1+c}{2}, w_r = \frac{\delta(1+c)}{2}} &= \frac{2c - \delta(1 + c - \beta(1 - c))}{1 - \delta\beta}. \end{aligned}$$

**Proof of Proposition 11.** (a) Note that with the coordinating contract, the retailer's profit is

$$\begin{aligned} \bar{\pi}_r(p_r, q_r) &= p_r [1 - \nu_r(p_r, p_G^*)] + [2\delta(1 - q_r) - \delta p_r] [\nu_r(p_r, p_G^*) - \nu_G(p_G^*)] - C(p_r, q_r) \\ &= p_r \left( \frac{1 - p_r - \delta\beta q_r}{1 - \delta\beta} \right) + [2\delta(1 - q_r) - \delta p_r] \left( \frac{p_r - 1 + q_r}{1 - \delta\beta} \right) - C(p_r, q_r). \end{aligned}$$

Taking the first derivatives with respect to  $p_r$  and  $q_r$  leads to

$$\begin{aligned} \frac{\partial \bar{\pi}_r(p_r, q_r)}{\partial p_r} &= \frac{1 - 2p_r - \delta\beta q_r - 2\delta p_r + 3\delta(1 - q_r)}{1 - \delta\beta} - \frac{\partial C(p_r, q_r)}{\partial p_r}, \\ \frac{\partial \bar{\pi}_r(p_r, q_r)}{\partial q_r} &= \frac{-\delta\beta p_r + 2\delta(1 - q_r) - \delta p_r - 2\delta(p_r - 1 + q_r)}{1 - \delta\beta} - \frac{\partial C(p_r, q_r)}{\partial q_r}. \end{aligned}$$

The result is, then, obtained by plugging the values of  $p_r^c$  and  $q_r^c = 1 - \frac{\delta p_r^c + w_r^c}{2\delta}$  from (4) and (5). The second order conditions are enforced to ensure uniqueness of optimal parameter values.

(b) To prove this part of the proposition, we first show that  $p_r^c = (1 + c)/2$  is the optimal retail price given order quantity of  $q_r^c = (1 - c)/2$  and vice versa. To see this, we consider the retailer profit in two regions: (i) when  $p_r \geq 1 - q_r$ , and the retailer diverts the gray market; (ii) when  $p_r \leq 1 - q_r$ , and the retailer does not divert to the gray market. We show that, given the conditions specified in part (b) of the proposition, the optimal solution in the first region is always  $p_r = p_r^{cN} = (1 + c)/2$ ,  $q_r = q_r^{cN} = (1 - c)/2$ , and therefore the gray market does not emerge. Given the coordinating contract, the retailer's profit is

$$\begin{aligned} \bar{\pi}_r(p_r, q_r) &= p_r [1 - \nu_r(p_r, p_G^*)] + [2\delta(1 - q_r) - \delta p_r] [\nu_r(p_r, p_G^*) - \nu_G(p_G^*)] - \widehat{C}(p_r, q_r) \\ &= p_r \left( \frac{1 - p_r - \delta\beta q_r}{1 - \delta\beta} \right) + [2\delta(1 - q_r) - \delta p_r] \left( \frac{p_r - 1 + q_r}{1 - \delta\beta} \right) - \widehat{C}(p_r, q_r). \end{aligned}$$

Taking the first derivatives with respect to  $p_r$  and  $q_r$  results in

$$\begin{aligned} \frac{\partial \bar{\pi}_r(p_r, q_r)}{\partial p_r} &= \frac{1 - 2p_r - \delta\beta q_r - 2\delta p_r + 3\delta(1 - q_r)}{1 - \delta\beta} - \frac{\partial \widehat{C}(p_r, q_r)}{\partial p_r}, \\ \frac{\partial \bar{\pi}_r(p_r, q_r)}{\partial q_r} &= \frac{-\delta\beta p_r + 2\delta(1 - q_r) - \delta p_r - 2\delta(p_r - 1 + q_r)}{1 - \delta\beta} - \frac{\partial \widehat{C}(p_r, q_r)}{\partial q_r}. \end{aligned}$$

The result is, then, obtained by plugging the values of  $p_r^{c^N} = \frac{1+c}{2}$  and  $q_r^{c^N} = \frac{1-c}{2}$ . The second order conditions are enforced to ensure uniqueness of optimal decisions.

For the case of  $p_r \leq 1 - q_r$ , we note that the optimal retail price is always  $p_r = 1 - q_r$  as retailer's sales is  $q_r$  and therefore setting any retail price below  $1 - q_r$  will be sub-optimal. Therefore, the retailer's profit can be written as a function of  $q_r$  only:

$$\bar{\pi}_r(q_r) = q_r(1 - q_r) - \tilde{C}(q_r).$$

Taking the first derivative with respect to  $q_r$  leads to

$$\frac{d\bar{\pi}_r(q_r)}{dq_r} = 1 - 2q_r - \frac{d\tilde{C}(q_r)}{dq_r}.$$

The result is, then, followed by plugging the value  $q_r^{c^N} = \frac{1-c}{2}$ . The second order condition is enforced to ensure uniqueness of optimal decisions.

**Proof of Proposition 12.** Define  $\Delta\gamma = \gamma^c - \gamma^{w^*}$  for  $\delta = 1$ . Then

$$\begin{aligned} \frac{d^2}{d\rho^2}\Delta\gamma &= \frac{7\beta^2 + 194\beta + 55}{64(1-\beta)(\beta+7)^2} > 0, \\ \Delta\gamma|_{\rho=0} &= \frac{3(1-c)^2(7\beta+9)}{8(\beta+7)^2} > 0, \\ \frac{d}{d\rho}\Delta\gamma|_{\rho=0} &= \frac{3(19-3\beta)(1-c)}{16(\beta+7)^2} > 0, \\ \Delta\gamma|_{\rho=\frac{(1-\beta)(1-c)}{2}} &= \frac{(55-7\beta)(1-c)^2}{512} > 0, \\ \frac{d}{d\rho}\Delta\gamma|_{\rho=\frac{(1-\beta)(1-c)}{2}} &= \frac{(7\beta+73)(1-c)}{128(\beta+7)} > 0. \end{aligned}$$

Thus,  $\Delta\gamma$  is a convex function of  $\rho$  and is positive for  $\rho$  between 0 and  $\frac{(1-\beta)(1-c)}{2}$ . Therefore,  $\gamma^c > \gamma^{w^*}$ .