

Collaborative Work Dynamics in Projects with Co-Production

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Many knowledge-intensive projects such as new product and software design, research and high technology development have flexible scope and involve co-production between a client and a vendor. In such projects, it is often challenging to estimate how much progress can be achieved within a certain time window or how much time may be needed to achieve a certain degree of progress, especially because the client and vendor often adjust their efforts as a function of the project's progress, the time until the deadline, and the incentives in place. Effective contracts should therefore be flexible in scope and foster collaboration. In this paper, we characterize the collaborative work dynamics of a client and a vendor who are engaged in a multi-state, multi-period stochastic project with a finite deadline. We show that when the client can verify the vendor's effort, it is optimal that they both exert high effort in one of two situations: when either not enough progress has been made and the deadline is close (deadline effect), or conversely, when so much progress has been made that the project state is close to a completion state set by the client (milestone effect). Hence, in this case, progress will typically be faster when the project is about to be stopped, due to either reaching the deadline or reaching the client's desired completion state. However, when the client cannot verify the vendor's effort, the vendor is prone to free-riding. Considering a time-based contract, which pays the vendor a per-period fee and a fixed completion bonus, we show that the equilibrium completion state is decreasing in the per-period fee and increasing in the bonus, justifying the use of both incentive mechanisms in practice. Moreover, we show that, under such contracts, some form of milestone effect arises in equilibrium, but the deadline effect does not. Hence, in those cases, early progress will typically lead to early project conclusion at a high state; whereas, slow progress will typically make the project drag until the deadline while still at a low state.

Key words: collaboration, project management, co-production, moral hazard, dynamic stochastic game

1. Introduction

Many knowledge-intensive projects, such as new product and software design, research and high technology development are iterative with an uncertain outcome, must be completed by a certain

deadline, and involve some degree of “co-production” between a client and a vendor.¹ For instance, consider a startup company (“the client”) hiring a software company (“the vendor”) for the development of a new software application, as described in Eckfeldt and Madden (2005). At the project outset, the client’s intent was to have “a version of the system running with a group of alpha users” within 8 weeks to prove the methodology and establish a basic revenue stream (p. 3), but the exact scope remained flexible. Given the uncertainty in the project scope, the parties chose to adopt an agile development method, operating in iterations of 1-2 weeks. At the beginning of the project, the team identified “the most important features” (e.g., “adding images/animations” to specific user interfaces, p. 4) and prioritized their development (“alpha/proof of concept, beta version/pre-investors, and full market/post investors”, p. 4). Yet, scope was dynamically re-adjusted at the beginning of each iteration, based on the progress so far and the estimated remaining work. See Eckfeldt and Madden (2005) for details. Similar project features apply in rapid prototyping and agile new product development (Fujimoto et al. 1996, Pichler 2010, Layton 2012).

In such stochastic projects with flexible scope, it is often challenging to estimate how much progress can be accomplished over time, and ultimately, if enough progress will be achieved by the deadline. This is not only because of the innate project uncertainty, but also because the parties’ work intensity may vary over time, depending on what has been accomplished so far, the time remaining until the deadline, and the incentives in place. Thus, in order to effectively manage a project’s triple constraint of scope, time, and budget (PMI 2013), contracts must be flexible enough to “embrace changes” while “empowering the [vendor] and the client to work collaboratively” over the course of the project (Eckfeldt and Madden 2005, p. 1).

In this paper, we propose a stylized model of a co-productive project and the work dynamics between a client and a vendor to assess how much progress can be achieved within a certain time window, or how much time may be needed to achieve a certain degree of progress. Specifically, we investigate the following research questions: When is it economical for both the parties to exert high effort as a function of the project’s progress and time to deadline? When is it economical for the client to stop the project? Do these high-intensity collaboration phases and stopping rules arise in equilibrium when the vendor and the client choose their effort levels independently, and how are they affected by the contracts in place?

We model the co-productive work process as a stochastic game between a client and a vendor (Shapley 1953, Sobel 1971). As in the case study described above (Eckfeldt and Madden 2005), we consider a project with flexible scope, uncertain outcome, and finite deadline. We represent the state of the project as the progress achieved at a point in time; for instance, in the above case study,

¹ Throughout the paper, we refer to the client as “she” and to the vendor as “he”.

progress could be measured as the number of features implemented in the software application. Scope is flexible, in the sense that the client can generate revenue for any level of progress at the project conclusion. We naturally assume that the revenue generated is increasing in the project progress (e.g., more features leads to greater revenue), but that it exhibits decreasing marginal returns, as would be the case if work were prioritized to focus first on the highest value-adding features (Eckfeldt and Madden 2005, Layton 2012). Moreover, the client has the right to stop the project at any point in time if she judges that enough progress has been accomplished, as it often happens in practice (Eckfeldt and Madden (2005).

In each time period, the sequence of decisions is as follows: First, the client decides whether to continue or stop the project. If the client chooses to stop, she collects the project reward based on the progress (state) achieved so far and pays the vendor according to their contractual arrangements. If the client chooses to continue the project, the client and the vendor, independently and simultaneously, choose their effort levels, which we assume to be binary (i.e., high or low). Naturally, their choice of efforts depends on the time remaining (period) and the progress so far (state). In that case, progress occurs stochastically as a Markov chain, in which the transition probabilities depend on joint efforts.

Because of the knowledge-intensive nature of such projects, it is often difficult for the client to verify the vendor's effort, which potentially leads to free riding (Holmström 1982). To understand how work dynamics are affected by this lack of effort verifiability, we study the work dynamics under two cases: when the vendor's efforts are fully verifiable by the client, which we refer to as the first-best (FB) solution, and when the vendor's efforts are not verifiable by the client. In this latter case, we consider a time-based contract, which mimics in a stylized way, the most common types of contract used in practice (Bartrick 2013, Edwards et al. 2014), according to which the client pays the vendor (i) a per-period fee each period, and (ii) a fixed bonus upon the conclusion of the project.²

Our analysis generates the following two key results with certain managerial implications. First, we show that there exists a time-independent state threshold above which it is optimal for the client to stop the project. However, the threshold varies, depending on whether efforts are verifiable or not. Specifically, the per-period fee causes the client to stop the project at a lower state than when efforts are verifiable, but a large bonus can cause the client to stop the project at a higher state than when efforts are verifiable. This implies that, in practice, one may need to combine both a bonus and a per-period fee to set the project stopping state close to the FB stopping state

² Although there exist other contracts in practice (such as output-based contracts), they tend to be less common in contexts where output is difficult to verify (Bartrick 2013, Edwards et al. 2014), or when the project has no immediate financial impact, as in our motivating case study (Eckfeldt and Madden 2005).

threshold. However, if the client must use only one of the two levers, we observe from numerical studies that setting a fixed bonus is often a better choice than setting a per-period fee.

Second, we show that, when efforts are verifiable, it is optimal that both the client and the vendor exert high efforts in two situations: (i) when not enough progress has been made and the deadline is close, which we refer to as a *deadline effect*, and (ii) when so much progress has been made that the project state is close to the completion state chosen by the client, which we refer to as a *milestone effect*. The emergence of these phases of high-intensity work near the conclusion of the project (deadline- or milestone-induced) implies that using past progress to infer future progress, as is often the case in planning tools of agile projects, would underestimate future progress. However, these two effects do not necessarily survive when efforts are not verifiable. Specifically, we find that, under the time-based contracts we consider, the milestone effect is preserved, but the deadline effect disappears. Hence, if the milestone effect is reached, the project may reach a high state quickly and be concluded early, but otherwise, the project may drag on until the deadline with little additional progress. Consequently, predicting progress in projects governed by time-based contracts will be more challenging than if efforts were verifiable, and will be significantly dependent on how much progress has been made in the early periods of the project.

The paper is organized as follows. We review the related literature in the next section and present the model in §3. We characterize the results for the case with verifiable efforts in §4 and study the game dynamics for time-based contracts in §5. We present our conclusions in §6. All proofs are presented in the appendix.

2. Literature Review

This paper builds upon two major streams of research: the operations management literature on the management of knowledge-intensive projects and the economics literature on co-production and moral hazard in teams.

Knowledge-Intensive Project Management. According to Hopp et al. (2009), white-collar processes are inherently more knowledge-intensive and creative than blue-collar processes. Thus, they need to be managed differently. In particular, in project management, a distinction is often made between waterfall and agile project management approaches (Moran 2015). Whereas waterfall approaches tend to perform well when requirements can be reasonably defined upfront and resources can be dedicated to specific tasks, agile approaches tend to work well for managing unstructured knowledge-intensive projects (Moran 2015).

The bulk of the academic literature on project management to date has studied well-structured projects, as is common in waterfall approaches, with the goal of minimizing the time to complete a certain set of well-defined tasks through scheduling (e.g., using the Critical Path Method or the

Program Evaluation Review Technique; see PMI 2013), overlapping and crashing tasks (Krishnan et al. 1997, Terwiesch and Loch 1999, Roemer et al. 2000, Roemer and Ahmadi 2004), stochastic durations and rework (Smith and Eppinger 1997, Banerjee et al. 2007), and aligning contractual incentives among different project suppliers (Bayiz and Corbett 2005, Kwon et al. 2010, Chen et al. 2015). In contrast to well-structured waterfall projects, agile projects have flexible scope and are iterative and collaborative (Pichler 2010, Layton 2012). Although we do not aim to model all the specifics of agile projects (which may involve learning, information asymmetry, contract renegotiation, etc.), we consider here a co-productive project with flexible requirements and a finite deadline as often arises in agile project management.

The more recent academic literature on the operations of agile projects has primarily studied the *design* of agile projects. Taking a process design perspective, Ha and Porteus (1995) characterize the optimal progress review frequency between a product and a process development teams, Terwiesch and Loch (2004) and Loch et al. (2001) respectively characterize the optimal number of prototypes and optimal testing strategy before a final market release, and Loch and Terwiesch (2005) discuss how to incorporate additional information throughout the development process. Taking a team design perspective, Sting et al. (2012) compare various forms of team hierarchies, whereas Hong and Page (2001), Kavadias and Sommer (2009), and LiCalzi and Surucu (2012) establish the importance of team diversity and its relationship to problem structure.

We complement this project design literature by characterizing the *execution* of the project. Specifically, we assume that the project's design decisions (e.g., deadline, sequence of potential tasks to perform, team composition, iterations' cycle times) have been made and characterize the collaborative dynamics that occur within that setting. Adopting a similar perspective on project execution, Siensen et al. (2007) explore the design of optimal incentives that induce task-related effort, helping, and knowledge sharing, and Ozkan et al. (2015) characterize the optimal dynamics of knowledge development and transfer between a product design and a process design teams, as the project unfolds. In contrast to these two papers, which consider effort allocation between multiple tasks, we consider only the work effort towards the project, but study its dynamic adjustment as a function of the project progress. Similarly, Demirezen et al. (2013) characterize the optimal effort exertion between a client and a vendor. In their setting, the project reward is collected continuously, as a function of the cumulative effort since the beginning of the project, whereas in our setting, the reward is obtained only at the project conclusion, and its magnitude depends on the completion state, which is only a stochastic function of the cumulative effort. In contrast to Demirezen et al. (2013), who find that efforts should be decreasing over time, due to their assumption of continuous release (since earlier efforts have a longer impact on revenue than later efforts), we find that efforts should be increasing over time and state. We thus complement their approach by considering a different reward stream and modeling the stochastic evolution of the project.

Co-Production and Moral Hazard in Teams. Knowledge-intensive projects often involve a client and a vendor working toward a common output (Fuchs 1968, Karmarkar and Pitbladdo 1995). In that context, efforts may not be observable (Bapna et al. 2010), resulting in double moral hazard (Holmström 1982). Roels et al. (2010) study the trade-off between moral hazard and monitoring costs in a static setting and find that, without monitoring, simple contracts, such as fixed-fee and time-and-materials, are ineffective. In fact, output-based contracts are second-best in such co-productive settings (Bhattacharyya and Lafontaine 1995), and they are indeed common in certain settings, such as pharma licensing (Crama et al 2008, Xiao and Xu 2012, Bhattacharya et al. 2014, Savva and Taneri 2015). However, output quality may not always be sufficiently measurable to be contracted on.³ For instance, in strategy consulting, it is uncommon to tie the consultant’s reward to the “quality” of the consulting report, unless it leads to a specific venture. Similarly, in software development, most contracts are still based on time-and-materials or fixed-fee (Bartrick 2013, Edwards et al. 2014). In this paper, we show that such simple contracts can be quite effective in a dynamic setting, because the timing of the payments may provide powerful work incentives.

Our paper is related to the economics literature on public goods, according to which agents’ make successive contributions to a public good from which they derive some utility; see Admati and Perry (1991), Varian (1994), Teoh (1997), Marx and Matthews (2000) among others. Considering a breakthrough project with uncertain success, but opportunities to learn about the project success rate, Bonatti and Horner (2011) show that agents procrastinate, due to free-riding, but that their effort is decreasing over time due to growing pessimism about the chances of success of the project. In contrast, we find that, in the absence of learning, efforts should be increasing over time as the project gets closer to its deadline.

In contrast to these papers (e.g., Admati and Perry 1991, Varian 1994, Teoh 1997, Marx and Matthews 2000, Bonatti and Horner 2011), which assume either a deterministic setting (in which state and time are equivalent) or a one-state stochastic setting, we consider here a multi-state stochastic project. This finer-grained perspective allows us to characterize the work dynamics as a function of both time and state. In the public goods literature, Georgiadis (2014) characterizes the work dynamics as a function of state for a project that has a fixed requirements and no deadline. We complement his approach in two respects. First, we consider a different setting, with flexible requirements but a fixed timeline (as opposed to fixed requirements with no deadline), with the option for the client to stop the project early (as opposed to an exogenous completion state) and we establish the emergence of a deadline effect in addition to the “procrastination” effect (which we refer to as “milestone effect”). Second, and more fundamentally, Georgiadis (2014) studies project

³ See Hart and Moore (1988) for a distinction between perfunctory and consummate performance of a contract.

design issues (such as team size and incentives), whereas our focus is on project execution, so as to estimate the speed of progress in such collaborative projects and assess how the work dynamics depend on effort verifiability and the incentive structure in place.

3. Model

In this section, we introduce a model of co-productive knowledge-intensive projects. We consider a vendor (v) and a client (c) who are engaged in a multi-period and multi-state development phase of a project with a finite deadline. Let $\mathcal{T} \subseteq \mathbb{N}$ denote the set of time periods and $\mathcal{X}_t \subseteq \mathbb{N}$ be the set of project states, which represent project progress.⁴

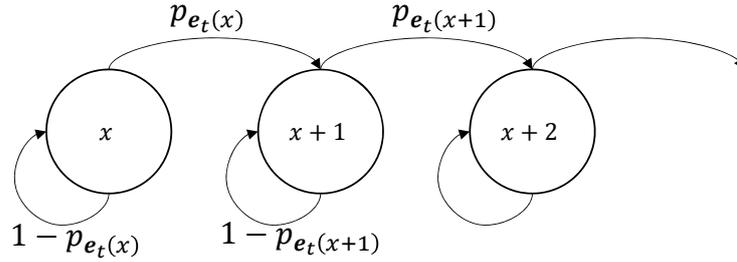
Deadline: We consider a project that has a fixed deadline, which might be imposed by external factors, such as trade shows, competition and time to market pressure (Highsmith 2010, VersionOne 2015) or internal factors, such as contractual obligations or accounting restrictions (Edwards et al. 2014). Specifically, we consider a deadline T for the project; hence, $\mathcal{T} = 0, \dots, T$. In addition, the client may choose to stop the project at any point in time before the deadline is reached. Thus in our model, the project can terminate in two ways: (i) when the client decides to stop the project, or (ii) when the deadline has been reached. Note that the deadline T can be loose or tight depending on the nature of the project.

Project value: When the project is stopped, it generates some value to the client (which may be financial or not). We denote the value associated with each state by $R(x)$ and assume that it is increasing concave in $x \in \mathcal{X}$. For instance, in a software application development project, the project state x may represent the number of features currently incorporated into the software application. Having more features is obviously desirable, but, under the assumption that work is prioritized to tackle first the highest value-adding components (Layton 2012), $R(x)$ has diminishing marginal returns; that is, $\Delta R(x) := R(x+1) - R(x)$ is decreasing in x .⁵ In addition, to capture the desire for early conclusion, we assume that rewards are discounted over time with discount factor $\delta \leq 1$.

Efforts and costs of effort: We assume that the project progress depends on the client's and the vendor's joint efforts. We denote by $e_t^c(x)$ and $e_t^v(x)$ the client's and vendor's effort in time t and state x , respectively. For simplicity, we consider binary efforts; that is, efforts can be either *high* or *low*, i.e., $e_t^c \in \{h, l\}$ and $e_t^v \in \{h, l\}$, where low effort captures the minimum effort committed by each party (e.g., assigning employees to work on development, attending meetings, providing and reviewing reports of progress), and high effort captures any additional non-committed effort

⁴ We consider progress at an aggregate level and do not model specific precedence constraints.

⁵ We acknowledge that, in practice, project rewards may not always have diminishing returns or may be time-dependent. Although we somewhat capture those effects by considering a fixed bonus and a time discount factor, we leave it for future research to explore in greater detail the work dynamics under such reward functions.

Figure 1 Evolution of state x in time t with efforts $\mathbf{e}_t(x) = (e_t^c(x), e_t^v(x))$ 

(e.g., forming creative teams to generate new ideas, assigning high skills employees to help with resolving technical issues).

We denote by $c^v(e_t^v)$ and $c^c(e_t^c)$ the vendor's and client's cost of effort, respectively, such that $c^v(e_t^v) \in \{c_h^v, c_l^v\}$ and $c^c(e_t^c) \in \{c_h^c, c_l^c\}$. In addition, we assume that the effort costs are increasing in efforts; that is, $c_h^v \geq c_l^v$ and $c_h^c \geq c_l^c$. Moreover, we assume, as is often the case in practice, that the vendor is more efficient than the client, i.e., $c_h^c \geq c_h^v$ and that $c_l^c \geq c_l^v$.

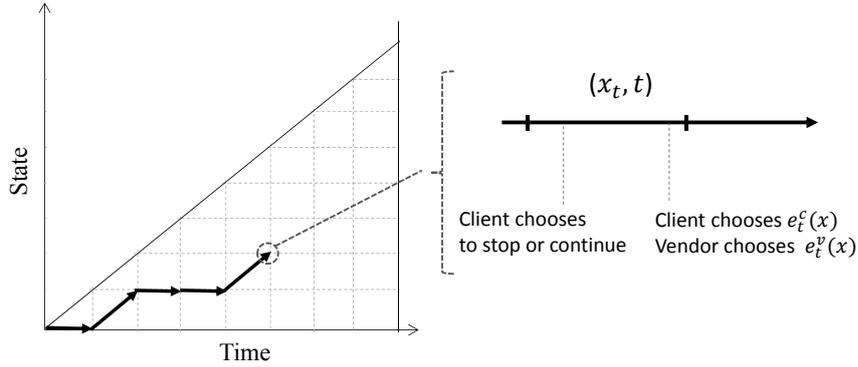
Transition probabilities: If the client and the vendor exert efforts $\mathbf{e}_t(x) = (e_t^c(x), e_t^v(x))$ in state x and time t , the state increases from x to $x+1$ with probability $p_{\mathbf{e}_t(x)}$ and remains at x with probability $1 - p_{\mathbf{e}_t(x)}$, as depicted in Figure 1. We denote by p_{hh} the success rate when both the client and the vendor exert high efforts, by p_{lh} (p_{hl}) the success rate when the client (vendor) exerts low effort and the vendor (client) exerts high effort,⁶ and by p_{ll} the success rate when both the client and the vendor exert low efforts. Naturally, we assume that the success rates are increasing in efforts; that is, $p_{hh} \geq \max\{p_{hl}, p_{lh}\} \geq p_{ll}$. Moreover, we assume that the vendor is more effective than the client and thus $p_{lh} \geq p_{hl}$. In our analytical characterization, we normalize p_{ll} to zero. That is, we assume that when both parties exert the minimum committed effort, the chances of making progress are so small that they can be neglected. For instance, if the client and the vendor only attend coordination meetings, but do not provide any additional background individual work, limited progress will be achieved in such meetings. However, we relax this assumption in our numerical analysis and show that all insights from our analytical characterization remain valid.

Note that, if the client and the vendor worked at a constant effort level, the sojourn time of each state would then follow a geometric distribution. However, because the client's and vendor's choices of effort are revised dynamically, the distribution of the sojourn time of each state is in general more complex.

Although we assume stationary success probabilities $\mathcal{P} = \{p_{hh}, p_{lh}, p_{hl}, p_{ll}\}$, we note that they could be time- and state-dependent in practice. For instance, they could increase over time as the

⁶ Consistent with the alphabetical order, the first index represents the client ("c") and the second index represents the vendor ("v").

Figure 2 State dynamics, sample path, and sequence of decisions



parties learn to work together and become more productive. Or, they could decrease over time if the chances of success are unknown, because of Bayesian updating on the viability of the project. They could also be state-dependent if some states are more challenging than others. Taking a middle approach, we assume that these probabilities are time- and state-independent, and leave the study of such dependencies for future research.

Because the project state increases by at most one step in each period, the set of reachable states in period t is $\mathcal{X}_t = \{0, 1, \dots, t\}$. Figure 2 depicts the set of reachable states along the time-state dimensions, together with a representative sample path. The horizontal axis denotes time, from the project start ($t = 0$) to the deadline ($t = T$), and the vertical axis represents the project state x as a function of time.

Sequence of Decisions: In each period t and state x , two decisions are sequentially made: first, whether the project should be continued or stopped and then, how much effort should be exerted by each party. As is common in practice, we assume that it is the client who decides whether to *Continue* or *Stop* the project at the beginning of each period (similar to Terwiesch and Loch 2004), while anticipating hers and the vendor's effort choices in case of continuation. If the client decides to continue the project, the client and the vendor choose whether to exert high or low efforts in that period. Figure 2 illustrates the sequence of decisions.

Contracts: To study the role of effort verifiability on the work dynamics, we consider two distinct situations: (i) when the vendor's effort is fully verifiable by the client, and (ii) when the vendor's effort is not verifiable by the client. In the first situation, the client can request that the vendor exert high effort through a contract. Thus, once the tasks are set, the vendor cannot free-ride. In the second situation, the vendor is prone to free riding, and the client and the vendor choose their effort levels simultaneously by maximizing their individual payoffs-to-go. Finally, if the client chooses to stop the project, she collects $R(x)$ and pays the vendor according to their pre-agreed contractual arrangement.

We consider here a situation where the project state is not contractible and thus the contract terms are time-based. Motivated by practice (e.g., Bajari and Tadelis 2001, Sheedy 2010, Edwards et al. 2014), we consider a time-based contract with two components: fixed bonus and per-period fee. Specifically, we assume that the client pays the vendor a bonus b upon stopping the project (i.e., the client collects $R(x) - b$ and pays b to the vendor), and that she pays a per-period fee f to the vendor every period until the project is stopped. In practice, there may exist other types of contracts, such as non-linear bonuses and penalties or, if the project state is verifiable (which we assume is not the case here), output-based contracts. Because time-based contracts, such as fixed-fee and time-and-materials are still the most common in many industries, such as in IT services (Bartrick 2013, Edwards et al. 2014), we focus here on such simple contracts and leave the study of work dynamics under more complex contracts for future research.

Dynamic stochastic game model: We model the problem with unverifiable efforts as a dynamic stochastic game between a client and a vendor. In our analysis, we focus on pure-strategy Markov equilibria, which are the most common type of equilibrium used in the analysis of dynamic games with simultaneous moves (Cachon and Netessine 2003, Fudenberg and Tirole 1991). That is, we assume that effort decisions depend on the state x and time period t , and not on the history of work decisions made in the past. Accordingly, we denote by $V_t^c(x | \mathbf{e}_t(x))$ and $V_t^v(x | \mathbf{e}_t(x))$ the client's and vendor's payoffs-to-go in time t and state x , respectively, when they exert effort levels $\mathbf{e}_t(x) = (e_t^c(x), e_t^v(x))$ such that

$$V_t^c(x | \mathbf{e}_t(x)) = -c^c(e_t^c(x)) - f + \delta [p_{\mathbf{e}_t(x)} V_{t+1}^c(x+1) + (1 - p_{\mathbf{e}_t(x)}) V_{t+1}^c(x)], \quad (1)$$

$$V_t^v(x | \mathbf{e}_t(x)) = -c^v(e_t^v(x)) + f + \delta [p_{\mathbf{e}_t(x)} V_{t+1}^v(x+1) + (1 - p_{\mathbf{e}_t(x)}) V_{t+1}^v(x)], \quad (2)$$

with $V_t^c(x | Stop) = R(x) - b$ and $V_t^v(x | Stop) = b$,

in which

$$V_t^c(x) = \max\{V_t^c(x | \mathbf{e}_t(x)), (R(x_t) - b)\} \quad (3)$$

$$V_t^v(x) = \begin{cases} b & \text{if } V_t^c(x) = R(x_t) - b \\ V_t^v(x | \mathbf{e}_t(x)) & \text{otherwise} \end{cases} \quad (4)$$

$$s.t. \ e_t^c(x) = \arg \max_{e \in \{h, l\}} V_t^c(x | (e, e_t^v(x))) \quad (4)$$

$$e_t^v(x) = \arg \max_{e \in \{h, l\}} V_t^v(x | (e_t^c(x), e)), \quad (5)$$

where $V_t^c(x | \mathbf{e}_t(x))$ and $V_t^v(x | \mathbf{e}_t(x))$ are presented in Figure 3 with $V_T^c(x) = R(x_T) - b$ and $V_T^v(x) = b$.⁷ In our analysis, we assume without loss of generality that $c_h^c/c_h^v = c_l^c/c_l^v = k$ with $k \geq 1$ by potentially rescaling the payoff functions b , f , and $R(x)$.

⁷ In Figure 3, we used the notation $\mathbb{E}[V_{t+1}^i(x + \xi) | \mathbf{e}_t(x)] := p_{\mathbf{e}_t(x)} V_{t+1}^i(x+1) + (1 - p_{\mathbf{e}_t(x)}) V_{t+1}^i(x)$ for $i \in \{c, v\}$ and effort levels $\mathbf{e}_t(x) \in \{(h, h), (h, l), (l, h), (l, l)\}$.

Figure 3 Payoff matrix $(V_t^c(x), V_t^v(x))$ in period t and state x .

$(\text{state, time})=(x_t, t)$

	Client Chooses to Continue	Client Chooses to Stop	
Client/ Vendor	<i>High</i>	<i>Low</i>	
			$V_t^c(x) = R(x_t) - b$
			$V_t^v(x) = b$
<i>High</i>	$-c_h^c - f + \delta \cdot \mathbb{E}[V_{t+1}^c(x + \xi) (h, h)],$ $-c_h^v + f + \delta \cdot \mathbb{E}[V_{t+1}^v(x + \xi) (h, h)]$	$-c_h^c - f + \delta \cdot \mathbb{E}[V_{t+1}^c(x + \xi) (h, l)],$ $-c_h^v + f + \delta \cdot \mathbb{E}[V_{t+1}^v(x + \xi) (h, l)]$	
<i>Low</i>	$-c_l^c - f + \delta \cdot \mathbb{E}[V_{t+1}^c(x + \xi) (l, h)],$ $-c_l^v + f + \delta \cdot \mathbb{E}[V_{t+1}^v(x + \xi) (l, h)]$	$-c_l^c - f + \delta \cdot \mathbb{E}[V_{t+1}^c(x + \xi) (l, l)],$ $-c_l^v + f + \delta \cdot \mathbb{E}[V_{t+1}^v(x + \xi) (l, l)]$	

The normal-form game depicted in Figure 3 is played each period until the client decides to stop the project or the deadline is reached. Note that because the parties' payoffs-to-go are functions of both the project state and time period, the game payoffs change dynamically; that is, the game played in period t and state x_t is different in terms of payoffs from the games played in other time periods (i.e., $t' > t$ or $t' < t$) and states (i.e., $x' > x_t$ or $x' < x_t$). Hence, the equilibrium outcome of the game $\mathcal{E}_t(x) \in \{(h, h), (h, l), (l, h), (l, l)\}$ is time- and state-dependent.

In the next section, we characterize the work dynamics when efforts are verifiable, thereby resulting in the first-best solution. We then study the case where efforts are not verifiable in §5, thereby resulting in a situation of double moral hazard (Holmström 1982).

4. Verifiable Efforts: First-Best Work Dynamics

In this section, we study the case where the vendor's efforts are verifiable by the client, which happens when the client requires the vendor to work on her site and make his effort observable (Roels et al. 2010, Eccles 2010), or when the client uses certain monitoring infrastructure, such as pair programming tools in agile software development (Williams and Kessler 2003). When efforts are verifiable, the client can request the vendor to exert a specific level of effort, and contractually enforce that, so as to maximize the total surplus. Accordingly, we refer to the work dynamics in this case as "first-best" (FB) solution. Although the FB solution may not always be fully attainable in practice, its characterization is important for this study, because it shows how effort levels should ideally be managed in the absence of free riding.

Using (1) and (2), we denote the total surplus in period t and state x by $V_t(x | \mathbf{e}_t(x)) \doteq V_t^c(x | \mathbf{e}_t(x)) + V_t^v(x | \mathbf{e}_t(x))$, which is independent of contract terms b and f . In addition, because the total surplus is maximized, the decisions of continuing or stopping the project can be made simultaneously to the choice of efforts during that period. Hence, five outcomes are attainable in period t and state x , i.e., $\mathcal{E}_t^{FB}(x) \in \{(h, h), (h, l), (l, h), (l, l), \text{Stop}\}$, and the optimal policy can be identified by solving a finite-horizon dynamic program.

Because by assumption the vendor is more effective and efficient than the client (i.e., $c_l^c \geq c_l^v$, $c_h^c \geq c_h^v$, and $p_{lh} \geq p_{hl}$), the work outcome (l, h) always dominates the work outcome (h, l) . In addition, the work outcome (l, l) turns out to be never optimal when $p_{ll} = 0$ (see Lemma A-3 in appendix). Thus, the decision in any state x and period t can be reduced to choosing among three working modes, namely (h, h) , (l, h) , and *Stop* with the following payoff to-go functions:

$$V_t(x | (h, h)) = -c_h^c - c_h^v + \delta[p_{hh}V_{t+1}(x+1) + (1-p_{hh})V_{t+1}(x)], \quad (6)$$

$$V_t(x | (l, h)) = -c_l^c - c_h^v + \delta[p_{lh}V_{t+1}(x+1) + (1-p_{lh})V_{t+1}(x)], \quad (7)$$

$$V_t(x | \text{Stop}) = R(x).$$

We first characterize, in Proposition 1, the condition under which it is optimal to stop the project and collect the reward. We then study, in Proposition 2, the work dynamics in the time periods and states for which it is optimal to continue the project. Finally, at the end of this section, we provide numerical illustrations for the case where $p_{ll} > 0$ and show that the overall insights and results, derived analytically, hold true in general.

In the next proposition, we show that there exists a time-independent state threshold x_s^{FB} above which it is optimal to stop the project and below which it is optimal to keep working.

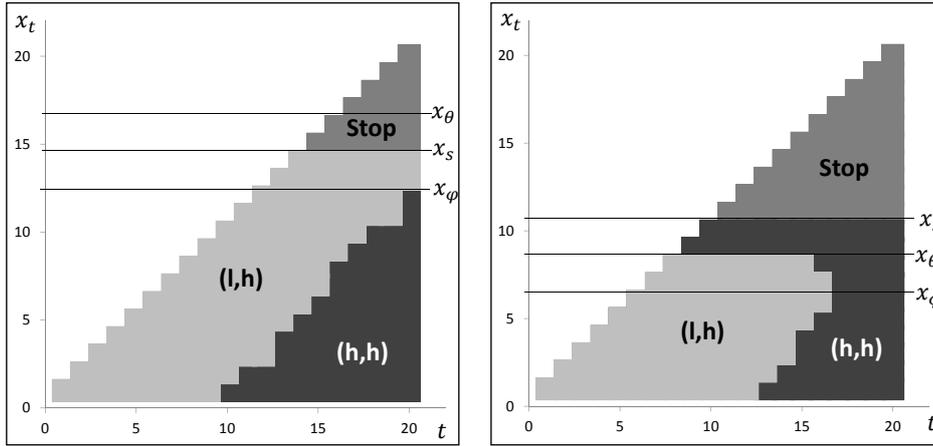
PROPOSITION 1. *There exists a state threshold x_s^{FB} such that it is optimal to Stop the project if and only if $x \geq x_s^{FB}$ for all t .*

The threshold x_s^{FB} can be interpreted as a completion state milestone. That is, it is optimal to stop the project as soon as the project state reaches milestone x_s^{FB} even if it is before the deadline T . The existence of this stopping threshold is driven by the assumption that $R(x)$ has diminishing marginal returns. Under that assumption, putting in more effort beyond a certain state is not worthwhile, because its cost of effort exceeds its marginal return. In addition, Proposition 1 shows that the stopping state threshold is time-independent. Because the reward function $R(x)$ is time-independent, when it becomes optimal to stop the project in state x and time t , it is also optimal to stop the project in state x and time $t+1$ as well as in state x and time $t-1$.

Hence, Proposition 1 indicates that when efforts are verifiable, there exists an optimal completion state, which is time-independent and should be determined based on success rates, costs of efforts, and discount factor. In particular, the stopping state threshold is increasing in success rates (p_{hh} and p_{lh}) and the discount factor (δ), and it is decreasing in effort costs (c_h^c , c_h^v , and c_l^c). That is, it is optimal to set a high aspirational goal for the project when its chances of success are high, the parties are patient, and their costs of opportunity are low, and the reverse holds otherwise.

Therefore, Proposition 1 implies that for any state below the stopping state threshold, either the (h, h) or the (l, h) working modes are optimal. Comparing the total surplus of (h, h) and

Figure 4 FB work dynamics when $\Delta R(x_s^{FB} - 1) \leq \frac{c_h^c - c_l^c}{\delta(p_{hh} - p_{lh})}$ (left) and when $\Delta R(x_s^{FB} - 1) > \frac{c_h^c - c_l^c}{\delta(p_{hh} - p_{lh})}$ (right)



Note. The parameters are: $T = 20$, $c_h^v = 5.4$, $c_l^v = 2$, $k = 1.5$, $p_{hh} = 0.82$, $p_{lh} = 0.78$, $p_{hl} = 0.76$, $p_{ll} = 0$, $R(x) = 950\sqrt{x}$, $\delta = 0.975$ (left) $\delta = 0.965$ (right). The state thresholds are: $x_s^{FB} = 15$, $x_\theta^{FB} = 17$, $x_\varphi^{FB} = 13$ (left) and $x_s^{FB} = 11$, $x_\theta^{FB} = 9$, $x_\varphi^{FB} = 6$ (right).

(l, h) working modes in (6)-(7) yields the following characterization of the optimal policy in states $x < x_s^{FB}$ and periods $t \leq T - 1$:

$$\mathcal{E}_t^{FB}(x) = (h, h) \text{ if } \Delta V_{t+1}(x) \geq \frac{c_h^c - c_l^c}{\delta(p_{hh} - p_{lh})} \text{ and } \mathcal{E}_t^{FB}(x) = (l, h) \text{ if } \Delta V_{t+1}(x) \leq \frac{c_h^c - c_l^c}{\delta(p_{hh} - p_{lh})}, \quad (8)$$

in which $\Delta V_{t+1}(x) := V_{t+1}(x+1) - V_{t+1}(x)$.

According to (8), if the payoff-to-go function $V_t(x)$ were concave in x , there would exist a threshold policy such that, if (h, h) were optimal in state x and period t , it would be optimal in all states $x' \leq x$ in period t . In general, however, the payoff-to-go function may exhibit increasing marginal returns. That is, (h, h) may be optimal in low and high states, but not in intermediate states.

The next proposition characterizes the optimal policy with respect to both time and states below the stopping threshold. We consider two cases, depending on whether the marginal reward below the stopping state, $\Delta R(x_s^{FB} - 1)$, is small or not, and Figure 4 illustrates the results for each of these two cases.

PROPOSITION 2. *Under the FB work dynamics, there exist state thresholds x_θ^{FB} and $x_\varphi^{FB} < \min\{x_\theta^{FB}, x_s^{FB}\}$ such that*

(i) *If $\Delta R(x_s^{FB} - 1) \leq \frac{c_h^c - c_l^c}{\delta(p_{hh} - p_{lh})}$, then*

(a) *for $x_\varphi^{FB} \leq x < x_s^{FB}$, the optimal policy is (l, h) for all t .*

(b) *for $1 \leq x < x_\varphi^{FB}$, there exists a time threshold $\tau(x)$, nondecreasing in x , such that the optimal policy is (h, h) for all $T > t > \tau(x)$ and it is (l, h) for all $x \leq t \leq \tau(x)$.*

- (ii) If $\Delta R(x_s^{FB} - 1) > \frac{c_h^c - c_i^c}{\delta(p_{hh} - p_{lh})}$, then
- (a) for all $x_\theta^{FB} \leq x < x_s^{FB}$, the optimal policy is (h, h) for all t ;
 - (b) for all $x_\varphi^{FB} \leq x < \min\{x_\theta^{FB}, x_s^{FB}\}$, there exists a time threshold $\tau(x)$, nonincreasing in x , such that the optimal policy is (h, h) for all $T > t > \tau(x)$ and it is (l, h) for all $x \leq t \leq \tau(x)$;
 - (c) for all $x < x_\varphi^{FB}$, there exists a time threshold $\tau(x)$, nondecreasing in x , such that the optimal policy is (h, h) for all $T > t > \tau(x)$.

At a broad level, Proposition 2 indicates efforts should be time- and state-dependent, which explains why it is difficult in practice for project managers to predict how much progress can be achieved within a certain time window or how much time is needed to reach a particular state.

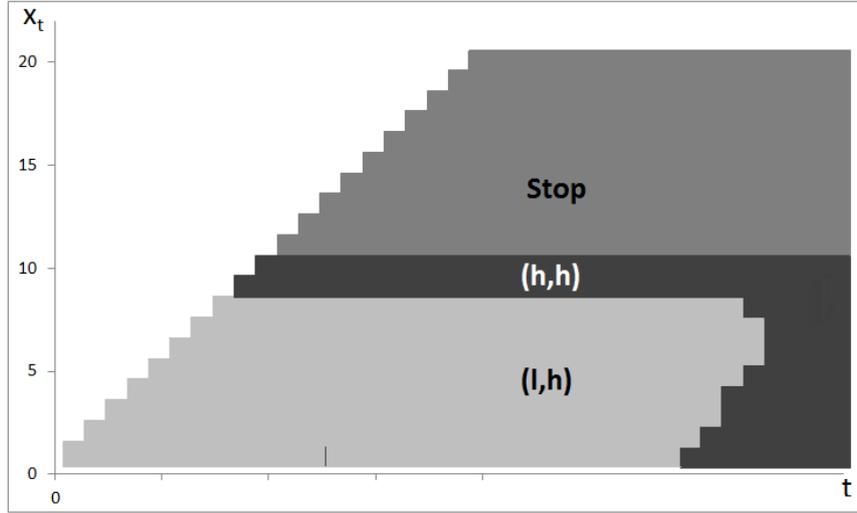
In particular, our results indicate that when efforts are verifiable, it is optimal that both the client and vendor exert high effort in two situations:

- (I) When not enough progress has been made and the deadline is close, as established in Proposition 2 (i.b) and (ii.c).
- (II) When good progress has been made early so that the project state is close to the stopping threshold (x_s^{FB}) , as established in Proposition 2 (ii.a) and (ii.b).

Case (I), which we call the *Deadline effect*, shows that the common effect of rush before the deadline (Repenning 2001, Chang 2007, Wu et al. 2014) can be rational and driven by the (negative) prospect of not ending the project at low value. In particular, in late periods, because it is very unlikely that the project reaches its stopping state threshold, it is optimal for both parties to exert high effort to increase the completion state of the project. Accordingly, project managers who use past progress to extrapolate future progress may underestimate the final completion state of the project. Specifically, when they observe that the project progressed at a rate p_{lh} in the past, they may incorrectly predict that the project will continue at the same rate, ignoring the deadline effect. In that sense, deadlines are beneficial to salvage projects that have not had good progress early on.

Case (II), which we call the *Milestone effect*, shows that it is optimal that both the client and vendor exert high effort when the project state is close to its stopping state threshold (x_s^{FB}) . To understand this milestone effect, note that in high states, it is very likely that the stopping state threshold x_s^{FB} will be reached by the deadline. Thus, there is little uncertainty about the completion state of the project; instead, what motivates the parties to exert high effort under the milestone effect is the (positive) prospect of reaching the stopping state threshold early and collecting their rewards sooner. Accordingly, project managers who use past progress to extrapolate future progress may overestimate the total time it takes to complete the project. In particular, if a project has had some progress early on, the milestone effect will make that progress even faster in the future.

Figure 5 FB work dynamics when $p_{ll} > 0$ and $\Delta R(x_s^{FB} - 1) > \frac{c_h^c - c_l^c}{\delta(p_{hh} - p_{lh})}$ and T is large



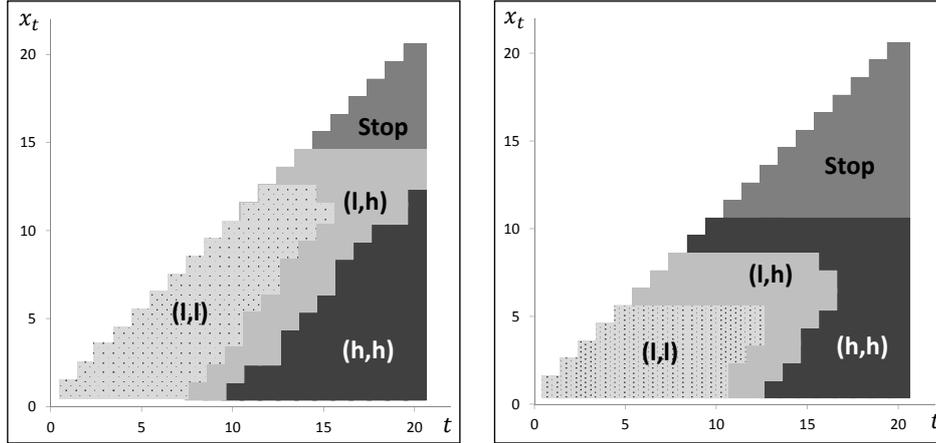
Note. The parameters are the same as in Figure 4 (right)

This milestone effect is due to the time-invariance of the reward function $R(x)$, as well as to the assumption that the reward can be collected as soon as the project is stopped. We conjecture that the milestone effect would disappear if the reward were obtainable only at the deadline or conversely, that it would be stronger if the reward were larger with early conclusion of the project (e.g., due to first-mover advantages of early time to market).

Studying comparative statics reveals that the milestone effect arises when the project stopping threshold is associated with a high marginal gain (i.e., when $R(x_s^{FB}) - R(x_s^{FB} - 1)$ is large), the effort costs (c_h^c , c_h^v , c_l^c) are low, and the success rate p_{hh} is high relative to the success rate p_{lh} . In addition, this effect tends to arise when the discount factor δ is small, because a lower discount factor makes the parties more impatient to collect their reward; therefore, creating higher incentives to exert high effort in the high states of the project. However, a lower discount factor also makes the parties less willing to push the project further, resulting in a lower stopping state threshold. The total effects of a change in the discount factor are illustrated in Figure 4: As δ decreases from $\delta = 0.975$ (left) to $\delta = 0.965$ (right), the optimal stopping state threshold decreases, the (h, h) region in the high states of the project appears, and the (h, h) region induced by the deadline shrinks.

The results of Propositions 1 and 2 naturally extend to the case where the project is not subject to an imminent deadline. Figure 5 illustrates the FB work dynamics when the deadline T is large. As T gets larger, the deadline effect becomes relatively less important, and the time-independent state thresholds x_s^{FB} and x_θ^{FB} essentially define the different phases of the project. Specifically, stretching the horizontal axis of Figure 4 to the right defines three (almost) time-independent phases of (l, h) , (h, h) , and *Stop*, as shown in Figure 5. This shows that the milestone effect is robust to any deadline.

Figure 6 FB work dynamics when $\Delta R(x_s^{FB} - 1) \leq \frac{c_h^c - c_l^c}{\delta(p_{hh} - p_{lh})}$ (left) and when $\Delta R(x_s^{FB} - 1) > \frac{c_h^c - c_l^c}{\delta(p_{hh} - p_{lh})}$ (right)



Note. The parameters are the same as in Figure 4 with $p_{ll} = 0.75$

We next assess the sensitivity of the characterization of the FB work dynamics presented in Propositions 1 and 2 to the case where the success probability in case of joint low efforts, p_{ll} , is strictly positive. Figure 6 presents a representative numerical illustration of the case where $p_{ll} > 0$. Although the (l,l) policy is feasible in that case, the work dynamics remain consistent with the analytical characterization presented in Propositions 1 and 2. Specifically, as shown in Figure 6, there appears to be a time-independent stopping state threshold, and the intensity of the client's and vendor's efforts increases when the project gets closer to its time-independent stopping threshold or when it gets closer to its deadline. Clearly, when the deadline is far away, the parties will exert low effort, in the hope that they will be successful without incurring a high cost of effort. But as the deadline looms, especially if little progress has been made, exerting higher effort at a higher cost becomes more justified. Thus, our managerial insights on the time independence of the completion milestone and the high-intensity collaboration arising due to the deadline and milestone effects appear to be robust to the assumption of no progress in case of joint low efforts, and we conjecture that the same effects will hold with more general levels of efforts.

In summary, our analysis in this section reveals that when efforts are verifiable, it is optimal to stop the project as soon as it reaches a time-independent stopping threshold. In addition, efforts should be time- and state-dependent; particularly, it is optimal that both parties exert high effort close to the stopping threshold (milestone effect) and deadline (deadline effect). In the next section, we study the case where efforts are not verifiable.

5. Unverifiable Efforts: Time-Based Contracts

In this section, we consider a situation where the vendor's efforts are not verifiable by the client, which may happen when the client does not have the infrastructure to monitor the vendor's efforts

(e.g., when the vendor is located remotely). In that case, the vendor is prone to free riding and this will in turn affect the client's choice of effort and stopping decision (Holmström 1982). We first analytically characterize the work dynamics for any type of time-based contract in §5.1, and then numerically investigate the work dynamics under contract parameters that maximize the client's payoff in §5.2.

5.1. Equilibrium Work Dynamics

Using the stochastic dynamic game model in (4)-(5) with payoffs depicted in the table of Figure 3, four equilibrium outcomes are attainable in period t and state x , i.e., $\mathcal{E}_t(x) \in \{(h, h), (h, l), (l, h), (l, l)\}$. We next determine conditions under which each pure-strategy equilibrium outcome can arise:

$$\begin{aligned} \mathcal{E}_t(x) &= (h, h) \text{ if } \Delta V_{t+1}^c(x) > \frac{c_h^c - c_l^c}{\delta(p_{hh} - p_{lh})} \text{ and } \Delta V_{t+1}^v(x) > \frac{c_h^v - c_l^v}{\delta(p_{hh} - p_{hl})} \\ \mathcal{E}_t(x) &= (h, l) \text{ if } \Delta V_{t+1}^c(x) > \frac{c_h^c - c_l^c}{\delta(p_{hl} - p_{ll})} \text{ and } \Delta V_{t+1}^v(x) \leq \frac{c_h^v - c_l^v}{\delta(p_{hh} - p_{hl})}, \\ \mathcal{E}_t(x) &= (l, h) \text{ if } \Delta V_{t+1}^c(x) \leq \frac{c_h^c - c_l^c}{\delta(p_{hh} - p_{lh})} \text{ and } \Delta V_{t+1}^v(x) > \frac{c_h^v - c_l^v}{\delta(p_{lh} - p_{ll})}, \\ \mathcal{E}_t(x) &= (l, l) \text{ if } \Delta V_{t+1}^c(x) \leq \frac{c_h^c - c_l^c}{\delta(p_{hl} - p_{ll})} \text{ and } \Delta V_{t+1}^v(x) \leq \frac{c_h^v - c_l^v}{\delta(p_{lh} - p_{ll})}. \end{aligned}$$

in which, $\Delta V_{t+1}^i(x) := V_{t+1}^i(x+1) - V_{t+1}^i(x)$ for $i \in \{c, v\}$. In the next technical lemma, we characterize conditions under which an equilibrium in pure strategies exists.

LEMMA 1. *In any period t and state x , if the client chooses to continue the project at the beginning of the period, an equilibrium in pure strategies exists if (i) $p_{hl} = p_{lh}$, or (ii) $p_{hh} = p_{lh} + p_{hl}$. Otherwise an equilibrium in pure strategies may not exist.*

Although a pure-strategy equilibrium is guaranteed to exist when the client is as effective as the vendor ($p_{hl} = p_{lh}$) or when efforts are additive ($p_{hh} = p_{lh} + p_{hl}$), it may not always be unique. When the game admits multiple equilibria in a particular time period t and state x , we assume that the client has the authority to select the equilibrium to be played, consistent with the equilibrium selection rule suggested by Gibbons and Roberts (2012, page 438). We denote by $\mathcal{E}_t(x)$ the selected pure-strategy Markov equilibrium played in period t and state x . For instance, we use the notations $\mathcal{E}_t(x) = (h, h)$ and $\mathcal{E}_t(x) \neq (h, h)$ to respectively mean that (h, h) is or is not the selected pure-strategy Markov perfect equilibrium in period t and state x .

In the next proposition, we show that similar to the FB solution, there exists a time-independent stopping state threshold above which it is optimal for the client to stop the project and below which it is optimal to keep working.

PROPOSITION 3. *Suppose an equilibrium in pure strategies exists in any period t and state x . Then, there exists a state threshold x_s such that it is optimal for the client to Stop the project for all t if and only if $x \geq x_s$. In addition, the threshold x_s is increasing in the bonus b and in the discount factor δ , and decreasing in the per-period fee f .*

Similar to Proposition 1, Proposition 3 shows that the stopping state threshold under time-based contracts is time-independent. In addition, the stopping threshold is increasing in the discount factor δ and in the bonus b , but it is decreasing in the per-period fee f . To explain these comparative statics, consider the following two scenarios: On one hand, when the parties are patient or the bonus is large, the client wants to postpone the time she would need to pay the bonus to the vendor; thus, it is optimal for her to set a higher completion state. On the other hand, when the parties are impatient and the per-period fee is large, the client wants to shorten the time she would need to pay the fee to the vendor; thus, it is optimal for her to set a lower completion state.⁸

Comparing the stopping thresholds under verifiable and unverifiable efforts, it can be shown that (i) when $b = 0$ and $f > 0$, the time-based contract always results in a lower stopping threshold than when efforts are verifiable, and (ii) when $b \gg 0$, $f = 0$, and $p_{hh} \gg p_{lh}$, the time-based contract results in a higher stopping threshold than when efforts are verifiable. This suggests that one may need, in practice, to design contracts with both a bonus and a per-period fee to set the project stopping state close to FB solution.

We next characterize the work dynamics in states $x < x_s$ by solving (4)-(5). For analytical tractability, we make the following assumptions in our analytical characterization:

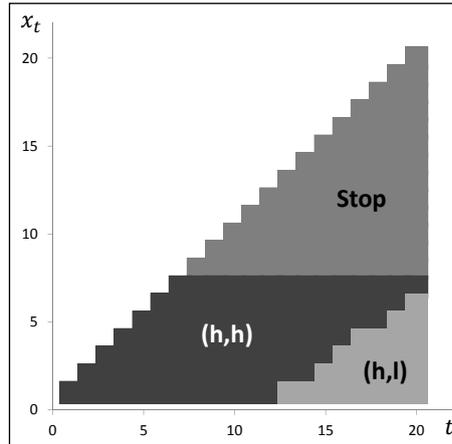
- *Assumption A:* The parties are symmetric (i.e., $p_{hl} = p_{lh}$ and $k = 1$), and their efforts are super-additive, i.e., $p_{hh} \geq 2p_{lh}$.
- *Assumption B:* The contract terms (b, f) are set such that $R(x_s) - b \geq b$ and $f < c_i^v$.

Although we make these assumptions in our analytical characterization of the dynamic equilibrium, we relax them in our numerical illustrations so as to assess the robustness of our analytical characterization. Assumption A requires symmetry between the parties and that high-effort intensity collaboration is very effective. Given our focus on co-productive processes, this is indeed the most interesting case to consider, because otherwise (i.e., with a high degree of asymmetry or ineffective high-effort intensity collaboration), the work dynamics may turn out to be such that one party never exerts high effort.⁹ Assumption B requires that the client's share of reward ($R(x_s) - b$)

⁸ Incidentally, fixed-fee contracts (which involve only a fixed completion bonus) are often criticized for leading to scope creep at the request of the client, whereas time-and-materials contracts (which involve only a per-period fee) are often criticized for leading to scope creep at the request of the vendor (Eckfeldt and Madden 2005).

⁹ Similarly, Marx and Matthews (2000), Bonatti and Horner (2011), and Georgiadis (2014) assume symmetric and perfectly substitutable efforts, which, in our context, is equivalent to assuming $p_{hh} = 2p_{lh}$.

Figure 7 Equilibrium work dynamics with unverifiable efforts



Note. Same parameters as in Figure 4 (right) with $p_{lh} = p_{hl} = 0.4$, $b = 750$, and $f = 1.5$. The completion state threshold is $x_s = 8$.

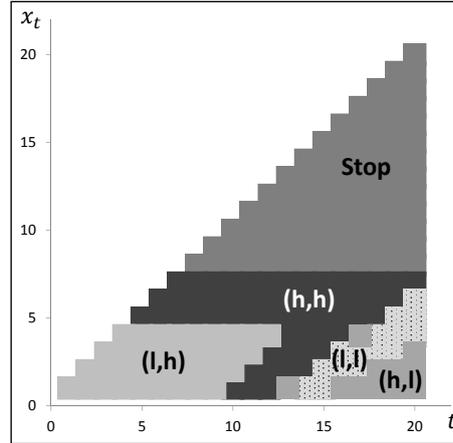
be larger than the vendor's bonus (b), and that the per-period fee (f) be less than the vendor's cost of low effort (c_l^v). In §5.2, we numerically evaluate the contract parameters that maximize the client's payoff and it appears that Assumption B is often satisfied under the optimal contract parameters.

The next proposition characterizes the equilibrium work dynamics in states below the equilibrium stopping state threshold, and Figure 7 illustrates the results.

PROPOSITION 4. *Suppose Assumptions A and B hold. For any $x < x_s$, there exists a time period $\tau(x)$, increasing in x , such that $\mathcal{E}_t(x) = (h, l)$ for $t \geq \tau(x)$ and $\mathcal{E}_t(x) = (h, h)$ for $t < \tau(x)$.*

Proposition 4 shows that, in contrast to the FB work dynamics (Proposition 2), the vendor does not exert high effort near the deadline under time-based contracts. That is, the *deadline effect* characterizing the FB work dynamics does not survive under such contracts. Specifically, in states below x_s and time periods close to the deadline, the equilibrium outcome is (h, l) , i.e., the client exerts high effort and the vendor only exerts his minimum committed effort. In contrast to the client, whose payoff depends on the project's reward $R(x)$, the vendor's payoff is not tied to the project completion state, and the vendor has therefore no incentive to exert more effort than the minimum committed effort, as is often observed in practice (Edwards et al. 2014).

Hence, the deadline effect does not survive in equilibrium when efforts are not verifiable. A project manager who would use the FB work dynamics as a guide for estimating the equilibrium work dynamics would thus ignore the vendor's free-riding behavior near the deadline and overestimate the project completion state by the deadline. Hence, unlike the FB solution, in which stalled projects may go through a final rejuvenation phase near the deadline, stalled projects under time-based contracts remain stalled.

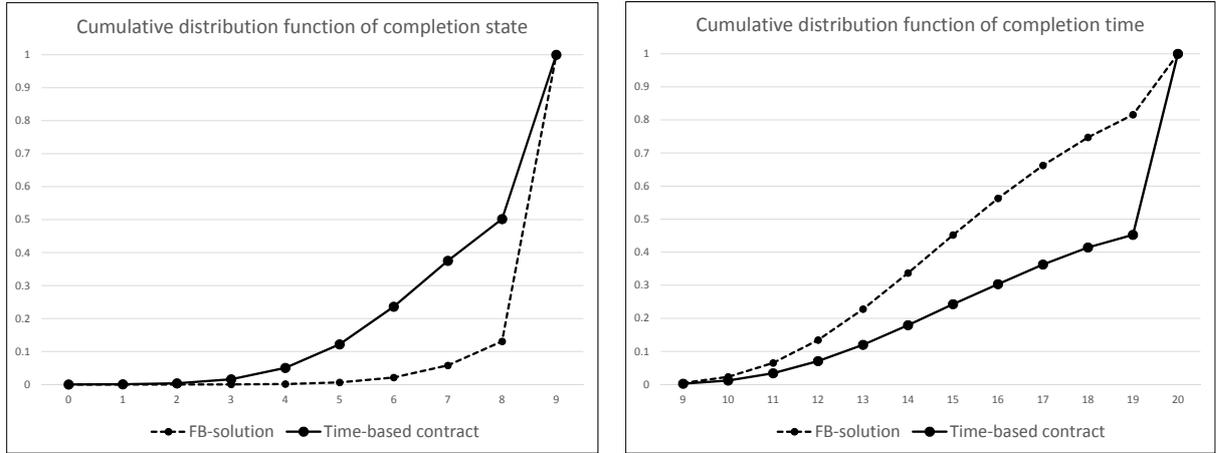
Figure 8 Equilibrium work dynamics with unverifiable efforts when $p_{ll} > 0$ and Assumptions A and B do not hold.

Note. Same parameters as in Figure 7 with $p_{lh} = 0.75$, $p_{ll} = 0.38$. The stopping state threshold is $x_s = 8$.

However, the vendor may choose to exert high effort in the early periods of the project especially when the project state is close to its equilibrium stopping state threshold. Because time is discounted, the vendor is indeed motivated to reach the stopping state threshold so as to collect his bonus b sooner and stop exerting effort that is compensated at a rate f lower than his cost of effort (per Assumption B). Effectively, the bonus b and the per-period fee f act as a carrot and a stick respectively, to induce the vendor to exert high effort so as to lead to early project conclusion. Accordingly, the emergence of (h, h) region under time-based contracts can be interpreted as a *milestone effect*, similar to that in the FB solution. Hence, in contrast to the deadline effect, which does not survive under unverifiable efforts, the milestone effect is robust, i.e., irrespective of whether efforts are verifiable or not, project managers can anticipate an acceleration phase as the project gets closer to its stopping state threshold.

We next numerically test the robustness of our analytical characterization to Assumptions A and B and to the assumption that no progress is made in the case of joint low efforts. As a representative example of such cases, Figure 8 illustrates that the (l, l) and (l, h) outcomes may also arise in equilibrium when those assumptions are relaxed. Although the client no longer exerts high effort in every state and every time period, we observe that, consistent with Proposition 3, there exists a time-independent stopping state threshold, and that, consistent with Proposition 4, the intensity of the vendor's efforts increases when the project gets closer to the equilibrium time-independent stopping state threshold (milestone effect), and decreases when it gets closer to deadline (absence of deadline effect). In contrast, the client has a more complex effort strategy, exerting high effort both near the stopping state threshold (in concert with the vendor) and near the deadline (in contrast to the vendor).

Figure 9 Cumulative distribution functions of completion state (left) and time (right) under the FB solution and a Time-based contract



Note. Same parameters as in Figure 7 with $p_{hh} = 0.55$, $\delta = 0.975$, and contract terms $(b, f) = (750, 1.5)$ maximize the client’s payoff (See §5.2). The stopping state thresholds are $x_s = x_s^{FB} = 9$. In the left figure, the mean completion states are 9.59 (FB) and 8.14 (time-based contract), and the variance in completion states are 0.93 (FB) and 4.06 (time-based contract). In the right figure, the mean completion times are 16.09 (FB) and 18.30 (time-based contract), and the variance in completion times are 9.61 (FB) and 10.90 (time-based contract).

Overall, our characterization of efforts under time-based contracts suggests that compared to the FB solution, the milestone effect is preserved under unverifiable efforts, but the deadline effect disappears. This implies that with time-based contracts, the project’s chances of success are lower, and the project completion state will fundamentally depend on how much progress is made in the early periods. To explain this managerial implication, consider the following two scenarios: (i) if some progress has been made early, there is a good chance that the milestone effect phase will be reached and that the project will reach the stopping state threshold. (ii) In contrast, if insignificant progress has been made early, there is a great chance that the project may stall, the vendor will free-ride, and even if the client puts in high effort near the deadline, the project completion state by the deadline will likely be lower than the client’s desired stopping state threshold. Hence, either projects converge quickly to the client’s desired stopping state threshold or they drag until the deadline and end at a low state. This suggests that both the variance in project completion states and the variance in project completion times are greater under unverifiable efforts than under verifiable efforts.

To confirm this intuition, Figure 9 (left) illustrates the cumulative probability distribution of the project completion state, i.e., the state at which the project is stopped either because the deadline has been reached or because the client has stopped the project, under the time-based contract (solid) and the FB solution (dashed). From the figure, it appears that the distribution of

completion states under FB stochastically dominates the distribution of completion states under the time-based contract, i.e., for any state x , the probability that the completion state is higher than x is higher under the FB solution than under the time-based contract.

Figure 9 (right) illustrates the cumulative probability distribution of the completion time, i.e., the time at which the project is stopped either because the stopping state threshold has been reached or because the deadline has been reached, under both the time-based contract (solid) and the FB-solution (dashed) for a particular example. From the figure, it appears that the distribution of completion times under the time-based contract stochastically dominates the distribution of completion times under the FB solution, i.e., for any time t , the probability that the completion time is higher than t is lower under the FB solution than under the time-based contract. These observations emphasize the criticality of making early progress when efforts are unverifiable.

In summary, our analysis of the work dynamics under time-based contracts reveals that, similar to the FB solution, it is optimal for the client to stop the project as soon as the project state reaches a time-independent stopping state threshold. However, that threshold can be lower or higher than the FB stopping threshold, depending on the contract parameters. In particular, the equilibrium stopping state threshold is decreasing in the per-period fee and increasing in the bonus, justifying the use of both incentive mechanisms in practice. In addition, we show that compared to the FB solution, the milestone effect is preserved, but the deadline effect disappears. This implies that with time-based contracts, the completion state and time of the project will fundamentally depend on how much progress is made in the early periods. In the next section, we investigate the optimal parameters of time-based contracts.

5.2. Optimal Contract Design

In this section, we investigate the contract terms that maximize the client's payoff subject to the vendor's participation. In addition, we investigate the robustness of our analytical results to our modeling assumptions (i.e., Assumptions A and B).

We denote the equilibrium payoff-to-go functions for the client (c) and the vendor (v) in state x and period t for given contract terms (b, f) by $V_t^c(x; b, f)$ and $V_t^v(x; b, f)$. Similar to Bhattacharyya and Lafontaine (1995), we consider an upfront transfer payment between the client and the vendor denoted by F . Thus, the client chooses the contract terms to maximize her payoff subject to a vendor's participation constraint, i.e.,

$$\begin{aligned} & \max_{b, f, F} V_0^c(0; b, f) - F \\ \text{s.t. } & V_0^v(0; b, f) + F \geq \bar{V} \\ & b \in \mathcal{B}, f \in \mathcal{F}, \end{aligned} \tag{9}$$

Table 1 Ranges of Parameters

Parameters	Ranges
Reward $R(x) = r \cdot x^\alpha$	$r = 950, \alpha \sim U[0.35, 1]$
Cost	$c_l^v \sim U[1, 5], c_h^v = 5, k = 2$
Transition Probabilities	$p_{hh} \sim U[0.4, 1], p_{lh} \sim U[0.2, p_{hh}], p_{hl} \sim U[0.2, p_{lh}], p_{ll} \sim U[0.2, p_{hl}]$
Discount Factor	$\delta = 0.975$
Contract Parameters	$\mathcal{B} = \{0, 50, 100, \dots, 2100\}, \mathcal{F} = \{0, 1, 2, \dots, 10\}$

Table 2 Equilibrium stopping state thresholds under time-based contracts

	$x_s < x_s^{FB}$	$x_s = x_s^{FB}$	$x_s > x_s^{FB}$
Percentage	78.8%	19.2%	2%

in which \mathcal{B} and \mathcal{F} denote the feasible sets for the bonus and the per-period fee. As is standard in such principal-agent models, it is optimal for the client to set the upfront payment F such that the vendor’s participation constraint binds (Bolton and Dewatripont 2005). Hence, with the optimal upfront payment, the vendor earns in expectation his reservation profit \bar{V} and the optimal contract parameters (b^*, f^*) maximize the total surplus.¹⁰

Parameters: We randomly generated 250 sets of parameter values over the ranges of parameters depicted in Table 1 setting $T = 20$ such that $R(x)$ is increasing concave, $c_h^v \geq c_l^v, p_{hh} \geq p_{lh} \geq p_{hl} \geq p_{ll}$, and the vendor is twice more efficient than the client ($k = 2$). For numerical tractability, we optimize the choice of contract parameters over discrete sets. We defined the sets \mathcal{B} and \mathcal{F} such that it is very unlikely that the client would choose to set the contract parameters to their upper bounds. In particular, we observed that in 97% of the instances, the optimal contract terms were less than their upper bounds.

Robustness: Because these problem instances are randomly generated, they offer us an opportunity to test the robustness of our analytical characterization with respect to the assumptions we made. In particular, there are several cases where Assumptions A or B may not hold and p_{ll} is strictly positive in most instances.¹¹ We found that in 69.6% of instances, the client chooses the per-period fee f^* such that $f^* < c_l^v$, and in 79.6% of instances, the client chooses the fixed bonus b^* and stopping threshold x_s such that $R(x_s) - b^* > b^*$, i.e., her reward upon conclusion of the project is higher than the vendor’s bonus. These two observations show that Assumption B is often satisfied.

Although these assumptions do not hold in general, we observed that, in 97.2% of the instances, there was a time-independent stopping state threshold, demonstrating the robustness of Proposition

¹⁰ The same contract parameters would be selected if it were the vendor who offered a take-it-or-leave-it contract or if the contract parameters were negotiated through the Nash bargaining solution.

¹¹ In case there is no pure-strategy equilibrium in a particular period and state, we consider the unique mixed-strategy equilibrium.

Table 3 Loss of efficiency of the time-based contracts relative to the FB solution

	Average	25% quantiles	50% quantiles	75% quantiles	maximum
Loss of efficiency	7.65%	1.38%	4.96%	10.62%	45.15%

Table 4 Optimal time-based contract terms

Optimal contract terms	$b^* > 0, f^* > 0$	$b^* > 0, f^* = 0$	$b^* = 0, f^* > 0$
Percentage	64.4%	34%	1.6%
Optimal per-period fee	$f^* \leq c_l^v$	$c_l^v < f^* \leq c_h^v$	$f^* > c_h^v$
Percentage	69.6%	12.8%	17.6%
Optimal fixed bonus	$R(x_s) - b^* > b^*$	$R(x_s) - b^* = b^*$	$R(x_s) - b^* < b^*$
Percentage	79.6%	0%	20.4%

Table 5 Loss of efficiency of time-based contract with only a bonus or only a per-period fee.

	Average	25% quantiles	50% quantiles	75% quantiles	maximum
Loss of efficiency of $(\hat{b}, 0)$ contracts	7.84%	1.58%	5.29%	11.04%	45.16%
Loss of efficiency of $(0, \hat{f})$ contracts	26.60%	13.73%	23.74%	38.40%	72.97%

3. Naturally, due to the presence of moral hazard in the game, this stopping state threshold is often lower than that under the FB solution, as shown in Table 2.

Efficiency of Time-Based Contracts: Table 3 depicts the average, quartiles, and maximum loss of efficiency of the total surplus (which can be used as a proxy for the client's payoff) associated with time-based contracts, relative to the FB solution, with the contract parameters that solve (9). We observe that the median loss of efficiency is only 4.96%, and therefore that time-based contracts perform generally well, except for a few cases, relative to the FB solution. This is in contrast to static settings (which can be modeled by setting $T = 1$) where such simple contracts perform poorly (Roels et al. 2010). This is because, in dynamic settings, the time-based contracts provide the vendor with incentives to exert higher effort in the upper states of the project (the milestone effect), similar to the FB solution.

Optimal Contract Parameters: Table 4 summarizes the characteristics of the optimal contract terms (b^*, f^*) . Consistent with our discussion of Proposition 4, which suggested that client may wish to use a combination of both a bonus and a per-period fee to set the equilibrium stopping state threshold x_s close to the FB stopping threshold, we observe that in 64.4% of the instances the optimal contract is associated with a positive bonus and a positive per-period fee (i.e., $b^* > 0$ and $f^* > 0$). If the client must choose between one of the two levers, then it appears that the bonus is a better choice for two reasons. First, it is more often optimal: in 34% of the instances, the optimal contract has only the fixed bonus (i.e., $b^* > 0$ and $f^* = 0$), and in 1.6% of the instances, the optimal contract has only the per-period fee (i.e., $b^* = 0$ and $f^* > 0$). Second, it is more robust: Table 5, which measures the loss of efficiency of time-based contracts with either only a bonus or only a per-period fee, indeed reveals that the average loss of efficiency of contracts with only a

bonus (denoted as $(\hat{b}, 0)$) (7.84%) is much lower than the average loss of efficiency of contracts with only a per-period fee (denoted as $(0, \hat{f})$) (26.6%). Hence, if the client must choose between one of the two levers, she would be better off choosing a bonus over a per-period fee and optimizing it to maximize her payoff.

What explains the good performance of the contracts that consist only of a bonus? When the project's chances of success when the vendor exerts low effort are small (i.e., when p_{hl} and p_{ll} are low), it is beneficial to induce the milestone effect ((h, h) region) to accelerate progress. Because the stopping state threshold is higher under a contract that has only a bonus than under a contract that has only a per-period fee (Proposition 3), the milestone effect is more prevalent under the former than under the latter type of contract. Hence, when p_{hl} and p_{ll} are low, it is preferable to adopt a contract that has only a bonus than a contract that has only a per-period fee.

In summary, our numerical analysis reveals that our analytical characterization of the equilibrium work dynamics under time-based contracts in Section 5.1 is robust to our modeling assumptions. In addition, it shows that, despite their simplicity, time-based contracts perform generally well relative to the FB solution, because in dynamic settings, they provide the vendor with incentives to exert high effort in the upper states of the project (the milestone effect). In addition, the client is better off using a combination of both a bonus and a per-period fee. However, if she must choose between one of the two levers, contracts with only a bonus outperform contracts with only a per-period fee.

6. Conclusion

Many knowledge-intensive projects are iterative and stochastic, must be completed within a certain timeline, and involve some degree of co-production between a client and a vendor. In such projects, one key challenge is to estimate progress and design a contract that aligns the parties' incentives to collaborate while keeping the scope flexible. In this paper, we characterize the collaborative work dynamics between a client and a vendor in a finite-horizon and multi-state stochastic project with co-production and benefits for early conclusion. We consider two extreme cases: when vendor's efforts are verifiable, allowing the client to enforce the first-best solution, and when the vendor's efforts are unverifiable, leading to moral hazard.

When efforts are verifiable, we first establish the existence of a time-independent stopping state threshold, which we refer to as a completion milestone. We then identify two phases in which the parties should both exert high effort: when there is limited time left before the deadline, the "deadline effect", and when the project state has reached close to the completion state, the "milestone effect". Project managers should anticipate these end-of-project high-intensity effort phases, for otherwise they may underestimate the completion state or overestimate the completion time of the project.

Considering a situation where efforts are not verifiable and the project reward is not contractible, we characterize the work dynamics under a time-based contract consisting of a per-period fee and a fixed bonus. Similar to the FB solution, there exists a time-independent stopping state threshold under time-based contracts, and this threshold increases in the fixed bonus and decreases in the per-period fee. Hence, in order to achieve the FB stopping state threshold, a client may need to use a combination of both incentive mechanisms, and our numerical simulations suggest that it is indeed the case.

Moreover, time-based contracts give rise to a high-intensity effort phase in the higher states of the project, i.e., near the equilibrium stopping state threshold, but they fail to induce high efforts from the vendor near the deadline; that is, the milestone effect survives in equilibrium, but the deadline effect cannot be sustained in equilibrium when efforts are unverifiable. Despite their simplicity, our numerical studies indicate that such time-based contracts perform relatively well, and that contracts that involve only a bonus but no per-period fee tend to outperform contracts that involve only a per-period fee but no bonus, because it is typically under the former type of contracts that the milestone effect emerges in equilibrium.

This work can be extended in several directions to incorporate a greater degree of realism. First, one can generalize several modeling assumptions, e.g., continuous efforts, state- and time-dependent success probabilities. More fundamentally, this model can be used as a backbone model to address other managerially relevant questions:

- How different would the work dynamics be if the structure of the project (e.g., reward function, transition probabilities) is unknown at the outset, as often happens in Request-for-Proposal processes (Kieliszewski et al. 2010)? Preliminary analysis suggests that the parties may need to exert high effort in the early stages of the project, in contrast to the deadline and milestone effects, which happen in later stages, to learn about the structure of the project (Rahmani 2013).

- If the vendor has not been hired yet, when should the client offer the contract to the vendor? Although we have not formally modeled the hiring process, our analysis of time-based contracts suggests that early progress is important. Thus, it may be optimal to hire the vendor early to increase the chances of early completion. In contrast, when efforts are verifiable, the client may want to postpone the hiring decision until reaching one of the (deadline- or milestone-induced) high-intensity work phases.

- If the client has the opportunity to renegotiate the contract once the project has started, should the client take advantage of that renegotiation opportunity, and if so, when should it take place? Although we haven't formally modeled the renegotiation process, our analysis of time-based contracts suggests that, if progress has not been made early, the project has low chances of being completed before the deadline and of reaching a high completion state. Thus, in such

situations, there may be a point where the client would want to abort or renegotiate the contract to avoid having the project drag until the deadline at a low state.

Given the ubiquity of collaborative work in today's business processes and the need to adapt traditional process analysis tools to knowledge-based work (Karmarkar and Pitbladdo 1995, Hopp et al. 2009, Kieliszewski et al. 2010, Staats and Upton 2011, Rahmani et al. 2016), investigating these questions would be worthwhile.

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Appendix. Proofs

A. FB Solution

Preliminaries:

Throughout the proofs, we define $\mathbb{E}_d[V_{t+1}(x + \xi)] := p_d V_{t+1}(x + 1) + (1 - p_d)V_{t+1}(x)$ and $\mathbb{E}_d[R(x + \xi)] := p_d R(x + 1) + (1 - p_d)R(x)$, for decision $d \in \{ll, hh, hl, lh\}$. In addition, we normalize $p_{ll} = 0$.

The FB optimal policy can be identified by solving the following finite-horizon dynamic program:

$$\begin{aligned} V_t(x) &= \max \{V_t(x | (h, h)), V_t(x | (h, l)), V_t(x | (l, h)), V_t(x | (l, l)), V_t(x | Stop)\} \quad \text{for } t < T, \quad (\text{A-1}) \\ V_T(x) &= V_T(x | Stop), \end{aligned}$$

We first define the following thresholds:

$$\theta_{sl}^{FB}(x) \doteq \frac{(1 - \delta)R(x) + c_l^c + c_h^v}{\delta p_{lh}}, \theta_{sh}^{FB}(x) \doteq \frac{(1 - \delta)R(x) + c_h^c + c_h^v}{\delta p_{hh}}, \theta_{lh}^{FB} \doteq \frac{c_h^c - c_l^c}{\delta(p_{hh} - p_{lh})}. \quad (\text{A-2})$$

The thresholds $\theta_{mj}^{FB}(x)$ allow for a pairwise comparison of decisions m and j , for any $m \in \{s, l\}$, $j \in \{l, h\}$, $m \neq j$, where subscript ‘‘s’’ denotes *Stop*, subscript ‘‘l’’ denotes (l, h) , and subscript ‘‘h’’ denotes (h, h) . Specifically, decision m is preferred over decision j in period $T - 1$ if and only if $R(x + 1) - R(x) \leq \theta_{mj}^{FB}(x)$. It turns out that for any state x , either $\theta_{sl}^{FB}(x) \leq \theta_{sh}^{FB}(x) \leq \theta_{lh}^{FB}$ or $\theta_{lh}^{FB} \leq \theta_{sh}^{FB}(x) \leq \theta_{sl}^{FB}(x)$. Let

$$x_\theta^{FB} := \min \left\{ x \in \mathbb{Z}^+ \mid \theta_{sh}^{FB}(x) \geq \theta_{lh}^{FB} \right\} = \min \left\{ x \in \mathbb{Z}^+ \mid R(x) \geq \frac{(c_h^c - c_l^c)p_{hh} - (c_h^c + c_h^v)(p_{hh} - p_{lh})}{(1 - \delta)(p_{hh} - p_{lh})} \right\}. \quad (\text{A-3})$$

By the definition of x_θ^{FB} and because $R(x)$ is increasing, we have that,

$$\theta_{sl}^{FB}(x) < \theta_{sh}^{FB}(x) < \theta_{lh}^{FB} \text{ if } x < x_\theta^{FB} \text{ and } \theta_{sl}^{FB}(x) \geq \theta_{sh}^{FB}(x) \geq \theta_{lh}^{FB} \text{ if } x \geq x_\theta^{FB}. \quad (\text{A-4})$$

We also define

$$x_s^{FB} := \min \{x \in \mathbb{Z}^+ \mid \Delta R(x) \leq \min \{\theta_{sl}^{FB}(x), \theta_{sh}^{FB}(x)\}\}. \quad (\text{A-5})$$

For each state $x < \min\{x_\theta^{FB}, x_s^{FB}\}$, let us define

$$\tau(x) := \max \{t \in \mathcal{T} \mid \Delta V_{t+1}(x) \leq \theta_{lh}^{FB}\} \quad (\text{A-6})$$

if there exists a t such that $\Delta V_{t+1}(x) \leq \theta_{lh}^{FB}$; otherwise, $\tau(x) := x - 1$. We define

$$x_\varphi^{FB} := \max \{x < \min\{x_\theta^{FB}, x_s^{FB}\} \mid \tau(x) > \tau(x - 1)\}, \quad (\text{A-7})$$

such that $\tau(x)$ is nondecreasing for all x , $x_\varphi^{FB} \leq x < \min\{x_\theta^{FB}, x_s^{FB}\}$.

Finally, we denote, for any t , the highest state smaller than or equal to x_φ^{FB} where $\mathcal{E}_t^{FB}(x) = (h, h)$, as

$$x_{h,t}^{FB} := \min \left\{ \max \{x \in \mathbb{Z}^+ \mid \Delta V_{t+1}(x) \geq \theta_{lh}^{FB}\}, x_\varphi^{FB} \right\}. \quad (\text{A-8})$$

Proofs:

LEMMA A-1. For any x , $V_t(x)$ is nonincreasing in t .

Proof. We prove the lemma by induction on t . When $t = T - 1$, $V_T(x) = R(x) \leq V_{T-1}(x)$ by (1). Fix $t < T - 1$ and suppose that $V_{t+1}(x) \leq V_t(x)$, $\forall x$.

$$\begin{aligned} V_t(x) &= \max \{R(x), -c_l^c - c_l^v + \delta \mathbb{E}_{lu} [V_{t+1}(x + \xi)], -c_h^c - c_h^v + \delta \mathbb{E}_{hh} [V_{t+1}(x + \xi)], \\ &\quad -c_l^c - c_h^v + \delta \mathbb{E}_{lh} [V_{t+1}(x + \xi)], -c_h^c - c_l^v + \delta \mathbb{E}_{hl} [V_{t+1}(x + \xi)]\} \\ &\leq \max \{R(x), -c_l^c - c_l^v + \delta \mathbb{E}_{lu} [V_t(x + \xi)], -c_h^c - c_h^v + \delta \mathbb{E}_{hh} [V_t(x + \xi)], \\ &\quad -c_l^c - c_h^v + \delta \mathbb{E}_{lh} [V_t(x + \xi)], -c_h^c - c_l^v + \delta \mathbb{E}_{hl} [V_t(x + \xi)]\} = V_{t-1}(x). \quad \square \end{aligned}$$

LEMMA A-2. For any t , $V_t(x)$ is nondecreasing in x .

Proof. We prove the lemma by induction on t . When $t = T$, $V_T(x + 1) = R(x + 1) \geq R(x) = V_T(x)$, because by assumption $R(x)$ is increasing in x . Fix $t < T - 1$ and suppose that $V_{t+1}(x + 1) \geq V_{t+1}(x)$, $\forall x$. Hence, for any decision $d \in \{ll, hh, lh, hl\}$, $\mathbb{E}_d[V_t(x + 1 + \xi)] = p_d V_{t+1}(x + 2) + (1 - p_d)V_{t+1}(x + 1) \geq p_d V_{t+1}(x + 1) + (1 - p_d)V_{t+1}(x) = \mathbb{E}_d[V_t(x + \xi)]$. Then,

$$\begin{aligned} V_t(x + 1) &= \max \{R(x + 1), -c_l^c - c_l^v + \delta \mathbb{E}_{lu} [V_{t+1}(x + 1 + \xi)], -c_h^c - c_h^v + \delta \mathbb{E}_{hh} [V_{t+1}(x + 1 + \xi)], \\ &\quad -c_l^c - c_h^v + \delta \mathbb{E}_{lh} [V_{t+1}(x + 1 + \xi)], -c_h^c - c_l^v + \delta \mathbb{E}_{hl} [V_{t+1}(x + 1 + \xi)]\} \\ &\geq \max \{R(x), -c_l^c - c_l^v + \delta \mathbb{E}_{lu} [V_{t+1}(x + \xi)], -c_h^c - c_h^v + \delta \mathbb{E}_{hh} [V_{t+1}(x + \xi)], \\ &\quad -c_l^c - c_h^v + \delta \mathbb{E}_{lh} [V_{t+1}(x + \xi)], -c_h^c - c_l^v + \delta \mathbb{E}_{hl} [V_{t+1}(x + \xi)]\} = V_t(x). \quad \square \end{aligned}$$

LEMMA A-3. $\mathcal{E}_t^{FB}(x) \neq (l, l)$ and $\mathcal{E}_t^{FB}(x) \neq (h, l)$.

Proof. By Lemma A-1, $V_t(x)$ is non-increasing in t . Thus, because $p_{ll} = 0$, $V_t(x) \geq V_{t+1}(x) > -c_l^c - c_l^v + \delta V_{t+1}(x)$, which implies that $\mathcal{E}_t^{FB}(x) \neq (l, l)$.

By Lemma A-2, $V_t(x)$ is non-decreasing in x , and by assumption, $p_{hl} \leq p_{lh}$. Hence, $\mathbb{E}_{hl}[V_t(x + \xi)] \leq \mathbb{E}_{lh}[V_t(x + \xi)]$. Moreover, by assumption $c_h^c = k \cdot c_h^v$ and $c_l^c = k \cdot c_l^v$ with $k \geq 1$ and $c_h^v \geq c_l^v$. Hence, $V_t(x | (h, l)) = -c_h^c - c_l^v + \mathbb{E}_{hl}[V_t(x + \xi)] \leq -c_h^c - c_l^v + \delta \mathbb{E}_{lh}[V_t(x + \xi)] \leq -c_h^v - c_l^c + \delta \mathbb{E}_{lh}[V_t(x + \xi)] = V_t(x | (l, h))$, which implies that $\mathcal{E}_t^{FB}(x) \neq (h, l)$. \square

Proof of Proposition 1. The proof uses Lemma A-1 in this appendix. Using (A-1) and (A-2), we obtain $\mathcal{E}_{T-1}^{FB}(x) = Stop$ if and only if $\Delta R(x) \leq \min \{\theta_{sl}^{FB}(x), \theta_{sh}^{FB}(x)\}$. Because $\Delta R(x)$ is decreasing and both $\theta_{sl}^{FB}(x)$ and $\theta_{sh}^{FB}(x)$ are increasing, $\mathcal{E}_{T-1}^{FB}(x) = Stop$ if $x \geq x_s^{FB}$ and $\mathcal{E}_{T-1}^{FB}(x) \neq Stop$ if $x < x_s^{FB}$ by (A-5).

We next show by induction on t that if $\mathcal{E}_{T-1}^{FB}(x) = Stop$, then $\mathcal{E}_t^{FB}(x + k) = Stop \forall k \geq 0$ and $t \leq T - 1$. Fix $x \geq x_s^{FB}$. When $t = T - 1$, $\mathcal{E}_{T-1}^{FB}(x + k) = Stop \forall k \geq 0$. For any $t < T - 1$, suppose that $\mathcal{E}_{t+1}^{FB}(x + k) = Stop \forall k \geq 0$. Applying the conditions $V_{t+1}(x) = R(x)$ and $V_{t+1}(x + 1) = R(x + 1)$ to (A-1) shows that $V_t(x) = V_{T-1}(x) = R(x)$, i.e., $\mathcal{E}_t^{FB}(x) = Stop$, completing the induction step.

We next show the converse: if $\mathcal{E}_t^{FB}(x) = Stop$ for some t , then $\mathcal{E}_{T-1}^{FB}(x) = Stop$. By Lemma A-1 and (A-1), we have $R(x) = V_t(x) \geq V_{T-1}(x) \geq R(x)$; therefore, $V_{T-1}(x) = R(x)$. \square

LEMMA A-4. If $x_\theta^{FB} < x_s^{FB}$, $\mathcal{E}_t^{FB}(x) = (h, h)$ for all $x_\theta^{FB} \leq x < x_s^{FB}$ and all t .

Proof. The proof uses Lemma A-1 in this appendix. Because $x_s^{FB} - 1 \geq x_\theta^{FB}$, $\theta_{lh}^{FB} \leq \theta_{sh}^{FB}(x_s^{FB} - 1) \leq \theta_{sl}^{FB}(x_s^{FB} - 1)$ by (A-4). Therefore for any $x < x_s^{FB}$, $\Delta R(x) \geq \Delta R(x_s^{FB} - 1) \geq \min\{\theta_{sh}^{FB}(x_s^{FB} - 1), \theta_{sl}^{FB}(x_s^{FB} - 1)\} \geq \theta_{lh}^{FB}$ given that $R(x)$ is concave in x and by (A-5). As a result, $\mathcal{E}_{T-1}^{FB}(x) = (h, h)$ for all $x < x_s^{FB}$ by (8) and (A-2).

Fix x , $x_\theta^{FB} \leq x \leq x_s^{FB}$ and $t < T - 1$. Suppose that $\mathcal{E}_{t+1}^{FB}(x) = (h, h)$, i.e., $V_{t+1}(x) = -c_h^c - c_h^v + \delta \mathbb{E}_{hh}[V_{t+2}(x + \xi)]$. By (A-1), $V_{t+1}(x + 1) \geq R(x + 1)$ and by Lemma A-1, $V_{t+1}(x + 1) \geq V_{t+2}(x + 1)$. Hence, $V_{t+1}(x + 1) \geq (1 - \delta)R(x + 1) + \delta V_{t+2}(x + 1)$, and $\Delta V_{t+1}(x) \geq (1 - \delta)R(x + 1) + \delta V_{t+2}(x + 1) + c_h^c + c_h^v - \delta \mathbb{E}_{hh}[V_{t+2}(x + \xi)] = (1 - \delta)R(x + 1) + \delta(1 - p_{hh})\Delta V_{t+2}(x) + c_h^c + c_h^v$. Because $\mathcal{E}_{t+1}^{FB}(x) = (h, h)$, $\Delta V_{t+2}(x) \geq \theta_{lh}^{FB}$ by (8) and (A-2). Similarly, because $\mathcal{E}_{T-1}^{FB}(x) = (h, h)$, $\Delta V_T(x) = \Delta R(x) \geq \theta_{lh}^{FB}$. Combining these results yields the following lower bound: $\Delta V_{t+1}(x) \geq (1 - \delta)R(x + 1) + \delta(1 - p_{hh})\theta_{lh}^{FB} + c_h^c + c_h^v \geq (1 - \delta)R(x) + (1 - \delta p_{hh})\theta_{lh}^{FB} + c_h^c + c_h^v$. Finally, because $x \geq x_\theta^{FB}$, $\theta_{sh}^{FB}(x) \geq \theta_{lh}^{FB}$ by (A-4), i.e., $(1 - \delta)R(x) + c_h^c + c_h^v \geq \delta p_{hh}\theta_{lh}^{FB}$. As a result, $\Delta V_{t+1}(x) \geq \theta_{lh}^{FB}$. By (8), $\mathcal{E}_t^{FB}(x) = (h, h)$. \square

LEMMA A-5. If $\mathcal{E}_{t+1}^{FB}(x + 1) = \mathcal{E}_{t+1}^{FB}(x) = (l, h)$, then $\mathcal{E}_t^{FB}(x) = (l, h)$.

Proof. Because $\mathcal{E}_{t+1}^{FB}(x) = \mathcal{E}_{t+1}^{FB}(x + 1) = (l, h)$, $\Delta V_{t+2}(x) \leq \theta_{lh}^{FB}$ and $\Delta V_{t+2}(x + 1) \leq \theta_{lh}^{FB}$ by (8) and (A-2). Moreover, $\Delta V_{t+1}(x) = -c_l^c - c_h^v + \delta \mathbb{E}_{lh}[V_{t+2}(x + 1 + \xi)] + c_l^c + c_h^v - \delta \mathbb{E}_{lh}[V_{t+2}(x + \xi)] = \delta p_{lh}\Delta V_{t+2}(x + 1) + \delta(1 - p_{lh})\Delta V_{t+2}(x)$. As a result, $\Delta V_{t+1}(x) \leq \delta p_{lh}\theta_{lh}^{FB} + \delta(1 - p_{lh})\theta_{lh}^{FB} \leq \theta_{lh}^{FB}$, and $\mathcal{E}_t^{FB}(x) = (l, h)$ by (8) and (A-2). \square

LEMMA A-6. If $\mathcal{E}_{t+1}^{FB}(x + 1) = \text{Stop}$ and $\mathcal{E}_{t+1}^{FB}(x) = (l, h)$, then $\mathcal{E}_t^{FB}(x) = (l, h)$.

Proof. The proof uses Lemma A-1 in this appendix. When $\mathcal{E}_{t+1}^{FB}(x + 1) = \text{Stop}$, $V_{t+1}(x + 1) = V_{t+2}(x + 1) = R(x + 1)$ by Proposition 1. By Lemma A-1, $V_{t+1}(x) \geq V_{t+2}(x)$. Moreover, because $\mathcal{E}_{t+1}^{FB}(x) = (l, h)$, $\Delta V_{t+2}(x) \leq \theta_{lh}^{FB}$ by (8) and (A-2). Hence, $\Delta V_{t+1}(x) = R(x + 1) - V_{t+1}(x) \leq R(x + 1) - V_{t+2}(x) = \Delta V_{t+2}(x) \leq \theta_{lh}^{FB}$, and $\mathcal{E}_t^{FB}(x) = (l, h)$ by (8). \square

LEMMA A-7. Fix \hat{t} , \tilde{t} and \hat{x} , with $\hat{t} < \tilde{t} < T - 1$. Suppose that $\mathcal{E}_t^{FB}(x) = (h, h)$ for all $t \geq \hat{t}$, $\hat{x} < x < x_s^{FB}$, $\mathcal{E}_t^{FB}(\hat{x}) = (l, h)$ for all $\hat{t} \leq t \leq \tilde{t}$ and $\mathcal{E}_t^{FB}(\hat{x}) = (h, h)$ for all $t > \tilde{t}$. Then, for all $t \geq \hat{t}$ and all $x \geq \hat{x}$, $V_t(x)$ has increasing differences in (x, t) , i.e., $\Delta V_t(x) \leq \Delta V_{t+1}(x)$.

Proof. The proof uses Lemma A-1 in this appendix. Consider first (x, t) such that $t \geq \hat{t}$ and $x > \hat{x}$. By Proposition 1, $\mathcal{E}_t^{FB}(x) = \text{Stop}$ for all $x \geq x_s^{FB}$. Because $\mathcal{E}_t^{FB}(x) = (h, h)$ for all $t \geq \hat{t}$, $\hat{x} < x < x_s^{FB}$, we thus need to consider three cases: (i) $\mathcal{E}_t^{FB}(x + 1) = \mathcal{E}_t^{FB}(x) = \text{Stop}$, (ii) $\mathcal{E}_t^{FB}(x + 1) = \text{Stop}$ and $\mathcal{E}_t^{FB}(x) = (h, h)$, and (iii) $\mathcal{E}_t^{FB}(x + 1) = \mathcal{E}_t^{FB}(x) = (h, h)$. We prove the results by induction on t . Consider first period $T - 1$. (i)-(ii) $\Delta V_{T-1}(x) = R(x + 1) - V_{T-1}(x) = V_T(x + 1) - V_{T-1}(x) \leq \Delta V_T(x)$ by Proposition 1 and Lemma A-1.

(iii) By concavity of $R(x)$, $\Delta V_{T-1}(x) = -c_h^c - c_h^v + \delta \mathbb{E}_{hh}[R(x + 1 + \xi)] + c_h^c + c_h^v - \delta \mathbb{E}_{hh}[R(x + \xi)] = \delta p_{hh}\Delta R(x + 1) + \delta(1 - p_{hh})\Delta R(x) \leq \delta p_{hh}\Delta R(x) + \delta(1 - p_{hh})\Delta R(x) = \delta \Delta R(x) \leq \Delta V_T(x)$.

Inductively applying the same argument as in period $T - 1$ shows that $\Delta V_t(x) \leq \Delta V_{t+1}(x)$, completing the induction step. A similar argument applies to all (x, t) such that $x = \hat{x}$ and $t > \tilde{t}$.

Consider next (x, t) such that $x = \hat{x}$ and $\tilde{t} \geq t \geq \hat{t}$. To initialize the proof, we first consider time \tilde{t} . Using (1) we obtain: $\Delta V_{\tilde{t}}(\hat{x}) \leq V_{\tilde{t}}(\hat{x} + 1 | (h, h)) - V_{\tilde{t}}(\hat{x} | (h, h)) = \delta p_{hh}\Delta V_{\tilde{t}+1}(\hat{x} + 1) + \delta(1 - p_{hh})\Delta V_{\tilde{t}+1}(\hat{x}) \leq$

$\delta p_{hh} \Delta V_{\hat{t}+2}(\hat{x} + 1) + \delta(1 - p_{hh}) \Delta V_{\hat{t}+2}(\hat{x}) \leq \Delta V_{\hat{t}+1}(\hat{x})$, in which the second inequality holds by the induction hypothesis and the first inequality holds because $V_{\hat{t}}(\hat{x}) \geq V_{\hat{t}}(\hat{x} | (h, h))$ by (A-1).

Consider next any time $t, \hat{t} \leq t < \tilde{t}$. Because $\mathcal{E}_t^{FB}(\hat{x}) = \mathcal{E}_{\hat{t}+1}^{FB}(\hat{x}) = (l, h)$ and $\mathcal{E}_t^{FB}(\hat{x} + 1) = \mathcal{E}_{\hat{t}+1}^{FB}(\hat{x} + 1) = (h, h)$, $\Delta V_t(\hat{x}) = c_l^c + c_h^v - c_h^c - c_h^v + \delta p_{hh} \Delta V_{\hat{t}+1}(\hat{x} + 1) + \delta(1 - p_{lh}) \Delta V_{\hat{t}+1}(\hat{x}) \leq c_l^c + c_h^v - c_h^c - c_h^v + \delta p_{hh} \Delta V_{\hat{t}+2}(\hat{x} + 1) + \delta(1 - p_{lh}) \Delta V_{\hat{t}+2}(\hat{x}) = \Delta V_{\hat{t}+1}(\hat{x})$, in which the inequality follows by the induction hypothesis. This completes the induction step. \square

LEMMA A-8. For all $x_\varphi^{FB} \leq x < \min\{x_\theta^{FB}, x_s^{FB}\}$, there exists a time threshold $\tau(x)$ defined by (A-6), nonincreasing in x , such that $\mathcal{E}_t^{FB}(x) = (h, h)$ for all $t > \tau(x)$ and $\mathcal{E}_t^{FB}(x) = (l, h)$ for all $x \leq t \leq \tau(x)$.

Proof. The proof uses Lemmas A-4, A-5, A-6, and A-7 in this appendix. Suppose first that $x_\theta^{FB} > x_s^{FB} - 1$. In that case, by (A-4) and (A-5), $\Delta R(x_s^{FB} - 1) > \min\{\theta_{sl}^{FB}(x_s^{FB} - 1), \theta_{sh}^{FB}(x_s^{FB} - 1)\} = \theta_{sl}^{FB}(x_s^{FB} - 1)$, and therefore $\mathcal{E}_{T-1}^{FB}(x_s^{FB} - 1) = (l, h)$ by (8). Hence, $\Delta V_{T-1}(x_s^{FB} - 1) \leq \theta_{lh}^{FB}$, i.e., $\tau(x_s^{FB} - 1) = T - 1$. Since, $\tau(x) \leq \tau(x - 1)$ for all $x \geq x_\varphi^{FB}$, we thus obtain that $\tau(x) = T - 1$ for all $x, x_\varphi^{FB} \leq x < x_s^{FB}$. Then, $\mathcal{E}_t^{FB}(x_s^{FB} - 1) = (l, h)$ for all $t \leq T - 1$ by Lemma A-6 and $\mathcal{E}_t^{FB}(x) = (l, h)$ for all t and all $x_\varphi^{FB} \leq x < x_s^{FB}$ by Lemma A-5.

Suppose next that $x_\theta^{FB} \leq x_s^{FB} - 1$. Fix $\hat{x}, x_\varphi^{FB} \leq \hat{x} < x_\theta^{FB}$, and consider first $\hat{t} > \tau(\hat{x} + 1)$. Because $\hat{t} > \tau(\hat{x} + 1)$ and $\tau(x)$ is nonincreasing when $x \geq x_\varphi^{FB}$ by (A-7), $\hat{t} > \tau(x)$ for all $\hat{x} + 1 \leq x \leq x_\theta^{FB}$. By Proposition 1, $\mathcal{E}_t^{FB}(x) \neq \text{Stop}$ for all $x < x_s^{FB}$; therefore, by (8), (A-2) and (A-6), $\mathcal{E}_t^{FB}(x) = (h, h)$ for all $t \geq \hat{t}$ and $\hat{x} < x < x_\theta^{FB}$. Moreover, by Lemma A-4, $\mathcal{E}_t^{FB}(x) = (h, h)$ for all t and all $x_\theta^{FB} \leq x < x_s^{FB}$. Thus, $\mathcal{E}_t^{FB}(x) = (h, h)$ for all $t \geq \hat{t}$ and $\hat{x} < x < x_s^{FB}$. Also, by definition of $\tau(\hat{x})$, $\mathcal{E}_t^{FB}(\hat{x}) = (h, h)$ for all $t > \tau(\hat{x})$ and $\mathcal{E}_{\tau(\hat{x})}^{FB}(\hat{x}) = (l, h)$ by (A-6) and (8).

Fix $\hat{t}, \tau(\hat{x}) \geq \hat{t} > \tau(\hat{x} + 1)$. Suppose that $\mathcal{E}_t^{FB}(\hat{x}) = (l, h)$ for all $\hat{t} \leq t \leq \tau(\hat{x})$; we will show that $\mathcal{E}_{\hat{t}-1}^{FB}(\hat{x}) = (l, h)$. By Lemma A-7, $\Delta V_{\hat{t}}(\hat{x}) \leq \Delta V_{\hat{t}+1}(\hat{x})$. Because $\mathcal{E}_{\hat{t}}^{FB}(\hat{x}) = (l, h)$ by assumption, $\Delta V_{\hat{t}+1}(\hat{x}) \leq \theta_{lh}^{FB}$ by (8); therefore, $\Delta V_{\hat{t}}(\hat{x}) \leq \theta_{lh}^{FB}$. This implies that $\mathcal{E}_{\hat{t}-1}^{FB}(\hat{x}) = (l, h)$ by (8). As a result, for any $x_\varphi^{FB} \leq x < x_\theta^{FB}$, $\mathcal{E}_t^{FB}(x) = (l, h)$ for all $\tau(x + 1) \leq t \leq \tau(x)$.

Consider next $\hat{t} \leq \tau(\hat{x} + 1) - 1$. The proof proceeds by induction. Fix \hat{x} and suppose that $\mathcal{E}_t^{FB}(\hat{x} + 1) = (l, h)$ for all $t \leq \tau(\hat{x} + 1)$ and $\mathcal{E}_t^{FB}(\hat{x} + 1) = (h, h)$ for all $t > \tau(\hat{x} + 1)$. By the previous argument, $\mathcal{E}_t^{FB}(\hat{x}) = (l, h)$ for all $\tau(\hat{x} + 1) \leq t \leq \tau(\hat{x})$. Applying Lemma A-5 yields that $\mathcal{E}_t^{FB}(\hat{x}) = (l, h)$ for all $t \leq \tau(\hat{x} + 1) - 1$. \square

LEMMA A-9. For any $t, x_{h,t}^{FB}$, defined in (A-8), is nondecreasing in t , i.e., $x_{h,t}^{FB} \leq x_{d,t+1}^{FB}$.

Proof. The proof uses Lemmas A-4, A-5, A-6, and A-8 in this appendix. By (A-8) and (8) and Proposition 1, $\mathcal{E}_t^{FB}(x) = (l, h)$ for all $x_{h,t}^{FB} < x \leq x_\varphi^{FB}$. Suppose for contradiction that $x_{h,t}^{FB} > x_{d,t+1}^{FB}$ for some t . Then, there must exist some $x, x_{h,t}^{FB} \geq x > x_{d,t+1}^{FB}$, such that $\mathcal{E}_t^{FB}(x) = (h, h)$ and $\mathcal{E}_{t+1}^{FB}(x) = (l, h)$. If either $\mathcal{E}_{t+1}^{FB}(x + 1) = (l, h)$ or $\mathcal{E}_{t+1}^{FB}(x + 1) = \text{Stop}$, we should then have had $\mathcal{E}_t^{FB}(x) = (l, h)$ by Lemmas A-5 and A-6, a contradiction. Hence, we must have $\mathcal{E}_{t+1}^{FB}(x + 1) = (h, h)$, which can only happen when $x + 1 = x_{h,t}^{FB} + 1 = x_\varphi^{FB} + 1$. Moreover, given that $\mathcal{E}_{t+1}^{FB}(x) = (l, h)$, $x < x_s^{FB}$ by Proposition 1 and $x < x_\theta^{FB}$ by Lemma A-4. By Lemma A-8, we should then have had $\mathcal{E}_t^{FB}(x) = (l, h)$ given that $\mathcal{E}_{t+1}^{FB}(x) = (l, h)$, a contradiction. \square

Proof of Proposition 2. (i.a) The proof uses Lemma A-6 and A-8 in this appendix. Given that $\Delta R(x_s^{FB} - 1) \leq \frac{c_h^c - c_l^c}{\delta(p_{hh} - p_{lh})} = \theta_{lh}^{FB}$, $\mathcal{E}_{T-1}^{FB}(x_s^{FB} - 1) = (l, h)$ by (8). By Proposition 1 and Lemma A-6, $\mathcal{E}_t^{FB}(x_s^{FB} - 1) = (l, h)$ for all $t < T$. Because $\tau(x)$, defined in (A-6), is nonincreasing when $x_\varphi^{FB} \leq x < x_s^{FB}$ by (A-4),

$\tau(x) \geq \tau(x_s^{FB} - 1) = T - 1$, i.e., $\tau(x) = T - 1$. Therefore, $\mathcal{E}_t^{FB}(x) = (l, h)$ for all $x_\varphi^{FB} \leq x < x_s^{FB}$ and for all $t < T$ by Lemma A-8.

(i.b) The proof uses Lemma A-9 in this appendix. Let $x_{h,t}^{FB}$ be defined as in (A-8) and $\tau(x)$ be defined as (A-6). By (i.a), $x_{h,t}^{FB} < x_\varphi^{FB}$ since $\mathcal{E}_t^{FB}(x_\varphi^{FB}) = (l, h) \forall t$.

We show next that $\Delta V_t(x-1) \geq \Delta V_t(x)$ for all $x \leq x_{h,t}^{FB}$, and that $\mathcal{E}_t^{FB}(x) = (h, h)$ for all $x \leq x_{h,t}^{FB}$. When $t = T$, $V_t(x)$ is concave because $V_T(x) = R(x)$, which is concave by assumption. In period $T - 1$, $\mathcal{E}_{T-1}^{FB}(x) = (h, h)$ for $x < x_{h,T-1}^{FB} + 1$ by (A-8) and (8). Thus, $V_{T-1}(x) = -c_h^e - c_h^v + \delta p_{hh} R(x+1) + \delta(1 - p_{hh})R(x)$, which is concave in x for $x < x_{h,T-1}^{FB} + 1$. By (i.a), $\mathcal{E}_{T-1}^{FB}(x_\varphi^{FB}) = (l, h)$ and therefore $x_\varphi^{FB} = x_{h,T-1}^{FB} + 1$ by (A-6). Hence, $V_{T-1}(x)$ is concave in x for $x < x_\varphi^{FB}$.

Fix $t < T - 1$ and suppose that $\Delta V_{t+2}(x-1) \geq \Delta V_{t+2}(x)$, $\forall 0 < x \leq x_{h,t+2}^{FB}$. Because $x_{h,t+1}^{FB} \leq x_{h,t+2}^{FB}$ by Lemma A-9, $\Delta V_{t+2}(x) \geq \Delta V_{t+2}(x_{h,t+1}^{FB})$ for all $x \leq x_{h,t+1}^{FB}$. Moreover, by (A-8), $\Delta V_{t+2}(x_{h,t+1}^{FB}) \geq \theta_{lh}^{FB}$. As a result, $\Delta V_{t+2}(x) \geq \theta_{lh}^{FB}$ for all $x \leq x_{h,t+1}^{FB}$, which, by (8), implies that $\mathcal{E}_{t+1}^{FB}(x) = (h, h)$ for all $x \leq x_{h,t+1}^{FB}$.

We next show that $\Delta V_{t+1}(x-1) \geq \Delta V_{t+1}(x)$. For any x , $0 < x \leq x_{h,t}^{FB}$, suppose that $\mathcal{E}_t^{FB}(x) = (h, h)$. By Lemma A-9, $x \leq x_{h,t+1}^{FB}$ and therefore by the above, $\mathcal{E}_{t+1}^{FB}(x) = \mathcal{E}_{t+1}^{FB}(x-1) = (h, h)$. We consider the following three cases: (1) $\mathcal{E}_{t+1}^{FB}(x+1) = (l, h)$, (2) $\mathcal{E}_{t+1}^{FB}(x+1) = Stop$, and (3) $\mathcal{E}_{t+1}^{FB}(x+1) = (h, h)$.

(1) When $\mathcal{E}_{t+1}^{FB}(x+1) = (l, h)$, using the induction hypothesis, given that $x \leq x_{h,t}^{FB} \leq x_{h,t+2}^{FB}$, Equation (2), and the identity $c_h^e - c_h^v = \delta(p_{hh} - p_{lh})\theta_{lh}^{FB}$ by (A-2), we obtain: $\Delta V_{t+1}(x-1) - \Delta V_{t+1}(x) = 2V_{t+1}(x | (h, h)) - V_{t+1}(x-1 | (h, h)) - V_{t+1}(x+1 | (l, h)) = -c_h^e + c_h^v + \delta(2p_{hh} - 1)\Delta V_{t+2}(x) + \delta(1 - p_{hh})\Delta V_{t+2}(x-1) - \delta p_{lh}\Delta V_{t+2}(x+1) \geq -c_h^e + c_h^v + \delta(2p_{hh} - 1 + 1 - p_{hh})\Delta V_{t+2}(x) - \delta p_{lh}\Delta V_{t+2}(x+1) \geq -\delta(p_{hh} - p_{lh})\theta_{lh}^{FB} + \delta(p_{hh} - p_{lh})\theta_{lh}^{FB} = 0$.

(2) When $\mathcal{E}_{t+1}^{FB}(x+1) = Stop$, using the induction hypothesis, given that $x \leq x_{h,t}^{FB} \leq x_{h,t+2}^{FB}$, Equation (2), the inequality $\theta_{lh}^{FB} > \theta_{sh}^{FB}(x+1)$ obtained from (A-4), and given that $x \leq x_{h,t}^{FB} < x_\varphi^{FB} < x_\theta^{FB}$, we obtain: $\Delta V_{t+1}(x-1) - \Delta V_{t+1}(x) = 2V_{t+1}(x | (h, h)) - V_{t+1}(x-1 | (h, h)) - V_{t+1}(x+1 | Stop) = -2(c_h^e + c_h^v) - (1 - \delta)R(x+1) + \delta(2p_{hh} - 1)[R(x+1) - V_{t+2}(x)] + \delta(1 - p_{hh})\Delta V_{t+2}(x-1) \geq -2(c_h^e + c_h^v) - (1 - \delta)R(x+1) + \delta p_{hh}[R(x+1) - V_{t+2}(x)] \geq -\delta p_{hh}\theta_{lh}^{FB} + \delta p_{hh}\theta_{lh}^{FB} = 0$.

(3) When $\mathcal{E}_{t+1}^{FB}(x+1) = (h, h)$, we must have that $x+1 \leq x_{h,t+2}^{FB}$. Suppose, for contradiction, that $x+1 > x_{h,t+2}^{FB}$. Because $x \leq x_{h,t}^{FB} \leq x_{h,t+1}^{FB} \leq x_{h,t+2}^{FB}$ by Lemma A-9, we must have that $x = x_{h,t+1}^{FB}$ and therefore $x = x_\varphi^{FB}$ by definition of $x_{h,t+1}^{FB}$ given that $\Delta V_{t+2}(x+1) \geq \theta_{lh}^{FB}$ by (2). By (i.a), we must then have $\mathcal{E}_t^{FB}(x) = (l, h)$, a contradiction. Hence, $x+1 \leq x_{h,t+2}^{FB}$, and one can apply twice the induction hypothesis to obtain: $\Delta V_{t+1}(x-1) - \Delta V_{t+1}(x) = 2V_{t+1}(x | (h, h)) - V_{t+1}(x-1 | (h, h)) - V_{t+1}(x+1 | (h, h)) = \delta p_{hh} [\Delta V_{t+2}(x) - \Delta V_{t+2}(x+1)] + \delta(1 - p_{hh}) [\Delta V_{t+2}(x-1) - \Delta V_{t+2}(x)] \geq 0$.

As a result, $\Delta V_{t+1}(x-1) \geq \Delta V_{t+1}(x)$ for all $x \leq x_{h,t}^{FB}$. Therefore if $\mathcal{E}_t^{FB}(x) = (h, h)$, i.e., $\Delta V_{t+1}(x) \geq \theta_{lh}^{FB}$ by (2), then $\Delta V_{t+1}(x-1) \geq \theta_{lh}^{FB}$, i.e., $\mathcal{E}_t^{FB}(x-1) = (h, h)$. Because $x_{h,t}^{FB}$ is nondecreasing in t by Lemma A-9, this also implies that if $\mathcal{E}_t^{FB}(x) = (h, h)$, i.e., if $x \leq x_{h,t}^{FB}$, then $x \leq x_{h,t+1}^{FB}$, i.e., $\mathcal{E}_{t+1}^{FB}(x) = (h, h)$. That is, $\tau(x)$ is nondecreasing in x for all $x \leq x_\varphi^{FB}$.

(ii.a) This is shown in Lemma A-4 in this appendix.

(ii.b) This is shown in Lemma A-8 in this appendix.

(ii.c) In order to show that there exist non-decreasing time thresholds $\tau(x)$ such that $\mathcal{E}_t^{FB}(x) = (h, h)$ for all $t \geq \tau(x)$, we show that there exist non-decreasing state thresholds $\bar{x}_{h,t}$ such that $\mathcal{E}_t^{FB}(x) = (h, h)$ for all $x \leq \bar{x}_{h,t}$. We define $\bar{x}_{h,t} = \max\{x \leq \bar{x}_{h,t+1} - 1 \mid \mathcal{E}_t^{FB}(x) = (h, h)\}$. The proof proceeds by induction on t . Because $\Delta R(x_s^{FB} - 1) > \theta_{lh}^{FB}$, $\mathcal{E}_{T-1}^{FB}(x_s^{FB} - 1) = (h, h)$ by (8). Because $\Delta R(x) \geq \Delta R(x_s^{FB} - 1) > \theta_{lh}^{FB}$ for all $1 \leq x < x_s^{FB}$, $\mathcal{E}_{T-1}^{FB}(x) = (h, h)$ for all $1 \leq x < x_s^{FB}$. Hence, $\bar{x}_{h,T-1} = x_s^{FB} - 1$ and $\Delta V_{T-1}(x)$ is nonincreasing in x for all $1 \leq x < x_s^{FB} - 1$ because $\Delta V_{T-1}(x) = \delta p_{hh} \Delta R(x+1) + \delta(1-p_{hh}) \Delta R(x) \leq \delta p_{hh} \Delta R(x) + \delta(1-p_{hh}) \Delta R(x-1) = \Delta V_{T-1}(x-1)$. Therefore, if $\mathcal{E}_{T-2}^{FB}(\bar{x}_{h,T-2}) = (h, h)$, then for all $x < \bar{x}_{h,T-2}$, $\Delta V_{T-1}(x) \geq \Delta V_{T-1}(\bar{x}_{h,T-2}) \geq \theta_{lh}^{FB}$, and thus $\mathcal{E}_{T-2}^{FB}(x) = (h, h)$ for all $x < \bar{x}_{h,T-2}$.

Fix $t < T - 1$ and $x = \bar{x}_{h,t}^{FB}$. As an induction hypothesis, suppose that $\mathcal{E}_\tau^{FB}(\xi) = (h, h)$ for all $\xi \leq \bar{x}_{h,\tau}^{FB}$ and $\tau > t$, and that $\Delta V_t(\xi)$ is nonincreasing in ξ for all $\tau > t$ and $\xi < \bar{x}_{h,\tau}^{FB}$. If $\mathcal{E}_t^{FB}(\bar{x}_{h,t}^{FB}) = (h, h)$, then for all $x < \bar{x}_{h,t}^{FB}$, $\Delta V_t(x) \geq \Delta V_t(\bar{x}_{h,t}^{FB}) \geq \theta_{lh}^{FB}$ and therefore $\mathcal{E}_t^{FB}(x) = (h, h)$. To complete the induction step, we show that $\Delta V_t(x)$ is nonincreasing in x for all $x < \bar{x}_{h,t}^{FB}$. Because $\Delta V_{t+1}(\xi)$ is nonincreasing in ξ for all $\xi < \bar{x}_{h,t+1}^{FB}$ and because $\bar{x}_{h,t}^{FB} < \bar{x}_{h,t+1}^{FB}$, we obtain for any $x < \bar{x}_{h,t}^{FB}$, $\Delta V_t(x) = V_t(x+1 \mid (h, h)) - V_t(x \mid (h, h)) = \delta p_{hh} \Delta V_{t+1}(x+1) + \delta(1-p_{hh}) \Delta V_{t+1}(x) \leq \delta p_{hh} \Delta V_{t+1}(x) + \delta(1-p_{hh}) \Delta V_{t+1}(x-1) = V_t(x \mid (h, h)) - V_t(x-1 \mid (h, h)) = \Delta V_t(x-1)$, completing the induction step.

B. Unverifiable Efforts

Preliminaries

Throughout the proofs, we define $\mathbb{E}_d[V_{t+1}^i(x + \xi)] := p_d V_{t+1}^i(x + 1) + (1 - p_d) V_{t+1}^i(x)$ and $\mathbb{E}_d[R(x + \xi)] := p_d R(x + 1) + (1 - p_d) R(x)$, for $i \in \{c, p\}$ and decisions $d \in \{ll, hh, lh, hl\}$. In addition, we normalize $p_{ll} = 0$.

We first define the following thresholds:

$$\theta_{lh}^c := \frac{c_h^c - c_l^c}{\delta(p_{hh} - p_{lh})}, \theta_{lh}^v := \frac{c_h^v - c_l^v}{\delta(p_{hh} - p_{hl})}, \theta_{ll}^c := \frac{c_h^c - c_l^c}{\delta p_{hl}}, \theta_{ll}^v := \frac{c_h^v - c_l^v}{\delta p_{lh}}. \quad (\text{B-1})$$

The thresholds θ_{ij}^i allow for comparison of party i 's decision l over h when party $-i$'s decision is $j \in \{l, h\}$, for $i \in \{c, p\}$. For instance, in period $T - 1$, the client prefers (l, h) over (h, h) if and only if $\Delta R(x) \leq \theta_{lh}^c$; she prefers (l, l) over (h, l) if and only if $\Delta R(x) \leq \theta_{ll}^c$.

At the beginning of each period, the client decides whether to stop or continue the project. Thus, at the beginning of each period t , in every state x , the client compares her payoff if she chooses *Stop* with the possible equilibrium outcomes if she chooses to continue the project. We define

$$\theta_{sl}(x) := \frac{c_h^c + f + (1 - \delta)[R(x) - b]}{\delta p_{hl}} \quad (\text{B-2})$$

In particular, in period $T - 1$, using the table in Figure 3, we obtain that the client prefers *Stop* over (h, l) if and only if $\Delta R(x) \leq \theta_{sl}(x)$. Accordingly, let us define

$$x_s := \min \{x \in \mathbb{Z}^+ \mid \Delta R(x) \leq \theta_{sl}(x)\} \quad (\text{B-3})$$

Proofs:

Proof of Lemma 1. We show the result for any t and x in the 2×2 subgame depicted in the table in Figure 3. By assumption, $c_h^c = k \cdot c_h^v$ and $c_l^c = k \cdot c_l^v$ for $k \geq 1$. Thus by (B-1), when $p_{hl} = p_{lh}$ or $p_{hh} = p_{hl} + p_{lh}$, $\theta_{lh}^c / \theta_{lh}^v = \theta_{ll}^c / \theta_{ll}^v$.

By (B-1), $V_t^c(x | (l, h)) \geq V_t^c(x | (h, h))$ if and only if $\Delta V_{t+1}^c \leq \theta_{lh}^c$, $V_t^c(x | (l, l)) \geq V_t^c(x | (h, l))$ if and only if $\Delta V_{t+1}^c \leq \theta_{ll}^c$, $V_t^v(x | (h, l)) \geq V_t^v(x | (h, h))$ if and only if $\Delta V_{t+1}^v \leq \theta_{lh}^v$, and $V_t^v(x | (l, l)) \geq V_t^v(x | (l, h))$ if and only if $\Delta V_{t+1}^v \leq \theta_{ll}^v$. Suppose there exists no equilibria in pure strategies in period t and state x if the client has chosen to continue the project. Then, it must be one of the following two cases:

- (a) $\theta_{ll}^c < \Delta V_{t+1}^c(x) < \theta_{lh}^c$ (i.e., $k \cdot \theta_{ll}^v < \Delta V_{t+1}^c(x) < k \cdot \theta_{lh}^v$) and $\theta_{lh}^v < \Delta V_{t+1}^v(x) < \theta_{ll}^v$, a contradiction.
- (b) $\theta_{lh}^c < \Delta V_{t+1}^c(x) < \theta_{ll}^c$ (i.e., $k \cdot \theta_{lh}^v < \Delta V_{t+1}^c(x) < k \cdot \theta_{ll}^v$) and $\theta_{ll}^v < \Delta V_{t+1}^v(x) < \theta_{lh}^v$, a contradiction.

We next show that when conditions (i) $p_{hl} = p_{lh}$ or (ii) $p_{hh} = p_{hl} + p_{lh}$ do not hold, an equilibrium in pure strategies may not exist. Suppose $p_{hl} < p_{lh}$ or $p_{hh} \neq p_{hl} + p_{lh}$. Then, $\frac{\theta_{lh}^c}{\theta_{lh}^v} = \frac{k \cdot (p_{hh} - p_{hl})}{(p_{hh} - p_{lh})} \doteq \beta$ and $\frac{\theta_{ll}^c}{\theta_{ll}^v} = \frac{k \cdot p_{lh}}{p_{hl}} \doteq \alpha$ by (B-1). Then, there exists no equilibria in pure strategies in period t and state x if one of the following conditions hold:

- (a) If $\beta > \alpha$, $\theta_{lh}^v < \theta_{ll}^v < \left(\frac{\beta}{\alpha}\right) \cdot \theta_{lh}^v$, $\theta_{ll}^c < \Delta V_{t+1}^c(x) < \theta_{lh}^c$ (i.e., $\alpha \cdot \theta_{ll}^v < \Delta V_{t+1}^c(x) < \beta \cdot \theta_{lh}^v$), and $\theta_{lh}^v < \Delta V_{t+1}^v(x) < \theta_{ll}^v$.
- (b) If $\beta < \alpha$, $\theta_{ll}^v < \theta_{lh}^v < \left(\frac{\alpha}{\beta}\right) \cdot \theta_{ll}^v$, $\theta_{lh}^c < \Delta V_{t+1}^c(x) < \theta_{ll}^c$ (i.e., $\beta \cdot \theta_{lh}^v < \Delta V_{t+1}^c(x) < \alpha \cdot \theta_{ll}^v$), and $\theta_{ll}^v < \Delta V_{t+1}^v(x) < \theta_{lh}^v$. \square

LEMMA B-1. $\mathcal{E}_{T-1}(x) = (h, l)$ for all $x < x_s$ and $\mathcal{E}_{T-1}(x) = Stop$ for all $x \geq x_s$, where x_s is defined in (B-3).

Proof. Because, $V_{T-1}^v(x | (l, h)) = -c_h^v + f + \delta p_{lh}b + \delta(1 - p_{lh})b < -c_l^v + f + \delta p_{lh}b + \delta(1 - p_{lh})b = V_{T-1}^v(x | (l, l))$, $\mathcal{E}_{T-1}(x) \neq (l, h)$. Moreover, because $V_{T-1}^v(x | (h, h)) = -c_h^v + f + \delta p_{hh}b + \delta(1 - p_{hh})b < -c_l^v + f + \delta p_{hl}b + \delta(1 - p_{hl})b = V_{T-1}^v(x | (h, l))$, $\mathcal{E}_{T-1}(x) \neq (h, h)$. Hence, $\Delta V_{t+1}^v(x) < \theta_{ll}^v$ and $\Delta V_{t+1}^v(x) < \theta_{lh}^v$, which imply if the client chooses to continue the project in time $T - 1$ and any state x , an equilibrium in pure strategies exists. In addition, that equilibrium is (h, l) if $\Delta R(x) \geq \theta_{ll}^c$ and it is (l, l) otherwise. However, given that the client chooses whether to stop the project by (4) and that $p_{ll} = 0$, she always prefers *Stop* over (l, l) in period $T - 1$ because $R(x) - b > -c_l^c - f + \delta(R(x) - b)$. In addition, she prefers *Stop* over (h, l) if and only if $\Delta R(x) < \theta_{sl}(x)$. Thus, $\mathcal{E}_{T-1}(x) = Stop$ for $x \geq x_s$ and $\mathcal{E}_{T-1}(x) = (h, l)$ for $x < x_s$. \square

LEMMA B-2. Suppose an equilibrium in pure strategies exists in any period t and state x . Then, for any $t \in \mathcal{T}$, $\mathcal{E}_t(x) = Stop$ if and only if $x \geq x_s$.

Proof. The proof proceeds by induction on t and uses Lemma B-1 in this appendix. We first show that $\mathcal{E}_t(x) = Stop$ for all $x \geq x_s$.

Consider period $T - 1$. By Lemma B-1, $\mathcal{E}_{T-1}(x) = Stop$ for $x \geq x_s$ and $\mathcal{E}_{T-1}(x) = (h, l)$ for $x < x_s$. Fix $t < T - 1$. Suppose that $\mathcal{E}_{t+1}(x) = Stop$ for $\forall x \geq x_s$. Then for any $i \in \{c, p\}$ and $x \geq x_s$, $V_t^i(x | \mathcal{E}) = V_{T-1}^i(x | \mathcal{E})$ for $\mathcal{E} \in \{(h, h), (h, l), (l, h), (l, l), Stop\}$. As a result, $\mathcal{E}_t(x) = \mathcal{E}_{T-1}(x) = Stop$ for all $x \geq x_s$.

We next show that $\mathcal{E}_t(x) \neq Stop$ for any $x < x_s$. Consider first period $T - 1$. By Lemma B-1, $\mathcal{E}_{T-1}(x) = (h, l)$ for all $x < x_s$, which implies $V_{T-1}^c(x) > R(x) - b$ for all $x < x_s$.

Fix $t < T - 1$. By (4), $V_{t+1}^c(x) \geq R(x) - b$ for all x . Then,

- (i) If $\mathcal{E}_t(x) = (h, l)$, $V_t^c(x) = -c_h^c - f + \delta p_{hl} V_{t+1}^c(x+1) + \delta(1-p_{hl})V_{t+1}^c(x) \geq -c_h^c - f + \delta p_{hl}(R(x+1) - b) + \delta(1-p_{hl})(R(x) - b) = V_{T-1}^c(x) > R(x) - b$.
- (ii) If $\mathcal{E}_t(x) = (h, h)$, $V_t^c(x | (h, h)) > V_t^c(x | (l, h)) > V_t^c(x | (h, l)) = -c_h^c - f + \delta p_{hl} V_{t+1}^c(x+1) + \delta(1-p_{hl})V_{t+1}^c(x) \geq -c_h^c - f + \delta p_{hl}(R(x+1) - b) + \delta(1-p_{hl})(R(x) - b) = V_{T-1}^c(x) > R(x) - b$, in which the second inequality holds because $c_h^c > c_l^c$ and $p_{lh} \geq p_{hl}$.
- (iii) If $\mathcal{E}_t(x) = (l, h)$, $V_t^c(x | (l, h)) > V_t^c(x | (h, l)) = -c_h^c - f + \delta p_{hl} V_{t+1}^c(x+1) + \delta(1-p_{hl})V_{t+1}^c(x) \geq -c_h^c - f + \delta p_{hl}(R(x+1) - b) + \delta(1-p_{hl})(R(x) - b) = V_{T-1}^c(x) > R(x) - b$, in which the first inequality holds because $c_h^c > c_l^c$ and $p_{lh} \geq p_{hl}$.
- (iv) If $\mathcal{E}_t(x) = (l, l)$, $V_t^c(x | (l, l)) > V_t^c(x | (h, l)) = -c_h^c - f + \delta p_{hl} V_{t+1}^c(x+1) + \delta(1-p_{hl})V_{t+1}^c(x) \geq -c_h^c - f + \delta p_{hl}(R(x+1) - b) + \delta(1-p_{hl})(R(x) - b) = V_{T-1}^c(x) > R(x) - b$.

This implies that $\mathcal{E}_t(x) \neq \text{Stop}$ for all $x < x_s$. \square

Proof of Proposition 3. The existence of a time-independent stopping threshold is shown in Lemma B-2 in this appendix. By (B-2), $\theta_{sl}(x)$ is increasing in f and decreasing in b and δ . Hence, by (B-3) and because $\Delta R(x)$ is decreasing in x , x_s is decreasing in f and increasing in b and δ . \square

LEMMA B-3. *Suppose $p_{hh} \geq 2p_{hl}$. Then, if $\mathcal{E}_{t+1}(x+1) = \mathcal{E}_{t+1}(x) = (h, l)$, then (i) $V_t^v(x | (h, l)) > V_t^v(x | (h, h))$, i.e., $\mathcal{E}_t(x) \neq (h, h)$ and (ii) $V_t^v(x | (l, l)) > V_t^v(x | (l, h))$, i.e., $\mathcal{E}_t(x) \neq (l, h)$.*

Proof. (i) By (B-1), we have $V_t^v(x | (h, l)) > V_t^v(x | (h, h))$ if and only if $\Delta V_{t+1}^v(x) < \theta_{lh}^v$. When $\mathcal{E}_{t+1}(x+1) = \mathcal{E}_{t+1}(x) = (h, l)$, $\Delta V_{t+2}^v(x+1) \leq \theta_{lh}^v$ and $\Delta V_{t+2}^v(x) \leq \theta_{lh}^v$. As a result, $\Delta V_{t+1}^v(x) < \theta_{lh}^v$ similar to the proof of Lemma A-5. Thus, $V_t^v(x | (h, l)) > V_t^v(x | (h, h))$.

(ii) By assumption, $p_{hh} \geq 2p_{hl}$, which implies that $\theta_{lh}^v \leq \theta_{ll}^v$. Using the proof of part (i), we obtain $\Delta V_{t+1}^v(x) < \theta_{lh}^v \leq \theta_{ll}^v$; thus, $V_t^v(x | (l, l)) > V_t^v(x | (l, h))$. \square

LEMMA B-4. *Suppose that $R(x_s) \geq 2b$. Suppose that, for all $x < x_s$, there exists a time period $\tau(x) \leq T-1$, nondecreasing in x , such that $\mathcal{E}_t(x) = (h, l)$ for $t \geq \tau(x)$ and $\mathcal{E}_t(x) = (h, h)$ for $t < \tau(x)$. Then, $\Delta V_t^c(x) > \Delta V_t^v(x)$ for all $x < x_s$.*

Proof. The proof proceeds by induction. Consider first state $x_s - 1$. In period T , $\Delta V_T^c(x_s - 1) = R(x_s) - R(x_s - 1) > 0 = b - b = \Delta V_{T-1}^v(x_s - 1)$. Fix $t < T$ and suppose that $\Delta V_{t+1}^c(x_s - 1) > \Delta V_{t+1}^v(x_s - 1)$. Then, for $t \geq \tau(x_s - 1)$, we have by Proposition 3 that $\Delta V_t^c(x_s - 1) = (1 - \delta)[R(x_s) - b] + c_h^c + f + \delta(1 - p_{hl})\Delta V_{t+1}^c(x_s - 1) > (1 - \delta)b + c_h^c - f + \delta(1 - p_{hl})\Delta V_{t+1}^v(x_s - 1) = \Delta V_t^v(x_s - 1)$ using the induction hypothesis and the facts that $R(x_s) \geq 2b$, $c_h^c \geq c_h^v > c_l^v$, and $f \geq 0$. Similarly, for $t < \tau(x_s - 1)$, we have $\Delta V_t^c(x_s - 1) = (1 - \delta)[R(x_s) - b] + c_h^c + f + \delta(1 - p_{hh})\Delta V_{t+1}^c(x_s - 1) > (1 - \delta)b + c_h^v - f + \delta(1 - p_{hh})\Delta V_{t+1}^v(x_s - 1) = \Delta V_t^v(x_s - 1)$.

Consider next any state $x < x_s - 1$ and suppose that $\Delta V_t^c(x+1) > \Delta V_t^v(x+1)$ for all t . In period T , $\Delta V_T^c(x) = \Delta R(x) > 0 = b - b = \Delta V_{T-1}^v(x)$. Fix $t < T$ and suppose that $\Delta V_{t+1}^c(x) > \Delta V_{t+1}^v(x)$. Given that $\tau(x)$ is assumed to be nondecreasing, we consider the following three cases: (i) $\mathcal{E}_t(x) = \mathcal{E}_t(x+1) = (h, l)$, (ii) $\mathcal{E}_t(x) = \mathcal{E}_t(x+1) = (h, h)$, and (iii) $\mathcal{E}_t(x) = (h, l)$ and $\mathcal{E}_t(x+1) = (h, h)$.

- (i) $\Delta V_t^c(x) = \delta p_{hl}\Delta V_{t+1}^c(x+1) + \delta(1-p_{hl})\Delta V_{t+1}^c(x) > \delta p_{hl}\Delta V_{t+1}^v(x+1) + \delta(1-p_{hl})\Delta V_{t+1}^v(x) = \Delta V_t^v(x)$.
- (ii) $\Delta V_t^c(x) = \delta p_{hh}\Delta V_{t+1}^c(x+1) + \delta(1-p_{hh})\Delta V_{t+1}^c(x) > \delta p_{hh}\Delta V_{t+1}^v(x+1) + \delta(1-p_{hh})\Delta V_{t+1}^v(x) = \Delta V_t^v(x)$.

$$(iii) \Delta V_t^c(x) = \delta p_{hh} \Delta V_{t+1}^c(x+1) + \delta(1-p_{hl}) \Delta V_{t+1}^c(x) > -c_h^v + c_i^v + \delta p_{hh} \Delta V_{t+1}^v(x+1) + \delta(1-p_{hl}) \Delta V_{t+1}^v(x) = \Delta V_t^v(x).$$

Hence, $\Delta V_t^c(x) > \Delta V_t^v(x)$, completing the induction step. \square

LEMMA B-5. *Suppose that, for all $x < x_s$, there exists a time period $\tau(x)$ such that $\mathcal{E}_t(x) = (h, l)$ for $t \geq \tau(x)$ and $\mathcal{E}_t(x) = (h, h)$ for $t < \tau(x)$. Then for all $x < x_s$, $V_{t-1}^c(x) > V_t^c(x)$.*

Proof. The proof uses Lemmas B-1 and B-2 in this appendix. By Lemma B-2, $V_t^c(x_s) = V_{t+1}^c(x_s) = R(x_s) - b$. Suppose next that $x < x_s$. The proof proceeds by induction on x and t . In period $T-1$ and state $x < x_s$, $V_{T-1}^c(x) > R(x) - b = V_T^c(x)$ because $\mathcal{E}_{T-1}(x) = (h, l)$ by Lemma B-1. Fix t and suppose that $V_t^c(x) > V_{t+1}^c(x)$ and $V_t^c(x+1) \geq V_{t+1}^c(x+1)$. If $t > \tau(x)$, $V_{t-1}^c(x) = -c_h^c - f + \delta \mathbb{E}_{hl}[V_t^c(x+\xi)] > -c_h^c - f + \delta \mathbb{E}_{hl}[V_{t+1}^c(x+\xi)] = V_t^c(x)$. If $t = \tau(x)$, $V_{t-1}^c(x) = V_{t-1}^c(x | (h, h)) \geq V_{t-1}^c(x | (l, h))$ given that $\mathcal{E}_{t-1}(x) = (h, h)$; moreover, $V_{t-1}^c(x | (l, h)) > V_{t-1}^c(x | (h, l))$ because $c_h^c > c_l^c$ and $p_{hl} = p_{lh}$; as a result, $V_{t-1}^c(x) > V_{t-1}^c(x | (h, l)) = -c_h^c - f + \delta \mathbb{E}_{hl}[V_t^c(x+\xi)] > -c_h^c - f + \delta \mathbb{E}_{hl}[V_{t+1}^c(x+\xi)] = V_t^c(x)$. Finally, if $t < \tau(x)$, $V_{t-1}^c(x) = -c_h^c - f + \delta \mathbb{E}_{hh}[V_t^c(x+\xi)] > -c_h^c - f + \delta \mathbb{E}_{hh}[V_{t+1}^c(x+\xi)] = V_t^c(x)$, completing the induction step. \square

LEMMA B-6. *Suppose $f < c_i^v$. Suppose that, for all $x < x_s$, there exists a time period $\tau(x)$ such that $\mathcal{E}_t(x) = (h, l)$ for $t \geq \tau(x)$ and $\mathcal{E}_t(x) = (h, h)$ for $t < \tau(x)$. Then for all $x < x_s$, $V_{t-1}^v(x) < V_t^v(x)$.*

Proof. The proof uses Lemmas B-1 and B-2 in this appendix. By Lemma B-2, $V_t^v(x_s) = V_{t+1}^v(x_s) = b$. The proof proceeds by induction on x and t . In period $T-1$ and state $x < x_s$, $V_{T-1}^v(x) = -c_l^v + f + \delta b < b = V_T^v(x)$ because $\mathcal{E}_{T-1}(x) = (h, l)$ by Lemma B-1 and $f < c_i^v$. Fix t and suppose that $V_t^v(x) < V_{t+1}^v(x)$ and $V_t^v(x+1) \leq V_{t+1}^v(x+1)$. If $t > \tau(x)$, $V_{t-1}^v(x) = -c_l^v + f + \delta \mathbb{E}_{hl}[V_t^v(x+\xi)] < -c_l^v + f + \delta \mathbb{E}_{hl}[V_{t+1}^v(x+\xi)] = V_t^v(x)$. If $t = \tau(x)$, $V_{t-1}^v(x) = V_{t-1}^v(x | (h, h)) = -c_h^v + f + \delta \mathbb{E}_{hh}[V_t^v(x+\xi)] < -c_h^v + f + \delta \mathbb{E}_{hh}[V_{t+1}^v(x+\xi)] = V_t^v(x | (h, h))$; moreover, $V_t^v(x | (h, h)) < V_t^v(x | (h, l))$ because $\mathcal{E}_t(x) = (h, l)$; as a result, $V_{t-1}^v(x) < V_t^v(x | (h, l)) = V_t^v(x)$. Finally, if $t < \tau(x)$, $V_{t-1}^v(x) = -c_h^v + f + \delta \mathbb{E}_{hh}[V_t^v(x+\xi)] < -c_h^v + f + \delta \mathbb{E}_{hh}[V_{t+1}^v(x+\xi)] = V_t^v(x)$, completing the induction step. \square

LEMMA B-7. *Suppose $p_{hh} \geq 2p_{hl}$ and $f < c_i^v$. Suppose that for all $x < x_s$, $\mathcal{E}_t(x) = (h, l)$ for all $t \geq \tau(x)$ and $\mathcal{E}_t(x) = (h, h)$ for all $t < \tau(x)$; moreover, suppose that $\tau(x)$ is nondecreasing. Then, $\Delta V_t^v(x) \geq \Delta V_{t+1}^v(x)$ for all $x < x_s$.*

Proof. The proof uses Lemmas B-2, B-3, and B-6 in this appendix and proceeds by induction on x . To initialize the induction, consider state $x_s - 1$. By Lemma B-2, $V_t^v(x_s) = V_{t-1}^v(x_s) = b$. By Lemma B-6, $V_{t-1}^v(x_s - 1) < V_t^v(x_s - 1)$. Hence, $\Delta V_{t-1}^v(x_s - 1) \geq \Delta V_t^v(x_s - 1)$.

Fix $x < x_s - 1$. The proof follows by induction on t . By Lemma B-1, $\mathcal{E}_{T-1}(x) = \mathcal{E}_{T-1}(x+1) = (h, l)$. Thus, $\Delta V_{T-1}^v(x) = (-c_l^v + f + \delta b) - (-c_l^v + f + \delta b) = 0 = \Delta V_T^v(x)$. Fix $t > \tau(x)$ and suppose $\Delta V_{t+1}^v(x) \geq \Delta V_{t+2}^v(x)$. Because $t > \tau(x)$, $\mathcal{E}_t(x) = (h, l)$ and $\mathcal{E}_{t+1}(x) = (h, l)$. Then, because $\tau(x)$ is nondecreasing, we need to consider three cases: (i) $\mathcal{E}_{t+1}(x+1) = \mathcal{E}_t(x+1) = (h, l)$, (ii) $\mathcal{E}_{t+1}(x+1) = (h, l)$ and $\mathcal{E}_t(x+1) = (h, h)$, and (iii) $\mathcal{E}_{t+1}(x+1) = \mathcal{E}_t(x+1) = (h, h)$.

$$(i) \Delta V_t^v(x) = V_t^v(x+1 | (h, l)) - V_t^v(x | (h, l)) = \delta p_{hl} \Delta V_{t+1}^v(x+1) + \delta(1-p_{hl}) \Delta V_{t+1}^v(x) \geq \delta p_{hl} \Delta V_{t+2}^v(x+1) + \delta(1-p_{hl}) \Delta V_{t+2}^v(x) = V_{t+1}^v(x+1 | (h, l)) - V_{t+1}^v(x | (h, l)) = \Delta V_{t+1}^v(x).$$

(ii) $\Delta V_t^v(x) = V_t^v(x+1 | (h, h)) - V_t^v(x+1 | (h, l)) \geq V_t^v(x+1 | (h, l)) - V_t^v(x+1 | (h, l)) = \delta p_{hl} \Delta V_{t+1}^v(x+1) + \delta(1-p_{hl}) \Delta V_{t+1}^v(x) \geq \delta p_{hl} \Delta V_{t+2}^v(x+1) + \delta(1-p_{hl}) \Delta V_{t+2}^v(x) = V_{t+1}^v(x+1 | (h, l)) - V_{t+1}^v(x | (h, l)) = \Delta V_{t+1}^v(x)$, in which the first inequality holds by equilibrium conditions that $\mathcal{E}_t(x+1) = (h, h)$ implies $V_t^v(x+1 | (h, h)) > V_t^v(x+1 | (h, l))$ and $V_t^c(x+1 | (h, h)) > V_t^c(x+1 | (l, h))$.

(iii) $\Delta V_t^v(x) = V_t^v(x+1 | (h, h)) - V_t^v(x | (h, l)) = -c_h^v + c_l^v + \delta p_{hh} \Delta V_{t+1}^v(x+1) + \delta(1-p_{hl}) \Delta V_{t+1}^v(x) \geq -c_h^v + c_l^v + \delta p_{hh} \Delta V_{t+2}^v(x+1) + \delta(1-p_{hl}) \Delta V_{t+2}^v(x) = V_{t+1}^v(x+1 | (h, h)) - V_{t+1}^v(x | (h, l)) = \Delta V_{t+1}^v(x)$.

Consider next period $\tau(x)$. Because $\mathcal{E}_{\tau(x)}(x) = (h, l)$, $V_{\tau(x)}^v(x | (h, h)) \leq V_{\tau(x)}^v(x | (h, l))$, i.e., $\Delta V_{\tau(x)+1}^v(x) \leq \theta_{lh}^v$. On the other hand, because $\mathcal{E}_{\tau(x)-1}(x) = (h, h)$, $V_{\tau(x)-1}^v(x | (h, h)) \geq V_{\tau(x)-1}^v(x | (h, l))$, i.e., $\Delta V_{\tau(x)}^v(x) \geq \theta_{lh}^v$. Hence, $\Delta V_{\tau(x)}^v(x) \geq \Delta V_{\tau(x)+1}^v(x)$.

Consider next any particular period $t < \tau(x)$ and suppose that $\Delta V_{t+1}^v(x+1) \geq \Delta V_{t+2}^v(x+1)$. Because $t < \tau(x) \leq \tau(x+1)$, $\mathcal{E}_t(x) = \mathcal{E}_t(x+1) = (h, h)$. Moreover, $\mathcal{E}_{t+1}(x+1) \neq (h, l)$, for otherwise we would have had $\mathcal{E}_{t+1}(x) = (h, l)$ given that $\tau(x) \leq \tau(x+1)$ and therefore $\mathcal{E}_t(x) \neq (h, h)$ by Lemma B-3, a contradiction. Hence, $\mathcal{E}_{t+1}(x+1) = (h, h)$. Applying the induction hypothesis, we obtain: $\Delta V_t^v(x) = V_t^v(x+1 | (h, h)) - V_t^v(x | (h, h)) = \delta p_{hh} \Delta V_{t+1}^v(x+1) + \delta(1-p_{hh}) \Delta V_{t+1}^v(x) \geq \delta p_{hh} \Delta V_{t+2}^v(x+1) + \delta(1-p_{hh}) \Delta V_{t+2}^v(x) = V_{t+1}^v(x+1 | (h, h)) - V_{t+1}^v(x | (h, h))$. Given that $\mathcal{E}_{t+1}(x+1) = (h, h)$, $V_{t+1}^v(x+1) = V_{t+1}^v(x+1 | (h, h))$. Moreover, because $\mathcal{E}_{t+1}(x) = (h, h)$ or $\mathcal{E}_{t+1}(x) = (h, l)$, $V_{t+1}^v(x) \geq V_{t+1}^v(x | (h, h))$. Therefore, $V_{t+1}^v(x+1 | (h, h)) - V_{t+1}^v(x | (h, h)) \geq \Delta V_{t+1}^v(x)$, completing the induction step. \square

LEMMA B-8. *Suppose Assumptions A and B hold. There exists a time threshold $\tau(x_s - 1)$ such that $\mathcal{E}_t(x_s - 1) = (h, h)$ for all $t < \tau(x_s - 1)$ and $\mathcal{E}_t(x_s - 1) = (h, l)$ for all $t \geq \tau(x_s - 1)$.*

Proof. The proof uses Lemmas B-1, B-2, B-5, B-6, and B-7 in this appendix. By Lemma B-2, $\mathcal{E}_t(x_s - 1) \neq \text{Stop}$ for all t . In addition, by Lemma 1, when $p_{hl} = p_{lh}$, there exists an equilibrium in pure strategies.

Let $\tau(x_s - 1) := \max\{t | \Delta V_t^v(x_s - 1) \geq \theta_{lh}^v\}$. We first show by induction that $\mathcal{E}_t(x_s - 1) = (h, l)$ for all $t \geq \tau(x_s - 1)$. In period $T - 1$, $\mathcal{E}_{T-1}(x_s - 1) = (h, l)$ by Lemma B-1. Hence, by (B-1), $\tau(x_s - 1) < T - 1$. Fix $t \geq \tau(x_s - 1)$ and suppose that $\mathcal{E}_{t+1}(x_s - 1) = (h, l)$. By Lemma B-5, $V_{t+1}^c(x_s - 1)$ is decreasing in t and by Proposition 3, $V_{t+1}^c(x_s) = V_{t+2}^c(x_s)$. Thus, $V_t^c(x_s - 1 | (h, l)) = -c_h^c - f + \delta \mathbb{E}_{hl}[V_{t+1}^c(x_s - 1 + \xi)] > -c_h^c - f + \delta \mathbb{E}_{hl}[V_{t+2}^c(x_s - 1 + \xi)] = V_{t+1}^c(x_s - 1) > -c_l^c - f + \delta V_{t+1}^c(x_s - 1)$, which implies that $\mathcal{E}_t(x_s - 1) \neq (l, l)$. Furthermore, because $t \geq \tau(x_s - 1)$, $\Delta V_t^v(x_s - 1) < \theta_{lh}^v$ by (B-1). Hence, $V_t^v(x_s - 1 | (h, l)) > V_t^v(x_s - 1 | (h, h))$ when $t \geq \tau(x_s - 1)$, which implies that $\mathcal{E}_t(x_s - 1) \neq (h, h)$. Finally, because $p_{hh} \geq 2p_{hl}$, $\theta_{lh}^v \leq \theta_{ll}^v$ by (B-1). Hence, $\Delta V_t^v(x_s - 1) < \theta_{lh}^v \leq \theta_{ll}^v$, i.e., $\mathcal{E}_t(x_s - 1) \neq (l, h)$. As a result, $\mathcal{E}_t(x_s - 1) = (h, l)$ for all $t \geq \tau(x_s - 1)$.

We next show by induction that $\mathcal{E}_t(x_s - 1) = (h, h)$ for all $t < \tau(x_s - 1)$. Consider first period $\tau(x_s - 1) - 1$. By definition of $\tau(x_s - 1)$, $\Delta V_{\tau}^v(x_s - 1) \geq \theta_{lh}^v$. Hence, $V_{\tau-1}^v(x_s - 1 | (h, l)) \leq V_{\tau-1}^v(x_s - 1 | (h, h))$. Because $V_t^c(x)$ is decreasing in t by Lemma B-5 and because $V_t^c(x_s) = V_{t+1}^c(x_s)$ by Proposition 3, $V_{\tau-1}^c(x_s - 1 | (h, l)) = -c_h^c - f + \delta \mathbb{E}_{hl}[V_{\tau}^c(x_s - 1 + \xi)] > -c_h^c - f + \delta \mathbb{E}_{hl}[V_{\tau+1}^c(x_s - 1 + \xi)] = V_{\tau}^c(x_s - 1) > -c_l^c - f + \delta V_{\tau}^c(x_s - 1) = V_{\tau-1}^c(x_s - 1 | (l, l))$, i.e., $\mathcal{E}_{\tau}(x_s - 1) \neq (l, l)$. Hence, by (B-1), $\Delta V_{\tau}^c(x_s - 1) > \theta_{ll}^c$, which implies, since $\theta_{ll}^c \geq \theta_{lh}^c$ by (B-1) when $p_{hh} \geq 2p_{hl}$, that $\Delta V_{\tau}^c(x_s - 1) > \theta_{lh}^c$, i.e., $\mathcal{E}_t(x_s - 1) \neq (l, h)$. As a result, (h, h) is an equilibrium. If (h, h) is an equilibrium and if $V_t^v(x_s - 1 | (h, h)) = V_t^v(x_s - 1 | (h, l))$, there is a possibility that (h, l) is also an equilibrium. However, because $p_{hh} > p_{hl}$, $V_t^c(x_s - 1 | (h, h)) > V_t^c(x_s - 1 | (h, l))$, i.e., the client will never select (h, l) as the equilibrium to be played. Hence, $\mathcal{E}_{\tau-1}(x_s - 1) = (h, h)$.

Next, fix $t < \tau(x_s - 1)$ and suppose $\mathcal{E}_{t+1}(x_s - 1) = (h, h)$. Using Figure 3 and Lemma 1, $\mathcal{E}_t(x_s - 1) = (h, h)$ if (i) $V_t^v(x_s - 1 | (h, h)) > V_t^v(x_s - 1 | (h, l))$, (ii) $V_t^c(x_s - 1 | (h, h)) > V_t^c(x_s - 1 | (l, h))$, (iii) In case of multiple equilibria $\{(h, h), (l, l)\}$, the client selects the (h, h) equilibrium, i.e., $V_t^c(x_s - 1 | (h, h)) > V_t^c(x_s - 1 | (l, l))$.

(i) By Lemma B-7, $\Delta V_t^v(x_s - 1)$ is nonincreasing in t , and therefore $\Delta V_t^v(x_s - 1) \geq \Delta V_\tau^v(x_s - 1) \geq \theta_{lh}^v$, i.e., $V_t^v(x_s - 1 | (h, h)) \geq V_t^v(x_s - 1 | (h, l))$ for all $t < \tau(x_s - 1)$. In addition, in case of multiple equilibria $\{(h, l), (h, h)\}$, which happens when $\Delta V_t^v(x_s - 1) = \theta_{lh}^v$, the client will select (h, h) because $V_t^c(x_s - 1 | (h, h)) > V_t^c(x_s - 1 | (h, l))$ given that $p_{hh} > p_{hl}$.

(ii) Because $\Delta V_{t+1}^v(x_s - 1) \geq \theta_{hl}^v$ for $t < \tau(x_s - 1)$ and because $\Delta V_{t+1}^c(x_s - 1) > \Delta V_{t+1}^v(x_s - 1)$ by Lemma B-4, $\Delta V_{t+1}^c(x_s - 1) > \theta_{lh}^c = \theta_{lh}^v$ when $k = 1$, i.e., $V_t^c(x_s - 1 | (h, h)) > V_t^c(x_s - 1 | (h, l))$.

(iii) By Lemma B-5, $V_t^c(x)$ is decreasing in t . Moreover by Proposition 3, $V_t^c(x_s) = V_{t+1}^c(x_s)$. Thus, using the induction hypothesis, $V_t^c(x_s - 1 | (h, h)) = -c_h^c - f + \delta \mathbb{E}_{hh}[V_{t+1}^c(x_s - 1 + \xi)] \geq -c_h^c - f + \delta \mathbb{E}_{hh}[V_{t+2}^c(x_s - 1 + \xi)] = V_{t+1}^c(x_s - 1) > -c_l^c - f + \delta V_{t+1}^c(x_s - 1) = V_t^c(x_s - 1 | (l, l))$. \square

Proof of Proposition 4. The proof uses Lemmas B-1, B-3, B-2, B-4, B-5, B-7, and B-8 in this appendix. By Lemma B-2, $\mathcal{E}_t(x) \neq \text{Stop}$ for all $x < x_s$ and all t . We therefore use the table in Figure 3 to characterize the equilibrium outcomes. The proof proceeds by induction on x . When $x = x_s - 1$, the result holds by Lemma B-8. Consider now any state $x < x_s - 1$ and suppose that the result holds for state $x + 1$. Because $\mathcal{E}_{T-1}(x) = (h, l)$ by Lemma B-1 and because $\mathcal{E}_t(x + 1) = (h, l)$ for all $t \geq \tau(x + 1)$, we obtain that $V_t^v(x | (h, l)) > V_t^v(x | (h, h))$ and $V_t^v(x | (l, l)) > V_t^v(x | (l, h))$ by Lemma B-3. In addition, by Lemma B-5, $V_{t+1}^c(x)$ and $V_{t+1}^c(x + 1)$ are decreasing in t . Thus, $V_t^c(x | (h, l)) = -c_h^c - f + \delta \mathbb{E}_{hl}[V_{t+1}^c(x + \xi)] > -c_h^c - f + \delta \mathbb{E}_{hl}[V_{t+2}^c(x + \xi)] = V_{t+1}^c(x) > -c_l^c - f + \delta V_{t+1}^c(x) = V_t^c(x | (l, l))$. Hence, $\mathcal{E}_t(x) = (h, l)$ for all $t \geq \tau(x + 1) - 1$.

Let $\tau(x) := \max\{t | \mathcal{E}_{t-1}(x) \neq (h, l)\}$. Because $\mathcal{E}_t(x) = (h, l)$ for all $t \geq \tau(x + 1) - 1$, we have $\tau(x) \leq \tau(x + 1) - 1$, i.e., $\tau(x)$ is increasing. Consider period $\tau(x) - 1$. By Lemma B-2, $\mathcal{E}_{\tau(x)-1}(x) \neq \text{Stop}$ and by Lemma 1, the equilibrium exists in pure strategy and is either (h, h) , (h, l) , (l, h) , or (l, l) . By definition of $\tau(x)$, $\mathcal{E}_{\tau(x)-1}(x) \neq (h, l)$. Therefore, because $V_t^c(x)$ and $V_t^c(x + 1)$ are decreasing in t by Lemma B-5, $V_{\tau(x)-1}^c(x | (h, l)) = -c_h^c - f + \delta \mathbb{E}_{hl}[V_{\tau(x)}^c(x + \xi)] > -c_h^c - f + \delta \mathbb{E}_{hl}[V_{\tau(x)+1}^c(x + \xi)] = V_{\tau(x)}^c(x) > -c_l^c - f + \delta V_{\tau(x)}^c(x) = V_t^c(x | (l, l))$, i.e., $\mathcal{E}_{\tau(x)-1}(x) \neq (l, l)$. Therefore, using (B-1), we obtain that $\Delta V_{\tau(x)}^c(x) > \theta_{lh}^c$. Since $\theta_{ll}^c \geq \theta_{lh}^c$ when $p_{hh} \geq 2p_{hl}$ by (B-1), this implies that $\Delta V_{\tau(x)}^c(x) > \theta_{lh}^c$, i.e., $\mathcal{E}_{\tau(x)}(x) \neq (l, h)$. Hence, $\mathcal{E}_{\tau(x)}(x) = (h, h)$.

Consider next any period $t < \tau(x) - 1$ and suppose that $\mathcal{E}_{t+1}(x) = (h, h)$ and $\mathcal{E}_{t'}(x + 1) = (h, h)$ for all $t' < \tau(x + 1)$. Because $\tau(x) \leq \tau(x + 1)$, $\mathcal{E}_{t+1}(x + 1) = (h, h)$. Because $\mathcal{E}_{t+1}(x) = (h, h)$, $V_{t+1}^v(x | (h, h)) \geq V_{t+1}^v(x | (h, l))$, which is equivalent to $\Delta V_{t+2}^v(x) \geq \theta_{lh}^v$ by (B-1). By Lemma B-7, $\Delta V_{t+1}^v(x) > \Delta V_{t+2}^v(x)$. Hence, $\Delta V_{t+1}^v(x) > \theta_{lh}^v$. Using Lemma B-4, $\Delta V_{t+1}^c(x) > \Delta V_{t+1}^v(x) > \theta_{lh}^v = \theta_{lh}^c$ when $k = 1$. Thus, $V_t^c(x | (h, h)) > V_t^c(x | (h, l))$. Hence, (h, h) is an equilibrium outcome. Because $V_t^c(x | (h, h)) > V_t^c(x | (h, l))$, even in case of multiple equilibria $\{(h, h), (h, l)\}$ (which may happen when $\Delta V_{t+1}^v(x) = \theta_{lh}^v$), the client would select (h, h) . Furthermore, by Lemma B-5, $V_t^c(x)$ and $V_{t+1}^c(x)$ are decreasing in t . Thus, using the induction hypothesis that $\mathcal{E}_{t+1}(x) = (h, h)$, $V_t^c(x) = -c_h^c - f + \delta \mathbb{E}_{hh}[V_{t+1}^c(x + \xi)] \geq -c_h^c - f + \delta \mathbb{E}_{hh}[V_{t+2}^c(x + \xi)] = V_{t+1}^c(x | (h, h)) = V_{t+1}^c(x) > -c_l^c - f + \delta V_{t+1}^c(x) = V_t^c(x | (l, l))$; thus, $\mathcal{E}_t(x) \neq (l, l)$, because even in case of multiple equilibria $\{(h, h), (l, l)\}$, the client chooses (h, h) over (l, l) . As a result, $\mathcal{E}_t(x) = (h, h)$. \square