Optimal Group Size in Joint Liability Contracts

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Abstract

We develop a model of repeated microcredit lending to study how group size affects optimal group-lending contracts with joint liability. In the setting being studied, a benevolent lender provides microcredit to a group of borrowers to invest in projects. The outcome of each risky project is not observable by the lender; therefore, if some of the borrowers default on their loan repayments, the lender cannot identify strategic default. The group will be entitled to a subsequent loan if total loan obligation is met.

We characterize the optimal contract and determine the optimal size of the borrowers' group endogenously. We find that, although joint liability contracts are feasible under a smaller set of parameter values than individual liability contracts, joint liability has positive effects on the borrowers' repayment amount and welfare. Our analysis also suggests that group size should increase with project risk.

Furthermore, we analyze the effect of partial joint liability, less severe punishment, and project correlation on the feasibility and characteristics of joint liability contracts. Our results show that, first, although partial joint liability has a negative effect on the borrowers' repayment amount and welfare, it can increase the loan ceiling of joint liability when collusion is not as likely, or when borrowers have high discount factors. Second, less severe punishment does not affect the borrowers' repayment amount or welfare, but decreases the loan ceiling of joint liability.

1

However, these negative effects created by partial joint liability and less severe punishment on the borrowers' repayment amount, borrowers' welfare, and loan ceiling can be offset by forming larger groups. Third, we also found that project correlation allows a higher loan ceiling in larger groups.

[*Keywords*: Microcredit, Joint Liability, Strategic default]

1 Introduction

Small-scale businesses are considered a major source of employment, particularly in developing countries where such businesses employ more than half of the economically active population (De Mel et al., 2008). One of the most efficient tools for microenterprise development is access to a source of finance (Berge et al., 2014). However, such enterprises are usually excluded from conventional financial services, either because they are unable to offer formal guarantees, or are located too far from financial networks (Prior and Argandoña, 2009). For example, in 2008, although 3,080,000 microenterprises in Peru employed 76% of the country's labor force and accounted for 42% of the country's GDP, these small businesses were never regarded as relevant by the formal banking system (Chu, 2015).

Microcredit may equip financial institutions with the right tools to carry out their specific social responsibilities of "integrating people in the active population and combating the cause of social and financial exclusion" (Lacalle-Calderón and Rico-Garrido, 2006) and reducing poverty (Littlefield et al., 2003) in developing countries. Microcredit is generally defined as small loans with little or no collateral offered to microentrepreneurs, who are usually excluded from conventional financial services. This type of lending has had some success in lending to the poor, and has been publicly viewed as one of the recent improvements in financial institutions to support development. Although it has also been reported that such lending only reaches moderately poor people and not the poorest of the poor (Scully, 2004; Marr, 2003).

In the present paper, we study the following microcredit lending scenario. A benevolent lender (she) wants to provide loans to a self-selected group of n microentrepreneur borrowers with *joint liability*. Although members in a jointly liable group are given individual loans, they are held jointly liable for repayment. In addition, the entire group qualifies for a subsequent loan on the condition that the total loan is repaid. The loans are then invested in n projects that have an equal

chance of success,[1] and can either be disjoint or correlated. We assume that the realized output of each project is known to the group members but unknown to the lender.[2] In this scenario, both sides (i.e., the lender and the borrowers) must make decisions. The lender decides on the optimal contract structure, the degree of joint liability, and the level of punishment in the case of strategic default, while the borrowers decide whether to repay the loan or to default.

Our study is primarily concerned with how large a group size should be in order to maximize the borrowers' benefit while leaving the lender to break even. On the one hand, a larger group size can have a positive effect on the expected repayment amount, as more people are liable for repaying defaulted payments, thus assuring a higher rate of repayment. On the other hand, a larger group can be a threat for repaying members, since they must repay all defaulting peers' repayment when everyone else in the group defaults on repayment. We propose a method to find the optimal group size that allows joint liability to reach its maximum loan ceiling.

We model the abovementioned lending situation as an infinitely repeated game. In our basic model, *simple joint liability (SJL),* we remain aligned with Bhole and Ogden's (2010) simple group-lending model designed for groups of two borrowers, and extend their results to groups of n borrowers. In our model, borrowers receive individual contracts that determine the amount of loan L and repayment R. They invest their loans on projects that are identical in terms of mean return and chance of success, and project returns are disjoint. Projects are either successful with a high return or unsuccessful with a low return. After the project outcomes are realized, each borrower decides whether or not to repay his loan. Only borrowers who have completed successful projects are supposed to repay. We differentiate between *strategic default* and *non-strategic default.* Borrowers default strategically when they refuse to repay while they have high outcomes from their projects. Borrowers default non-strategically when they do not repay due to little or no outcome from their projects. Although the lender cannot identify strategic default, members of the group can. The lender deprives a defaulting group from future loans in order to decrease incentives for strategic default. We assume that group members play a *grim-trigger* strategy amongst themselves, meaning that, group members keep repaying their loans, as well as the loans of their non-strategically

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[1]In the literature, it is widely believed that when borrowers can self-select their group members, joint liability can induce borrowers to pool with borrowers of similar risk (see e.g., Stiglitz, 1990; Ghatak, 1999). Thus, it is not unrealistic to assume that the chance of project success for all members of a self-selected group is equal.

[2]This is a plausible assumption, as group members are self-selected and have better information about each other than the lender.

defaulting peers, as long as no one defaults strategically. If an individual does default strategically, other members of the group will not repay his loan as well as their own loans, thus no further loans are granted to the group.

Our results show that, although an SJL contract is feasible under a smaller set of parameter settings than an *individual liability (IL) contract*, it has a higher performance than the IL contract in terms of the borrowers' welfare and repayment amount. Specifically, the SJL contract can also outperform the IL contract in terms of loan ceiling if the group size is not too large. In this paper, we calculate the optimal group size given the discount factor of borrowers for future loans, the chance of project success, and project returns.

Further, we consider relaxing some of our assumptions on the basic model. We seek to examine how much flexibility is possible in designing the SJL contract in terms of the degree of joint liability and the penalty function employed against the defaulting borrowers. Should successful group members be held liable for full repayment of their unsuccessful peers or should they only be responsible for a fraction of the repayment? Should the defaulting borrowers be excluded from lending forever or be given another chance? We compare the SJL contract, which holds successful group members responsible for their unsuccessful peers' full repayment, with a *partial joint liability (PJL) contract*, which demands successful group members to repay only a fraction of their unsuccessful peers' repayment. We also compare the SJL contract, which deprives the strategically defaulting members from future loans forever, with *a flexible joint liability (FJL) contract*, which deprives them for only T periods and enables them to rejoin the group afterwards.

We show that a PJL contract can have the same advantages of an SJL contract if the expected cost of auditing is not excessively high; when these costs are high, the loan ceiling of the PJL contract deteriorates. In addition, the borrowers' repayment amount rises and the borrowers' welfare declines. However, the PJL contract allows for larger group sizes, and forming larger groups can enhance the performance of the PJL contract by increasing the loan ceiling and allowing for a lower repayment amount, which also results in a higher borrowers' welfare. A disadvantage of the PJL contract is that when a small auditing probability exists, borrowers need a higher average discount factor to qualify for a loan.

We also show that although a less-severe punishment does not have any significant effect on the borrowers' repayment amount and welfare, it negatively affects the loan ceiling of joint liability. We

4

prove that enlarging the group of borrowers mitigates the negative effect of less-severe punishment employed under the FJL contract by enhancing repayment insurance.

As another extension on our basic model, we explore how project correlation can affect the lending outcome. It is natural to assume that the chance of project success is an increasing function of the group size, as jointly liable group members are likely to help each other to succeed. However, the marginal desirability of forming larger groups should decrease, as very large groups must confront higher tensions. We show that when any positive correlation exists between project returns, we could increase the loan ceiling of the SJL contract by enlarging the borrowers' group.

2   Related Literature

Our study, in general, is in line with the literature that explores conditions under which group lending can help alleviate informational asymmetry, incentivize risk-sharing, and reduce enforcement problems (see e.g., Ghatak, 1999; 2000; Armendáriz de Aghion and Gollier, 2000). Within this literature, our study is related to studies concerned with the optimal design of joint liability contracts that should be offered to a group of microcredit borrowers, who are subject to strategic default (see, e.g., Bhole and Ogden, 2010; Allen, 2016; Tedeschi, 2006).

The literature generally concludes that when borrowers are able to impose strong social sanctions on each other, a microcredit lender is better off offering a joint liability contract rather than an individual liability contract (see e.g., Besley and Coate, 1995; Armendáriz de Aghion, 1999).We assume a specific mechanism of social sanctions (i.e., a strong punishment strategy) and discuss that although joint liability contracts are feasible under a smaller set of parameters, such contracts can perform better than an IL contract. More specifically, joint liability contracts can positively affect the borrowers' welfare and repayment amount, as well as the loan ceiling.

The economic literature that has investigated different features of group lending with joint liability has paid little attention to group size as one of the potentially influential factors in the relative success of group lending. Theoretical studies have mostly analyzed the lending models of groups of two borrowers, while experimental and empirical studies have suggested the importance of group size (Abbink et al., 2006; Galak et al., 2011). We argue that the size of the borrowers' group is an important factor in decreasing the borrowers' repayment amounts and increasing the

borrowers' welfare in microcredit lending, and we determine the optimal group size endogenously.

Extant literature that has considered group size as an important factor has come to different conclusions. On the one hand, although arguments in favor of larger group sizes have been made, such as Conning (2004) and Ahlin (2015), they also (similar to our paper) suggest that group size cannot grow too large. Conning (2004) specifies that it becomes increasingly costly to contain free-riding as group size increases. Ahlin (2015) shows that the presence of local borrower information is necessary in order for large groups to have any impact. Our assumption—that borrowers play a grim-trigger strategy against each other—helps us to deal with free-riding inside a group of borrowers and it also replaces the local borrower information effect. On the other hand, there are arguments in the literature that favor a smaller group size. For example, Bourjade and Schindele (2012) prove that if group members have social ties, a rational lender should choose a group of limited size. They explain that a trade-off exists between raising profits through an increased group size and providing incentives for borrowers with less social ties. The findings of Baland et al. (2013) are more similar to ours, as they find that the optimal group size depends on project characteristics. We argue that, although a larger group can provide higher support for a defaulting member, it also increases the risk of being in charge of everybody else for a repaying member, especially when projects are risky. Therefore, although riskier projects should be handled by relatively large groups, the group size cannot become too large.[3]

It is believed in the literature that group lending improves risk-sharing between group members (see e.g., Allen and Babus, 2009). Part of this literature emphasizes the role of social networks on the informal insurance that group members provide for each other and argues that group lending, even without joint liability, improves risk-sharing (Feinberg et al., 2013). Another part of this literature, however, gives credit to the joint liability feature of microcredit lending for improving risk-sharing. Among the well-known examples are Ghatak (2000), Fischer (2013), and Allen (2016). In accordance with this latter literature, we argue that joint liability positively influences risk-sharing. More specifically, we prove that as the degree of joint liability increases, borrowers can be charged a lower repayment amount as a result of the higher repayment insurance they can provide.

Furthermore, the literature that studies the effect of correlation between projects is also relevant

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[3]Ahlin (2015) is relevant for our paper from an additional aspect, as the author also considers the possibility of a less-than-full repayment. He suggests an affordable joint liability contract and concludes that, whenever affordable, a contract with full liability is optimal. However, Ahlin's (2015) modeling assumptions are different from ours.

to our study, such as Ghatak (2000), and Katzur and Lensink (2012). These authors argue that when projects are likely to succeed (or fail) at the same time, the joint liability part of the contract is not applied as often. Based on this argument, Ghatak (2000) proves that joint liability contracts are less feasible when project outcomes are positively correlated. We argue, in contrast, that joint liability contracts may be even more feasible when a positive correlation exists between project outcomes, because jointly liable groups are more likely to contribute to each other's success. In other words, joint liability can cause the formation of project externalities. Katzur and Lensink (2012) argue, similar to our argument, that positive correlation of project returns may improve the efficiency of group lending contracts, but our model differs from theirs in various ways.

3 Model

Consider an infinitely repeated lending game, with a benevolent lender and n borrowers, in which each period of the game has the following three steps.

$s = 0$ Each borrower receives an individual contract (L, R), specifying the amount of loan L and the repayment $R > L$ (primary loan plus interest).

$s = 1$ Each borrower invests L in his project that will either succeed with a chance of $\alpha \in [0, 1]$ and yields a high return $H > 0$, or not succeed with a chance of $1 - \alpha$ and yields a zero return. Group members can observe each other's project returns, but the lender cannot.

$s = 2$ Borrowers simultaneously decide whether to repay their repayment R. If i members default, the other $n - i$ members will be asked to pay additional amount $\frac{iR}{n-i}$ to the lender for their defaulting peers. If the total repayment is equal to nR or more, the group receives future financing; otherwise, the lender will exclude the entire group from future loans.

These three steps are repeatedly played until the lender realizes that the borrowers are not entitled to financing for the next period. For each period of not receiving a loan, the borrowers' utility will be zero. We assume that projects do not differ in their riskiness (i.e., α is the same for all borrowers). We also assume that each borrower always invests in the same project.

7

Two types of defaults are possible: *strategic default,* in which the borrower does not repay although he had high outcome H, and *nonstrategic default* as a result of obtaining zero outcome resulting from bad luck. Although, the lender is unable to observe whether a borrower defaults strategically or nonstrategically, borrowers are able to observe strategic defaults of their peers without any cost.

The benevolent lender strives to maximize the payoff of each borrower contingent on the following criteria. First, each borrower must be willing to accept a loan (the repayment amount must be affordable); second, each successful borrower must have the incentive to repay for himself and for each defaulting peer (in the worst case that all other members default, he must still be willing to repay for the entire group); and third, the lender must break even, meaning that she must maintain a sustainable lending operation over the entire loan portfolio by charging the appropriate repayment amount.

We initially make a set of simplifying assumptions that we will relax when discussing extensions of the model. In our basic model, we assume that borrowers are fully liable for each other; that is, the repaying borrowers are asked to repay the total repayment nR in order to qualify for further loans. We also assume that borrowers play a grim-trigger strategy amongst themselves, meaning that, if some members default strategically on their repayments at a certain point, other members will stop repaying themselves and stop repaying the defaulting players' shares. The group will therefore not be eligible to obtain a loan in the next period. Finally, we assume that no correlation exists between project returns.

4   Simple Joint Liability (SJL) Contract

In this section, we formalize the basic model assuming that the borrowers' group is liable for the total group loan (i.e., nR), group members play a grim-trigger strategy between themselves, and projects are disjointed. In each period in which i members default, the expected repayment for a borrower, who plays the repayment strategy, can be calculated as

$$\sum_{i=0}^{n-1} \binom{n-1}{i} \alpha^{n-i} (1-\alpha)^i \left(R + \frac{i}{n-i} R \right) = \left[1 - (1-\alpha)^n \right] R.$$

Thus, the expected utility of a borrower who plays a repayment strategy at any period in which he obtains financing is determined as

$$V_{SJL}^R = \alpha H - [1 - (1 - \alpha)^n] R + \delta [1 - (1 - \alpha)^n] V_{SJL}^R,$$

where $0 \leq \delta \leq 1$ is the borrowers' discount factor. The expected utility of a repaying borrower can be rewritten as

$$V_{SJL}^R = \frac{\alpha H - R [1 - (1 - \alpha)^n]}{1 - \delta [1 - (1 - \alpha)^n]}. \tag{1}$$

The benevolent lender wants to maximize the lifetime utility of a borrower who plays the repayment strategy. Therefore, considering the stationary nature of the model, the lender's optimization problem can be stated as $\max\limits_{L,R} V_{SJL}^R$, subject to the following.

1. The stipulated repayment amount for a successful borrower cannot exceed his output (i.e., the repayment amount must be affordable) even in the worst case when everyone else's project has failed,

$$nR \leq H. \tag{2}$$

2. A strategy in which each borrower repays when his project is successful is a subgame perfect Nash equilibrium if for each successful borrower, the payoff of strategically defaulting cannot be larger than the payoff of repaying and being refinanced,

$$H \leq H - nR + \delta V_{SJL}^R, \tag{3}$$

which also implies $R < \delta V_{SJL}^R$, thus guaranteeing that a successful borrower pays the repayment R when all of his partners are successful.

3. The lender must be able to sustain the lending game over periods and at least break even. Thus, the expected repayment amount of each borrower has to be at least as large as $L(1 + \epsilon)$, where ϵ is the interest rate,

$$R [1 - (1 - \alpha)^n] \geq L(1 + \epsilon). \tag{4}$$

If there are some (L, R) that satisfy constraints (2), (3), and (4), then the SJL contract is feasible

and these constraints define its feasibility region. Note that individual lending can be considered as a special type of group lending with $n = 1$.

Proposition 1. *There exist $\tilde{\delta}_{SJL}(n, \alpha)$ such that:*

a) if $\delta \geq \tilde{\delta}_{SJL}(n, \alpha)$, the SJL contract is feasible iff $L \leq \tilde{L}_{SJL}(n, \alpha, H)$;

b) if $\delta \leq \tilde{\delta}_{SJL}(n, \alpha)$, the SJL contract is feasible iff $L \leq \hat{L}_{SJL}(n, \alpha, \delta, H)$.

Moreover, whenever the SJL contract is feasible, for any $\alpha \neq 0$, the lender demands an optimal repayment $R_{SJL} = \frac{L(1+\epsilon)}{[1-(1-\alpha)^n]}$ from each borrower, and the expected lifetime utility for each borrower will amount to $V_{SJL}^R = \frac{\alpha H - L(1+\epsilon)}{1 - \delta[1-(1-\alpha)^n]}$.

Proposition 1 suggests that the SJL contract is feasible if and only if any offered loan under the SJL is limited to the following upper bound,

$$\mathscr{L}_{SJL}(n, \alpha, \delta, H) = \min\left\{\hat{L}_{SJL}, \tilde{L}_{SJL}\right\} = \begin{cases} \hat{L}_{SJL} & \delta \leq \tilde{\delta}_{SJL} \\ \tilde{L}_{SJL} & \delta \geq \tilde{\delta}_{SJL} \end{cases}. \tag{5}$$

The loan ceiling under the SJL contract (i.e., $\mathscr{L}_{SJL}$) depends positively on the borrowers' discount factor, as $\hat{L}_{SJL}$ is strictly increasing and $\tilde{L}_{SJL}$ is constant in δ. Intuitively, borrowers who highly value receiving future loans would have a higher incentive to repay their loans and thus could be trusted to repay larger loans.

From Proposition 1, it can also be inferred that optimal repayment R_{SJL} decreases in group size n. As a larger group with joint liability can offer a higher repayment insurance, a larger group can be charged less, which in turn increases the borrowers' welfare V_{SJL}^R. Corollary 1 is a direct result from Proposition 1 and follows from substituting $n = 1$.

Corollary 1. *Individual lending is feasible iff $L \leq \frac{\alpha^2 \delta H}{1+\epsilon}$. For any $\alpha \neq 0$, the lender optimally demands repayment $R_{IL} = \frac{L(1+\epsilon)}{\alpha}$. The borrowers' expected lifetime utility will be $V_{IL}^R = \frac{\alpha H - L(1+\epsilon)}{1 - \alpha\delta}$.*

We continue this section determining how the loan ceiling of the SJL contract is affected by group size as well as the optimal group size that allows the maximum loan ceiling. In Lemma 1 we take a closer look at the changes of the determinants of the loan ceiling of the SJL contract (i.e., $\tilde{\delta}_{SJL}$, $\hat{L}_{SJL}$, and $\tilde{L}_{SJL}$) with respect to changes of group size n when other parameters (α, δ, H) are given.

Lemma 1. *Assume $\tilde{\delta}_{SJL}$, $\hat{L}_{SJL}$ and $\tilde{L}_{SJL}$ are functions defined in Proposition 1.*

1) There exists $\hat{\delta}_{SJL}(n, \alpha)$ such that $\hat{L}_{SJL}$ is strictly decreasing in n if $0 < \delta < \hat{\delta}_{SJL}$, and $\hat{L}_{SJL}$ is strictly increasing in n if $\hat{\delta}_{SJL} < \delta < \tilde{\delta}_{SJL}$.

2) $\tilde{L}_{SJL}(n, \alpha, H)$ is strictly decreasing in n.

3) $\hat{\delta}_{SJL}$ is strictly increasing, and $\tilde{\delta}_{SJL}$ is strictly decreasing in n and α.

Lemma 1 shows that a larger group size can affect the loan ceiling in two different ways. On the one hand, this Lemma proves that a larger group size can increase the loan ceiling of the SJL contract for mid-range borrowers' discount factors, and more specifically for $\delta \in \left(\hat{\delta}_{SJL}, \tilde{\delta}_{SJL}\right)$. On the other hand, the Lemma argues that the interval for feasible discount factors becomes tighter by increasing the group size.

The reason that a larger group size can increase the loan ceiling of the SJL contract only for $\delta \in \left(\hat{\delta}_{SJL}, \tilde{\delta}_{SJL}\right)$ can be explained as follows. We recall from the proof of Proposition 1 that for $\delta \leq \tilde{\delta}_{SJL}$, the loan ceiling is determined by the incentive constraint (3), which allows for larger group sizes if the borrowers' discount factor is not too low. If additionally $\delta \leq \hat{\delta}_{SJL}$, the incentive constraint (3) can be satisfied only for very small group sizes. For large discount factors, particularly for $\delta \geq \tilde{\delta}_{SJL}$, the loan ceiling is determined by the individual rationality constraint (2). This constraint becomes tighter for larger group sizes.

The third part of Lemma 1 additionally proves that the interval $\left(\hat{\delta}_{SJL}, \tilde{\delta}_{SJL}\right)$ becomes also tighter by increasing the chance of project success. Therefore, it can be inferred from this lemma that in order to offer larger loans to borrowers with mid-range discount factors, projects with a higher chance of success should be handled in smaller groups, while riskier projects should be handled in larger groups. In Proposition 2, we formally prove that for projects with a small chance of success, the loan ceiling increases in the group size for a wider range of discount factors; that is, in such situations the interval $\left(\hat{\delta}_{SJL}, \tilde{\delta}_{SJL}\right)$ might actually be wide enough to allow for more than one feasible group size.

Proposition 2. *Assume $\tilde{\delta}_{SJL}(n, \alpha)$ and $\hat{\delta}_{SJL}(n, \alpha)$ are functions defined respectively in Proposition 1 and Lemma 1.*

1) There exists $\bar{\alpha}$ such that:

a) if $\alpha < \bar{\alpha}$, the loan ceiling of the SJL contract is increasing in $n \in [2, N_{\alpha,\delta}]$, where

$$N_{\alpha,\delta} = \min\left\{ \lfloor \hat{\delta}_{SJL}^{-1}(\alpha,\delta) \rfloor, \lfloor \tilde{\delta}_{SJL}^{-1}(\alpha,\delta) \rfloor \right\};$$ (6)

b) if $\alpha > \bar{\alpha}$, the maximum loan ceiling of the SJL contract happens at $n = 2$.

2) For very large n, the loan ceiling of the SJL contract will be decreasing.

Proposition 2 suggests that the group size should be limited to an upper bound and cannot grow too large. Intuitively, group size has two countervailing effects. On the one hand, a larger group can provide stronger repayment insurance, handle riskier projects, and repay successfully. On the other hand, a larger group can be a threat toward the feasibility of joint liability. The threat comes from the fact that each successful member is in charge of all defaulting peers: if one group member succeeds and everybody else fails, the one successful member is in charge of the entire group loan. Obviously, this becomes very difficult if the group is too large. This proposition also presents a characterization of the optimal group size, as a function of all other parameters.

Proposition 2 additionally looks at this result from another angle recalling that although groups of only two borrowers always allow for the widest range of acceptable discount factors for any chance of project success, the loan ceiling of the SJL contract can increase only in larger groups.

Using the proof of Proposition 2 and assuming that $H = 1$, one could simply verify that for $\alpha < 0.477$, the SJL contract is feasible for $n = 2, 3, 4$, while for $\alpha < 0.718$, the SJL contract is feasible for $n = 2$. In other words, larger group sizes are feasible for a smaller chance of project success. Table 1 provides more examples of this kind when $H = 1$. As shown in this table, the largest α, for which more than one n is feasible, is $\bar{\alpha} = 0.568$. Thus, according to Proposition 2, the loan ceiling of the SJL contract can increase in group size only if $\alpha < 0.568$ and the given δ is such that $\hat{\delta}_{SJL} < \delta < \tilde{\delta}_{SJL}$. In terms of Proposition 2, $\bar{\alpha}$ is approximately 0.5 if $H = 1$.

Figures 1 and 2 help to derive better intuition about Proposition 2. For simplicity, it is assumed that $H = 1$. Figure 1 depicts $\tilde{\delta}_{SJL}$ and $\hat{\delta}_{SJL}$ with respect to group size n both when the chance of project success α is small (e.g., $\alpha = 0.3$) and when it is large (e.g., $\alpha = 0.8$). As shown in Figure 1a, for $\alpha = 0.3$, there are some n for which the given $\delta = 0.85$ belongs to the interval $\left(\hat{\delta}_{SJL}, \tilde{\delta}_{SJL} \right)$, and the largest of such n lies at the $\min\left\{ \hat{\delta}_{SJL}^{-1}, \tilde{\delta}_{SJL}^{-1} \right\} = \lfloor \hat{\delta}_{SJL}^{-1} \rfloor = \lfloor 7.136 \rfloor = 7$. Note that if the given δ approaches 1, the largest n lies at $\lfloor \tilde{\delta}_{SJL}^{-1} \rfloor = 10$; while, in Figure 1b, for $\alpha = 0.8$, the interval

Table 1: The Range of Feasible Group Sizes for the SJL Contract for Some Given α

	$n=2$	$n=3$	$n=4$	$n=5$	$\cdots$	$n=10$	$\cdots$	$n=50$
$\alpha < 0.718$	✓							
$\alpha < 0.568$	✓	✓						
$\alpha < 0.477$	✓	✓	✓					
$\alpha < 0.415$	✓	✓	✓	✓				
$\alpha < 0.269$	✓	✓	✓	✓	$\cdots$	✓		
$\alpha < 0.082$	✓	✓	✓	✓	$\cdots$	✓	$\cdots$	✓

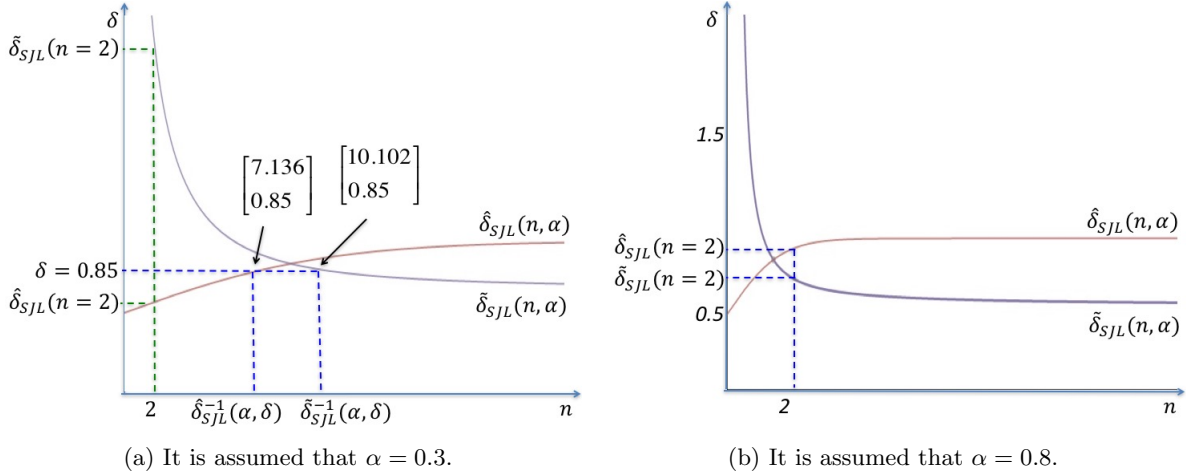(a) It is assumed that $\alpha = 0.3$. (b) It is assumed that $\alpha = 0.8$.

Figure 1: Larger group sizes are possible only for small α.

$\left(\hat{\delta}_{SJL}, \tilde{\delta}_{SJL}\right)$ is empty for any $n > 1$.

Figure 2 presents the changes of $\hat{L}_{SJL}$ and $\tilde{L}_{SJL}$ with respect to group size n both when the chance of project success α is small (e.g., $\alpha = 0.3$) and when it is large (e.g., $\alpha = 0.8$). When α is small (Figure 2a), $\hat{L}_{SJL}$ defines the loan ceiling, and it reaches its maximum at $n = 7$. Therefore, $n = 7$ is the group size that maximizes the loan ceiling of the SJL contract. While, when α is large (Figure 2b), $\tilde{L}_{SJL}$ defines the loan ceiling, which decreases in n. Therefore, $n = 2$ is the optimum group size that maximizes the loan ceiling of the SJL contract.

Up to this point, we have discussed that the loan ceiling of the SJL contract can increase in group size when the chance of project success is small. However, we do not yet know if the SJL contract can perform more efficiently than the IL contract. Are there circumstances under which an SJL contract outperforms an IL contract? Proposition 3 proves formally that the lender can charge borrowers less under the SJL contract compared to the IL contract while still breaking even. One explanation for this may be that "no repayment" is something that happens less often under an
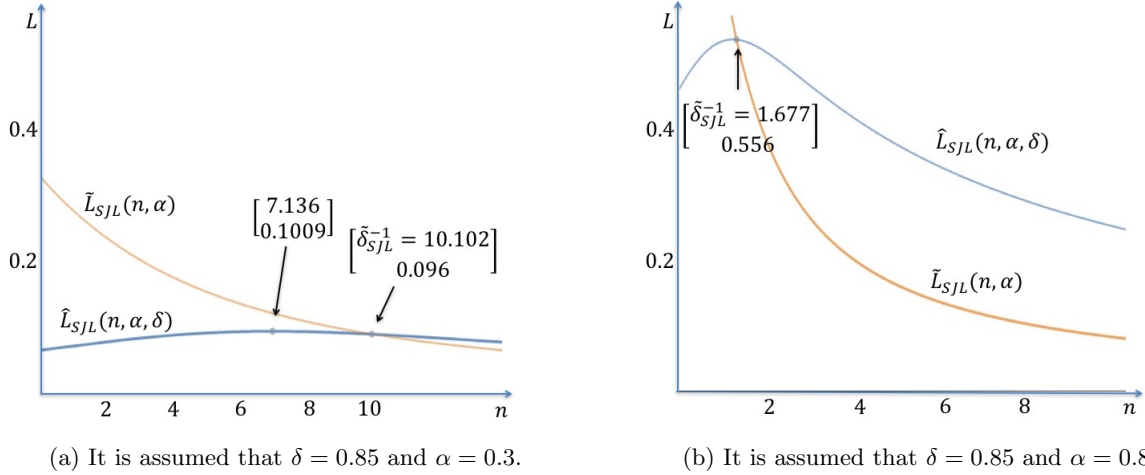
(a) It is assumed that $\delta = 0.85$ and $\alpha = 0.3$.

(b) It is assumed that $\delta = 0.85$ and $\alpha = 0.8$.

Figure 2: The loan ceiling of the SJL contract increases in group size as long as the discount factor belongs to the interval $\left(\hat{\delta}, \tilde{\delta}\right)$.

SJL contract compared to an IL contract. In turn, a smaller repayment amount leads to a higher welfare level for the borrowers.

Proposition 3. *The following statements hold when both the IL and the SJL contracts are feasible.*

1) The borrowers' repayment amount is lower and his welfare is higher under the SJL contract than the IL contract.

2) There exists $\underline{\alpha}$, such that:

a) if $\alpha < \underline{\alpha}$, the loan ceiling of the SJL contract is higher than the IL contract only if at least for some n,

$$\frac{\alpha n - [1 - (1-\alpha)^n]}{\alpha (n-1)[1-(1-\alpha)^n]} < \delta < \frac{[1-(1-\alpha)^n]}{n\alpha^2}; \tag{7}$$

b) if $\alpha > \underline{\alpha}$, the loan ceiling of the IL contract is higher than the SJL contract.

3) For very large n, the loan ceiling of the IL contract is higher than the SJL contract.

Proposition 3 shows that the SJL contract has a positive effect on the borrowers' welfare and repayment amount when compared with the IL contract. This proposition also proves that although the loan ceiling is higher under the IL contract if the chance of project success is high, in case of projects with a lower chance of success, the loan ceiling is higher under the SJL contract, with the condition that (7) is satisfied. Inequality (7) requires the group size to be moderately large.

The results of Proposition 3 relate to the results of Proposition 2, as the IL contract can be

14

Table 2: The Range of Group Sizes for Which the SJL Contract Outperforms the IL Contract for Some Given α

	$n = 2$	$n = 3$	$n = 4$	$n = 5$	$\cdots$	$n = 10$	$\cdots$	$n = 50$
$\alpha < 0.764$	✓							
$\alpha < 0.634$	✓	✓						
$\alpha < 0.552$	✓	✓	✓					
$\alpha < 0.495$	✓	✓	✓	✓				
$\alpha < 0.349$	✓	✓	✓	✓		✓		
$\alpha < 0.150$	✓	✓	✓	✓		✓		✓

mathematically interpreted as a special case of the SJL contract in which $n = 1$. As previously explained in Proposition 2, a larger group can create the certainty of enjoying a stronger repayment insurance, as well as the threat of being in charge of the total repayment. Therefore, when borrowers are investing in highly safe projects, strong repayment insurance is not necessary, and loans can be given to individuals. At the same time, when borrowers invest in highly risky projects, strong repayment insurance is urgent.

Table 2 underlines the main point of Proposition 3 by providing numerical examples of group sizes for which an SJL contract can be better than an IL contract. According to Table 2, for $\alpha = 0.764$, condition (7) can be satisfied by some δ only if $n = 2$. Thus, for such α and any δ that satisfies (7), the SJL contract can do better than the IL contract only in the particular case of $n = 2$. For smaller α, more than one group size exists, for which (7) can be satisfied for some δ and thus the SJL contract can do better. Therefore, $\underline{\alpha} = 0.764$ is the maximum α, for which the SJL contract can outperform the IL contract when the given δ is such that condition (7) holds. For any $\alpha > 0.764$, the lender should offer the IL contract.

Figure 3 illustrates the results of Proposition 3. For the sake of simplicity, it is assumed that $H = 1$. As shown in Figure 3a, when the chance of project success is small enough so that (7) is satisfied (e.g., $\alpha = 0.4$), for any $0 < \delta < 1$, $\hat{L}$ defines the loan ceiling of the SJL contract, which is always equal to or larger than the loan ceiling of the IL contract. However, when the chance of project success is large (e.g., $\alpha = 0.9$), the loan ceiling of the IL contract is strictly higher than the loan ceiling of the SJL contract.
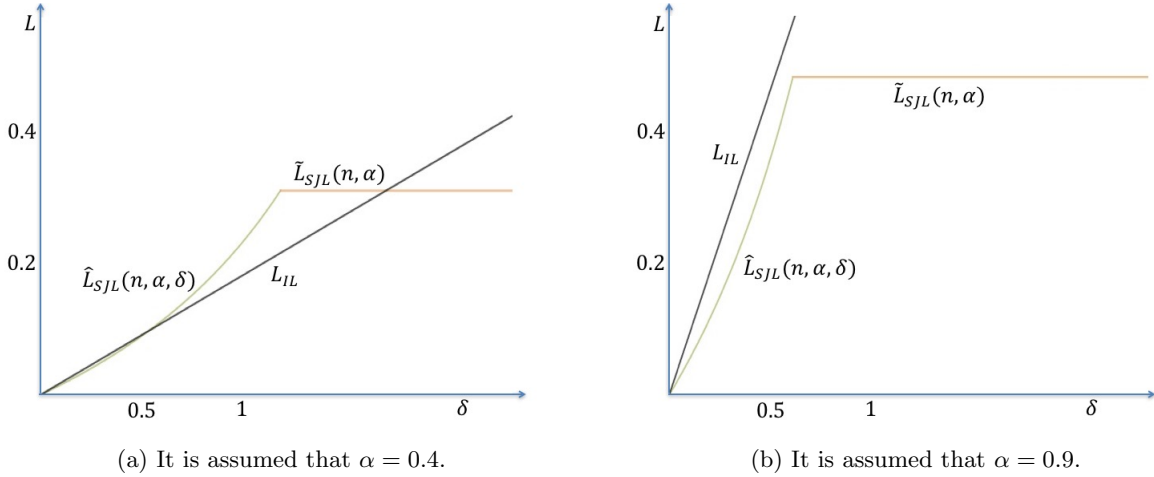
(a) It is assumed that $\alpha = 0.4$.   (b) It is assumed that $\alpha = 0.9$.

Figure 3: For smaller α, the loan ceiling of the SJL contract is higher than of the IL contract, while for larger α, the loan ceiling of the IL contract is higher than of the SJL contract.

5 Partial Joint Liability (PJL) Contract

In our basic model, we assume that repaying group members must pay the total loan to qualify for a loan in the next period. This constraint may be too conservative. It may be the case that repaying members could pay a portion of the total loan but not the entire total loan, while the lender's net present value is still positive. In this section, we assume that a successful member has to pay γR for each defaulting member of his group, where $0 \leq \gamma \leq 1$; that is, he has to pay only a fraction of his defaulting peers' repayment. Note that $\gamma = 0$ and $\gamma = 1$ resemble the case in which they are not liable and the case in which group members are fully liable for each other's repayment, respectively.

However, partial joint liability may give rise to collusion as a new type of strategic default, because successful members may be better off if they pretend that only one of them is successful and only pay the minimum repayment acceptable by the lender (i.e., $R + (n-1)\gamma R$). Collusion can particularly benefit borrowers when the degree of joint liability is low. Thus, our partial joint liability mechanism also has to guard against collusion. To tackle this problem, we assume that if total group repayment is less than nR, the lender audits the states of projects at a cost $c > 0$, with probability q. If the lender verifies any collusion, she excludes them from future financing. Under such joint liability, the lender can ensure that each successful borrower is better off repaying than

16

colluding if

$$H - \frac{1 + (n-1)\gamma}{n-i}R + (1-q)\delta V_{PJL}^R \leq H - \left[1 + \frac{i\gamma}{n-i}\right]R + \delta V_{PJL}^R, \tag{8}$$

where $n-i$ is the number of successful members, and V_{PJL}^R denotes the expected utility of a repaying borrower in each period of receiving loan under the PJL contract. Note that collusion can exist if there are at least two successful members in the group (i.e., $n-i \geq 2$), and thus (8) can be simplified to

$$R \leq \frac{(n-i)q\delta V_{PJL}^R}{(n-1-i)(1-\gamma)}.$$

For a high probability of auditing or a high degree of joint liability, collusion is unlikely and the incentive constraint (8) is slack. Under partial joint liability, the expected repayment for a borrower who plays the repayment strategy can be calculated in each period as follows:

$$\alpha R + \sum_{i=0}^{n-1} \binom{n-1}{i}\alpha^{n-i}(1-\alpha)^i \left(\frac{i}{n-i}\gamma R\right) = \alpha R + \gamma\left[(1-\alpha) - (1-\alpha)^n\right]R.$$

Thus, the expected utility of a borrower who plays a repayment strategy at any period in which he obtains financing is determined as

$$V_{PJL}^R = \alpha H - \left[\alpha + \gamma(1-\alpha) - \gamma(1-\alpha)^n\right]R + \delta\left[1 - (1-\alpha)^n\right]V_{PJL}^R$$

or

$$V_{PJL}^R = \frac{\alpha H - \left[\alpha + \gamma(1-\alpha) - \gamma(1-\alpha)^n\right]R}{1 - \delta\left[1 - (1-\alpha)^n\right]}.$$

In Proposition 4, we will solve the benevolent lender's optimization problem, $\max\limits_{L,R} V_{PJL}^R$, in the case of partial joint liability.

Proposition 4. *There exist* $\bar{q}(n,\gamma,i)$, $\tilde{\delta}_{PJL}(n,\alpha,\gamma)$, *and* $\bar{\delta}_{PJL}(n,\alpha,\gamma,q,i)$ *such that:*

if $q \geq \bar{q}(n,\gamma,i)$,

a) for any $\delta \leq \tilde{\delta}_{PJL}(n,\alpha,\gamma)$, *the PJL contract is feasible iff* $L \leq \hat{L}_{PJL}(n,\alpha,\delta,\gamma,c,q,H)$,

b) for any $\delta \geq \tilde{\delta}_{PJL}(n,\alpha,\gamma)$, *the PJL contract is feasible iff* $L \leq \tilde{L}_{PJL}(n,\alpha,\gamma,c,q,H)$;

if $q < \bar{q}(n,\gamma,i)$,

a) for any $\delta \leq \bar{\delta}_{PJL}(n,\alpha,\gamma,q,i)$, *the PJL contract is feasible iff* $L \leq \bar{L}_{PJL}(n,\alpha,\delta,\gamma,c,q,H,i)$,

17

b) for any $\delta \geq \bar{\bar{\delta}}_{PJL}(n, \alpha, \gamma, q, i)$, the PJL contract is feasible iff $L \leq \tilde{L}_{PJL}(n, \alpha, \gamma, c, q, H)$.

Moreover, whenever the PJL contract is feasible for any $\alpha \neq 0$, the lender demands optimal repayment $R_{PJL} = \frac{L(1+\epsilon)+cq}{[\alpha+\gamma(1-\alpha)-\gamma(1-\alpha)^n]}$ from each borrower, and the expected lifetime utility for each borrower will amount to $V_{PJL}^R = \frac{\alpha H - L(1+\epsilon) - cq}{1-\delta[1-(1-\alpha)^n]}$.

A comparison between Proposition 4 and Proposition 1 shows that the PJL contract compared to the SJL contract has disadvantages in terms of the borrowers' repayment amount and lifetime benefit. Under the PJL contract, the lender has to ask for a higher repayment amount to compensate for the expected cost of auditing, and this higher repayment amount results in a lower lifetime benefit for the borrower. In addition to the expected cost of auditing, the degree of joint liability also influences the borrowers' repayment amount. Under the PJL contract, as the degree of joint liability lowers, the higher the borrowers' repayment amount should be in order to enhance the expected total repayment.

However, under the PJL contract, similar to the SJL contract, the borrowers' repayment amount decreases and the borrowers' welfare increases with group size. Thus, enlarging the group can reverse the negative effect of partial joint liability on the borrowers' welfare and repayment amount at least to some extent.

Proposition 4 suggests that the PJL contract is feasible if and only if the loan ceiling under the PJL contract is limited to the following upper bound,

$$
\mathscr{L}_{SJL}(n, \alpha, \delta, \gamma, q, i) = \min\left\{\hat{L}_{PJL}, \tilde{L}_{PJL}, \bar{L}_{PJL}\right\} = \begin{cases} \hat{L}_{PJL} & q \geq \bar{q} \text{ and } \delta \leq \tilde{\delta}_{PJL} \\ \tilde{L}_{PJL} & q \geq \bar{q} \text{ and } \delta \geq \tilde{\delta}_{PJL} \\ \bar{L}_{PJL} & q < \bar{q} \text{ and } \delta \leq \bar{\bar{\delta}}_{PJL} \\ \tilde{L}_{PJL} & q < \bar{q} \text{ and } \delta \geq \bar{\bar{\delta}}_{PJL} \end{cases}. \quad (9)
$$

The loan ceiling of the PJL contract, as in the SJL contract, depends positively on the borrowers' discount factor, as with the increase of δ, both $\hat{L}_{PJL}$ and $\bar{L}_{PJL}$ strictly increase and $\tilde{L}_{PJL}$ is constant. Intuitively, borrowers who highly value receiving future loans have a higher incentive to repay their loans. Such borrowers could be trusted to repay larger loans whether offered an SJL or a PJL contract.

What is the effect of the degree of joint liability on the loan ceiling of a PJL contract? How should group size be adjusted when partial joint liability replaces full joint liability? The answer to these questions depend on whether the probability of auditing is sufficiently high. In Lemmas 2 and 3, we look at the changes in the determinants of the loan ceiling of the PJL contract (i.e., $\tilde{L}_{PJL}$, $\hat{L}_{PJL}$, $\tilde{\delta}_{PJL}$ when $q \geq \bar{q}$, and $\tilde{L}_{PJL}$, $\bar{L}_{PJL}$, $\bar{\delta}_{PJL}$ when $q < \bar{q}$), with respect to changes of group size n and degree of joint liability γ, respectively.

Lemma 2. *Assume $\tilde{L}_{PJL}$, $\hat{L}_{PJL}$, and $\tilde{\delta}_{PJL}$ are functions defined in Proposition 4.*

1) When $q \geq \bar{q}$, there exists $\hat{\delta}_{PJL}(n, \alpha, \gamma)$ such that $\hat{L}_{PJL}$ is strictly decreasing in n if $0 < \delta < \hat{\delta}_{PJL}$, and $\hat{L}_{PJL}$ is strictly increasing in n if $\hat{\delta}_{PJL} < \delta < \tilde{\delta}_{PJL}$. Moreover, $\tilde{L}_{PJL}$ and $\tilde{\delta}_{PJL}$ are strictly decreasing in n.

2) When $q < \bar{q}$, there exists $\hat{\delta}'_{PJL}(n, \alpha, \gamma, i)$ such that $\bar{L}_{PJL}$ is strictly decreasing in n if $0 < \delta < \hat{\delta}'_{PJL}$, and $\bar{L}_{PJL}$ is strictly increasing in n if $\hat{\delta}'_{PJL} < \delta < \bar{\delta}_{PJL}$. Moreover, $\tilde{L}_{PJL}$ and $\bar{\delta}_{PJL}$ are strictly decreasing in n.

From Lemma 2, it can be inferred that the loan ceiling of the PJL contract increases in group size for some mid-range discount factors, as in the SJL contract. The exact range of such discount factors depends on the probability of auditing. As suggested by Lemma 2, such discount factors, should lie in $\left(\hat{\delta}_{PJL}, \tilde{\delta}_{PJL}\right)$ when the probability of auditing is high, and should lie in $\left(\hat{\delta}'_{PJL}, \bar{\delta}_{PJL}\right)$ when the probability of auditing is low.

Lemma 3. *Assume $\tilde{\delta}_{SJL}$, $\tilde{L}_{SJL}$, and $\hat{L}_{SJL}$ are functions defined in Proposition 1, and $\tilde{\delta}_{PJL}$, $\tilde{L}_{PJL}$, and $\hat{L}_{PJL}$ are functions defined in Proposition 4, and finally, that $\hat{\delta}_{PJL}$ is the function defined in Lemma 2.*

1) If $q \geq \bar{q}$, $\tilde{L}_{PJL}$ and $\hat{L}_{PJL}$ are strictly decreasing in γ. $\hat{\delta}_{PJL}$ is strictly increasing and $\tilde{\delta}_{PJL}$ strictly decreasing in γ.

2) If $q < \bar{q}$, $\tilde{L}_{PJL}$ is strictly decreasing and $\bar{L}_{PJL}$ is strictly increasing in γ. $\hat{\delta}'_{PJL}$ and $\bar{\delta}_{PJL}$ are both strictly decreasing in γ.

When collusion is unlikely, according to the first part of Lemma 3, a lower degree of joint liability increases the loan ceiling of the PJL contract, and allows for a wider range of discount factors under which the loan ceiling of the PJL contract increases in group size (the interval $\left(\hat{\delta}_{PJL}, \tilde{\delta}_{PJL}\right)$ becomes wider).

When collusion is more likely, the second part of Lemma 3 shows that as the degree of joint liability decreases, the interval $\left(\hat{\delta}'_{PJL}, \bar{\delta}_{PJL}\right)$ shifts increasingly to the right; that is, borrowers should have a higher average discount factor to qualify for loans that are increasing in group size. This may be explained by the fact that borrowers with a high valuation for future loans find it less interesting to default strategically. Proposition 5 is a direct result of Lemma 3 and presents a comparison between the PJL and the SJL contracts.

Proposition 5. *The following statements hold when both the SJL and the PJL contracts are feasible.*

1) $\tilde{L}_{PJL} \geq \tilde{L}_{SJL} - cq$, $\hat{L}_{PJL} \geq \hat{L}_{SJL} - cq$, and $\bar{L}_{PJL} \geq \hat{L}_{SJL} - cq$.

2) If $q \geq \bar{q}$, then $\hat{\delta}_{PJL} \leq \hat{\delta}_{SJL}$ and $\tilde{\delta}_{PJL} \geq \tilde{\delta}_{SJL}$.

3) If $q < \bar{q}$, then $\hat{\delta}'_{PJL} > \hat{\delta}_{SJL}$ and $\bar{\delta}_{PJL} > \tilde{\delta}_{SJL}$.

The first part of Proposition 5 proves that the PJL contract is feasible for, at least, the same loan ceiling as the SJL contract minus the expected cost of auditing. Thus, when the expected cost of auditing is low can the PJL contract have a higher loan ceiling than the SJL contract.

The last two parts of Proposition 5 show that, irrespective of the likelihood of collusion, the range of discount factors, for which the loan ceiling of the PJL contract increases in group size, is at least as wide as that of the SJL contract. In other words, the optimal group size under the PJL contract should be larger than it was under the SJL contract. However, when collusion is more likely, the average borrowers' discount factor must be higher than was necessary under the SJL.

6 Flexible Joint Liability (FJL) Contract

In some instances, grim-trigger strategy that group members play against each other under the SJL contract may be too harsh. Consider a situation in which some borrowers default strategically, but it is still beneficial for the other group members to pay the entire repayment and thus be able to obtain a new loan in the next round. Moreover, strategically defaulting members may provide convincing explanations for their behavior, making other group members less willing to punish them severely. In this section, we examine joint liability under a strategy that is less severe than grim-trigger. Players start in the lending phase and cooperate until someone defaults strategically. They then go to the punishment phase and exclude the defaulter for T periods of receiving loans. After

the T period of punishment, members allow the defaulter to re-enter the game. Timing remains the same as in the SJL contract.

The lender's optimization problem in this case is similar to the one encountered in the case of the SJL contract, except for a change in the incentive constraint (3). Each successful borrower must have an incentive not to default strategically and to repay not only for himself, but also for all of the defaulting peers (even for the entire group, in the worst case). Thus we must have,

$$H + \delta^{T+1} V_{FJL}^{R} < H - nR + \delta V_{FJL}^{R},$$

in which V_{FJL}^{R} denotes the expected utility of a repaying borrower at any period of obtaining loan under the FJL contract. The above inequality can be simplified to

$$nR < \left(1 - \delta^{T}\right) \delta V_{FJL}^{R}. \tag{10}$$

Clearly, if T is a large constant, then $1 - \delta^{T} \to 1$, and (10) will be equal to the incentive constraint of the SJL contract. Note that (10) is more difficult to satisfy compared with (3), as it is less costly to default strategically under the FJL contract.

Proposition 6. *There exist* $\tilde{\delta}_{FJL}\left(n, \alpha, \delta, T, H\right)$, $\tilde{L}_{FJL}\left(n, \alpha, H\right)$, *and* $\hat{L}_{FJL}\left(n, \alpha, \delta, T, H\right)$ *such that:*

a) for any $\delta \geq \tilde{\delta}_{FJL}\left(n, \alpha, \delta, T, H\right)$, *the FJL contract is feasible iff* $L \leq \tilde{L}_{FJL}\left(n, \alpha, H\right)$;

b) for any $\delta \leq \tilde{\delta}_{FJL}\left(n, \alpha, \delta, T, H\right)$, *the FJL contract is feasible iff* $L \leq \hat{L}_{FJL}\left(n, \alpha, \delta, T, H\right)$.

Moreover, whenever the FJL contract is feasible, for any $\alpha \neq 0$, *the lender demands the optimal repayment* $R_{FJL} = \frac{L(1+\epsilon)}{[1-(1-\alpha)^{n}]}$ *from each borrower, and the expected lifetime utility for each borrower will amount to* $V_{FJL}^{R} = \frac{H - L(1+\epsilon)}{1 - \delta[1-(1-\alpha)^{n}]}$.

A comparison between Propositions 6 and 1 shows that $R_{FJL} = R_{SJL}$ and $V_{FJL}^{R} = V_{SJL}^{R}$. In other words, the optimal repayment amount as well as borrowers' welfare are not affected by the length of the punishment phase and remain the same under less-severe punishment. Therefore, under the FJL contract, similar to the SJL contract, the optimal repayment R_{FJL} decreases and borrowers' welfare V_{FJL}^{R} increases in group size n. In other words, a larger group is required to pay less, and as a result, enjoy a higher level of welfare.

Moreover, $\tilde{L}_{FJL}$ is independent from T; that is, $\tilde{L}_{FJL} = \tilde{L}_{SJL}$, and for very large T, also

$\tilde{\delta}_{FJL} = \tilde{\delta}_{SJL}$ and $\hat{L}_{FJL} = \hat{L}_{SJL}$. Thus, when the punishment phase is very long, the FJL contract is the same as the SJL contract in terms of loan ceiling. But, what happens if the punishment phase is not very long? Is the FJL contract still as attractive as the SJL contract? Before we answer this question, we should recall that, as suggested by Proposition 6, the FJL contract is feasible if and only if its loan ceiling under is limited to the following upper bound,

$$\mathscr{L}_{FJL}(n, \alpha, \delta, T, H) = \min\left\{\hat{L}_{FJL}, \tilde{L}_{FJL}\right\} = \begin{cases} \hat{L}_{FJL} & \delta \leq \tilde{\delta}_{FJL} \\ \tilde{L}_{FJL} & \delta \geq \tilde{\delta}_{FJL} \end{cases}. \tag{11}$$

The loan ceiling of the FJL contract, similar to that of the SJL contract, depends positively on the borrowers' discount factor, as with the increase of δ, $\hat{L}_{FJL}$ strictly increases and $\tilde{L}_{FJL}$ is constant. Intuitively, borrowers who highly value receiving future loans have a higher incentive to repay their loans whether they are offered an SJL or an FJL contract.

We continue this section by investigating if and how the length of the punishment phase and the group size can affect the loan ceiling of the FJL contract. To this end, we need to know how the determinants of the loan ceiling of the FJL contract (i.e., $\hat{L}_{FJL}$, $\tilde{L}_{FJL}$, and $\tilde{\delta}_{FJL}$) react to the changes in group size n and length of punishment period T. Note that $\tilde{L}_{FJL} = \tilde{L}_{SJL}$, and thus, $\tilde{L}_{FJL}$ does not depend on T and is strictly decreasing in n. In Lemmas 4 and 5, we look at the changes of $\hat{L}_{FJL}$ and $\tilde{\delta}_{FJL}$ with respect to changes of n and T, respectively.

Lemma 4. *Assume* $\hat{L}_{FJL}(n, \alpha, \delta, T, H)$ *and* $\tilde{\delta}_{FJL}(n, \alpha, \delta, T, H)$ *are functions defined in Proposition 6.*

1) There exists a $\hat{\delta}_{FJL}(n, \alpha)$ *such that* $\hat{L}_{FJL}(n, \alpha, \delta, T, H)$ *is strictly decreasing in* n *if* $0 < \delta < \hat{\delta}_{FJL}(n, \alpha)$; $\hat{L}_{FJL}(n, \alpha, \delta, T, H)$ *is strictly increasing in* n *if* $\hat{\delta}_{FJL}(n, \alpha) < \delta < \tilde{\delta}_{FJL}(n, \alpha, \delta, T, H)$.

2) $\tilde{\delta}_{FJL}(n, \alpha, \delta, T, H)$ *is strictly decreasing in* n.

Lemma 4 proves that the loan ceiling of the FJL contract increases in group size for mid-range discount factors, as in the SJL contract. Feasible discount factors in case of the FJL contract have to lie in the interval $\left(\hat{\delta}_{FJL}, \tilde{\delta}_{FJL}\right)$. A comparison between Lemma 4 and Lemma 1 shows that the loan ceiling of the FJL and the SJL contracts respond similarly to the changes in group size. More specifically, the changes of $\hat{L}_{FJL}$ and $\tilde{\delta}_{FJL}$ are in the same direction of changes of $\hat{L}_{SJL}$ and $\tilde{\delta}_{SJL}$, with respect to the changes of n (note that $\tilde{L}_{FJL} = \tilde{L}_{SJL}$ and $\hat{\delta}_{FJL} = \hat{\delta}_{SJL}$, and thus both $\tilde{L}_{FJL}$ and $\hat{\delta}_{FJL}$ are independent from T).

Lemma 5. *Assume $\hat{L}_{FJL}(n, \alpha, \delta, T, H)$ and $\tilde{\delta}_{FJL}(n, \alpha, \delta, T, H)$ are functions defined in Proposition 6.*

    1) $\hat{L}_{FJL}(n, \alpha, \delta, T, H)$ is strictly increasing in T.

    2) $\tilde{\delta}_{FJL}(n, \alpha, \delta, T, H)$ is strictly decreasing in T.

Lemma 5 proves that decreasing the length of the punishment period leads to two different effects on the loan ceiling of the FJL contract. Although the first result of this lemma argues that decreasing the length of the punishment period decreases $\hat{L}_{FJL}$, the second result of this lemma promises that decreasing the length of the punishment period increases the range of discount factors for which the loan ceiling of the PJL contract increases in group size. In other words, when the punishment phase is not that long—and thus it is less costly for members to default strategically—larger groups are needed to assure repayment. Proposition 7 builds on the findings of Lemma 5 and compares the FJL and SJL contracts.

Proposition 7. *Assume $\hat{L}_{SJL}(n, \alpha, \delta, H)$ and $\tilde{\delta}_{SJL}(n, \alpha, H)$ are functions defined in Proposition 1, and $\hat{L}_{FJL}(n, \alpha, \delta, T, H)$ and $\tilde{\delta}_{FJL}(n, \alpha, \delta, T, H)$ are functions defined in Proposition 6. The following statements hold when both of the SJL and the FJL contracts are feasible.*

    1) $\hat{L}_{FJL}(n, \alpha, \delta, T, H) \leq \hat{L}_{SJL}(n, \alpha, \delta, H)$.

    2) $\tilde{\delta}_{FJL}(n, \alpha, \delta, T, H) \geq \tilde{\delta}_{SJL}(n, \alpha, H)$.

According to the first part of Proposition 7, reducing the length of the punishment period creates a disadvantage for joint liability contracts, in terms of the loan ceiling. The FJL contract is feasible for, at most, the same loan ceiling as the SJL contract. Intuitively, the more forgiving the lender is towards strategic default, the less likely the group members are to fulfill their repayment obligations.

However, the second part of Proposition 7 proves that this disadvantage could be offset by forming larger groups. More specifically, the interval $\left(\hat{\delta}_{FJL}, \tilde{\delta}_{FJL}\right)$ is wider than the interval $\left(\hat{\delta}_{SJL}, \tilde{\delta}_{SJL}\right)$; that is, under the FJL contract, a larger range of discount factors allow the loan ceiling to increase in group size, as compared with the SJL contract. In other words, the FJL contract accommodates larger group sizes, as compared with the SJL contract, and larger group sizes enhance the loan ceiling of the FJL contract.

7 Project Correlation

Although most of the literature on microfinance group lending assumes independence between projects,[4] it may be more realistic to assume that projects are correlated, especially when group members self-select. In this section, we examine the effect of project correlation on the lending outcome. We include project correlation in our model by assuming that α is not a constant, but an increasing function of n. This is a realistic assumption because being in charge of each other's repayment would increase cooperation among group members, and thus group members may actually contribute to each other's chances of project success. We assume that $\alpha(n)$ is a concave function, which is also a sensible assumption, as a very large group would also deal with higher tensions that could have an adverse effect on projects' success.

Since assuming project correlation does not affect our model, and except for the fact that α depends on n, the results of Proposition 1 remain valid. Therefore, if $\tilde{\delta}'_{SJL}$, $\tilde{L}'_{SJL}$, and $\hat{L}'_{SJL}$ are as defined in Proposition 1 with the only difference being that α is now a function of n instead of a constant, the following statements hold. For any $\delta \geq \tilde{\delta}'_{SJL}$, the SJL contract is feasible if and only if $L \leq \tilde{L}'_{SJL}$, and for any $\delta \leq \tilde{\delta}'_{SJL}$, the SJL contract is feasible if and only if $L \leq \hat{L}'_{SJL}$. Moreover, whenever the SJL contract is feasible, for any $\alpha \neq 0$, the lender demands optimal repayment $R'_{SJL} = \frac{L(1+\epsilon)}{[1-(1-\alpha(n))^n]}$ from each borrower, and the expected lifetime utility for each borrower will amount to $V'^R_{SJL} = \frac{\alpha H - L(1+\epsilon)}{1-\delta[1-(1-\alpha(n))^n]}$.

In what follows, we examine the effect of project correlation on the loan ceiling of the SJL contract. The following lemma provides us with some insight into the conditions under which the loan ceiling can increase in the group size. To further simplify the calculation, it is assumed that $H = 1$.

Lemma 6. *Assume $\tilde{\delta}'_{SJL}(n, \alpha(n))$, $\tilde{L}'_{SJL}(n, \alpha(n))$, and $\hat{L}'_{SJL}(n, \alpha(n), \delta)$ are functions defined in Proposition 1 with the only difference being that α is now a function of n instead of a constant.*

1) If $\frac{d\alpha(n)}{dn} < \frac{\alpha(n)}{n-1}$, then there exists $\hat{\delta}'_{SJL}\left(n, \alpha(n), \frac{d\alpha(n)}{dn}\right)$ such that:

a) for any $0 < \delta < \hat{\delta}'_{SJL}\left(n, \alpha(n), \frac{d\alpha(n)}{dn}\right)$, $\hat{L}'_{SJL}(n, \alpha(n), \delta)$ is strictly decreasing in n;

b) for any $\hat{\delta}'_{SJL}\left(n, \alpha(n), \frac{d\alpha(n)}{dn}\right) < \delta < \tilde{\delta}'_{SJL}(n, \alpha, \delta)$, $\hat{L}'_{SJL}(n, \alpha(n), \delta)$ is strictly increasing in n.

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[4]There are some exceptions that consider project correlations. As notable examples, we mention Laffont (2003) and Ahlin and Townsend (2007).

2) If $\frac{d\alpha(n)}{dn} < \frac{1-(1-\alpha(n))^n+(1-\alpha(n))^n \ln(1-\alpha(n))^n}{n(1-\alpha(n))^{n-1}}$, then $\tilde{L}'_{SJL}(n, \alpha(n))$ is strictly decreasing in n.

Lemma 6 demonstrates that the determinants of the loan ceiling of the SJL contract (i.e., $\hat{\delta}'_{SJL}$, $\hat{L}'_{SJL}$, and $\tilde{L}'_{SJL}$) exhibit the same behavior, both in the case of projects being independent as well as weakly correlated (i.e., the correlation degree, $\frac{d\alpha(n)}{dn}$, is lower than the boundaries suggested in Lemma 6). More specifically, when projects are weakly correlated, the loan ceiling of the SJL contract could increase in the group size for some mid-range discount factors that lie in $\left(\hat{\delta}'_{SJL}, \tilde{\delta}'_{SJL}\right)$. Therefore, if projects are weakly correlated, whether the loan ceiling of the SJL contract increases in n depends on $\left(\hat{\delta}'_{SJL}, \tilde{\delta}'_{SJL}\right)$ being non-empty. Contrary to the case in which projects are independent, this is not necessarily the only time that the loan ceiling of the SJL contract could increase in n.

From Lemma 6, one could also infer that a strong correlation (i.e., when the correlation degree, $\frac{d\alpha(n)}{dn}$, is higher than both boundaries suggested in Lemma 6) reverses the behavior of $\hat{L}'_{SJL}$ and $\tilde{L}'_{SJL}$. To be precise, for any $0 < \delta < \hat{\delta}'_{SJL}$, $\hat{L}'_{SJL}$ will strictly increase in n, and for any $\hat{\delta}'_{SJL} < \delta < \tilde{\delta}'_{SJL}$, $\hat{L}'_{SJL}$ will strictly decrease in n. Moreover, $\tilde{L}'_{SJL}$ will always strictly increase in n. Put simply, when projects are strongly correlated, the loan ceiling of the SJL contract always increases in the group size, with the exception of mid-range discount factors that lie in $\left(\hat{\delta}'_{SJL}, \tilde{\delta}'_{SJL}\right)$. Proposition 8 proves that $\left(\hat{\delta}'_{SJL}, \tilde{\delta}'_{SJL}\right)$ is non-empty when projects are weakly correlated, but is empty when projects are strongly correlated.

Proposition 8. *When projects are correlated, the loan ceiling of the SJL contract increases in group size.*

According to Proposition 8, a larger feasible range of borrowers' discount factor allow the loan ceiling to increase in group size when there is any project correlation. This proposition is related to our assumption that jointly liable group members positively contribute to the success of each other's project and therefore are more reliable in fulfilling their repayment obligations.

8 Conclusion

The original question that motivated our study is determining how group size affects the efficiency of microcredit lending under joint liability contracts and what is the optimal group size.

Most of the existing theoretical papers on microfinance discuss joint liability lending in groups of only two members, although experimental and empirical studies have emphasized the importance of group size (Abbink et al., 2006; Galak et al., 2011). Our results show that group size can be an influential factor in improving lending efficiency, and the assumption $n = 2$, which has been pervasively used in the literature, may result in neglecting the effect of group size.

Our results show that although joint liability contracts are feasible under a smaller set of parameter values than individual lending contracts, for riskier projects, the loan ceiling of joint liability contracts can increase as the group enlarges. We conclude that larger groups are more reliable in repaying loans when projects are risky. However, the group size cannot become too large. Our findings confirm the results of Conning (2004) and Ahlin (2015), who focus on different issues (i.e., free-riding and local borrower information, respectively) and conclude that the size of borrowers' groups must be limited.

Furthermore, we discuss that joint liability, when feasible, has a positive effect on the borrowers' repayment amount as well as borrowers' welfare compared to individual liability. Joint liability can also outperform individual liability in terms of the loan ceiling. We derive this result under the assumption that group members play a strong punishment strategy against any strategically defaulting member; that is, they exclude the strategically defaulting member from the lending game at least for a T period of time and maybe forever. From this perspective, we are in line with the existing literature that has found that a lender may be better off choosing the individual liability contract over the joint liability contract, unless borrowers are able to impose strong social sanctions on one another (Besley and Coate, 1995; Armendáriz de Aghion, 1999).

In addition to comparing joint liability and individual liability contracts, our study examined the possibility of relaxing some of the simplifying assumptions that we made in our basic model of joint liability. First, we examined how flexible a lender could be about the degree of joint liability. Is full joint liability the only possible mechanism or could a lender let the group repay only a fraction of the defaulting members' repayment? We demonstrated that any degree less than full joint liability creates disadvantages in a joint liability contract.

We next sought to learn how much remission is possible to the defaulting borrowers. Should they be excluded from future loans forever or for only T periods? Although we prove that a less-severe punishment creates a disadvantage for joint liability in terms of loan ceiling, this negative effect

could be offset by forming larger groups. Finally, we investigated the effect of project correlation on the loan ceiling of joint liability contracts.

Our model is simple and could be generalized in several ways. An immediate direction for extending our model is relaxing the assumption that microentrepreneur borrowers are identical. Borrowers may differ in the level of risk they are willing to take when they are choosing their projects or in the level of effort they are willing to exert in their projects. A more general setting that allows for different types of borrowers could be used to examine, for example, the effect of risk pooling on the outcome of joint liability.

Another interesting direction for extending our model is to include the structure of the borrowers' social network. Risk-sharing and risk-pooling agreements among borrowers can be influenced by the structure of their social networks (see e.g., Allen and Babus, 2009; Feinberg, 2013). We leave these issues for future research.

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9 Appendix

Proof of Proposition 1. In the case of an SJL contract, the optimal contract determines (L, R) and is a solution to the following problem:

$$\max_{L,R} \quad V_{SJL}^R \quad = \quad \frac{\alpha H - R\left[1 - (1-\alpha)^n\right]}{1 - \delta\left[1 - (1-\alpha)^n\right]} \tag{12}$$

$$s.t. \quad nR \quad \leq H \tag{13}$$

$$nR \quad \leq \delta V_{SJL}^R \tag{14}$$

$$R \quad \geq \frac{L\,(1+\epsilon)}{1 - (1-\alpha)^n}. \tag{15}$$

As V_{SJL}^R decreases in R, the lender sets R as low as possible. Constraint (15) gives the minimum R required for breaking even; that is $R_{SJL} = \frac{L(1+\epsilon)}{[1-(1-\alpha)^n]}$, for any $\alpha \neq 0$. As long as R is limited to the upper limits expressed in constraints (13) and (14), the SJL contract is feasible; otherwise it is not feasible. Replacing R_{SJL} in the optimization problem, we will have

$$\max_{L,R} \quad V_{SJL}^R \quad = \quad \frac{\alpha H - L\,(1+\epsilon)}{1 - \delta\left[1 - (1-\alpha)^n\right]}$$

$$s.t. \quad L \quad \leq \left(\frac{1}{1+\epsilon}\right) \times \frac{\left[1 - (1-\alpha)^n\right] H}{n} \equiv \tilde{L}_{SJL}\,(n, \alpha, H)$$

$$L \quad \leq \left(\frac{1}{1+\epsilon}\right) \times \frac{\delta \alpha H \left[1 - (1-\alpha)^n\right]}{n - \delta\,(n-1)\left[1 - (1-\alpha)^n\right]} \equiv \hat{L}_{SJL}\,(n, \alpha, \delta, H).$$

Any feasible solution for the above problem must satisfy both constraints; that is, we must have $L \leq \min\left\{\hat{L}_{SJL}, \tilde{L}_{SJL}\right\}$. There are two cases: either $\tilde{L}_{SJL} \leq \hat{L}_{SJL}$ or $\tilde{L}_{SJL} \geq \hat{L}_{SJL}$.

i) $\tilde{L}_{SJL} \leq \hat{L}_{SJL}$ if and only if

$$\delta \quad \geq \quad \frac{1}{\alpha + \frac{n-1}{n}\left[1 - (1-\alpha)^n\right]} \equiv \tilde{\delta}_{SJL}\,(n, \alpha)\,;$$

ii) and $\hat{L}_{SJL} \leq \tilde{L}_{SJL}$ if and only if $\delta \leq \tilde{\delta}_{SJL}$.

Thus, for $\delta \leq \tilde{\delta}_{SJL}$, the SJL contract is feasible for any $L \leq \hat{L}_{SJL}$, and for $\delta \geq \tilde{\delta}_{SJL}$, the SJL contract is feasible for any $L \leq \tilde{L}_{SJL}$. □

Proof of Lemma 1. 1)

$$\frac{\partial \hat{L}_{SJL}}{\partial n} = \left(\frac{1}{1+\epsilon}\right) \times \frac{\delta \alpha H \left[-(1-\alpha)^n \ln(1-\alpha)^n + \delta \left[1-(1-\alpha)^n\right]^2 - \left[1-(1-\alpha)^n\right]\right]}{\left[n - \delta(n-1)\left[1-(1-\alpha)^n\right]\right]^2}.$$

$\frac{\partial \hat{L}_{SJL}}{\partial n} < 0$ if and only if

$$-(1-\alpha)^n \ln(1-\alpha)^n + \delta \left[1-(1-\alpha)^n\right]^2 - \left[1-(1-\alpha)^n\right] < 0,$$

or

$$\delta < \frac{\left[1-(1-\alpha)^n\right] + (1-\alpha)^n \ln(1-\alpha)^n}{\left[1-(1-\alpha)^n\right]^2} \equiv \hat{\delta}_{SJL}(n,\alpha).$$

Thus, for any $0 < \delta < \hat{\delta}_{SJL}$, $\hat{L}_{SJL}$ strictly decreases in n, and if for some δ, $\hat{\delta}_{SJL} < \delta < \tilde{\delta}_{SJL}$, $\hat{L}_{SJL}$ strictly increases in n.

2) $\tilde{L}_{SJL}$ is strictly decreasing in n, because

$$\frac{\partial \tilde{L}_{SJL}}{\partial n} = \left(\frac{1}{1+\epsilon}\right) \times \frac{(1-\alpha)^n \left[1 - \ln(1-\alpha)^n\right] H - H}{n^2} < 0.$$

It could be simply verified that $(1-\alpha)^n \left[1 - \ln(1-\alpha)^n\right] < 1$.

3) First, derivatives of $\hat{\delta}_{SJL}$ with respect to n and α after simplification are as follows:

$$\frac{\partial \hat{\delta}_{SJL}}{\partial n} = (1-\alpha)^n \ln(1-\alpha)^n \times \frac{\ln(1-\alpha)^n \left[1 + (1-\alpha)^n\right] + 2\left[1 - (1-\alpha)^n\right]}{\left[1 - (1-\alpha)^n\right]^3}$$

$$\frac{\partial \hat{\delta}_{SJL}}{\partial \alpha} = -n(1-\alpha)^{n-1} \times \frac{\ln(1-\alpha)^n \left[1 + (1-\alpha)^n\right] + 2\left[1 - (1-\alpha)^n\right]}{\left[1 - (1-\alpha)^n\right]^3}.$$

Clearly, $\hat{\delta}_{SJL}$ is strictly increasing in n and in α if and only if

$$K(n,\alpha) \equiv \ln(1-\alpha)^n + \frac{2\left[1 - (1-\alpha)^n\right]}{\left[1 + (1-\alpha)^n\right]} < 0.$$

31

One could simply verify that $K(n, \alpha) < 0$.

Second, $\tilde{\delta}_{SJL}$ strictly decreases in both n and α simply because $[1 - (1 - \alpha)^n]$ strictly increases in both n and α. $\square$

Proof of Proposition 2. According to Lemma 1, for given α and δ, if there are some n such that $\hat{\delta}_{SJL} < \delta < \tilde{\delta}_{SJL}$, the loan ceiling $\mathscr{L}_{SJL}(n, \alpha, \delta, H)$ will then be strictly increasing in n. Everywhere else, $\mathscr{L}_{SJL}(n, \alpha, \delta, H)$ will be strictly decreasing in n.

1) It can be verified that

i. $\lim\limits_{\alpha \to 0} \hat{\delta}_{SJL}(n, \alpha) = -\infty$ and $\lim\limits_{\alpha \to 0} \tilde{\delta}_{SJL}(n, \alpha) = \infty$;

ii. $\lim\limits_{\alpha \to 1} \hat{\delta}_{SJL}(n, \alpha) = 1$ and $\lim\limits_{\alpha \to 1} \tilde{\delta}_{SJL}(n, \alpha) = \frac{n}{2n-1}$.

That is, for very small α, we always have $\hat{\delta}_{SJL}(n, \alpha) < \tilde{\delta}_{SJL}(n, \alpha)$, and for very large α, we always have $\hat{\delta}_{SJL}(n, \alpha) > \tilde{\delta}_{SJL}(n, \alpha)$. We also know from Lemma 1 that $\hat{\delta}_{SJL}$ and $\tilde{\delta}_{SJL}$ are strictly monotonic in α. Therefore, we should conclude that $\hat{\delta}_{SJL}$ and $\tilde{\delta}_{SJL}$ coincide only once at some critical $\bar{\alpha} \neq 0$. So for any given n, if $\alpha \in (0, \bar{\alpha})$, then $\hat{\delta}_{SJL} < \tilde{\delta}_{SJL}$. Consequently, for $\alpha \in (0, \bar{\alpha})$, the loan ceiling will be increasing in n and maximized at

$$N_{\alpha, \delta} = \max \left\{ n \mid \quad \hat{\delta}_{SJL} < \delta < \tilde{\delta}_{SJL} \right\}.$$

Therefore, when α and δ are given, the maximum n can be calculated as follows. Start with $n = 2$ and increase n one by one until one of the $\hat{\delta}_{SJL}$ or $\tilde{\delta}_{SJL}$ equals the given δ, and n cannot be increased further. If $\hat{\delta}_{SJL} = \delta$, then $N_{\alpha, \delta} = \lfloor \hat{\delta}_{SJL}^{-1} \rfloor$ (note that we are only interested in n that is a natural number) and if $\tilde{\delta}_{SJL} = \delta$, then $N_{\alpha, \delta} = \lfloor \tilde{\delta}_{SJL}^{-1} \rfloor$ (see Fig. 1 for intuition). Thus, $N_{\alpha, \delta}$ can be rewritten as $N_{\alpha, \delta} = \min \left\{ \lfloor \hat{\delta}_{SJL}^{-1} \rfloor, \lfloor \tilde{\delta}_{SJL}^{-1} \rfloor \right\}$. A direct result from the abovementioned discussion is that for $\alpha > \bar{\alpha}$, the loan ceiling will decrease in n, and thus the maximum loan ceiling of the SJL contract can be achieved for $n = 2$.

2) We know from Lemma 1 that as n grows larger, the interval $\left(\hat{\delta}_{SJL}, \tilde{\delta}_{SJL} \right)$ becomes smaller. Let's see how small the interval can become when $n \to \infty$.

$$\lim\limits_{n \to \infty} \hat{\delta}_{SJL} = \lim\limits_{n \to \infty} \frac{[1 - (1 - \alpha)^n] + (1 - \alpha)^n \ln(1 - \alpha)^n}{[1 - (1 - \alpha)^n]^2} = \lim\limits_{n \to \infty} 1 - (1 - \alpha)^n = 1$$

and

$$\lim_{n\to\infty} \tilde{\delta}_{SJL} = \lim_{n\to\infty} \frac{1}{\alpha + \frac{n-1}{n}\left[1-(1-\alpha)^n\right]} = \frac{1}{\alpha+1} < 1.$$

Therefore, when $n \to \infty$, $\tilde{\delta}_{SJL} < \hat{\delta}_{SJL}$ and the interval $\left(\hat{\delta}_{SJL}, \tilde{\delta}_{SJL}\right)$ is empty. As a result, when $n \to \infty$, the loan ceiling of the SJL contract decreases in n. □

Proof of Proposition 3. 1) When both the IL and SJL contracts are feasible, each borrower is supposed to repay $R_{SJL} = \frac{L(1+\varepsilon)}{[1-(1-\alpha)^n]}$ under the SJL contract and $R_{IL} = \frac{L(1+\varepsilon)}{\alpha}$ under the IL contract. Clearly, a borrower pays less under the SJL contract. Each borrower's expected lifetime utility under the SJL contract is $V_{SJL}^R = \frac{\alpha H - L(1+\varepsilon)}{1-\delta[1-(1-\alpha)^n]}$, while under the IL contract, it is $V_{IL}^R = \frac{\alpha H - L(1+\varepsilon)}{1-\alpha\delta}$. Obviously, the expected lifetime utility of each borrower is higher under the SJL contract.

2) Which of the SJL or the IL contract can offer larger loans? There are two cases:

i) If $\delta \leq \tilde{\delta}_{SJL}$, then the SJL contract is feasible for all $L \leq \hat{L}_{SJL} = \frac{\alpha^2 \delta H}{1+\epsilon} \times \frac{[1-(1-\alpha)^n]}{\alpha[n-\delta(n-1)[1-(1-\alpha)^n]]} - \varepsilon$, and the IL contract is feasible for all $L \leq \frac{\alpha^2 \delta H}{1+\epsilon}$. Thus, the loan ceiling of the SJL contract is higher than the IL contract if and only if

$$\frac{[1-(1-\alpha)^n]}{\alpha\left[n-\delta\left(n-1\right)\left[1-(1-\alpha)^n\right]\right]} > 1,$$

which can be rewritten as

$$\delta > \frac{\alpha n - [1-(1-\alpha)^n]}{\alpha\left(n-1\right)\left[1-(1-\alpha)^n\right]}.$$

Since we already have an upper bound for δ, it must be true that

$$\frac{\alpha n - [1-(1-\alpha)^n]}{\alpha\left(n-1\right)\left[1-(1-\alpha)^n\right]} < \delta \leq \tilde{\delta}_{SJL}.$$

ii) If $\delta \geq \tilde{\delta}_{SJL}$, then the SJL contract is feasible for all $L \leq \tilde{L}_{SJL} = \frac{H[1-(1-\alpha)^n]}{n(1+\epsilon)}$ and again the IL contract is feasible if $L \leq \frac{\alpha^2 \delta H}{1+\epsilon}$. Thus, the loan ceiling of the SJL contract is higher than the IL contract if and only if

$$\delta < \frac{[1-(1-\alpha)^n]}{n\alpha^2}.$$

33

As we already have a lower bound for δ, it must be true that

$$\tilde{\delta}_{SJL} \leq \delta < \frac{[1-(1-\alpha)^n]}{n\alpha^2}.$$

Comparing the results of parts i and ii, the loan ceiling of the SJL contract is higher than the IL contract if and only if

$$\frac{\alpha n - [1-(1-\alpha)^n]}{\alpha(n-1)[1-(1-\alpha)^n]} < \delta < \frac{[1-(1-\alpha)^n]}{n\alpha^2}; \tag{16}$$

otherwise, the loan ceiling of the IL contract is higher than the SJL contract. Both the right-hand side and the left-hand side of (16) are strictly monotonic in $\alpha \in (0,1)$. If α is very large, it is easy to see that (16) never holds, because its left-hand side becomes larger than its right-hand side for any n:

$$\lim_{\alpha \to 1} \frac{\alpha n - [1-(1-\alpha)^n]}{\alpha(n-1)[1-(1-\alpha)^n]} = 1$$

$$\lim_{\alpha \to 1} \frac{[1-(1-\alpha)^n]}{n\alpha^2} = \frac{1}{n}.$$

If α is very small, (16) can hold, because for any n, its left-hand side remains smaller than its right-hand side:

$$\lim_{\alpha \to 0} \frac{\alpha n - [1-(1-\alpha)^n]}{\alpha(n-1)[1-(1-\alpha)^n]} = \frac{1}{2}$$

$$\lim_{\alpha \to 0} \frac{[1-(1-\alpha)^n]}{n\alpha^2} = \infty.$$

Therefore, we should conclude that there exists a critical $\underline{\alpha}$ such that for any $\alpha < \underline{\alpha}$, (16) can hold, and for any $\alpha > \underline{\alpha}$, (16) never holds.

3) For very large n, (16) never holds, because its left-hand side becomes larger than its right-hand side independent from the magnitude of α:

$$\lim_{n \to \infty} \frac{\alpha n - [1-(1-\alpha)^n]}{\alpha(n-1)[1-(1-\alpha)^n]} = \lim_{n \to \infty} \frac{\alpha n - 1}{\alpha(n-1)} = 1$$

$$\lim_{n \to \infty} \frac{[1 - (1 - \alpha)^n]}{n\alpha^2} = 0. \quad \square$$

Proof of Proposition 4. The lender's optimization problem in the case of partial joint liability can be stated as follows:

$$\max_{L,R} \quad V_{PJL}^R = \frac{\alpha H - [\alpha + \gamma(1-\alpha) - \gamma(1-\alpha)^n]R}{1 - \delta[1 - (1-\alpha)^n]} \tag{17}$$

$$s.t. \quad R \leq \frac{H}{1 + (n-1)\gamma} \tag{18}$$

$$R \leq \frac{\delta V_{PJL}^R}{1 + (n-1)\gamma} \tag{19}$$

$$R \leq \frac{(n-i)q\delta V_{PJL}^R}{(n-i-1)(1-\gamma)} \tag{20}$$

$$R \geq \frac{L(1+\epsilon) + cq}{[\alpha + \gamma(1-\alpha) - \gamma(1-\alpha)^n]}, \tag{21}$$

where $0 \leq q, \gamma, \alpha, \delta \leq 1$. Clearly, if $\gamma \to 1$, this maximization problem will be the same as that of the SJL contract. As V_{PJL}^R is decreasing in R, from constraint (21), the optimal repayment will be $R_{PJL} = \frac{L(1+\epsilon)+cq}{[\alpha+\gamma(1-\alpha)-\gamma(1-\alpha)^n]}$. The optimization problem after replacing R_{PJL} simplifies to

$$\max_{L,R} \quad V_{PJL}^R = \frac{\alpha H - L(1+\epsilon) - cq}{1 - \delta[1 - (1-\alpha)^n]}$$

$$s.t. \quad L \leq \left(\frac{1}{1+\epsilon}\right) \frac{H[\alpha + \gamma(1-\alpha) - \gamma(1-\alpha)^n]}{1 + (n-1)\gamma} - cq$$

$$\equiv \tilde{L}_{PJL}(n, \alpha, \gamma, c, q, H) \tag{22}$$

$$L \leq \left(\frac{1}{1+\epsilon}\right) \frac{\delta \alpha H[\alpha + \gamma(1-\alpha) - \gamma(1-\alpha)^n]}{[1 + (n-1)\gamma][1 - \delta + \delta(1-\alpha)^n] + \delta[\alpha + \gamma(1-\alpha) - \gamma(1-\alpha)^n]} - cq$$

$$\equiv \hat{L}_{PJL}(n, \alpha, \delta, \gamma, c, q, H) \tag{23}$$

$$L \leq \left(\frac{1}{1+\epsilon}\right) \frac{\delta \alpha H[\alpha + \gamma(1-\alpha) - \gamma(1-\alpha)^n]}{\frac{(n-i-1)(1-\gamma)}{q(n-i)}[1 - \delta + \delta(1-\alpha)^n] + \delta[\alpha + \gamma(1-\alpha) - \gamma(1-\alpha)^n]} - cq$$

$$\equiv \bar{L}_{PJL}(n, \alpha, \delta, \gamma, c, q, i, H), \tag{24}$$

in which depending on the magnitude of q, one of the incentive constraints (23) or (24) is slack. More specifically, when

$$[1 + (n-1)\gamma] > \frac{(n-i-1)(1-\gamma)}{q(n-i)}$$

or

$$q > \frac{(n-i-1)(1-\gamma)}{(n-i)[1+(n-1)\gamma]} \equiv \bar{q}(n,\gamma,i),$$

(24) is slack; otherwise (23) is slack.

1) If $q \geq \bar{q}$ and collusion is unlikely, any feasible solution for the optimization problem must satisfy constraints (22) and (23); that is, we must have $L \leq \min\left\{\hat{L}_{PJL}, \tilde{L}_{PJL}\right\}$.

i) $\hat{L}_{PJL} \leq \tilde{L}_{PJL}$ if and only if

$$\delta \leq \frac{1}{\alpha + [1 - (1-\alpha)^n] - \dfrac{[\alpha + \gamma(1-\alpha) - \gamma(1-\alpha)^n]}{[1+(n-1)\gamma]}}$$
$$\equiv \tilde{\delta}_{PJL}(n,\alpha,\gamma),$$

ii) and $\tilde{L}_{PJL} \leq \hat{L}_{PJL}$ if and only if $\delta \geq \tilde{\delta}_{PJL}$.

2) If $q < \bar{q}$ and collusion is more likely, any feasible solution for the optimization problem must satisfy constraints (22) and (24); that is, we must have $L \leq \min\left\{\bar{L}_{PJL}, \tilde{L}_{PJL}\right\}$.

i) $\bar{L}_{PJL} \leq \tilde{L}_{PJL}$ if and only if

$$\delta \leq \frac{1}{[1 - (1-\alpha)^n] - \dfrac{(n-i)q\gamma}{(n-i-1)(1-\gamma)}[1 - n\alpha - (1-\alpha)^n]}$$
$$\equiv \bar{\delta}_{PJL}(n,\alpha,\gamma,q,i).$$

ii) $\tilde{L}_{PJL} \leq \bar{L}_{PJL}$ if and only if $\delta \geq \bar{\delta}_{PJL}$. $\square$

Proof of Lemma 2. 1) Assume, $q \geq \bar{q}$. First,

$$\frac{\partial \hat{L}_{PJL}}{\partial n} = -\left(\frac{\delta \alpha H}{1+\epsilon}\right) \frac{[1+(n-1)\gamma][\gamma + \alpha\delta - \gamma\alpha\delta](1-\alpha)^n \ln(1-\alpha)}{([1+(n-1)\gamma][1-\delta+\delta(1-\alpha)^n]+\delta[\alpha+\gamma(1-\alpha)-\gamma(1-\alpha)^n])^2}$$

$$-\left(\frac{\delta \alpha H}{1+\epsilon}\right) \frac{\gamma[1-\delta+\delta(1-\alpha)^n][\alpha+\gamma(1-\alpha)-\gamma(1-\alpha)^n]}{([1+(n-1)\gamma][1-\delta+\delta(1-\alpha)^n]+\delta[\alpha+\gamma(1-\alpha)-\gamma(1-\alpha)^n])^2}.$$

$\frac{\partial \hat{L}_{PJL}}{\partial n} < 0$ if and only if

$$[1+(n-1)\gamma][\gamma+\alpha\delta-\delta\gamma\alpha](1-\alpha)^n \ln(1-\alpha)$$

$$+\gamma[1-\delta+\delta(1-\alpha)^n][\alpha+\gamma(1-\alpha)-\gamma(1-\alpha)^n] > 0,$$

which can be rewritten as

$$\delta < \frac{[\alpha+\gamma(1-\alpha)-\gamma(1-\alpha)^n]+[1+(n-1)\gamma](1-\alpha)^n \ln(1-\alpha)}{[1-(1-\alpha)^n][\alpha+\gamma(1-\alpha)-\gamma(1-\alpha)^n]-\frac{\alpha(1-\gamma)}{\gamma}[1+(n-1)\gamma](1-\alpha)^n \ln(1-\alpha)}$$

$$\equiv \hat{\delta}_{PJL}(n,\alpha,\gamma).$$

Thus, for any $0 < \delta < \hat{\delta}_{PJL}$, $\hat{L}_{PJL}$ is strictly decreasing in n, and if for some δ, $\hat{\delta}_{PJL} < \delta < \tilde{\delta}_{PJL}$, then $\hat{L}_{PJL}$ will be strictly increasing in n.

Second,

$$\frac{\partial \tilde{L}_{PJL}}{\partial n} = -\left(\frac{\gamma H}{1+\epsilon}\right) \frac{[1+(n-1)\gamma](1-\alpha)^n \ln(1-\alpha)+[\alpha+\gamma(1-\alpha)-\gamma(1-\alpha)^n]}{[1+(n-1)\gamma]^2}.$$

$\frac{\partial \tilde{L}_{PJL}}{\partial n} < 0$ if and only if

$$G(n,\alpha,\gamma) = [1+(n-1)\gamma](1-\alpha)^n \ln(1-\alpha)+[\alpha+\gamma(1-\alpha)-\gamma(1-\alpha)^n] > 0.$$

It can be simply verified that $G(n,\alpha,\gamma=0) > 0$ and

$$\frac{\partial G}{\partial \gamma} = (n-1)(1-\alpha)^n \ln(1-\alpha)+[(1-\alpha)-(1-\alpha)^n] > 0.$$

Thus, $G(n, \alpha, \gamma)$ strictly increases in γ, and thus, $G(n, \alpha, \gamma) > 0$. As a result, $\tilde{L}_{PJL}$ strictly decreases in n.

Third,

$$\frac{\partial \tilde{\delta}_{PJL}}{\partial n} = \frac{\left[1 + (2n - 3)\gamma + (n - 1)(n - 2)\gamma^2\right](1 - \alpha)^n \ln(1 - \alpha) - \gamma\left[\alpha + \gamma(1 - \alpha) - \gamma(1 - \alpha)^n\right]}{\left[1 + (n - 1)\gamma\right]^2 \left[\alpha + \left[1 - (1 - \alpha)^n\right] - \dfrac{1}{\left[1 + (n - 1)\gamma\right]}\left[\alpha + \gamma(1 - \alpha) - \gamma(1 - \alpha)^n\right]\right]^2}.$$

$\frac{\partial \tilde{\delta}_{PJL}}{\partial n} < 0$. Thus, $\tilde{\delta}_{PJL}$ is strictly decreasing in n.

2) Assume $q < \bar{q}$. First,

$$\frac{\partial \bar{L}_{PJL}}{\partial n} = -\frac{\left(\dfrac{1}{1 + \epsilon}\right) q(1 - \gamma)\delta\alpha H\left[\alpha + \gamma(1 - \alpha) - \gamma(1 - \alpha)^n\right]\left[1 - \delta\left[1 - (1 - \alpha)^n\right]\right]}{\left((n - i - 1)(1 - \gamma)\left[1 - \delta\left[1 - (1 - \alpha)^n\right]\right] + (n - i)q\delta\left[\alpha + \gamma(1 - \alpha) - \gamma(1 - \alpha)^n\right]\right)^2}$$
$$- \frac{\left(\dfrac{1}{1 + \epsilon}\right) q(1 - \gamma)\delta\alpha H\left[\gamma + \delta\alpha - \delta\gamma\alpha\right](n - i)(n - i - 1)(1 - \alpha)^n \ln(1 - \alpha)}{\left((n - i - 1)(1 - \gamma)\left[1 - \delta\left[1 - (1 - \alpha)^n\right]\right] + (n - i)q\delta\left[\alpha + \gamma(1 - \alpha) - \gamma(1 - \alpha)^n\right]\right)^2}.$$

$\frac{\partial \bar{L}_{PJL}}{\partial n} < 0$ if and only if

$$\left[\alpha + \gamma(1 - \alpha) - \gamma(1 - \alpha)^n\right]\left[1 - \delta\left[1 - (1 - \alpha)^n\right]\right]$$
$$+ \left[\gamma + \delta\alpha - \delta\gamma\alpha\right](n - i)(n - i - 1)(1 - \alpha)^n \ln(1 - \alpha) > 0,$$

which can be rewritten as

$$\delta < \frac{\left[\alpha + \gamma(1 - \alpha) - \gamma(1 - \alpha)^n\right] + (n - i)(n - i - 1)\gamma(1 - \alpha)^n \ln(1 - \alpha)}{\left[1 - (1 - \alpha)^n\right]\left[\alpha + \gamma(1 - \alpha) - \gamma(1 - \alpha)^n\right] - \alpha(1 - \gamma)(n - i)(n - i - 1)(1 - \alpha)^n \ln(1 - \alpha)}$$
$$\equiv \hat{\delta}'_{PJL}(n, \alpha, \gamma, i).$$

Thus, for any $0 < \delta < \hat{\delta}'_{PJL}$, $\bar{L}_{PJL}$ strictly decreases in n, and if $\hat{\delta}'_{PJL} < \delta < \tilde{\delta}_{PJL}$, then $\bar{L}_{PJL}$ will strictly increase in n.

Second,

$$\bar{\delta}_{PJL} = \frac{1}{\left[1 - (1 - \alpha)^n\right] - \dfrac{(n - i)q\gamma}{(n - i - 1)(1 - \gamma)}\left[1 - n\alpha - (1 - \alpha)^n\right]}.$$

38

Clearly, $\bar{\delta}_{PJL}$ strictly decreases in n if and only if its denominator, which we call it D, strictly increases in n,

$$
\begin{aligned}
\frac{\partial D}{\partial n} &= \frac{(n-i)\,q\gamma\alpha\,(1-\gamma)}{(n-i-1)\,(1-\gamma)^2} - \frac{(n-i)\,q\,[1-n\alpha-(1-\alpha)^n]}{(n-i-1)\,(1-\gamma)^2} \\
&\quad + \frac{(n-i)\,q\gamma - (n-i-1)\,(1-\gamma)}{(n-i-1)\,(1-\gamma)}\,(1-\alpha)^n \ln(1-\alpha).
\end{aligned}
$$

$\frac{\partial D}{\partial n} > 0$ if and only if $(n-i)\,q\gamma - (n-i-1)\,(1-\gamma) < 0$, which is always true because $q < \frac{(n-i-1)(1-\gamma)}{(n-i)[1+(n-1)\gamma]}$. $\square$

Proof of Lemma 3. 1) When $q \geq \bar{q}$, determinants of the loan ceiling of the PJL contract are $\hat{L}_{PJL}$, $\tilde{L}_{PJL}$, $\hat{\delta}_{PJL}$, and $\tilde{\delta}_{PJL}$. First, we look at the changes of $\hat{L}_{PJL}$ and $\tilde{L}_{PJL}$ with respect

$$
\frac{\partial \hat{L}_{PJL}}{\partial \gamma} = \frac{\left(\dfrac{1}{1+\epsilon}\right)\delta\alpha H\,[1-\delta+\delta(1-\alpha)^n]\,[1-n\alpha-(1-\alpha)^n]}{([1+(n-1)\gamma]\,[1-\delta[1-(1-\alpha)^n]]+\delta[\alpha+\gamma(1-\alpha)-\gamma(1-\alpha)^n])^2}
$$

$$
\frac{\partial \tilde{L}_{PJL}}{\partial \gamma} = \left(\frac{1}{1+\epsilon}\right)\frac{H\,[1-n\alpha-(1-\alpha)^n]}{[1+(n-1)\gamma]^2}.
$$

$\frac{\partial \hat{L}_{PJL}}{\partial \gamma} < 0$ and $\frac{\partial \tilde{L}_{PJL}}{\partial \gamma} < 0$ if and only if $1-n\alpha-(1-\alpha)^n < 0$, which is true for any $\alpha \in (0,1)$. Thus, $\hat{L}_{PJL}$ and $\tilde{L}_{PJL}$ are both strictly decreasing in γ.

Second, assume the denominator of $\hat{\delta}_{PJL}$ is called A to improve the readability of equations:

$$
\begin{aligned}
\frac{\partial \hat{\delta}_{PJL}}{\partial \gamma} &= -\frac{\alpha\,[1+(n-1)\gamma]^2\,(1-\alpha)^{2n}\ln^2(1-\alpha)}{\gamma^2 A^2} \\
&\quad - \frac{(1-\alpha)^n \ln(1-\alpha)\,[1+(n-1)\gamma]\,[(1-\alpha)-(1-\alpha)^n]\,\dfrac{\alpha(1-\gamma)}{\gamma}}{A^2} \\
&\quad - \frac{(1-\alpha)^n \ln(1-\alpha)\,[1+(n-1)\gamma]\,[(1-\alpha)-(1-\alpha)^n]\,[1-(1-\alpha)^n]}{A^2} \\
&\quad - \frac{(1-\alpha)^n \ln(1-\alpha)\,[\alpha+\gamma(1-\alpha)-\gamma(1-\alpha)^n]\,\dfrac{\alpha[1+(n-1)\gamma^2]}{\gamma^2}}{A^2} \\
&\quad + \frac{(1-\alpha)^n \ln(1-\alpha)\,(n-1)\,[1-(1-\alpha)^n]\,[\alpha+\gamma(1-\alpha)-\gamma(1-\alpha)^n]}{A^2},
\end{aligned}
$$

which simplifies to

$$
\begin{aligned}
\frac{\partial \hat{\delta}_{PJL}}{\partial \gamma} &= -(1-\alpha)^n \ln(1-\alpha) \frac{\alpha \left[1+(n-1)\gamma\right]^2 (1-\alpha)^n \ln(1-\alpha)}{\gamma^2 A^2} \\
&\quad -(1-\alpha)^n \ln(1-\alpha) \frac{\left[\alpha+\gamma(1-\alpha)-\gamma(1-\alpha)^n\right]^2}{\gamma^2 A^2}.
\end{aligned}
$$

$\frac{\partial \hat{\delta}_{PJL}}{\partial \gamma} > 0$ if and only if

$$
M(n,\alpha,\gamma) \equiv \alpha\left[1+(n-1)\gamma\right]^2(1-\alpha)^n \ln(1-\alpha) + \left[\alpha+\gamma(1-\alpha)-\gamma(1-\alpha)^n\right]^2 > 0.
$$

$M(n,\alpha,\gamma)$ is monotonic in γ, and for any $\alpha \in (0,1)$,

$$
\begin{aligned}
\lim_{\gamma \to 0} M(n,\alpha,\gamma) &= \alpha(1-\alpha)^n \ln(1-\alpha) + \alpha^2 > 0 \\
\lim_{\gamma \to 1} M(n,\alpha,\gamma) &= n^2\alpha(1-\alpha)^n \ln(1-\alpha) + \left[1-(1-\alpha)^n\right]^2 > 0.
\end{aligned}
$$

Thus, $M(n,\alpha,\gamma)$ is always positive, and as a result, $\hat{\delta}_{PJL}$ is strictly increasing in γ.

Third,

$$
\frac{\partial \tilde{\delta}_{PJL}}{\partial \gamma} = \frac{1-n\alpha-(1-\alpha)^n}{\left(\left[\alpha+1-(1-\alpha)^n\right]\left[1+(n-1)\gamma\right]-\left[\alpha+\gamma(1-\alpha)-\gamma(1-\alpha)^n\right]\right)^2}.
$$

$\frac{\partial \tilde{\delta}_{PJL}}{\partial \gamma} < 0$ if and only if $1-n\alpha-(1-\alpha)^n < 0$, which is true for any $\alpha \in (0,1)$. Thus, $\tilde{\delta}_{PJL}$ is strictly decreasing in γ.

2) Assume $q < \bar{q}$. First,

$$
\frac{\partial \bar{L}_{PJL}}{\partial \gamma} = \frac{\left(\dfrac{1}{1+\epsilon}\right)(n-i)(n-i-1)q\delta\alpha H\left[1-\delta+\delta(1-\alpha)^n\right]\left[1-(1-\alpha)^n\right]}{\left(\left[1-\delta+\delta(1-\alpha)^n\right]+\dfrac{(n-i)q\delta}{(n-i-1)(1-\gamma)}\left[\alpha+\gamma(1-\alpha)-\gamma(1-\alpha)^n\right]\right)^2}.
$$

$\frac{\partial \bar{L}_{PJL}}{\partial \gamma} > 0$ for any $\alpha \in (0,1)$, therefore, $\bar{L}_{PJL}$ is strictly increasing in γ for any $\alpha \in (0,1)$.

Second, as was proved previously, $\tilde{L}_{PJL}$ is strictly decreasing in γ.

Third, to improve the readability of equations, the denominator of $\hat{\delta}'_{PJL}$ is called B,

$$\frac{\partial \hat{\delta}'_{PJL}}{\partial \gamma} = -\frac{\alpha (n-i)^2 (n-i-1)^2 (1-\alpha)^{2n} \ln^2 (1-\alpha)}{B^2}$$
$$-\frac{(n-i)(n-i-1)(1-\alpha)^n \ln(1-\alpha) \alpha (1-\gamma) [(1-\alpha)-(1-\alpha)^n]}{B^2}$$
$$-\frac{(n-i)(n-i-1)(1-\alpha)^n \ln(1-\alpha) \gamma [1-(1-\alpha)^n] [(1-\alpha)-(1-\alpha)^n]}{B^2}$$
$$-\frac{(n-i)(n-i-1)(1-\alpha)^n \ln(1-\alpha) \alpha [\alpha + \gamma(1-\alpha) - \gamma(1-\alpha)^n]}{B^2}$$
$$+\frac{(n-i)(n-i-1)(1-\alpha)^n \ln(1-\alpha) [1-(1-\alpha)^n] [\alpha + \gamma(1-\alpha) - \gamma(1-\alpha)^n]}{B^2},$$

which simplifies to

$$\frac{\partial \hat{\delta}'_{PJL}}{\partial \gamma} = -\frac{\alpha (n-i)^2 (n-i-1)^2 (1-\alpha)^{2n} \ln^2 (1-\alpha)}{B^2}.$$

$\frac{\partial \hat{\delta}'_{PJL}}{\partial \gamma} < 0$ for any $\alpha \in (0,1)$. Thus, $\hat{\delta}'_{PJL}$ strictly decreases in γ for any $\alpha \in (0,1)$.

Fourth,

$$\frac{\partial \bar{\delta}_{PJL}}{\partial \gamma} = \frac{(n-i-1)(n-i) q [1 - n\alpha - (1-\alpha)^n]}{((n-i-1)(1-\gamma)[1-(1-\alpha)^n] - (n-i) q\gamma [1 - n\alpha - (1-\alpha)^n])^2}.$$

$\frac{\partial \bar{\delta}_{PJL}}{\partial \gamma} < 0$ if and only if $1 - n\alpha - (1-\alpha)^n < 0$, which is always true for any $\alpha \in (0,1)$. Thus, $\bar{\delta}_{PJL}$ decreases in γ. $\square$

Proof of Proposition 5. According to Lemma 3, $\hat{L}_{PJL}$, $\tilde{L}_{PJL}$, $\tilde{\delta}_{PJL}$, $\hat{\delta}'_{PJL}$, and $\bar{\delta}_{PJL}$ are all strictly decreasing in γ, while $\bar{L}_{PJL}$ and $\hat{\delta}_{PJL}$ are strictly increasing in γ:

$$\hat{L}_{PJL} > \lim_{\gamma \to 1} \hat{L}_{PJL} = \left(\frac{1}{1+\epsilon}\right) \frac{\delta \alpha H [1 - (1-\alpha)^n]}{n - \delta(n-1)[1-(1-\alpha)^n]} - cq = \hat{L}_{SJL} - cq$$
$$\tilde{L}_{PJL} > \lim_{\gamma \to 1} \tilde{L}_{PJL} = \left(\frac{1}{1+\epsilon}\right) \frac{H [1 - (1-\alpha)^n]}{n} - cq = \tilde{L}_{SJL} - cq$$
$$\bar{L}_{PJL} < \lim_{\gamma \to 1} \bar{L}_{PJL} = \left(\frac{1}{1+\epsilon}\right) \alpha H - cq > L_{IL} - cq$$
$$\hat{\delta}_{PJL} > \lim_{\gamma \to 1} \hat{\delta}_{PJL} = \frac{[1-(1-\alpha)^n] + (1-\alpha)^n \ln(1-\alpha)}{[1-(1-\alpha)^n]^2} = \hat{\delta}_{SJL}$$

$$\tilde{\delta}_{PJL} \;>\; \lim_{\gamma \to 1} \tilde{\delta}_{PJL} = \frac{1}{\alpha + \frac{n-1}{n}\left[1 - (1-\alpha)^n\right]} = \tilde{\delta}_{SJL}$$

$$\hat{\delta}'_{PJL} \;>\; \lim_{\gamma \to 1} \hat{\delta}'_{PJL} = \frac{\left[1 - (1-\alpha)^n\right] + (n-i)(n-i-1)(1-\alpha)^n \ln(1-\alpha)}{\left[1 - (1-\alpha)^n\right]^2} > \hat{\delta}_{SJL}$$

$$\bar{\delta}_{PJL} \;>\; \lim_{\gamma \to 1} \bar{\delta}_{PJL} = \lim_{\gamma \to 1} \frac{1}{\left[1 - (1-\alpha)^n\right] + \dfrac{(n-i)\,q\gamma}{(n-i-1)(1-\gamma)}\left[n\alpha - 1 + (1-\alpha)^n\right]}$$

$$>\; \lim_{\gamma \to 1} \frac{1}{\left[1 - (1-\alpha)^n\right] + \dfrac{\gamma}{\left[1 + (n-1)\gamma\right]}\left[n\alpha - 1 + (1-\alpha)^n\right]} = \tilde{\delta}_{SJL}. \quad \square$$

Proof of Proposition 6. In the case of the FJL contract, the lender's optimization problem turns to the following problem,

$$\max_{L,R} \; V^R_{FJL} = \frac{\alpha H - R\left[1 - (1-\alpha)^n\right]}{1 - \delta\left[1 - (1-\alpha)^n\right]} \tag{25}$$

$$s.t. \quad R \le \frac{H}{n} \tag{26}$$

$$R \le \frac{\left(1 - \delta^T\right)\delta V^R_{FJL}}{n} \tag{27}$$

$$R \ge \frac{L(1+\varepsilon)}{1 - (1-\alpha)^n}. \tag{28}$$

It is clear that V^R_{FJL} decreases in R, so the bank would like to set R as low as possible. Constraint (28) gives the minimum R required for breaking even $R_{FJL} = \frac{L(1+\varepsilon)}{[1-(1-\alpha)^n]}$. Replacing R_{FJL} in the above problem, we will have:

$$\max_{L,R} \; V^R_{FJL} = \frac{\alpha H - L(1+\varepsilon)}{1 - \delta\left[1 - (1-\alpha)^n\right]}$$

$$s.t. \quad L \le \left(\frac{1}{1+\varepsilon}\right)\frac{H\left[1 - (1-\alpha)^n\right]}{n} \equiv \tilde{L}_{FJL}(n,\alpha,H)$$

$$L \le \left(\frac{1}{1+\varepsilon}\right)\frac{\delta\left(1-\delta^T\right)\alpha H\left[1 - (1-\alpha)^n\right]}{n - \left[n\delta - \delta\left(1-\delta^T\right)\right]\left[1 - (1-\alpha)^n\right]} \equiv \hat{L}_{FJL}(n,\alpha,\delta,T,H).$$

A feasible L must satisfy $L \le Min\left\{\hat{L}_{FJL}, \tilde{L}_{FJL}\right\}$ in order to satisfy both constraints. There are two cases, either $\hat{L}_{FJL} \le \tilde{L}_{FJL}$ or $\tilde{L}_{FJL} > \hat{L}_{FJL}$:

i) $\hat{L}_{FJL} \leq \tilde{L}_{FJL}$ if and only if

$$\delta \leq \frac{1}{\left(1 - \delta^T\right)\alpha + \frac{n - \left(1 - \delta^T\right)}{n}\left[1 - (1-\alpha)^n\right]} \equiv \tilde{\delta}_{FJL}\left(n, \alpha, \mu, H\right);$$

ii) $\tilde{L}_{FJL} \leq \hat{L}_{FJL}$ if and only if $\delta \geq \tilde{\delta}$.

Therefore, if $0 \leq \delta \leq \tilde{\delta}_{FJL}$, the FJL contract is feasible for any $L \leq \hat{L}_{FJL}$, and if $\tilde{\delta}_{FJL} \leq \delta < 1$, the FJL contract is feasible for any $L \leq \tilde{L}_{FJL}$. $\square$

Lemma 4. 1)

$$\frac{\partial \hat{L}_{FJL}}{\partial n} = \left(\frac{1}{1+\varepsilon}\right)\frac{\delta\left(1-\delta^T\right)\alpha H\left[-(1-\alpha)^n \ln(1-\alpha)^n + \delta\left[1 - (1-\alpha)^n\right]^2 - \left[1 - (1-\alpha)^n\right]\right]}{\left[n - \delta\left[n - \left(1 - \delta^T\right)\right]\left[1 - (1-\alpha)^n\right]\right]^2}.$$

Clearly, $\frac{\partial \hat{L}_{FJL}}{\partial n} < 0$ if and only if

$$-(1-\alpha)^n \ln(1-\alpha)^n + \delta\left[1 - (1-\alpha)^n\right]^2 - \left[1 - (1-\alpha)^n\right] < 0$$

or

$$\delta < \frac{(1-\alpha)^n \ln(1-\alpha)^n + \left[1 - (1-\alpha)^n\right]}{\left[1 - (1-\alpha)^n\right]^2} \equiv \hat{\delta}\left(n, \alpha\right).$$

Therefore, $\hat{L}_{FJL}$ is strictly decreasing in n for any $0 < \delta < \hat{\delta}\left(n, \alpha\right)$, and $\hat{L}_{FJL}$ is strictly increasing in n for any $\hat{\delta}\left(n, \alpha\right) < \delta < \tilde{\delta}_{FJL}$.

2) $\tilde{\delta}_{FJL}$ is strictly decreasing in n, because its denominator is strictly increasing in n. $\square$

Proof of Lemma 5. 1) The derivative of $\hat{L}_{FJL}$ with respect to T after simplification is as follows:

$$\frac{\partial \hat{L}_{FJL}}{\partial T} = \left(\frac{1}{1+\varepsilon}\right)\frac{n\delta^{T+1}\left(\ln\delta\right)\alpha H\left[1 - (1-\alpha)^n\right]\left(\delta\left[1 - (1-\alpha)^n\right] - 1\right)}{\left(n - \delta\left[n - \left(1 - \delta^T\right)\right]\left[1 - (1-\alpha)^n\right]\right)^2}.$$

Clearly, $\frac{\partial \hat{L}_{FJL}}{\partial T}$ is always positive. Thus $\hat{L}_{FJL}$ is strictly increasing in T.

2) The derivative of $\partial \tilde{\delta}_{FJL}$ with respect to T is as follows:

$$\frac{\partial \tilde{\delta}_{FJL}}{\partial T} = \frac{\delta^T \ln \delta \left[\frac{1}{n} \left[1 - (1 - \alpha)^n \right] - \alpha \right]}{\left[\left(1 - \delta^T \right) \alpha + \frac{1}{n} \left[n - (1 - \delta^T) \right] \left[1 - (1 - \alpha)^n \right] \right]^2}.$$

Clearly, $\frac{\partial \hat{L}_{FJL}}{\partial T} < 0$ if and only if $\frac{1}{n} \left[1 - (1 - \alpha)^n \right] - \alpha < 0$, which is always true. $\square$

Proof of Proposition 7. 1) $\hat{L}_{FJL} \leq \hat{L}_{SJL}$ if and only if

$$\frac{\delta \left(1 - \delta^T \right) \mathbb{E}(Y) \left[1 - (1 - \alpha)^n \right]}{n - \delta \left[n - (1 - \delta^T) \right] \left[1 - (1 - \alpha)^n \right]} \leq \frac{\delta \mathbb{E}(Y) \left[1 - (1 - \alpha)^n \right]}{n - \delta \left(n - 1 \right) \left[1 - (1 - \alpha)^n \right]}. \tag{29}$$

Equation (29) can be simplified to $\left(1 - \delta^T \right) \leq 1$, which is always true.

2) $\tilde{\delta}_{FJL} \geq \tilde{\delta}_{SJL}$ if and only if

$$\left(1 - \delta^T \right) \alpha H + \frac{n - \left(1 - \delta^T \right)}{n} \left[1 - (1 - \alpha)^n \right] H \leq \alpha H + \frac{n - 1}{n} \left[1 - (1 - \alpha)^n \right] H. \tag{30}$$

Equation (30) can be simplified to $\frac{[1 - (1 - \alpha)^n]}{n} \leq \alpha$, which is always true.

3) First, we recall from Propositions 1 and 6, the maximum loan that can be offered to each borrower each time is the minimum of $\hat{L}$ and $\tilde{L}$, which is dependent on the magnitude of $\tilde{\delta}$. We also know that $\tilde{\delta}_{SJL} \leq \tilde{\delta}_{FJL}$. In order to compare the loan ceilings of the FJL and the SJL contracts, three different cases should be considered:

i) For any $\delta < \tilde{\delta}_{SJL}$, the SJL contract is feasible for all $L \leq \hat{L}_{SJL}$, and the FJL contract is feasible for all $L \leq \hat{L}_{FJL}$. We have already proved that $\hat{L}_{FJL} \leq \hat{L}_{SJL}$; that is, for any $\delta < \tilde{\delta}_{SJL}$, the loan ceiling of the SJL contract is higher than the FJL contract.

ii) For any $\tilde{\delta}_{SJL} \leq \delta \leq \tilde{\delta}_{FJL}$, the FJL contract is feasible for all $L \leq \hat{L}_{FJL}$ and the SJL contract is feasible for all $L \leq \tilde{L}_{SJL}$. From the proof of Proposition 6, we know that for any $\delta \leq \tilde{\delta}_{FJL}$, $\hat{L}_{FJL} \leq \tilde{L}_{FJL} = \tilde{L}_{SJL}$. Therefore, for any $\tilde{\delta}_{SJL} \leq \delta \leq \tilde{\delta}_{FJL}$, the loan ceiling of the SJL contract is higher than the FJL contract.

iii) For any $\delta > \tilde{\delta}_{FJL}$, both the SJL and the FJL contracts are feasible for any $L \leq \tilde{L}_{FJL} = \tilde{L}_{SJL}$; that is, for any $\delta > \tilde{\delta}_{FJL}$, both the SJL and the FJL contracts have the same loan ceiling. $\square$

Proof of Lemma 6. 1)

$$\frac{\partial \hat{L}'_{SJL}}{\partial n} = \left(\frac{1}{1+\varepsilon}\right) \frac{\delta H \frac{d\alpha}{dn} \left[1 - (1-\alpha)^n\right] \left[n - \delta (n-1) \left[1 - (1-\alpha)^n\right]\right]}{\left[n - \delta (n-1) \left[1 - (1-\alpha)^n\right]\right]^2}$$

$$+ \left(\frac{1}{1+\varepsilon}\right) \frac{\delta H n\alpha \left[n\frac{d\alpha}{dn}(1-\alpha)^{n-1} - (1-\alpha)^n \ln(1-\alpha)\right]}{\left[n - \delta (n-1) \left[1 - (1-\alpha)^n\right]\right]^2}$$

$$- \left(\frac{1}{1+\varepsilon}\right) \frac{\delta H\alpha \left[1 - (1-\alpha)^n\right] \left(1 - \delta \left[1 - (1-\alpha)^n\right]\right)}{\left[n - \delta (n-1) \left[1 - (1-\alpha)^n\right]\right]^2}.$$

Clearly, $\frac{\partial \hat{L}'_{SJL}}{\partial n} < 0$ if and only if

$$\frac{d\alpha}{dn}\left[1 - (1-\alpha)^n\right]\left(n - \delta [n-1]\left[1 - (1-\alpha)^n\right]\right)$$

$$+ n\alpha \left[n\frac{d\alpha}{dn}(1-\alpha)^{n-1} - (1-\alpha)^n \ln(1-\alpha)\right]$$

$$- \alpha \left[1 - (1-\alpha)^n\right]\left(1 - \delta \left[1 - (1-\alpha)^n\right]\right) < 0. \tag{31}$$

Assuming that $\frac{d\alpha}{dn} < \frac{\alpha}{n-1}$, (31) can be rewritten as

$$\delta < \frac{\left(n\frac{d\alpha}{dn} - \alpha\right)\left[1 - (1-\alpha)^n\right] + n\alpha \left[n\frac{d\alpha}{dn}(1-\alpha)^{n-1} - (1-\alpha)^n \ln(1-\alpha)\right]}{\left(n\frac{d\alpha}{dn} - \frac{d\alpha}{dn} - \alpha\right)\left[1 - (1-\alpha)^n\right]^2} \equiv \hat{\delta}'_{JL}\left(n, \alpha, \frac{d\alpha}{dn}\right).$$

Thus, for any $0 < \delta < \hat{\delta}'_{FJL}$, $\hat{L}'_{FJL}$ is strictly decreasing in n, and for any $\hat{\delta}'_{FJL} < \delta < \tilde{\delta}'_{FJL}$, $\hat{L}'_{FJL}$ is strictly increasing in n.

2) The derivative of $\partial \tilde{L}'_{FJL}$ with respect to n after simplification is as follows:

$$\frac{\partial \tilde{L}'_{FJL}}{\partial n} = \left(\frac{H}{1+\varepsilon}\right) \frac{n\frac{d\alpha}{dn}(1-\alpha)^{n-1} - (1-\alpha)^n \ln(1-\alpha)^n - \left[1 - (1-\alpha)^n\right]}{n^2}.$$

Clearly, $\frac{\partial \tilde{L}'_{FJL}}{\partial n} < 0$ if and only if its numerator is positive; that is,

$$\frac{d\alpha}{dn} < \frac{1 - (1-\alpha)^n + (1-\alpha)^n \ln(1-\alpha)^n}{n(1-\alpha)^{n-1}}. \tag{32}$$

Thus, $\tilde{L}'_{FJL}$ is strictly decreasing in n if α is such that (32) is satisfied. $\square$

Proof of Proposition 8. When projects are weakly correlated, $\frac{d\alpha}{dn} < \frac{\alpha}{n-1}$, and the loan ceiling of the SJL contract increases in n as far as

$$\hat{\delta}'_{SJL}\left(n, \alpha\left(n\right), \frac{d\alpha}{dn}\right) < \tilde{\delta}'_{SJL}\left(n, \alpha\left(n\right)\right). \tag{33}$$

Thus, replacing $\hat{\delta}'_{SJL}$ and $\tilde{\delta}'_{SJL}$ into (33) and simplifying, we will have

$$\frac{d\alpha}{dn} > \frac{\alpha\left[1 - \left(1-\alpha\right)^n\right] + \alpha\left(1-\alpha\right)^n \ln\left(1-\alpha\right)^n - \dfrac{\alpha\left[1 - \left(1-\alpha\right)^n\right]^2}{\alpha + \frac{n-1}{n}\left[1 - \left(1-\alpha\right)^n\right]}}{n\left[1 - \left(1-\alpha\right)^n\right] + n^2\alpha\left(1-\alpha\right)^{n-1} - \dfrac{\left(n-1\right)\left[1 - \left(1-\alpha\right)^n\right]^2}{\alpha + \frac{n-1}{n}\left[1 - \left(1-\alpha\right)^n\right]}}. \tag{34}$$

For large n, $\lim\limits_{n\to\infty}\left[1 - \left(1-\alpha\right)^n\right] = 1$, and thus, the right-hand side of (34) can be simplified to

$$\lim_{n\to\infty} \frac{\alpha + \alpha\left(1-\alpha\right)^n \ln\left(1-\alpha\right)^n - \dfrac{\alpha}{\alpha + \frac{n-1}{n}}}{n + n^2\alpha\left(1-\alpha\right)^{n-1} - \dfrac{\left(n-1\right)}{\alpha + \frac{n-1}{n}}} = \lim_{n\to\infty} \frac{\alpha - \dfrac{\alpha}{\alpha+1}}{n - \dfrac{n-1}{\alpha+1}} = 0.$$

For small n, $\lim\limits_{n\to\infty}\left[1 - \left(1-\alpha\right)^n\right] = \alpha$, and thus, the right-hand side of (34) can be simplified to

$$\lim_{n\to 1} \frac{\alpha^2 + \alpha\left(1-\alpha\right)^n \ln\left(1-\alpha\right)^n - \dfrac{\alpha^2}{\alpha + \frac{n-1}{n}\alpha}}{n\alpha + n^2\alpha\left(1-\alpha\right)^{n-1} - \dfrac{\left(n-1\right)\alpha^2}{\alpha + \frac{n-1}{n}\alpha}} = \frac{\left(1-\alpha\right)\ln\left(1-\alpha\right) - \left(1-\alpha\right)}{2} < 0.$$

Therefore, the right-hand side of (34) is always negative, while the left-hand side is always positive (we assume that α is increasing in n). Therefore, (34) is always valid.

When projects are strongly correlated, $\frac{d\alpha}{dn} > \frac{\alpha}{n-1}$, and the loan ceiling of the SJL contract increases in n if and only if

$$\hat{\delta}'_{SJL}\left(n, \alpha\left(n\right), \frac{d\alpha}{dn}\right) > \tilde{\delta}'_{SJL}\left(n, \alpha\left(n\right)\right). \tag{35}$$

Replacing $\hat{\delta}'_{SJL}$ and $\tilde{\delta}'_{SJL}$ into (35) results in

$$\frac{d\alpha}{dn} > \frac{\alpha\left[1-(1-\alpha)^n\right] + \alpha(1-\alpha)^n \ln(1-\alpha)^n - \dfrac{\alpha\left[1-(1-\alpha)^n\right]^2}{\alpha + \frac{n-1}{n}\left[1-(1-\alpha)^n\right]}}{n\left[1-(1-\alpha)^n\right] + n^2\alpha(1-\alpha)^{n-1} - \dfrac{(n-1)\left[1-(1-\alpha)^n\right]^2}{\alpha + \frac{n-1}{n}\left[1-(1-\alpha)^n\right]}}.$$

This equation is the same equation (34) that we have already proved to hold always. $\square$