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Portfolio Claustrophobia: Asset Pricing in Markets with Illiquid Assets

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Many classes of assets are illiquid or nonmarketable in that they cannot always be traded immediately. Thus, a portfolio position in these becomes at least temporarily irreversible. We study the asset-pricing implications of this type of illiquidity in an exchange economy with heterogeneous agents. In this market, one asset is always liquid. The other asset can be traded initially, but then not again until after a "blackout" period. Illiquidity has a dramatic effect. Agents abandon diversification and choose polarized portfolios instead. The value of liquidity can represent a large portion of the equilibrium price of an asset. (JEL G11, G12)

One of the cornerstones of traditional asset-pricing theory is the assumption that all assets are liquid and readily tradable by economic agents. In reality, however, many important classes of assets are not readily marketable and agents often cannot buy and sell them immediately. For example, a large percentage of the wealth of the typical household is held in the form of illiquid assets such as human capital, sole proprietorships, partnerships, equity in other closely held firms, deferred compensation, pension plans, tax-deferred retirement accounts, savings bonds, annuities, trusts, inheritances, and residential real estate. On the institutional side, an increasing amount of wealth is being allocated to illiquid asset classes such as private equity, emerging markets, venture capital, commercial real estate, and the rapidly growing hedge fund sector. In each of these examples, an investor might have to wait months, years, or even decades before being able to unwind a position.[1]

The problem of asset illiquidity or nonmarketability has also become daily headline news in the financial press as the current subprime crisis continues. In an October 15, 2007, speech at the Economic Club of New York, Federal Reserve Chairman Ben Bernanke noted, "Moreover, in the absence of an active syndication market for the leveraged loans they had committed to underwrite and without a well-functioning securitization market for the nonconforming mortgages they had issued, many large banks might be forced to hold those assets on their books rather than sell them to investors as planned." As the subprime crisis deepened in the fall of 2007, many financial institutions found that the structured-credit market almost completely dried up and that it

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[1] A familiar example is the decision of a student to invest in his or her human capital by earning a degree in economics. This investment (which typically requires a large financial commitment and hopefully pays dividends throughout the student's life) is completely illiquid and cannot be unwound or reversed.

was all but impossible for them to liquidate positions in subprime collateralized debt obligations. In response, a number of major Wall Street banks, with Treasury Department backing, proposed the creation of a "super SIV" (structured investment vehicle) which would allow them to unwind their illiquid or nonmarketable off-balance-sheet positions in mortgage-related structured-credit products. The nonmarketability of compromised or downgraded asset-backed securities has also led to a major contraction in the multi-trillion-dollar commercial paper market. Another recent example of nonmarketability is the auction-rate debt market: "Last week, the estimated $330 billion auction-rate-securities market became the latest casualty of the global credit crunch when auctions of an estimated $60 billion of such debt failed to generate enough bidders. Now, investors are finding themselves locked into investments they can't cash out of."[2] Currently, the Treasury is proposing to allow financial market participants to temporarily "swap" as much as $200 billion of illiquid structured credit securities for much more marketable Treasury securities in order to mitigate funding problems. During the past year, many hedge funds with credit-related losses have suspend redemptions of assets, leaving investors in these hedge funds with completely illiquid positions. The problem of asset illiquidity or nonmarketability may prove to be one of most severe challenges faced by the financial markets in the past decade.

Asset illiquidity or nonmarketability has major implications for asset pricing since it changes the economics of portfolio choice in a fundamental way. When there are assets that cannot be bought or resold immediately, portfolio decisions take on an important dimension of permanence or irreversibility. In this sense, illiquidity in financial investment parallels the role that irreversibility of physical investment plays in the real options literature. Intuition suggests that agents' financial decisions could be very different in a market where, once they invested, the agents might find themselves "trapped" in a risky portfolio that they cannot exit (the portfolio equivalent of claustrophobia). In turn, the resulting changes in investment strategies and consumption plans could affect equilibrium asset prices. Thus, asset illiquidity raises a number of key asset-pricing issues: How does illiquidity affect optimal portfolio decisions? What risks are created by asset illiquidity? How are those risks shared among agents? How does the marketability of an asset affect its own price and the prices of other assets?

To address these issues, this paper examines the asset-pricing implications of nonmarketability within a continuous-time exchange economy with multiple assets and heterogeneous agents. In this framework, agents have identical logarithmic preferences, but differ in their subjective time discount factors. Thus, we can characterize the agents as either more patient or less patient. Liquid assets can always be traded. Illiquid assets can be traded initially, but then cannot be traded again until after a fixed horizon or trading "blackout" period. Although stylized, this simple approach to modeling illiquidity is actually consistent with a number of examples of illiquidity observed in the markets, such as IPO lockups, Rule 144 restrictions on security transfers, and trading blackouts around earnings announcements. This approach also has the advantage of capturing the intuitive notion of illiquidity as the absence of immediacy.[3]

As a benchmark for comparison, we first present results for the traditional case where both assets are fully liquid. In this fully liquid case, we obtain the standard result that all agents choose to hold the market portfolio. Furthermore, assets with identical cash flow dynamics have identical prices.

With illiquidity, however, asset pricing becomes dramatically different. Calibrating the model using National Income and Product Account (NIPA) data for the cash flows generated by

[2] Jane J. Kim and Shefali Anand, *Wall Street Journal*, February 21, 2008, D1.

[3] Furthermore, this approach to modeling illiquidity has a number of commonalities with the notion of irreversible purchases of durable consumption goods that appears in the literature. For example, see Sanford Grossman and Guy Laroque (1990) and Ayman Hindy and Chi-fu Huang (1993).

aggregate US public and private equity from 1935 to 2005, we show that the logarithmic agents cease being myopic and no longer choose to hold the market portfolio. Rather, the less patient agent tilts his portfolio toward the liquid asset and holds less of the illiquid asset. In some cases, the agents completely abandon diversification as a portfolio strategy; the portfolio for the less patient agent consists almost entirely of the liquid asset, while the opposite is true for the more patient agent. The reason for this polarization in portfolio choice stems from the interplay of two key factors. First, the less patient agent has a strong incentive to accelerate his consumption by selling his portfolio over time to the more patient agent. To do so, however, the less patient agent needs to hold more of the liquid asset than is optimal in the fully liquid case. Second, illiquidity creates a new need for intertemporal risk sharing not present in traditional asset pricing models with fully liquid assets. In particular, even though the agents have identical attitudes toward instantaneous risk, the less patient agent is better able to bear the longer-horizon portfolio risk induced by illiquidity because his higher subjective discount rate dampens the effect of noninstantaneous risk on his utility. In equilibrium, these two effects can either reinforce or offset each other.

Illiquidity also has major implications for asset prices. In general, the more polarized the optimal portfolios, the greater is the effect on asset prices. In the calibrated model, illiquidity has the effect of making the liquid asset more valuable, and the illiquid asset less valuable, relative to what prices would be in the fully liquid equilibrium. Surprisingly, however, the opposite can be true for other scenarios in which illiquid assets are in relatively small supply. Intuitively, this is because the intertemporal risk-sharing effect described above dominates when the market consists almost entirely of risky liquid assets. In the calibrated model, asset-pricing effects are typically less than 1 percent when the illiquidity horizon is only one or two years. For longer illiquidity horizons, however, asset prices can change by 10 percent or more. For other calibrations in which the vast majority of the assets in the economy are illiquid, asset-pricing effects can be much larger, with the liquid asset almost doubling in value. With illiquidity, assets with identical cash flow dynamics need not have the same value, thus providing a rational liquidity-based rationale for what otherwise might appear as a violation of the law of one price.

Illiquidity affects not only current prices, but also the distribution of future returns for the assets in the economy. A key driving factor is that the portfolio effects resulting from illiquidity have long-term implications for the distribution of wealth among agents. Thus, even after the illiquidity horizon lapses and all assets become fully liquid again, their prices need not revert to what they would have been otherwise. We explore the implications of the model for expected returns, return volatilities, and return correlations.

Finally, we examine the effect of illiquidity on the total trading turnover in the market. When there are liquidity restrictions, the liquid asset becomes the "only game in town" and agents trade much more of it, both at time zero and during the blackout period. In the calibrated model, total turnover in the liquid asset can nearly quadruple for longer illiquidity horizons. The total turnover of the illiquid asset is typically less when there is a blackout, but may be only 10 or 20 percent less than it would be in the fully liquid case. Thus, the presence of illiquid assets in the market tends to lead to much higher overall trading activity.

In summary, this paper makes four key points about asset pricing and illiquidity. First, illiquidity affects optimal portfolio choice profoundly. In particular, agents with identical attitudes toward risk may rationally choose to hold very different types of portfolios. These results may help explain a number of well-known puzzles about the way actual agents choose to invest such as limited stock market participation (for example, Greg Mankiw and Stephen Zeldes 1991; Shlomo Benartzi and Richard Thaler 2001). Second, the type of illiquidity we consider in this paper can have first-order effects on equilibrium asset prices. Third, these results demonstrate that the value of a liquid asset can be greater than the "simple" present value of its cash flows.

For example, a liquid asset can be worth significantly more that an illiquid asset with the identical cash flow dynamics. Thus, in asset pricing, it is not directly dividend cash flows that matter, but rather the consumption stream that asset ownership generates. Finally, these results illustrate that differences in patience across investors can have major asset-pricing effects; heterogeneity in patience may provide an important (but underresearched) channel for understanding financial markets and resolving asset-pricing puzzles. Furthermore, differences in patience map well into the familiar notions of short-horizon financial institutions and longer-horizon investors often found in the literature.

This paper contributes to the growing literature on the asset-pricing effects of illiquidity.[4] Important empirical work in this area includes Yakov Amihud and Haim Mendelson (1986, 1991), Bradford Cornell and Alan C. Shapiro (1990), Jacob Boudoukh and Robert F. Whitelaw (1991), William L. Silber (1992), Philip Daves and Michael Ehrhardt (1993), Avrahma Kamara (1994), Mark Grinblatt and Longstaff (2000), Menachem Brenner, Rafi Eldor, and Shmuel Hauser (2001), Tarun Chordia, Richard Roll, and Avanidhar Subrahmanyam (2001), Laura Field and Gordon Hanka (2001), Arvind Krishnamurthy (2002), Lubos Pastor and Robert Stambaugh (2003), Ashley Wang (2003), Longstaff (2004), and many others. Similarly, important theoretical contributions include David Mayers (1972, 1973, 1976), Steven Lippman and John McCall (1986), George M. Constantinides (1986), Amihud and Mendelson (1986), Grossman and Laroque (1990), Jacob Boudoukh and Robert F. Whitelaw (1993), Longstaff (1995, 2001), Dimitri Vayanos (1998, 2003), Bengt Hölmstrom and Jean Tirole (2001), Pierre-Oliver Weill (2008), Matthias Kahl, Jun Liu, and Longstaff (2003), Ming Huang (2003), Maureen O'Hara (2003), Vayanos and Tan Wang (2007), Andrea Eisfeldt (2004), Viral Acharya and Lasse H. Pedersen (2005), Darrell Duffie, Nicolae Gârleanu, and Pedersen (2005, 2007), and many others. The approach taken in this paper of modeling illiquidity as portfolio irreversibility has a number of parallels in the literature on asset pricing and uninsurable labor income. Important papers in this literature include S. Rao Aiyagari and Mark Gertler (1991), John Heaton and Deborah J. Lucas (1992, 1996), Chris I. Telmer (1993), Constantinides and Duffie (1996), and others. Finally, this paper has implications for the rapidly growing limited-participation literature, since assets that are not readily marketable clearly restrict an agent's ability to participate in financial markets.

The remainder of this paper is organized as follows. Section I describes the basic modeling framework used throughout the paper. Section II presents the fully liquid benchmark case. Section III presents the illiquid-asset case. Section IV compares the two cases and discusses the asset-pricing implications of asset illiquidity. Section V summarizes the results and make concluding remarks.

I. The Model

The key challenge in exploring the asset-pricing implications of illiquidity is developing a modeling framework in which agents endogenously want to trade and, therefore, care about whether they can buy or sell individual assets. This task is complicated by the fact that many standard paradigms in asset pricing, such as the familiar single-asset representative-agent model, imply that there is no trading in equilibrium. Thus, to capture the incentives that agents have for trading assets over time, we need to move beyond the usual types of asset-pricing models. To this end, we develop a simple two-asset version of the standard Robert E. Lucas (1978) pure exchange economy in which there are two heterogeneous agents. When both assets can be traded, the market is effectively dynamically complete. When one asset is illiquid, however, the market becomes

[4] For an excellent survey of the literature, see Amihud, Mendelson, and Pedersen (2005).

dynamically incomplete. All models are stylized to some degree, and the model we present is no exception to the rule. Despite this, however, the model does capture the economics of asset illiquidity in an intuitive way and is consistent with many actual forms of asset illiquidity observed in the markets.

The basic structure of the model can be viewed as the extension of John H. Cochrane, Longstaff, and Pedro Santa-Clara (2008) to a heterogeneous agent economy. There are two assets or "trees" in this economy. Each asset produces a stream of dividends in the form of the single consumption good. Let X_t and Y_t denote the dividends generated by the assets. The dividends follow simple i.i.d. geometric Brownian motions,

$$\frac{dX}{X} = \mu_X \, dt + \sigma_X \, dZ_X, \tag{1}$$

$$\frac{dY}{Y} = \mu_Y \, dt + \sigma_Y \, dZ_Y, \tag{2}$$

where the correlation between dZ_X and dZ_Y is ρdt. Although we refer to the assets as risky throughout the discussion, nothing prevents one of the assets from being a riskless bond since σ_X or σ_Y can equal zero.[5] We normalize the number of shares of each asset in the economy to be one. To keep notation as simple as possible, expectations and variables without time subscripts (such as X and Y) will denote initial or time-zero values.

There are two agents in this model. The first agent is endowed with w shares of the first and second assets, respectively. Thus, the second agent is endowed with $1 - w$ shares of the two assets. Denote the agents' consumption streams by C_t and D_t, respectively. Since the effect of illiquidity in this model is that agents may have to wait to rebalance their portfolios, intuition suggests that attitudes about waiting may play a central role in the economics of illiquidity. Accordingly, we allow for heterogeneity in the agents' level of patience. Specifically, we assume that preferences at time zero are given by

$$\ln(C) + E\left[\int_0^\infty e^{-\beta t} \ln(C_t) \, dt\right], \tag{3}$$

$$\ln(D) + E\left[\int_0^\infty e^{-\delta t} \ln(D_t) \, dt\right], \tag{4}$$

where the subjective time discount rates β and δ may differ, implying that one agent is less patient that the other. For concreteness, assume that the first agent is less patient than the second, $\beta > \delta$. We refer to the first and second agents as the less patient and more patient agents, respectively.[6]

[5] More specifically, one of the assets can be a riskless consol bond in positive net supply by setting either σ_X or σ_Y equal to zero.

[6] These preferences imply that consumption at time zero takes the form of a discrete gulp, while consumption at all future times occurs as a flow. Thus, X and Y represent discrete dividends, while X_t and Y_t represent dividend flows. This feature simplifies the exposition of the model, but has no effect on any of the results. In particular, the model could easily be modified to allow only continuous consumption, or only discrete consumption, without any loss of generality.

II. The Fully Liquid Case

As a preliminary, we present results for the case where both assets are fully liquid. These results will be used as a benchmark for comparison to the illiquid-asset case in the next section.

Let P_t and Q_t denote the equilibrium prices for the first and second assets. The first-order conditions for the first agent imply

$$P_t = E_t\left[\int_0^\infty e^{-\beta s}\left(\frac{C_t}{C_{t+s}}\right)X_{t+s}\,ds\right], \tag{5}$$

$$Q_t = E_t\left[\int_0^\infty e^{-\beta s}\left(\frac{C_t}{C_{t+s}}\right)Y_{t+s}\,ds\right]. \tag{6}$$

Similarly, the first-order conditions for the second agent imply

$$P_t = E_t\left[\int_0^\infty e^{-\delta s}\left(\frac{X_t + Y_t - C_t}{X_{t+s} + Y_{t+s} - C_{t+s}}\right)X_{t+s}\,ds\right], \tag{7}$$

$$Q_t = E_t\left[\int_0^\infty e^{-\delta s}\left(\frac{X_t + Y_t - C_t}{X_{t+s} + Y_{t+s} - C_{t+s}}\right)Y_{t+s}\,ds\right], \tag{8}$$

after substituting in the market-clearing condition $D_t = X_t + Y_t - C_t$.

In this setting, the first agent chooses his consumption C_t, and a portfolio consisting of N_t and M_t shares of the two assets, respectively, and similarly for the second agent. The Web Appendix (available at http://www.aeaweb.org/articles.php?doi=10.1257/aer.99.4.1119) shows that equilibrium consumption of the first agent is given by

$$C_t = \frac{e^{-\beta t}\,\beta(1+\delta)w}{e^{-\beta t}\,\beta(1+\delta)w + e^{-\delta t}\,\delta(1+\beta)\,(1-w)}\,(X_t + Y_t). \tag{9}$$

From equation (9), optimal consumption depends directly on the distribution of wealth in the economy as measured by the fraction w of total assets initially held by the first agent.

Similarly, the optimal portfolio for the first agent is given by

$$N_t = M_t = \frac{e^{-\beta t}\,(1+\delta)w}{e^{-\beta t}\,(1+\delta)w + e^{-\delta t}\,(1+\beta)\,(1-w)}. \tag{10}$$

This optimal portfolio rule has several important aspects. First, each agent holds a portfolio that has the same number of shares of each of the two assets. Since there are equal numbers of shares of the two assets in the economy, however, this means that the optimal portfolio for each agent is simply the market portfolio. Thus, a one-fund separation result holds; agents would be indifferent between trading the assets individually or trading the shares of a stock index fund. Second, trading occurs in equilibrium since the number of shares held by the two agents changes over time. In particular, since the first agent is less patient than the second, the first agent systematically sells his portfolio to the second agent over time.[7] This enables the first agent to consume more than the total dividends he receives initially. Over time, however, the share of dividends consumed by the first agent declines while the share of dividends consumed by the second agent increases. Third, the portfolio rule is a deterministic function of time; the portfolio rule does not vary with changes in the state variables X_t and Y_t. Finally, the optimal portfolio rule is also affected by the initial distribution of wealth as measured by w.

Asset prices can now be obtained by substituting the optimal consumption process into the first-order equations and evaluating the expectations. The Web Appendix shows that the prices for the two assets are given by

$$P_t = C_t A(\beta, X_t, Y_t) + (X_t + Y_t - C_t) A(\delta, X_t, Y_t), \tag{11}$$

$$Q_t = C_t B(\beta, X_t, Y_t) + (X_t + Y_t - C_t) B(\delta, X_t, Y_t), \tag{12}$$

where we define the functions $A(\cdot, X_t, Y_t)$ and $B(\cdot, X_t, Y_t)$ as

$$A(\cdot, X_t, Y_t) = k_1 \left(\frac{X_t}{Y_t}\right) F\left(1, 1-\gamma; 2-\gamma; -\frac{X_t}{Y_t}\right) + k_2 F\left(1, \theta; 1+\theta; -\frac{Y_t}{X_t}\right), \tag{13}$$

$$B(\cdot, X_t, Y_t) = k_3 \left(\frac{Y_t}{X_t}\right) F\left(1, 1+\theta; 2+\theta; -\frac{Y_t}{X_t}\right) - k_4 F\left(1, -\gamma; 1-\gamma; -\frac{X_t}{Y_t}\right), \tag{14}$$

where k_1, k_2, k_3, k_4, γ, and θ are constants defined in the Web Appendix. The function $F(a, b; c; z)$ is the standard hypergeometric function (see Milton Abramowitz and Irene A. Stegum 1970, ch. 15). The hypergeometric function is defined by the power series

$$F(a, b; c; z) = 1 + \frac{ab}{c \times 1} z + \frac{a(a+1)\, b(b+1)}{c(c+1) \times 1 \times 2} z^2 + \frac{a(a+1)(a+2)\, b(b+1)(b+2)}{c(c+1)(c+2) \times 1 \times 2 \times 3} z^3 + \cdots \tag{15}$$

The hypergeometric function has an integral representation, which can be used for numerical evaluation and as an analytic continuation beyond $\| z \| < 1$,

$$F(a, b; c; z) = \frac{\Gamma(c)}{\Gamma(b)\Gamma(c-b)} \int_0^1 w^{b-1}(1-w)^{c-b-1}(1-wz)^{-a}\, dw, \tag{16}$$

where $\mathrm{Re}(c) > \mathrm{Re}(b) > 0$.

Interestingly, the functions $A(\cdot, X_t, Y_t)$ and $B(\cdot, X_t, Y_t)$ have intuitive interpretations as the equilibrium prices for the first and second assets (normalized by total consumption) that would exist in a representative-agent version of this economy, where the subjective time discount rate for the representative agent equals the first argument. Thus, the price functions P_t and Q_t in equations (11) and (12) are simple consumption-weighted (or, equivalently, wealth-weighed) averages of the prices in a representative-agent economy with subjective discount rates of β and δ, respectively.

[7] Note that this trading activity occurs because of the heterogeneity of the agents; trading does not occur in the standard single-representative-agent setting.

III. The Illiquid-Asset Case

In this section, we relax the assumption that both assets are always liquid. Specifically, we assume that the first asset can always be traded. Thus, there is always a way for agents to smooth their intertemporal consumption by trading in the first asset. Note that the first asset can play the role of a riskless asset in this framework since the volatility of its dividends can be set equal to zero and the agents are allowed to take whatever long or short position in this asset they choose.[8] The second asset can be traded at time zero, but then becomes illiquid and cannot be traded again until time T. After time T, the second asset reverts back to being fully liquid. Thus, this approach models the illiquidity of the second asset as a trading blackout period. By contrasting this case with the benchmark fully liquid case, we can examine directly how asset prices and portfolio choice are affected by illiquid assets in the market.

In this setting, the first agent chooses an initial consumption level C and a portfolio of assets N and M at time zero, and similarly for the second agent. Once the portfolio is chosen, however, the number of shares of the second asset cannot be changed until time T. Thus, $M_t = M$ for $t < T$. The first agent's initial consumption C consists of his endowment less the value of the portfolio chosen,

$$C = w(P + X + Q + Y) - NP - MQ. \tag{17}$$

Subsequent consumption equals the dividends on the portfolio net of changes in the holdings of the first asset,

$$C_t = N_t X_t + MY_t - H_t P_t, \tag{18}$$

for $0 < t < T$, where H_t is the rate at which shares of the first asset are sold. By definition,

$$N_t = N + \int_0^t H_s \, ds. \tag{19}$$

At time T, both assets become fully liquid again and the framework reverts back to the benchmark fully liquid case, although with one slight difference. Specifically, the first agent arrives at time T with N_T (before rebalancing the portfolio) and M shares of the two assets, respectively, rather than with w shares of each asset. With this modification, the Appendix shows that optimal consumption at time T is given by

$$C_T = \frac{(N_T A(\delta, X_T, Y_T) + MB(\delta, X_T, Y_T))(X_T + Y_T)}{1/\beta + N_T(A(\delta, X_T, Y_T) - A(\beta, X_T, Y_T)) + M(B(\delta, X_T, Y_T) - B(\beta, X_T, Y_T))}. \tag{20}$$

Similarly, the prices of the two assets at time T are again given by

$$P_T = C_T A(\beta, X_T, Y_T) + (X_T + Y_T - C_T)\, A(\delta, X_T, Y_T), \tag{21}$$

[8] The fact that the asset is in positive net supply rather than in zero net supply has little effect on the results. In equilibrium, market clearing requires that the total amount of the first asset must be held by market participants. Thus, the existence of this market-clearing condition is important in determining the qualitative nature of the equilibrium. Whether the market clears at a level of zero, one, two, etc. shares is qualitatively less important than the fact that some market-clearing condition is imposed.

$$Q_T = C_T\, B(\beta, X_T, Y_T) + (X_T + Y_T - C_T)\, B(\delta, X_T, Y_T). \tag{22}$$

but where C_T is defined as in equation (20).

Solving for the values of the assets at time zero in this illiquid-asset case requires an recursive approach in which we first solve for the value of the liquid asset and the optimal number of shares of the first asset to hold at times $T - \Delta t$, $T - 2\Delta t$, $T - 3\Delta t, \ldots, \Delta t$, conditional on the corresponding state variables. In doing this, we make sequential use of the first-order conditions implied by the agents' optimal portfolio and consumption choices

$$P_t = E_t\left[\int_0^{T-t} e^{-\beta s}\left(\frac{C_t}{C_{t+s}}\right) X_{t+s}\, ds + e^{-\beta(T-t)}\left(\frac{C_t}{C_T}\right) P_T\right], \tag{23}$$

$$P_t = E_t\left[\int_0^{T-t} e^{-\delta s}\left(\frac{X_t + Y_t - C_t}{X_{t+s} + Y_{t+s} - C_{t+s}}\right) X_{t+s}\, ds \right.$$
$$\left. + e^{-\delta(T-t)}\left(\frac{X_t + Y_t - C_t}{X_T + Y_T - C_T}\right) P_T\right]. \tag{24}$$

The Appendix describes the recursive numerical approach that is used to solve these pairs of equations for the values of P_t and N_t.

Once this recursive process is complete, we need to solve for the time-zero values N, M, P, and Q. At time zero, the first-order conditions for the first agent imply

$$P = E\left[\int_0^{T} e^{-\beta t}\left(\frac{C}{C_t}\right) X_t\, dt + e^{-\beta T}\left(\frac{C}{C_T}\right) P_T\right], \tag{25}$$

$$Q = E\left[\int_0^{T} e^{-\beta t}\left(\frac{C}{C_t}\right) Y_t\, dt + e^{-\beta T}\left(\frac{C}{C_T}\right) Q_T\right], \tag{26}$$

where consumption values are given as above. Similarly, the first-order conditions for the second agent imply

$$P = E\left[\int_0^{T} e^{-\delta t}\left(\frac{X + Y - C}{X_t + Y_t - C_t}\right) X_t\, dt + e^{-\delta T}\left(\frac{X + Y - C}{X_T + Y_T - C_T}\right) P_T\right], \tag{27}$$

$$Q = E\left[\int_0^{T} e^{-\delta t}\left(\frac{X + Y - C}{X_t + Y_t - C_t}\right) Y_t\, dt + e^{-\delta T}\left(\frac{X + Y - C}{X_T + Y_T - C_T}\right) Q_T\right]. \tag{28}$$

The Appendix shows how the equilibrium values of N, M, P, and Q can be determined from these four equations numerically.[9] A key difference between the equilibrium trading rules in the fully

[9] As shown by Peter Diamond (1967); John Geanakoplos and Herakles Polemarchakis (1986); Geanakoplos et al. (1990), and others, an unconstrained Pareto optimal equilibrium need not exist in an incomplete market. This occurs in our framework when one of the agents is at a corner by choosing zero shares of an asset. In this situation, the agent's

liquid case and in this illiquid-asset case is that trading becomes state dependent when only the liquid asset can be traded. In fact, illiquidity can have large effects on the amount of trading that occurs in equilibrium in this framework.

IV. Asset-Pricing Implications

A. *Calibrating the Model*

The illiquid asset in the model can be viewed as representing any one of a wide variety of illiquid asset classes. For example, one could interpret the illiquid asset as private equity such as a sole proprietorship, a general or limited partnership interest, or even a venture capital investment. One could also interpret the illiquid asset as being representative of pension or retirement accounts, real estate holdings, trusts, human capital, etc.

To make the intuition as clear as possible, however, we will explore the asset-pricing implications of the model by calibrating to a specific market. We then illustrate comparative statics results by varying individual parameters of the baseline calibration. In particular, we will interpret the liquid asset in the model as representing aggregate corporate equity, and the illiquid asset as representing aggregate noncorporate equity (sole proprietorships, partnerships, other forms of private equity, etc.).

To calibrate the dividend process for corporate equity, we follow Longstaff and Monika Piazzesi (2004) and we use the real per capita growth rates for corporate profits after tax using annual NIPA data for the 1929–2005 period.[10] Similarly, to calibrate the dividend process for the noncorporate equity, we use annual NIPA data for proprietors' income (with inventory valuation and capital consumption allowance) for the same time period. Both series are deflated using the GDP implicit price deflator. We use corporate profits after tax rather than before tax to eliminate one layer of the double taxation of corporate profits, making this series more comparable to proprietors' income. During the 1929–2005 period, the standard deviation of annual growth rates for corporate profits and proprietors' income are 21.6 and 10.3 percent, respectively. The correlation between the two growth rates is 52.1 percent (which we round down to 50 percent). The mean annual growth rates for corporate profits and proprietors' income are 4.40 and 1.96 percent, respectively. The average ratio of proprietors' income to the sum of corporate profits and proprietors' income from 1929 to 2005 is 61.1 percent (rounded down to 60 percent). In the baseline calibration, we assume that the less patient agent has 50 percent of aggregate wealth. Finally, we assume that the agents have subjective time-discount rates of $\beta = 0.10$ and $\delta = 0.01$.[11] The parameter values used in the baseline calibration are summarized in Table 1.

Throughout this section, we report results for this baseline calibration as the top line in each panel of the tables. To show how the baseline results vary with the parameters, however, the other

first-order condition for that asset is not required to hold. The Appendix shows how the equilibrium portfolio holdings for the other asset and the prices can be determined from the remaining first-order conditions in this situation.

[10] Longstaff and Piazzesi (2004) argue that because corporations tend to smooth dividend payouts over time, measures tied to corporate earnings are likely to provide better information about the actual cash flows generated by these firms. Since corporate profits after tax are negative during 1931–1932, we drop these two years in computing statistics for this time series.

[11] These subjective time-discount rates are consistent with both experimental and empirical research in the area. For example, in an experimental study, Uri Benzion, Ammon Rapoport, and Yosef Yagil (1989) find that when a large amount of money was involved ($5,000), subjects behaved as if their personal discount rates were between 10 and 20 (or higher) percent. The high personal discount rates apparently used by economic agents in making decisions over short horizons is one of the main justifications for the use of hyperbolic discounting in utility functions (for example, see Thaler 1981; David Laibson 1996, 1997; Ariel Rubinstein 2003). These parameter values are also broadly consistent with the range of values estimated in recent empirical work using data from consumer credit markets. For example, see Haiyan Shui, and Lawrence Ausubel (2005) and Laibson, Andrea Repetto, and Jeremy Tobacman (2005).

TABLE 1—PARAMETERS USED IN THE BASELINE CALIBRATION

Parameter	Symbol	Value
Time discount rate for less-patient agent	β	0.1000
Time discount rate for more-patient agent	δ	0.0100
Distribution of initial wealth	w	0.5000
Dividend growth rate for liquid asset	μ_X	0.0440
Dividend growth rate for illiquid asset	μ_Y	0.0196
Dividend volatility for liquid asset	σ_X	0.2160
Dividend volatility for illiquid asset	σ_Y	0.1030
Correlation of dividends	ρ	0.5000
Ratio of illiquid dividend to total dividends	$Y_0/(X_0 + Y_0)$	0.6000

lines in each panel report results where one parameter at a time is varied from its baseline value (all other parameters held fixed). Specifically, in addition to the results for the baseline calibration, we report results when the volatility of the illiquid asset is 5.0 and 21.6 percent. These values represent realistic alternative calibrations for the model. For example, a volatility of 5.0 percent for the illiquid asset's dividend is consistent with the annualized volatility of short- to intermediate-term fixed income portfolios similar to those that fund many pension plans and retirement accounts. This level also approximates the 4.79 percent volatility of the annual percentage change in real per capita compensation of employees received for the 1929–2005 period reported in the NIPA tables. Similarly, the 21.6 percent volatility for the illiquid asset reflects the situation in which the two assets are essentially equally risky, differing only in terms of their liquidity.

In like manner, we also report results for the cases where the size of the illiquid asset (in terms of the ratio of the illiquid asset's dividends to total dividends) is varied to 10 percent and to 90 percent. The former case can be viewed as the situation in which illiquid assets are a relatively small segment of the market (such as restricted stock relative to unrestricted stock). The latter case can be viewed as the scenario in which the illiquid asset represents private equity, real estate holdings, pension assets, human capital, and the other types of not readily marketable investments that constitute the vast majority of the typical US household's balance sheet.[12] Similarly, we also report results for the cases where the correlation of the dividend processes is 10 and 90 percent, and for the cases where the less patient agent is endowed with 10 percent and 90 percent of initial wealth.

In each table, we report the results for illiquidity horizons T of zero (the fully liquid case), 1, 2, 5, 10, and 30 years. These values reflect degrees of illiquidity ranging from private equity, IPO, or hedge-fund lockups all the way to the extended illiquidity of pensions and retirement assets, or even human capital.

B. *Calibration Diagnostics*

As a diagnostic for the reasonableness of the baseline calibration, it is also useful to examine the implications for model outputs such as return moments. As will be shown in subsequent tables, the baseline calibration implies that the expected one-year return and return volatility of the liquid asset are 12.03 and 22.08 percent, respectively. These values are very consistent with historical stock index return moments. For example, the mean return and return volatility of the CRSP value-weighted index during the 1927–2007 period (data provided courtesy of Ken

[12] For example, Robert Shiller (1998) estimates that 73.7 percent of US national wealth is in the form of human capital, 7.9 percent in real estate, and 2.2 percent in the form of durable assets.

French) are 12.01 and 20.11 percent, respectively. During the more recent 1950–2007 period, the same measures are 13.02 and 17.02 percent, respectively. For the S&P 500 index, the mean one-year return and return volatility during the 1950–2007 period are 12.57 and 15.38 percent, respectively. Since the S&P 500 index includes only larger firms, the volatility of the S&P 500 index may understate the volatility of all stocks since smaller firms tend to have higher return volatilities than larger firms.

The baseline calibration also implies that the expected return and return volatility of the illiquid asset (when the illiquidity horizon is zero) are 11.50 and 11.21 percent, respectively. While it is difficult to obtain data on historical returns for aggregate noncorporate equity, several recent studies provide some results for private equity and venture capital. Important examples include Alexander Ljungqvist and Matthew Richardson (2003), Cochrane (2005), and Steven N. Kaplan and Antoinette Schoar (2005). The most directly comparable results, however, are those presented in Tobias Moskowitz and Annette Vissing-Jorgensen (2002), who compute equity return series for proprietors and partnerships for the 1953–1999 period. Table 6 of their paper reports that the average return and return volatility for proprietors and partnerships equity is 13.1 and 6.9 percent, respectively. For the 1963–1999 period, the corresponding values are 13.2 and 7.7 percent, respectively. These results clearly indicate that the return moments implied by the baseline calibration for the illiquid asset are in the right "ballpark." Furthermore, Moskowitz and Vissing-Jorgensen report that the correlation of proprietors and partnerships equity returns with the corresponding measure for public equity returns during the 1963–1999 period is 0.70. This closely matches the correlation value of 0.668 implied by the baseline calibration.

C. *Portfolio Choice*

We turn first to the implications of the model for the portfolio choices of the two agents. Table 2 reports the optimal portfolio holdings of the less patient agent; the optimal portfolio holdings for the more patient agent are simply one minus those for the less patient agent.

The top line in each panel shows the results for the baseline calibration. As expected, both agents find it optimal to hold the market portfolio in the fully liquid case in which $T = 0$. In particular, the less patient agent chooses a portfolio consisting of 0.480 shares of each of the two assets (implying that the more patient agent chooses a portfolio consisting of 0.520 shares of each of the two assets).

When one asset is illiquid, however, optimal portfolio behavior becomes strikingly different. Instead of holding the diversified market portfolio, the less patient agent now begins to tilt his portfolio toward the liquid asset and away from the illiquid asset. This effect becomes more pronounced as the length of the illiquidity horizon increases. Thus, with illiquidity, the logarithmic agents no longer behave myopically, as in the case in the standard Robert C. Merton (1969, 1971) portfolio-choice paradigm. For example, when the illiquidity horizon is ten years in the baseline calibration, the less patient agent's optimal portfolio consists of 0.713 shares and 0.363 shares of the liquid and illiquid assets, respectively. Going further, when the illiquidity horizon is 30 years, the less patient agent owns all of the liquid asset in the economy, leaving the more patient agent with a portfolio consisting entirely of only one of the two assets. Thus, the agents abandon portfolio diversification as a strategy and choose highly polarized portfolios instead. In the extreme, an almost complete separation in equilibrium portfolio choice may occur among agents with different degrees of patience.[13]

[13] These results, of course, depend on the assumption that there are no trading costs. If there were costs to rebalancing a portfolio, the degree of polarization in optimal portfolios would likely decrease. For an in-depth analysis of the effects of transaction costs in incomplete markets with heterogeneous agents, see Heaton and Lucas (1996).

TABLE 2—EFFECTS OF ILLIQUIDITY ON OPTIMAL PORTFOLIO WEIGHTS

					Portfolio weight for illiquidity horizon				
Volatility	Size	Correlation	Wealth	Fully liquid weight	1	2	5	10	30
Panel A. Liquid asset									
0.103	0.60	0.50	0.50	0.480	0.492	0.503	0.549	0.713	1.000
0.050	—	—	—	0.480	0.496	0.512	0.566	0.745	1.000
0.216	—	—	—	0.480	0.487	0.493	0.520	0.594	1.000
—	0.10	—	—	0.480	0.482	0.483	0.490	0.501	0.554
—	0.90	—	—	0.480	0.538	0.652	1.000	1.000	1.000
—	—	0.10	—	0.480	0.494	0.508	0.556	0.699	1.000
—	—	0.90	—	0.480	0.488	0.495	0.532	0.732	1.000
—	—	—	0.10	0.093	0.097	0.102	0.131	0.188	0.270
—	—	—	0.90	0.893	0.898	0.900	0.915	0.935	1.000
Panel B. Illiquid asset									
0.103	0.60	0.50	0.50	0.480	0.474	0.469	0.446	0.363	0.183
0.050	—	—	—	0.480	0.472	0.465	0.440	0.353	0.193
0.216	—	—	—	0.480	0.475	0.471	0.451	0.397	0.123
—	0.10	—	—	0.480	0.472	0.467	0.441	0.395	0.167
—	0.90	—	—	0.480	0.474	0.463	0.421	0.396	0.389
—	—	0.10	—	0.480	0.473	0.466	0.441	0.365	0.180
—	—	0.90	—	0.480	0.476	0.473	0.458	0.360	0.186
—	—	—	0.10	0.093	0.091	0.089	0.076	0.052	0.015
—	—	—	0.90	0.893	0.890	0.888	0.880	0.869	0.827

Notes: This table presents the optimal portfolio weights chosen by the less-patient agent for the indicated illiquidity horizons. Panel A reports the portfolio weights for the liquid asset; panel B reports the portfolio weights for the illiquid asset. The illiquidity horizons are measured in years. The top line in each panel reports the weights for the baseline calibration. The remaining lines report the weights when the indicated parameter is varied from the baseline calibration. Volatility denotes the volatility of the dividend growth rate of the illiquid asset. Size denotes the ratio of the dividends of the illiquid asset to total dividends. Correlation denotes the correlation between the growth rates of the two assets' dividends. Wealth denotes the initial proportion of total wealth with which the less-patient agent is endowed. Fully liquid weight is the optimal portfolio weight in the absence of a liquidity restriction.

While these results differ dramatically from those that hold when assets are fully liquid, the intuition behind them is not hard to understand. In this economy, the less patient agent has a strong incentive to accelerate his consumption. His only mechanism for doing so, however, is to systematically sell off his portfolio to the more patient agent over time. When one of the assets is illiquid, the less patient agent needs to hold more of the liquid asset to be able to do so. The longer the illiquidity horizon, the larger is the position in the liquid asset that the first agent chooses. By tilting his portfolio toward the liquid asset, however, the less patient first agent suffers the welfare costs of holding an undiversified portfolio consisting of the higher-volatility liquid asset. Ultimately, the optimal portfolio represents a trade-off between the benefits of being able to sell assets to accelerate consumption and the costs of being undiversified.

Turning now to the additional results in which the parameters of the baseline calibrations are varied, Table 2 shows that the qualitative implications for optimal portfolio choice are the same in all cases. Specifically, the less patient agent tilts his portfolio toward the liquid asset and away from the illiquid asset. The degree to which this occurs depends, however, on the parameter values.

For example, Table 2 shows that the degree of portfolio polarization is generally an increasing function of volatility of the illiquid asset. Specifically, the first agent tends to deviate more from the market portfolio when the volatility of the illiquid asset is 5.0 percent, and vice versa when the volatility of the illiquid asset is 21.6 percent. The intuition for this is simply that, holding the volatility of the liquid asset fixed, the total risk in the market decreases as the volatility of the illiquid asset decreases. Thus, the welfare costs of not holding the diversified market portfolio decrease with the volatility of the illiquid assets, leading to larger deviations from the market portfolio. This effect may help explain a number of well-known puzzles about the way actual agents choose investment portfolios. For example, Mankiw and Zeldes (1991) find evidence that equity market participation of the typical US household is far less than predicted by classical models of portfolio choice (Merton 1969, 1971). Our results suggest that if households view stockholdings as riskier and more liquid than less volatile but illiquid assets such as pension and retirement assets, human capital, or home equity, it may be rational for them to hold far less stock than short-horizon institutional investors.[14]

Varying the size of the illiquid asset can have a large effect on the degree of portfolio polarization. For example, when the illiquid asset's dividend represent 90 percent of total dividends, the less patient agent holds 100 percent of the liquid asset for illiquidity horizons of five years or more. Intuitively, this follows because the less patient agent has strong incentives to sell his portfolio over time to accelerate his consumption. To do this, he needs to hold some of the liquid asset. If the liquid asset represents only a small portion of the total assets in the economy, then he needs to sell more of it to generate the same amount of accelerated consumption. In turn, he needs to hold more of it to be able to sell more of it. In essence, he needs a larger slice of a "smaller pie."

Table 2 shows that the optimal portfolio holdings become slightly more polarized as the correlation between the dividend processes increases. The intuition for this is that the welfare benefit from holding a diversified portfolio decreases as the correlation increases. Thus, as the welfare costs of holding a less diversified portfolio decrease, the agents find it optimal to deviate further from holding the market portfolio. While varying the distribution of the initial wealth endowment across agents naturally changes the actual number of shares held by the agents, it does not alter the qualitative nature of the optimal portfolio decisions of the agents.

Finally, Figures 1 and 2 plot the less patient agent's consumption share and portfolio holdings over time for both the fully liquid case and the ten-year illiquidity horizon. Recall that the consumption share and portfolio holding are time varying but deterministic in the fully liquid case. In contrast, both consumption and the number of shares of the liquid asset become stochastic in the illiquid case. Accordingly, Figures 1 and 2 plot the expected consumption share and expected portfolio holdings for the illiquid case along with the corresponding tenth and ninetieth percentile values.

D. *Asset Valuation*

The introduction of illiquidity in the form of temporary portfolio irreversibility can have important effects on equilibrium asset values. To illustrate this, Table 3 reports the percentage change in the value of each asset (relative to what the asset's price would be in the fully liquid or $T = 0$ case) for the indicated illiquidity horizons.

Table 3 shows that even though the liquidity constraint is imposed on only one of the assets, the equilibrium values of both assets are affected. In the baseline calibration, the liquid asset becomes more valuable when illiquidity is introduced, while the illiquid asset becomes less valuable. The magnitude of the valuation effect varies significantly, however, with the length of the illiquidity horizon. In particular, the valuation effect on the liquid asset is only 0.24 percent for a one-year horizon, but increases monotonically to 10.52 percent for a 30-year horizon. Similarly, the valuation effect on the illiquid asset is -0.12 percent for a one-year horizon, but becomes

[14] This assumes, of course, that institutional investors do have shorter horizons than individual investors, an assumption that is clearly debatable.

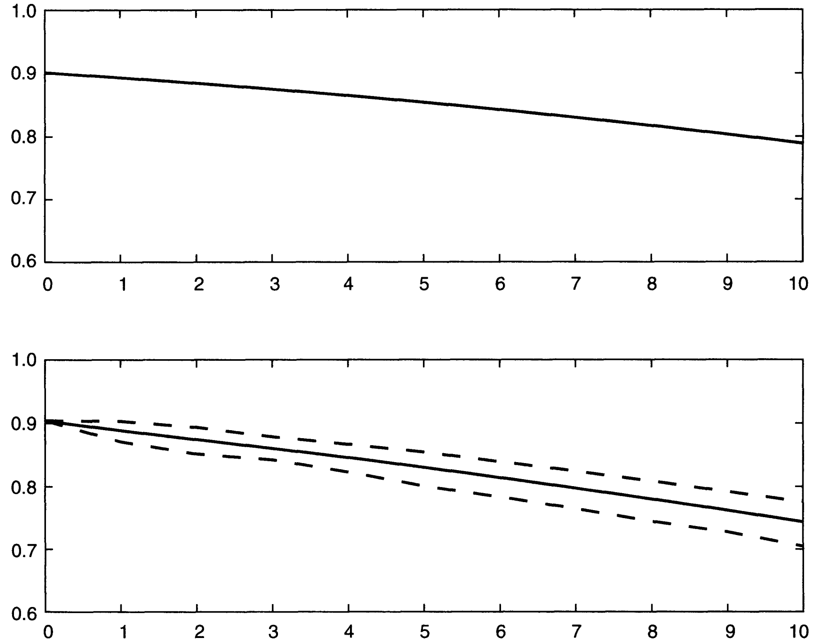

FIGURE 1. CONSUMPTION SHARE OF THE LESS-PATIENT AGENT

Notes: The top panel plots the consumption share of the less-patient agent over the indicated horizon measured in years. The solid line in the lower panel plots the expected consumption share of the less-patient agent over the indicated horizon given a ten-year illiquidity period. The dashed lines in the lower panel represent the tenth and ninetieth percentiles of the consumption share. Both panels present results for the baseline calibration.

−5.13 percent for a 30-year horizon.[15] Thus, while the valuation effects are modest for shorter illiquidity horizons, they can represent an important portion of the price of an asset for longer illiquidity horizons.

For the longer illiquidity horizons, the magnitude of the valuation differences implied by the model may seem implausibly large. These valuation differences, however, are broadly consistent with the extensive recent empirical evidence about the effects of illiquidity on otherwise identical securities. For example, Silber (1992) finds that the difference in value between privately placed illiquid stock and otherwise identical publicly traded stock averages 35 percent. Karen Wruck (1989) documents valuation differences of about 15 percent for the illiquid privately placed shares of major NYSE firms. Boudoukh and Whitelaw (1991) show that illiquid Japanese government bonds traded at as much as a 5 percent discount to virtually identical liquid benchmark Japanese government bonds. Amihud and Mendelson (1991), Kamara (1994), Krishnamurthy (2002), Longstaff (2004), and others find similar large differences in the values of otherwise identical liquid and illiquid Treasury bonds. Brenner, Eldor, and Hauser (2001)

[15] Our results are consistent with those in Anthony Lynch and Sinan Tan (forthcoming) who use a partial equilibrium framework to estimate the utility cost to an agent who faces redemption costs in liquidating stockholdings, and may even not be allowed to sell at all for a number of periods. When returns are predictable (as they are in our general equilibrium setting), Tables 3 and 4 of Lynch and Tan show that the utility cost of a one-year redemption period can exceed 1 percent of the agent's wealth. These results also parallel those in Longstaff (2001), who finds similar utility costs when agents are restricted in their ability to trade by being able to follow only those trading strategies of bounded variation.

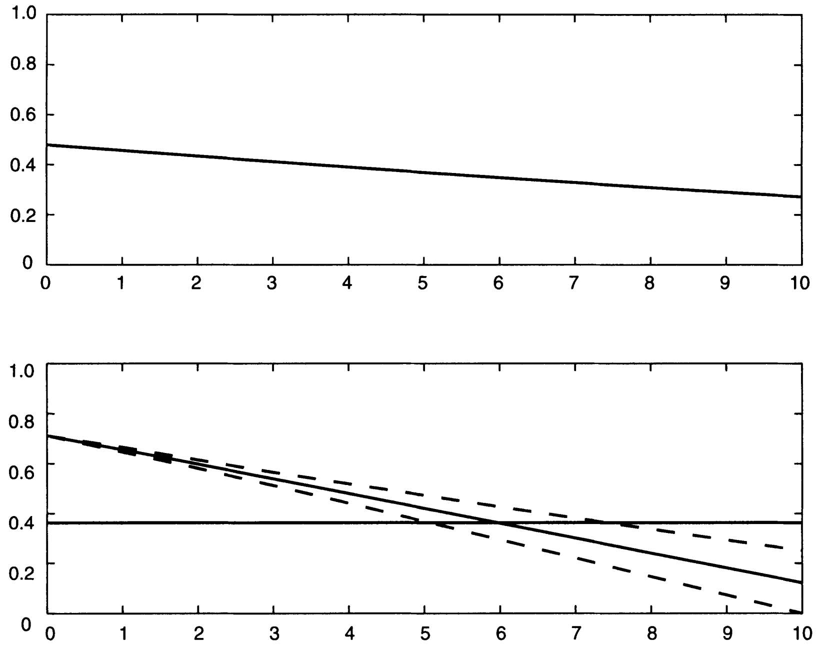

Figure 2. Portfolio Holdings of the Less-Patient Agent

Notes: The top panel plots the number of shares of the two securities held by the less-patient agent over the indicated horizon measured in years. The horizontal solid in the lower panel plots the less-patient agent's holdings of the illiquid asset for the indicated horizon given a ten-year illiquidity period while the other solid line plots the agent's holdings of the liquid asset. The dashed lines in the lower panel represent the tenth and ninetieth percentiles of the holdings of the liquid asset. Both panels present results for the baseline calibration.

show that the valuation difference between liquid and illiquid foreign currency options can be as large as 30 percent. It is important to observe, however, that the illiquidity horizons considered in these papers are typically on the order of six months to two years. Thus, we acknowledge that our calibrated model is not able to capture the full magnitude of these empirical results. The model does hold out the hope, however, that what may appear as irrationally large illiquidity discounts may in fact be reconcilable with some rational general equilibrium model.[16]

Turning now to the comparative statics results, Table 3 shows that varying the volatility of the illiquid asset has only a minor effect on the magnitude of the valuation effects for all but the longest illiquidity horizon. To illustrate, the valuation effects for the two assets are within several percentage points, as the illiquid asset's volatility varies from 5.0 to 21.6 percent for all values of $T < 10$ years.

The same is not true as the size of the illiquid asset varies, however. Given the results for the calibrated model, it might be tempting to infer that effects of illiquidity on asset prices are straightforward: that liquid assets become more valuable, and vice versa for illiquid assets. In actuality, the asset-pricing effects of illiquidity are much more subtle in general equilibrium. In particular, when the dividend of the illiquid asset is only 10 percent of total dividends, the sign of the asset-pricing effect is reversed; the liquid asset can actually become *less* valuable as

[16] Partial equilibrium models include Longstaff (2001), and Kahl, Liu, and Longstaff (2003).

TABLE 3—EFFECTS OF ILLIQUIDITY ON ASSET PRICES

					Percentage price change for illiquidity horizon				
Volatility	Size	Correlation	Wealth	Fully liquid price	1	2	5	10	30
Panel A. Liquid asset									
0.103	0.60	0.50	0.50	5.854	0.24	0.50	0.54	1.81	10.52
0.050	—	—	—	5.599	0.13	0.26	0.30	1.58	12.87
0.216	—	—	—	7.407	0.23	0.47	0.51	1.14	−2.25
—	0.10	—	—	14.360	−0.12	−0.25	−0.24	−0.22	0.14
—	0.90	—	—	1.584	1.04	2.24	14.20	55.53	78.08
—	—	0.10	—	6.029	0.15	0.31	0.36	1.64	7.94
—	—	0.90	—	5.628	0.39	0.78	0.81	2.04	14.22
—	—	—	0.10	12.733	0.28	0.53	0.62	0.94	1.96
—	—	—	0.90	3.945	0.23	0.46	0.47	0.49	5.33
Panel B. Illiquid asset									
0.103	0.60	0.50	0.50	11.919	−0.12	−0.24	−0.26	−0.87	−5.13
0.050	—	—	—	12.175	−0.06	−0.12	−0.13	−0.72	−5.88
0.216	—	—	—	10.367	−0.16	−0.33	−0.36	−0.80	1.65
—	0.10	—	—	3.414	0.55	1.09	1.09	1.07	−0.27
—	0.90	—	—	16.190	−0.10	−0.22	−1.39	−5.43	−7.65
—	—	0.10	—	11.744	−0.08	−0.16	−0.18	−0.83	−4.04
—	—	0.90	—	12.145	−0.18	−0.36	−0.37	−0.94	−6.56
—	—	—	0.10	29.468	−0.11	−0.22	−0.23	−0.32	−0.47
—	—	—	0.90	7.048	−0.13	−0.26	−0.26	−0.24	−2.97

Notes: This table presents the percentage change in the price of the asset relative to its fully liquid value for the indicated illiquidity horizons. Panel A reports the percentge price changes for the liquid asset; panel B reports the percentage price changes for the illiquid asset. The illiquidity horizons are measured in years. The top line in each panel reports the weights for the baseline calibration. The remaining lines report the weights when the indicated parameter is varied from the baseline calibration. Volatility denotes the volatility of the dividend growth rate of the illiquid asset. Size denotes the ratio of the dividends of the illiquid asset to total dividends. Correlation denotes the correlation between the growth rates of the two assets' dividends. Wealth denotes the initial proportion of total wealth with which the less-patient agent is endowed. Fully liquid price is the asset price in the absence of a liquidity restriction.

illiquidity is introduced, and vice versa for the illiquid asset. This result parallels Vayanos (1998), who finds that the price of an asset can be an increasing function of its transaction costs.

The intuition behind this perplexing result is perhaps best understood by recognizing that there are two major factors at work in determining the agents' optimal portfolios. The first is the strong incentive that the less patient agent has to hold the liquid asset. As discussed previously, this leads the less patient agent to accelerate his consumption by selling his portfolio over time. The second effect is more subtle and has to do with intertemporal risk sharing. The two agents have the same attitude toward instantaneous risk because they both have logarithmic preferences. When illiquidity is introduced, however, patience emerges as an important additional factor driving portfolio choice. In particular, the less patient agent is now better suited to bear the risk of holding the riskier asset during the illiquid period, since his future utility is more heavily discounted and, consequently, less sensitive to noninstantaneous risk. This aspect introduces a strong demand for intertemporal risk sharing that is not present when markets are fully liquid. Thus, equilibrium prices adjust to give the less patient agent an incentive to tilt his portfolio toward the risky asset when the market is illiquid. Usually, the first effect dominates and the less patient agent has to pay a premium to buy the liquid asset. However, when the market is awash with the liquid asset (for example, when it constitutes 90 percent of the market), the more patient agent is willing to pay a small premium to the less patient agent to hold more of the riskier asset, which then allows the more patient agent hold more of the less-volatile asset in his portfolio.

At the other end of the liquidity spectrum, Table 3 also reports the valuation effects when the liquid asset represents only 10 percent of the total market. In this polar extreme case, the valuation effects can be enormous. For example, when $T = 5$ years, the liquid asset trades at a 14.20 percent premium to its price in the fully liquid case, while the illiquid asset trades at a 1.39 percent discount. When the illiquidity horizon is $T = 30$ years, the liquid asset has a price that is 78.08 percent higher than in the fully liquid case, while the illiquid asset trades at a 7.65 percent discount. The intuition behind these results follows simply from the limited supply of the liquid asset. The less patient agent is willing to pay an increasingly large premium for the liquid asset as it becomes more scarce in the market.

The other comparative statics results illustrate that the valuation effects are only modestly affected by variation in the correlation parameter. Similarly, wide variation in the distribution of initial endowments does not lead to major differences in the valuation effects.

In summary, these results illustrate that the irreversibility of portfolios created by asset illiquidity introduces an important additional dimension into asset pricing. As another way of seeing this, consider a symmetric case where both assets have identical dividend dynamics and are equal in terms of the current dividends. In the fully liquid case, the two assets would have identical prices. In the presence of illiquidity, however, it is easily demonstrated that the two assets could differ in price by as much as 10 percent or more for, say, a 30-year illiquidity horizon. On its face, this result might appear to be a violation of the law of one price given that two otherwise identical assets with identical distribution of future dividends are trading at different prices. In reality, however, this example drives home the principle that while the value of an asset comes from the present value of its cash flows, the cash flows that an agent receives from an asset are not simply the dividends the asset pays. Rather, the cash flows from an asset consist of the dividends plus the market value of shares bought and sold over time. Thus, the consumption stream from a liquid asset can be very different from that generated by an illiquid asset, even when both assets have identical dividend dynamics. In asset pricing, it is the actual consumption stream generated by asset ownership that matters, not just the stream of dividends.

E. *Expected Returns*

In this section, we explore the implications of illiquidity on the expected returns of the two assets. Since the liquid asset is traded continuously, its return over any holding period is well defined. Although the illiquid asset does not trade during the illiquidity horizon, it is straightforward to calculate its T-period return over the period from time zero to the end of the illiquidity horizon. To solve for the annualized T-period returns for both assets, we simulate one million paths of the two dividend processes, follow the optimal consumption and trading policies along each path, solve for the corresponding values of the two assets at time T, and then compute the annualized arithmetic return for each asset implied by the T-period return. Table 4 reports the difference between the annualized expected T-period return in the presence of the illiquidity constraint and the annualized expected T-period return in the absence of the illiquidity constraint.

To put the expected return effects into perspective, we observe that for the fully liquid case in the baseline calibration, the expected return for a one-year horizon is 12.03 percent for the liquid asset and 11.50 percent for the illiquid asset. Turning now to the results in Table 4, we see that for shorter horizons, the effect of illiquidity on expected returns is roughly equal to the (negative of the) valuation effect divided by T.

For longer horizons, however, this intuitive symmetry disappears. For example, when $T = 30$ in the baseline calibration, the valuation effect for the liquid asset is 10.52 percent, while the expected return effect is $-\$1.65$ percent. In some of the alternative calibrations, the expected return actually changes in sign as the length of the illiquidity horizon increases. Intuitively, the

TABLE 4—EFFECTS OF ILLIQUIDITY ON EXPECTED RETURNS

					Change in percentage expected return for illiquidity horizon				
Volatility	Size	Correlation	Wealth	Fully liquid expected return	1	2	5	10	30
Panel A. Liquid asset									
0.103	0.60	0.50	0.50	12.03	−0.24	−0.23	−0.09	−0.24	−1.65
0.050	—	—	—	12.17	−0.13	−0.12	−0.04	−0.23	−1.79
0.216	—	—	—	11.41	−0.27	−0.23	−0.10	−0.15	−0.99
—	0.10	—	—	11.81	−1.18	0.14	0.06	0.04	−0.88
—	0.90	—	—	11.65	0.15	−1.11	−3.03	−6.77	−4.59
—	—	0.10	—	12.23	−0.16	−0.15	−0.06	−0.25	−1.59
—	—	0.90	—	12.16	−0.36	−0.37	−0.13	−0.21	−1.75
—	—	—	0.10	7.15	−0.27	−0.25	−0.12	−0.11	−0.23
—	—	—	0.90	11.17	−0.22	−0.23	−0.09	−0.05	−3.67
Panel B. Illiquid asset									
0.103	0.60	0.50	0.50	11.50	0.18	0.16	0.08	0.03	−1.25
0.050	—	—	—	11.46	0.10	0.09	0.05	−0.02	−1.33
0.216	—	—	—	11.81	0.21	0.18	0.07	0.04	1.11
—	0.10	—	—	10.65	0.15	−0.55	−0.20	−0.09	−0.99
—	0.90	—	—	11.74	−0.65	0.15	−0.13	−1.86	−2.59
—	—	0.10	—	11.57	0.12	0.10	0.05	−0.01	−1.28
—	—	0.90	—	11.40	0.26	0.23	0.12	0.10	−1.23
—	—	—	0.10	8.94	0.15	0.13	0.06	0.01	−0.16
—	—	—	0.90	10.13	0.14	0.14	0.06	0.04	−3.82

Notes: This table presents the change in the percentage expected return of the asset relative to its fully liquid value for the indicated illiquidity horizons. Panel A reports the changes for the liquid asset; panel B reports the changes for the illiquid asset. The illiquidity horizons are measured in years. The top line in each panel reports the weights for the baseline calibration. The remaining lines report the weights when the indicated parameter is varied from the baseline calibration. Volatility denotes the volatility of the dividend growth rate of the illiquid asset. Size denotes the ratio of the dividends of the illiquid asset to total dividends. Correlation denotes the correlation between the growth rates of the two assets' dividends. Wealth denotes the initial proportion of total wealth with which the less-patient agent is endowed. Fully liquid expected return is the percentage expected return in the absence of a liquidity restriction.

reason for this disconnect is that even after the illiquidity period lapses and both assets become fully liquid again, the economy is not the same as if there had been no trading blackout in the first place. Specifically, the distribution of time-T wealth between the two agents becomes stochastic when there is an illiquidity horizon (in contrast to the fully liquid case in which the distribution of wealth at time T is deterministic). This creates an additional source of randomness in the distribution of time-T prices which then drives a wedge between the valuation effect and the effect on expected returns.[17]

In an important recent paper, Acharya and Pedersen (2005) present a model in which time-varying liquidity costs affect equilibrium asset prices. Calibrating their model to stock market data, they find that the combined effects of illiquidity adds about 1.1 percent to the difference in expected returns between liquid and illiquid stocks. Their results extend those of Amihud and Mendelson (1986), Constantinides (1986), and others by demonstrating how transaction and other types of liquidity costs can affect equilibrium expected returns. Our results complement this important earlier work by showing that even in the absence of transaction costs or other explicit liquidity costs, equilibrium asset pricing effects may arise through the distortions in agents' consumption plans induced by illiquidity.

[17] In principle, it would be possible for the illiquidity-induced randomness in the time-T distribution of wealth to affect expected returns even in the absence of a valuation effect at time zero.

TABLE 5—EFFECTS OF ILLIQUIDITY ON RETURN VOLATILITY

					Change in percentage return volatility for illiquidity horizon				
Volatility	Size	Correlation	Wealth	Fully liquid return volatility	1	2	5	10	30
Panel A. Liquid asset									
0.103	0.60	0.50	0.50	22.08	−0.08	−0.05	−0.03	−0.06	0.07
0.050	—	—	—	22.13	−0.13	−0.07	−0.00	−0.10	0.09
0.216	—	—	—	21.38	−0.05	−0.03	0.03	0.03	−0.07
—	0.10	—	—	23.43	−0.36	0.01	0.01	0.01	−0.03
—	0.90	—	—	21.58	0.01	−0.26	−0.16	−0.19	−0.03
—	—	0.10	—	21.28	−0.14	−0.07	0.01	−0.04	0.07
—	—	0.90	—	22.69	−0.04	−0.04	0.07	−0.07	0.05
—	—	—	0.10	21.72	−0.08	−0.05	−0.05	−0.08	−0.02
—	—	—	0.90	21.31	−0.05	−0.04	0.01	0.05	0.27
Panel B. Illiquid asset									
0.103	0.60	0.50	0.50	11.21	0.01	0.01	0.02	−0.01	0.01
0.050	—	—	—	6.65	−0.08	−0.04	0.01	−0.05	0.08
0.216	—	—	—	21.96	0.04	0.02	−0.03	−0.04	0.01
—	0.10	—	—	12.33	0.01	−0.05	−0.01	0.04	−0.02
—	0.90	—	—	10.91	−0.09	0.00	0.02	0.10	−0.03
—	—	0.10	—	10.25	0.03	0.01	0.01	0.01	−0.04
—	—	0.90	—	12.06	0.07	0.04	0.09	−0.05	0.08
—	—	—	0.10	11.14	0.01	0.01	−0.01	−0.03	−0.01
—	—	—	0.90	10.54	0.01	0.01	0.01	0.02	0.06

Notes: This table presents the change in the percentage return volatility of the asset relative to its fully liquid value for the indicated illiquidity horizons. Panel A reports the changes for the liquid asset; panel B reports the changes for the illiquid asset. The illiquidity horizons are measured in years. The top line in each panel reports the weights for the baseline calibration. The remaining lines report the weights when the indicated parameter is varied from the baseline calibration. Volatility denotes the volatility of the dividend growth rate of the illiquid asset. Size denotes the ratio of the dividends of the illiquid asset to total dividends. Correlation denotes the correlation between the growth rates of the two assets' dividends. Wealth denotes the initial proportion of total wealth with which the less-patient agent is endowed. Fully liquid weight is the percentage return volatility in the absence of a liquidity restriction.

F. *Volatility of Returns*

Following the approach used to calculate expected returns, we use the simulated distribution of returns to compute the volatility of the annualized T-period return for the assets with and without liquidity restrictions. Table 5 reports the differences. For reference, the volatilities of the one-year returns for the liquid and illiquid assets are 22.08 and 11.21 percent, respectively, in the fully liquid case when the baseline calibration is used.

Table 5 shows that the introduction of illiquidity generally has only a modest effect on asset return volatility. Furthermore, the sign of the effect on the return volatility of the assets is indeterminant. For the baseline calibration, the volatility of the liquid asset is slightly lower with the illiquidity restriction than without it, while the volatility of the illiquid asset is almost identical.

G. *Return Correlation*

To calculate the correlation between the annualized T-period returns, we again use the distribution of returns implied by the simulated paths of the dividend processes. Table 6 reports the change in the correlation of the assets induced by the introduction of an illiquidity horizon in the model.

TABLE 6—EFFECTS OF ILLIQUIDITY ON RETURN CORRELATIONS

					Change in percentage return correlation for illiquidity horizon				
Volatility	Size	Correlation	Wealth	Fully liquid return correlation	1	2	5	10	30
0.103	0.60	0.50	0.50	66.82	−0.21	−0.14	0.66	−0.81	3.47
0.050	—	—	—	79.81	−0.96	−0.69	−0.07	−2.41	5.52
0.216	—	—	—	62.30	0.01	0.00	−0.04	−0.06	−0.62
—	0.10	—	—	80.21	−0.76	−0.09	0.02	0.11	−0.23
—	0.90	—	—	53.98	−0.12	−1.06	2.12	10.41	3.86
—	—	0.10	—	38.84	−0.93	−0.68	0.36	−0.68	3.68
—	—	0.90	—	92.98	0.07	0.07	0.34	−0.38	1.06
—	—	—	0.10	71.18	−0.12	−0.07	−0.46	−1.30	−0.38
—	—	—	0.90	60.59	−0.06	−0.04	0.31	−1.45	9.80

Notes: This table presents the change in the percentage return correlation of the assets relative to its fully liquid value for the indicated illiquidity horizons. The illiquidity horizons are measured in years. The top line reports the weights for the baseline calibration. The remaining lines report the weights when the indicated parameter is varied from the baseline calibration. Volatility denotes the volatility of the dividend growth rate of the illiquid asset. Size denotes the ratio of the dividends of the illiquid asset to total dividends. Correlation denotes the correlation between the growth rates of the two assets' dividends. Wealth denotes the initial proportion of total wealth with which the less-patient agent is endowed. Fully liquid return correlation is the percentage return correlation in the absence of a liquidity restriction.

As shown in Cochrane, Longstaff, and Santa-Clara (2008), the correlation between the returns on the two assets can be significantly higher than the correlation between dividend processes even when both assets are fully liquid. The source of this additional correlation is the common variation in the pricing kernel (marginal utility of the representative agent) applied to both assets. In the baseline calibration, the correlation of dividends is 50 percent. When both assets are fully liquid, this implies a correlation of 66.82 percent between the annualized two-year returns on the assets.

Table 6 shows that the introduction of illiquidity in the two-asset model can have a significant incremental effect on the correlation of returns. For example, in the baseline calibration, the correlation between the annualized 30-year returns of the two assets is 3.47 percent higher when there is a 30-year illiquidity horizon than in the fully liquid case. On the other hand, Table 6 shows that the effect of illiquidity on asset return correlations can be in the negative direction.

H. *Trading Turnover*

In the fully liquid model, the cumulative number of shares traded over time is deterministic. In the presence of a liquidity restriction, the cumulative number of shares traded becomes a random process. Furthermore, the illiquid asset is traded only at time zero and then not again until time T.

Despite these difficulties, however, it is still possible to provide a measure of the effect of illiquidity on trading activity in the following way. First, from Table 2, we know the initial number of shares of each asset traded at time zero. Next, again using the simulated paths of the dividend processes, we calculate the mean change in the agents' holdings of the liquid asset during the illiquidity horizon. Finally, we compute the ratio of the total number of shares of each asset traded from time zero to time T in the presence of illiquidity restrictions to the same measure for the fully liquid case. Table 7 reports these trading turnover ratios.

Consistent with the implications of the model for optimal portfolio choice, the introduction of illiquidity into the market can have a huge effect on trading activity. The top panel of Table 7 shows that the total turnover in shares of the liquid asset in the baseline calibration is typically

TABLE 7—EFFECTS OF ILLIQUIDITY ON TRADING TURNOVER RATIOS

				Trading turnover ratio for illiquidity horizon				
Volatility	Size	Correlation	Wealth	1	2	5	10	30
Panel A. Liquid asset								
0.103	0.60	0.50	0.50	2.07	2.14	3.13	3.94	3.00
0.050	—	—	—	2.29	2.40	3.43	4.25	3.00
0.216	—	—	—	1.73	1.74	2.21	2.67	2.91
—	0.10	—	—	1.10	1.11	1.14	1.17	1.13
—	0.90	—	—	9.58	10.37	12.50	7.36	3.01
—	—	0.10	—	2.09	2.16	3.12	3.78	2.98
—	—	0.90	—	2.18	2.26	3.09	4.18	3.01
—	—	—	0.10	2.24	2.29	3.68	4.57	4.35
—	—	—	0.90	1.89	1.99	2.86	3.33	1.24
Panel B. Illiquid asset								
0.103	0.60	0.50	0.50	0.53	0.53	0.46	0.67	0.64
0.050	—	—	—	0.59	0.58	0.51	0.71	0.62
0.216	—	—	—	0.49	0.49	0.42	0.50	0.76
—	0.10	—	—	0.57	0.64	0.51	0.52	0.67
—	0.90	—	—	0.57	0.63	0.68	0.51	0.24
—	—	0.10	—	0.58	0.58	0.51	0.66	0.64
—	—	0.90	—	0.46	0.45	0.38	0.69	0.63
—	—	—	0.10	0.55	0.55	0.67	0.88	0.86
—	—	—	0.90	0.50	0.50	0.39	0.28	0.08

Notes: This table presents the ratio of the expected number of shares traded from time zero to the end of the illiquidity horizon in the illiquid case to the corresponding measure for the fully liquid case. Panel A reports the ratios for the liquid asset; panel B reports the ratios for the illiquid asset. The illiquidity horizons are measured in years. The top line in each panel reports the weights for the baseline calibration. The remaining lines report the weights when the indicated parameter is varied from the baseline calibration. Volatility denotes the volatility of the dividend growth rate of the illiquid asset. Size denotes the ratio of the dividends of the illiquid asset to total dividends. Correlation denotes the correlation between the growth rates of the two assets' dividends. Wealth denotes the initial proportion of total wealth with which the less-patient agent is endowed.

multiples of the turnover in the fully liquid case. To see the intuition behind this result, observe that when there is an illiquidity horizon, the liquid asset essentially has to do "double duty" in financing the less patient agent's consumption plan. Roughly speaking, the less patient agent now has to sell $2X$ shares of the liquid asset (rather than selling X shares of each of the two assets) to approximate the consumption level that he would achieve in the absence of liquidity restrictions. To be ready to sell more shares of the liquid asset, however, the less patient agent has to "stock up" on these shares at time zero. Thus, the introduction of illiquidity into the model results in both more shares of the liquid asset being traded at time zero and more shares being traded over time than in the fully liquid case.

The bottom panel of Table 7 shows that illiquidity results in total trading turnover for the illiquid asset that is less than for the fully liquid case. As the length of the illiquidity horizon increases, however, the turnover ratio increases and begins to approach one. For example, the turnover ratio for the illiquid asset in the baseline calibration is 0.67 when $T = 10$ years. The intuition for these results can be understood by thinking of the time-zero trading in the illiquid asset as a one-shot attempt to approximate the average number of shares that would be held by the less patient agent over the T-period horizon. For example, the less patient agent might begin with 0.50 shares at time zero in the fully liquid case, and then reduce his holdings to 0.30 over the next ten years, resulting in an average holding of about 0.40 shares over this time frame. With the introduction of illiquidity, however, the agent has only one chance to trade and so would sell a lump sum 0.10 shares at time zero, resulting in average holdings of 0.40 shares during the illiquid period.

V. Conclusion

We study the asset-pricing implications of relaxing the standard assumption that assets can always be traded whenever investors would like to. To do this, we develop a two-asset heterogeneous agent model in which one asset can always be traded. The other asset, however, can be traded initially, but not again until after a trading blackout period. Thus, during the blackout period, only the liquid asset can be traded by the agents.

Because portfolio decisions become temporarily irreversible in this setting, the economics of portfolio choice are much more complex than in the traditional portfolio-choice framework. We show that when one of the assets is illiquid, the agents no longer find it optimal to hold the market portfolio. Instead, they abandon diversification as a strategy and tend to hold highly polarized portfolios. The reason for this stems from the effects of two factors not present in the traditional portfolio-choice problem. First, because the less patient agent needs to sell assets over time to accelerate his consumption, he needs to hold more of the liquid asset going into the trading blackout period. Second, illiquidity introduces an additional demand for intertemporal risk sharing which induces the less patient agent to hold more of the riskier asset. These two factors can reinforce or offset each other. In general, the less patient agent tilts his portfolio significantly toward the liquid asset.

The illiquidity-induced changes in the optimal portfolios can have major effects on asset prices. In our baseline calibration, expected returns can differ by roughly 20 basis points when the illiquidity horizon is one year, but can differ by more than 50 or 100 basis points when the illiquidity horizon is 30 years. Furthermore, assets with identical cash flow distributions, which otherwise would have the same value, can differ by 10 percent or more in their prices when one is liquid and the other is not. These results complement important recent work by Acharya and Pedersen (2005) and others showing that transaction-cost-based types of illiquidity can induce asset-pricing effects that are similar in magnitude to those we illustrate. In reality, both types of illiquidity effects may influence asset prices and a complete model of illiquidity may need to incorporate both transaction costs and marketability frictions.

Finally, our results indicate that heterogeneity in investor patience can have large asset-pricing implications. Heterogeneity in patience has been largely unexplored as a channel for resolving asset-pricing puzzles. Our results indicate that this may be a promising direction well worth pursuing in future research.

REFERENCES

Abramowitz, Milton, and Irene A. Stegun. 1970. *Handbook of Mathematical Functions*. New York: Dover Publications.

Acharya, Viral V., and Lasse Heje Pedersen. 2005. "Asset Pricing with Liquidity Risk." *Journal of Financial Economics*, 77(2): 375–410.

Aiyagari, S. Rao, and Mark Gertler. 1991. "Asset Returns with Transactions Costs and Uninsured Individual Risk." *Journal of Monetary Economics*, 27(3): 311–31.

Amihud, Yakov, and Haim Mendelson. 1986. "Asset Pricing and the Bid-Ask Spread." *Journal of Financial Economics*, 17(2): 223–49.

Amihud, Yakov, and Haim Mendelson. 1991. "Liquidity, Maturity, and the Yields on U.S. Treasury Securities." *Journal of Finance*, 46(4): 1411–25.

Amihud, Yakov, Haim Mendelson, and Lasse Heje Pedersen. 2005. "Liquidity and Asset Prices." *Foundations and Trends in Finance*, 1(4): 1–96.

Benartzi, Shlomo, and Richard H. Thaler. 2001. "Naive Diversification Strategies in Defined Contribution Saving Plans." *American Economic Review*, 91(1): 79–98.

Benzion, Uri, Ammon Rapoport, and Yosef Yagil. 1989. "Discount Rates Inferred from Decisions: An Experimental Study." *Management Science*, 35: 270–84.

Boudoukh, Jacob, and Robert F. Whitelaw. 1991. "The Benchmark Effect in the Japanese Government Bond Market." *The Journal of Fixed Income*, 1(2): 52–59.

Boudoukh, Jacob, and Robert F. Whitelaw. 1993. "Liquidity as a Choice Variable: A Lesson from the Japanese Government Bond Market." *Review of Financial Studies*, 6(2): 265–92.

Brenner, Menachem, Rafi Eldor, and Shmuel Hauser. 2001. "The Price of Options Illiquidity." *Journal of Finance*, 56(2): 789–805.

Chordia, Tarun, Richard Roll, and Avanidhar Subrahmanyam. 2001. "Market Liquidity and Trading Activity." *Journal of Finance*, 56(2): 501–30.

Cochrane, John H. 2005. "The Risk and Return of Venture Capital." *Journal of Financial Economics*, 75(1): 3–52.

Cochrane, John H., Francis A. Longstaff, and Pedro Santa-Clara. 2008. "Two Trees." *Review of Financial Studies*, 21(1): 347–85.

Constantinides, George M. 1986. "Capital Market Equilibrium with Transaction Costs." *Journal of Political Economy*, 94(4): 842–62.

Constantinides, George M., and Darrell Duffie. 1996. "Asset Pricing with Heterogeneous Consumers." *Journal of Political Economy*, 104(2): 219–40.

Cornell, Bradford, and Alan C. Shapiro. 1990. "The Mispricing of U. S. Treasury Bonds: A Case Study." *The Review of Financial Studies*, 2: 297–310.

Daves, Phillip R., and Michael C. Ehrhardt. 1993. "Liquidity, Reconstitution, and the Value of U.S. Treasury Strips." *Journal of Finance*, 48(1): 315–29.

Diamond, Peter. 1967. "The Role of a Stock Market in a General Equilibrium Model with Technological Uncertainty." *American Economic Review*, 57(4): 759–76.

Duffie, Darrell, Nicolae Gârleanu, and Lasse Heje Pedersen. 2005. "Over-the-Counter Markets." *Econometrica*, 73(6): 1815–47.

Duffie, Darrell, Nicolae Gârleanu, and Lasse Heje Pedersen. 2007. "Valuation in over-the-Counter Markets." *Review of Financial Studies*, 20(6): 1865–1900.

Eisfeldt, Andrea L. 2004. "Endogenous Liquidity in Asset Markets." *Journal of Finance*, 59(1): 1–30.

Field, Laura Casares, and Gordon Hanka. 2001. "The Expiration of IPO Share Lockups." *Journal of Finance*, 56(2): 471–500.

Geanakoplos, John, and Herakles Polemarchakis. 1986. "Existence, Regularity, and Constrained Suboptimization of Competitive Allocations when Markets are Incomplete." In *Uncertainty, Information and Communication: Essays in Honor of Kenneth Arrow, Volume 3*, ed. W. P. Heller, R. M. Ross, and D. A. Starrett, 65–95. Cambridge: Cambridge University Press.

Geanakoplos, John D., Michael Magill, Martine Quinzii, and Jacques Dreze. 1990. "Generic Inefficiency of Stock Market Equilibrium when Markets Are Incomplete." *Journal of Mathematical Economics*, 19(1–2): 113–51.

Gradshteyn, Izrail S., and Iosif M. Ryzhik. 2000. *Tables of Integrals, Series, and Products.* 6th ed. New York: Academic Press.

Grinblatt, Mark, and Francis A. Longstaff. 2000. "Financial Innovation and the Role of Derivative Securities: An Empirical Analysis of the Treasury Strips Program." *Journal of Finance*, 55(3): 1415–36.

Grossman, Sanford J., and Guy Laroque. 1990. "Asset Pricing and Optimal Portfolio Choice in the Presence of Illiquid Durable Consumption Goods." *Econometrica*, 58(1): 25–51.

Heaton, John, and Deborah J. Lucas. 1992. "The Effects of Incomplete Insurance Markets and Trading Costs in a Consumption-Based Asset Pricing Model." *Journal of Economic Dynamics and Control*, 16(3–4): 601–20.

Heaton, John, and Deborah J. Lucas. 1996. "Evaluating the Effects of Incomplete Markets on Risk Sharing and Asset Pricing." *Journal of Political Economy*, 104(3): 443–87.

Hindy, Ayman, and Chi-fu Huang. 1993. "Optimal Consumption and Portfolio Rules with Durability and Local Substitution." *Econometrica*, 61(1): 85–121.

Holmström, Bengt, and Jean Tirole. 2001. "LAPM: A Liquidity-Based Asset Pricing Model." *Journal of Finance*, 56(5): 1837–67.

Huang, Ming. 2003. "Liquidity Shocks and Equilibrium Liquidity Premia." *Journal of Economic Theory*, 109(1): 104–29.

Kahl, Matthias, Jun Liu, and Francis A. Longstaff. 2003. "Paper Millionaires: How Valuable Is Stock to a Stockholder Who Is Restricted from Selling It?" *Journal of Financial Economics*, 67(3): 385–410.

Kamara, Avraham. 1994. "Liquidity, Taxes, and Short-Term Treasury Yields." *Journal of Financial and Quantitative Analysis*, 29(3): 403–17.

Kaplan, Steven N., and Antoinette Schoar. 2005. "Private Equity Performance: Returns, Persistence, and Capital Flows." *Journal of Finance*, 60(4): 1791–1823.

Krishnamurthy, Arvind. 2002. "The Bond/Old-Bond Spread." *Journal of Financial Economics*, 66: 463–506.

Laibson, David I. 1996. "Hyperbolic Discount Functions, Undersaving, and Savings Policy." National Bureau of Economic Research Working Paper 5635.

Laibson, David. 1997. "Golden Eggs and Hyperbolic Discounting." *Quarterly Journal of Economics*, 112(2): 443–77.

Laibson, David, Andrea Repetto, and Jeremy Tobacman. 2005. "Estimating Discount Functions with Consumption Choices over the Lifecyle." Unpublished.

Lippman, Steven A., and John J. McCall. 1986. "An Operational Measure of Liquidity." *American Economic Review*, 76(1): 43–55.

Ljungqvist, Alexander, and Matthew Richardson. 2003. "The Cash Flow, Return and Risk Characteristics of Private Equity." National Bureau of Economic Research Working Paper 9454.

Longstaff, Francis A. 1995. "How Much Can Marketability Affect Security Values?" *Journal of Finance*, 50(5): 1767–74.

Longstaff, Francis A. 2001. "Optimal Portfolio Choice and the Valuation of Illiquid Securities." *Review of Financial Studies*, 14(2): 407–31.

Longstaff, Francis A. 2004. "The Flight-to-Liquidity Premium in U.S. Treasury Bond Prices." *Journal of Business*, 77(3): 511–26.

Longstaff, Francis A., and Monika Piazzesi. 2004. "Corporate Earnings and the Equity Premium." *Journal of Financial Economics*, 74(3): 401–21.

Lucas, Robert E, Jr. 1978. "Asset Prices in an Exchange Economy." *Econometrica*, 46(6): 1429–45.

Lynch, Anthony, and Sinan Tan. Forthcoming. "Multiple Risky Assets, Transaction Costs and Return Predictability: Allocation Rules and Implications for U.S. Investors." *Journal of Financial and Quantitative Analysis*.

Mankiw, N. Gregory, and Stephen P. Zeldes. 1991. "The Consumption of Stockholders and Nonstockholders." *Journal of Financial Economics*, 29(1): 97–112.

Mayers, David. 1972. "Nonmarketable Assets and Capital Market Equilibrium Under Uncertainty." In *Studies in the Theory of Capital Markets* ed. M. C. Jensen, 223–48. New York: Praeger.

Mayers, David. 1973. "Nonmarketable Assets and the Determination of Capital Asset Prices in the Absence of a Riskless Asset." *Journal of Business*, 46(2): 258–67.

Mayers, David. 1976. "Nonmarketable Assets, Market Segmentation, and the Level of Asset Prices." *Journal of Financial and Quantitative Analysis*, 11(1): 1–12.

Merton, Robert C. 1969. "Lifetime Portfolio Selection under Uncertainty: The Continuous-Time Case." *Review of Economics and Statistics*, 51(3): 247–57.

Merton, Robert C. 1971. "Optimum Consumption and Portfolio Rules in a Continuous-Time Model." *Journal of Economic Theory*, 3(4): 373–413.

Moskowitz, Tobias J., and Annette Vissing-Jorgensen. 2002. "The Returns to Entrepreneurial Investment: A Private Equity Premium Puzzle?" *American Economic Review*, 92(4): 745–78.

O'Hara, Maureen. 2003. "Presidential Address: Liquidity and Price Discovery." *Journal of Finance*, 58(4): 1335–54.

Pastor, Lubos, and Robert F. Stambaugh. 2003. "Liquidity Risk and Expected Stock Returns." *Journal of Political Economy*, 111(3): 642–85.

Rubinstein, Ariel. 2003. ""Economics and Psychology?" The Case of Hyperbolic Discounting." *International Economic Review*, 44(4): 1207–16.

Shiller, Robert. 1998. "Institutions for Managing Risks to Living Standards." *NBER Reporter*, Spring. http://www.nber.org/reporter/spring98/shiller_spring98.html.

Shui, Haiyan, and Lawrence Ausubel. 2005. "Time Inconsistency in the Credit Card Market." Unpublished.

Silber, William L. 1992. "Discounts on Restricted Stock: The Impact of Illiquidity on Stock Prices." *Financial Analysts Journal*, 47(4): 60–64.

Spiegel, Murray. 1967. *Applied Differential Equations*. 2nd ed. Englewood Cliffs: Prentice-Hall Inc.

Telmer, Chris I. 1993. "Asset-Pricing Puzzles and Incomplete Markets." *Journal of Finance*, 48(5): 1803–32.

Thaler, Richard. 1981. "Some Empirical Evidence on Dynamic Inconsistency." *Economic Letters*, 8: 201–07.

Vayanos, Dimitri. 1998. "Transaction Costs and Asset Prices: A Dynamic Equilibrium Model." *Review of Financial Studies*, 11(1): 1–58.

Vayanos, Dimitri. 2003. "Flight to Quality, Flight to Liquidity, and the Pricing of Risk." Unpublished.

Vayanos, Dimitri, and Tan Wang. 2007. "Search and Endogenous Concentration of Liquidity in Asset Markets." *Journal of Economic Theory*, 136(1): 66–104.

Wang, Ashley. 2003. "Institutional Equity Flows, Liquidity Risk, and Asset Pricing." Unpublished.

Weill, Pierre-Oliver. 2008. "Liquidity Premia in Dynamic Bargaining Markets." *Journal of Economic Theory*, 140(1): 66–96.

Wruck, Karen Hopper. 1989. "Equity Ownership Concentration and Firm Value: Evidence from Private Equity Financings." *Journal of Financial Economics*, 23(1): 3–28.