IMPLEMENTATION OF THE LONGSTAFF-SCHWARTZ INTEREST RATE MODEL

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Our two-factor model of the term structure of interest rates [1992] has a number of attractive features from the point of view of practical implementation.[1] It allows for twists in the term structure. It explicitly takes into account the stochastic nature of interest rate volatility. Closed-form solutions for discount bonds and options on discount bonds can be obtained. It allows for consistent pricing and hedging of all interest rate-contingent claims.

Here we address a number of important issues related to practical implementation of the model. First, we develop a simple procedure to estimate the stationary parameters of the model. These parameters are obtained from a time series of interest rates and the corresponding time series of variances of changes in interest rates estimated using a GARCH procedure.

Second, we examine whether forecasts of future interest rates and volatilities implied by these parameters are reasonable. Third, we show how to estimate the term structure parameter of the model from the current discount function. When only one term structure parameter is estimated, the discount function obtained does not fit exactly all the discount bonds used in the estimation.

We then show how the model can be made to fit exactly any number of discount bonds by estimating one term structure parameter ("time-varying parameter") for every discount bond used in the estimation.[2] We use this last version of the model to price interest rate caps. Finally, we show how cap prices can be used to extract an implied volatility.

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I. THE LONGSTAFF-SCHWARTZ MODEL

We are concerned here only with the aspects of the model that are required for its practical implementation. (For a complete presentation see Longstaff and Schwartz [1992].) The model starts from assumptions about the stochastic evolution of two unspecified positive state variables that follow uncorrelated mean-reverting square root processes with unit variance:

$$dx = (\gamma - \delta x)dt + \sqrt{x}dw$$
$$dy = (\eta - \xi y)dt + \sqrt{y}dz \quad (1)$$

where dw and dz are uncorrelated increments to Gauss-Wiener processes, and the Greek letters are parameters of the model.

The fact that these two state variables or factors are unspecified will turn out to be unimportant, because one of the important implications of the model is that both the short rate of interest, r, and the variance of changes in this rate, V, are linear functions of x and y:

$$r = \alpha x + \beta y$$
$$V = \alpha^2 x + \beta^2 y \quad (2)$$

where the parameters α and β come from the production process assumed in the general equilibrium framework.

The important thing for our purposes is that there are so far six parameters to estimate. Later on we show how to estimate these parameters from an historical time series of interest rates and volatilities, and we then refer to them as "stationary parameters" of the model.

As both r and V are linear functions of x and y, from (2) it is trivial to solve for x and y:

$$x = \frac{\beta r - V}{\alpha(\beta - \alpha)}$$
$$y = \frac{V - \alpha r}{\beta(\beta - \alpha)} \quad (3)$$

Without loss of generality we can assume that $\beta > \alpha$. Then, for the original state variables to be positive we must have, from (3):

$$\alpha < \frac{V}{r} < \beta \quad (4)$$

We will make use of this relation when we estimate the parameters α and β.

From Equations (1), (2), and (3) we can obtain the stochastic process for changes in r and V. As opposed to the processes for the original state variables, the processes for r and V will be correlated, and it can be shown that this correlation is positive and can vary from 0 to 1. It can also be shown that r and V have long-run stationary unconditional distributions with means:

$$E[r] = \frac{\alpha\gamma}{\delta} + \frac{\beta\eta}{\xi}$$
$$E[V] = \frac{\alpha^2\gamma}{\delta} + \frac{\beta^2\eta}{\xi} \quad (5)$$

and variances:

$$Var[r] = \frac{\alpha^2\gamma}{2\delta^2} + \frac{\beta^2\eta}{2\xi^2}$$
$$Var[V] = \frac{\alpha^4\gamma}{2\delta^2} + \frac{\beta^4\eta}{2\xi^2} \quad (6)$$

These expressions will be used to estimate the other four stationary parameters of the model.

Another important implication of the general equilibrium framework is that the value of any default-free interest rate-contingent claim follows the same fundamental partial differential equation, which can be written in terms of the original state variables as:

$$\frac{x}{2}H_{xx} + \frac{y}{2}H_{yy} + (\gamma - \delta x)H_x + (\eta - \nu y)H_y$$
$$- (\alpha x + \beta y)H - H_\tau = 0 \quad (7)$$

and subject to the appropriate boundary conditions of the particular claim.

Note that one of the stationary parameters, ξ, does not appear in the PDE. It is replaced by a new parameter, υ, which we later call the term structure parameter, and which corresponds to the original parameter plus a market price of risk.

Partial differential equation (7) has a closed-form solution for some claims such as discount bonds and options on discount bonds, and can be solved numerically for claims where closed-form solutions are unavailable.

II. ESTIMATION OF INTEREST RATE VOLATILITY USING GARCH PROCEDURE

Unfortunately, one of the two factors of the model, V, is not directly observable. Thus, our first task in implementation of the approach is to obtain a time series of volatilities. This can be easily done using the GARCH (Generalized Autoregressive Conditionally Heteroscedastic) framework developed by Bollerslev [1986].

More specifically, the econometric specification we use to model discrete changes in the riskless interest rate is given by:

$$r_{t+1} - r_t = \alpha_0 + \alpha_1 r_t + \alpha_2 V_t + \varepsilon_{t+1}$$

$$\varepsilon_{t+1} \sim N(0, V_t)$$

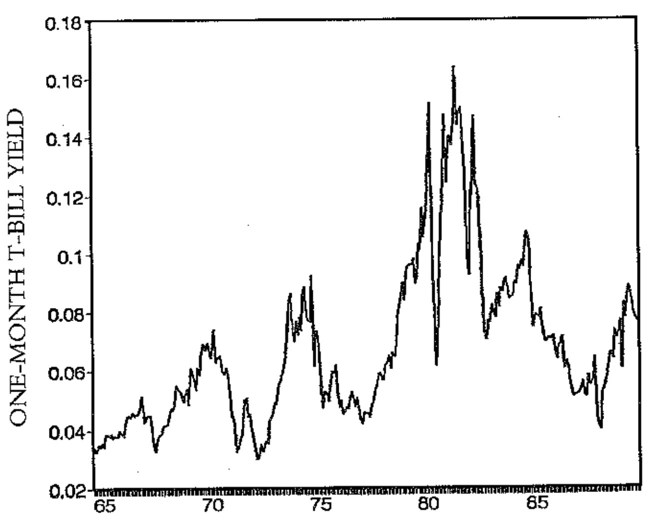
$$V_t = \beta_0 + \beta_1 r_t + \beta_2 V_{t-1} + \beta_3 \varepsilon_t^2 \qquad (8)$$

This specification closely parallels the continuous time dynamics of r and V, and it allows unexpected changes in r to be conditionally heteroscedastic and V to follow an autoregressive process.

The coefficients of system (8) and a time series of volatilities can be obtained from a time series of interest rates using an econometric package, such as TSP, that computes GARCH models. The coefficients depend in a complicated way on the six stationary parameters of the model and could potentially be used to estimate these parameters. We suggest instead an easier procedure to estimate these parameters based only on the time series of interest rates and volatilities.

Exhibit 1 shows the time series of one-month U.S. Treasury bill rates plotted at monthly intervals for the period January 1964 through December 1989. Exhibit 2 presents the GARCH estimates of the time series of volatilities over the same sample period. By comparing the two figures we can see that r and V are positively, although not perfectly, correlated.

EXHIBIT 1 ■ One-Month Treasury Bill Rate



If the last observation of the time series of interest rates is at the time when the model is to be applied to value contingent claims, the last volatility obtained using the GARCH procedure is the value of the volatility factor to be used in the model. As a second approach to compute this volatility, in Section VI we will show how to obtain an "implied volatility" factor from cap prices.

EXHIBIT 2 ■ Variance of Changes in the One-Month Treasury Bill Rate

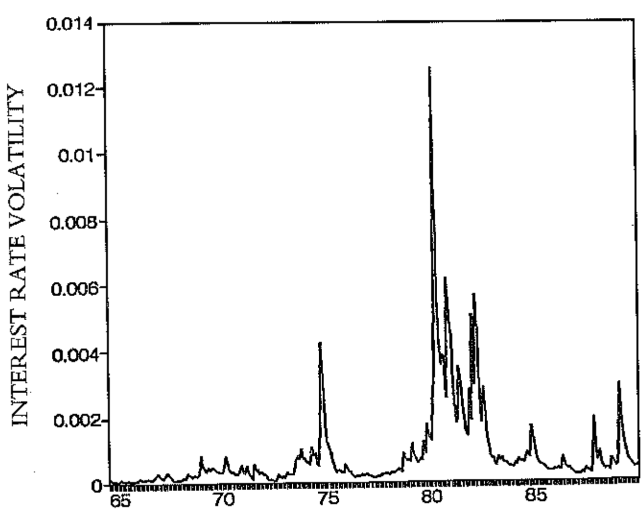


EXHIBIT 3 ■ Stationary Parameters

Obtained Using Monthly U.S. Treasury Bill Rates from 1964 to 1989

Mean Value of r	0.06717
Variance of r	0.0007157
Mean Value of V	0.0007658
Variance of V	0.000001526
Minimum Ratio of V/r	0.001149
Maximum Ratio of V/r	0.1325
α	0.001149
β	0.1325
γ	3.0493
δ	0.05658
η	0.1582
ξ	3.998

III. ESTIMATION OF STATIONARY PARAMETERS

A simple way to estimate all six stationary parameters of the model using only the original time series of interest rates and the GARCH estimated time series of volatilities is as follows:

1. Compute the mean and variance of the time series of interest rates.

2. Compute the mean and variance of the time series of volatilities.

3. Compute the maximum and the minimum value of the ratio V/r for contemporaneous values of these factors.

With these six parameters of the long-run joint distribution of r and V, the stationary parameters of the model can be obtained from (4), (5), and (6) as:

$$\alpha = \min\left(\frac{V_t}{r_t}\right)$$

$$\beta = \max\left(\frac{V_t}{r_t}\right)$$

$$\delta = \frac{\alpha(\alpha+\beta)(\beta E[r] - E[V])}{2(\beta^2 Var[r] - Var[V])}$$

$$\gamma = \frac{\delta(\beta E[r] - E[V])}{\alpha(\beta-\alpha)}$$

$$\xi = \frac{\beta(\alpha+\beta)(E[V] - \alpha E[r])}{2(Var[V] - \alpha^2 Var[r])}$$

$$\eta = \frac{\xi(E[V] - \alpha E[r])}{\beta(\beta-\alpha)}$$

Note that the way α and β are estimated assures that (4) will hold for the data used; these values are clearly related to the historical covariation between r and V.

In Exhibit 3 we report the six parameters of the time series of r and V and the corresponding six parameters of the model obtained using these formulas. Note that there is an exact mapping between the six parameters of the model and the six parameters of the distribution of r and V, so we could think about the parameters of the model as the first two moments of the distribution of r and V, and the maximum and minimum ratios of V/r.

To test whether the parameters obtained imply reasonable dynamics for future interest rates and volatilities, we use Equations (1) and (2) to simulate daily changes (360 times per year) in interest rates and volatilities up to a thirty-year horizon. Exhibit 4 reports the means and variances of the simulated values of r and V thirty years in the future using 10,000 simulations, and compares them with the corresponding historical parameters used to estimate the model parameters. As we can see from the table, the distribution of r and V obtained closely approximates its historical counterpart.

This is a test to which all interest rate models should be subject. It is quite possible to have an inter-

EXHIBIT 4 ■ Comparison Between Historical and Simulated Values

	Historical Value	Simulated Value
Mean Value of r	0.06717	0.06766
Variance of r	0.0007157	0.0006781
Mean Value of V	0.0007658	0.0008681
Variance of V	0.000001526	0.000001715

est model that nicely fits the initial term structure, but that forecasts very unreasonable future term structures giving, for example, negative interest rates.

IV. ESTIMATION OF THE TERM STRUCTURE PARAMETER

To price any interest rate-contingent claim, we need to solve PDE (7), subject to the boundary conditions determining the payoffs to the particular claim. But to do this, we need first to estimate the term structure parameter ν that replaces the stationary parameter ξ of the model in the equation (ν is simply ξ plus the market price of interest rate risk).

An easy way to estimate ν is to make use of the fact that the PDE has a simple analytical solution for a discount bond (see Longstaff and Schwartz [1992, p. 1266]). Given any number of discount bond prices (or STRIP prices) at any time, a grid search can very rapidly find the value of ν that minimizes the squared deviation between model and market prices.

To illustrate the procedure, we obtained from the *Wall Street Journal* fifteen note and bond principal U.S. Treasury STRIPs prices on November 9, 1992 (midpoint between bid and ask prices). Maturities ranged from one to twenty-nine years. The first two columns of Exhibit 5 report the maturities and prices, respectively, for the fifteen discount bonds. The discount function is also graphed in Exhibit 6.

To apply the discount bond formula, we also need the current value of the factors and five of the six stationary parameters. On November 9, 1992, the one-month Treasury bill rate was 0.0304, and we assumed a volatility factor of 0.0003. Because our interest rate data to estimate volatilities end in 1989, we could not get the volatility factor as the last element in the time series of volatilities estimated with the GARCH procedure. (Later we show how to imply the volatility factor from cap prices.) The stationary parameters used are the first five from Exhibit 3.

The value of ν obtained in the grid search is 0.335. The model discount bond prices obtained using this value of the term structure parameter and the corresponding error between market and model prices are given, respectively, in columns 3 and 4 of Exhibit 5. For some applications the magnitude of the errors reported in the table will be too large. Notice, however, that we are estimating one stationary term structure parameter for a discount function that goes twenty-nine years into the future.

In Exhibit 7, we can see that the model discount function calculated using the stationary term structure parameter obtained in the grid search does not exactly fit the STRIP prices used in the estimation of ν. The model spot rates and one-year forward rates shown in Exhibit 8, however, seem to be very reasonable.[3]

EXHIBIT 5 ■ Prices and Maturities of STRIPs Used to Estimate Term Structure Parameter

Maturity (Years)	STRIP Price	Model Value	Error
1	96.172	96.050	0.122
2	91.125	90.843	0.282
3	85.453	85.059	0.394
4	79.016	79.133	-0.117
5	73.203	73.312	-0.109
6	67.313	67.726	-0.413
7	62.188	62.438	-0.250
8	56.813	57.471	-0.658
9	52.484	52.831	-0.347
12	40.156	40.791	-0.635
17	25.563	26.107	-0.544
23	15.516	15.020	0.496
24	14.266	13.680	0.586
26	12.328	11.337	0.991
29	9.859	8.537	1.322

EXHIBIT 6 ■ Discount Function on November 9, 1992

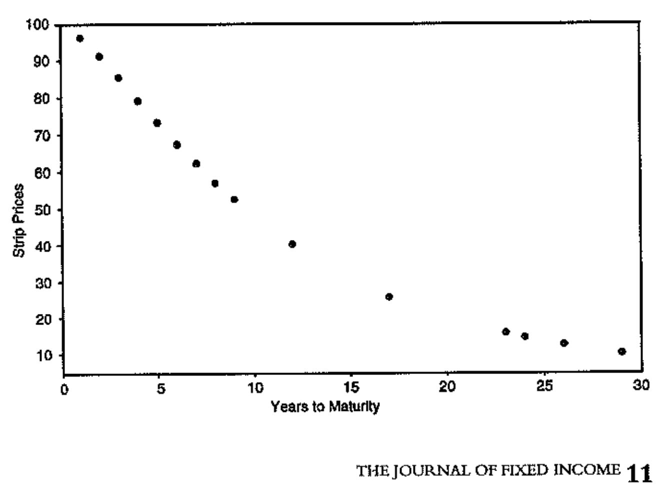
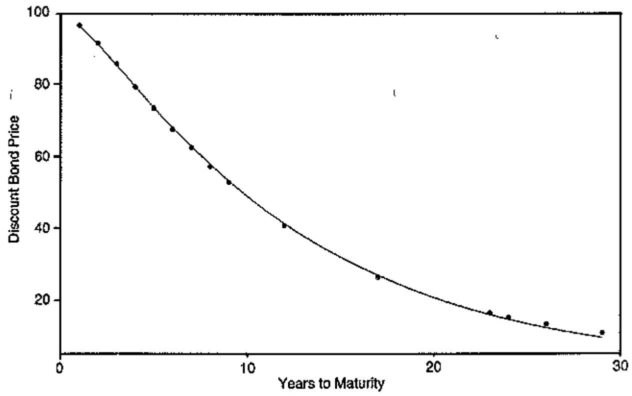


EXHIBIT 7 ■ Discount Function: Stationary Term Structure Parameter



V. TIME-VARYING TERM STRUCTURE PARAMETER

For some applications it may be important that the model discount bond prices exactly fit the STRIP prices used in the estimation. This can be accomplished in our model by making the term structure parameter time-varying. In this case there will be the same number of term structure parameters as the number of discount bond prices used in the estimation.

To implement the procedure, we first consider the discount bond closest to maturity: in our previous illustration the one-year STRIP. We determine the first term structure parameter, $\nu(1)$, as the value of ν that fits this first bond. As now we have only one bond to fit, this can be done exactly.

We then consider the next discount bond closest to maturity: the two-year STRIP in our example. We determine the value of the second term structure parameter, $\nu(2)$, which will hold from the maturity of the first bond to the maturity of the second, by determining the value of ν that will exactly price the second bond (taking into account that up to the maturity of the first bond $\nu(1)$ already estimated in the first step will be used). As this procedure involves values of ν that change with time to maturity, PDE (7) must be solved iteratively using numerical methods.

The next step in the procedure involves the determination of $\nu(3)$ from the third discount bond price, but taking into account that up to the maturity of the first bond $\nu(1)$ will be used, and between the maturity of the first and second bond $\nu(2)$ will be used. The procedure continues in this way until all the desired discount bonds have been used.

Exhibit 9 shows the values for the fifteen time-varying term structure parameters obtained by exactly fitting the fifteen STRIP prices reported in Exhibit 5. These time-varying term structure parameters are also graphed in Exhibit 10. On average their value is close to the value of the stationary term structure parameter

EXHIBIT 8 ■ Spot and One-Year Forward Interest Rates: Stationary Parameter

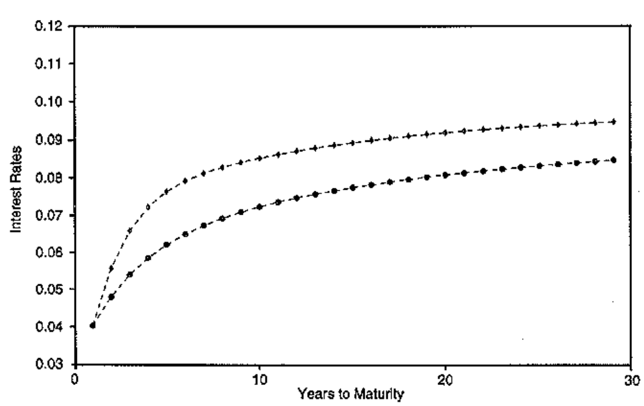
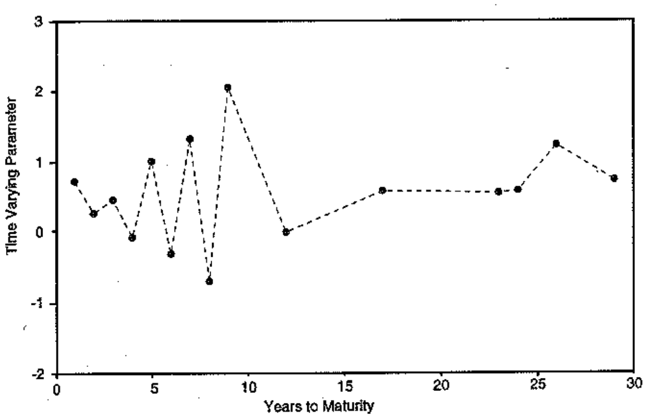


EXHIBIT 9 ■ Time-Varying Term Structure Parameter

Time to Maturity of Discount Bond	Term Structure Parameter: $\nu(t)$
1	0.713
2	0.253
3	0.443
4	-0.097
5	0.993
6	-0.332
7	1.305
8	-0.724
9	2.029
12	-0.036
17	0.544
23	0.510
24	0.537
26	1.188
29	0.685

EXHIBIT 10 ■ Time-Varying Term Structure Parameter on November 9, 1992



obtained in the previous section.

The discount function obtained using the time-varying term structure parameter is shown in Exhibit 11. The fifteen discount bonds used in the estimation of these parameters correspond exactly to those shown in Exhibit 6. In Exhibit 12 we plot the spot rates and one-year forward rates implied by this discount function.

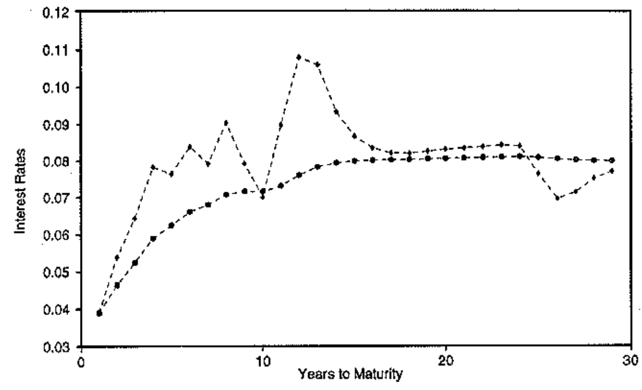
It is interesting to compare these fluctuating rates with the smooth rates obtained using a single stationary term structure parameter (see Exhibit 8). We see that fitting the discount function exactly has its drawbacks. Small errors in prices can have a big effect on yields.

It is hard to give a good theoretical justification for the pattern of forward rates observed in Exhibit 12. The result might just as well be a consequence of slight errors in the prices or non-synchronous trading as opposed to a time-varying term structure parameter.

VI. VALUATION OF CAPS

We are now in a position to value any interest-contingent claim. Using the stationary parameters estimated in Section III, the time-varying term structure parameters for November 9, 1992, estimated in Section V, and the values of the factors r and V for that day, PDE (7) can be solved numerically by imposing the appropriate boundary conditions of the particular claim. As an illustration, in this section we apply the methodology to value caps.

EXHIBIT 12 ■ Spot and One-Year Forward Interest Rates: Time-Varying Parameters



Consider a five-year cap tied to the short rate of interest in which the potential payments occur every six months. Each payment is the largest of the difference between the prevailing short-term interest rate and the cap rate or zero, multiplied by the face value of the contract. Exhibit 13 shows the value of the cap for different values of the cap rate on November 9, 1992, assuming a face value of 100.

In Exhibit 14 we graph the same cap just described, but fixing the cap rate at 0.06 and changing the volatility factor from 0.0001 to 0.0011. Note that when we change the current volatility factor we need to recalculate the time-varying term structure parameters as indicated in Section V before we solve for the

EXHIBIT 11 ■ Discount Function: Time-Varying Parameters

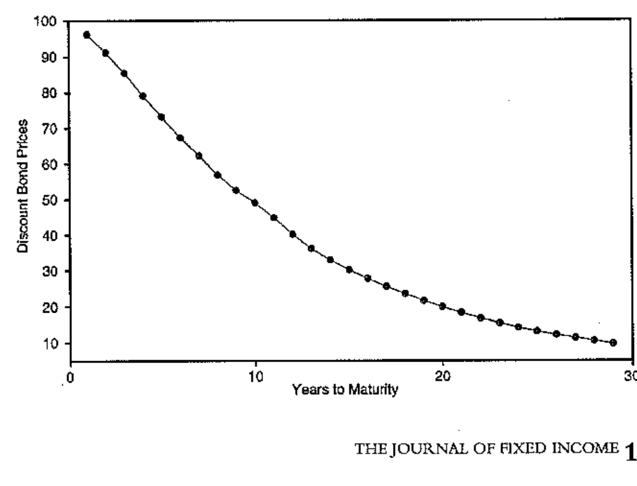


EXHIBIT 13 ■ Cap Prices as a Function of Cap Rates

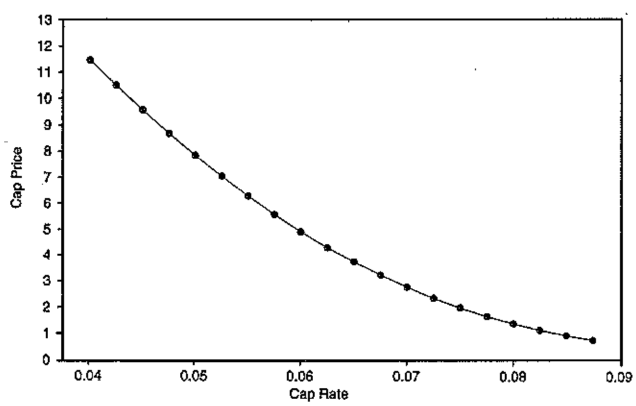
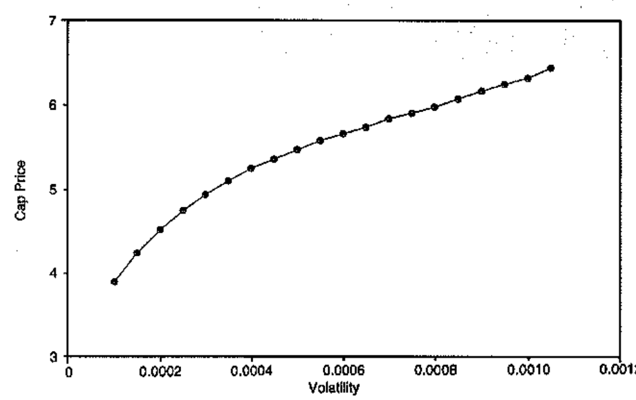


EXHIBIT 14 ■ Cap Prices as a Function of Volatility



value of the cap, because these parameters depend on the assumed volatility.

This exercise suggests a procedure to obtain an "implied volatility factor" from cap prices. If the market price of a cap is known, volatility can be easily inferred from Exhibit 14.

VII. SUMMARY AND CONCLUSIONS

We have shown how to implement our two-factor model of the term structure of interest rates. The parameters of the model are obtained from the time series of the two factors (the five stationary parameters) and from the initial term structure of interest rates (the same number of parameters as points given on the term structure).

Thirty-year simulations show that this estimation procedure provides reasonable future dynamics of the factors. When the model is required to fit the initial term structure, no analytical solutions exist, but efficient numerical algorithms have been implemented that can take into account American features as often as desired and can fit any number of points on the term structure.

Once the parameters have been estimated, the model can be applied to value any interest-contingent claim. A byproduct of the analysis is the sensitivities of the claim to changes in the two factors. These sensitivities can then be used to hedge simultaneously all default-free interest-contingent claims.

In spite of the clear theoretical superiority of two-factor models, their adoption has been slow. We believe this is because they are usually difficult to estimate and implement. Our work shows one way that our model can be implemented.

ENDNOTES

[1]One factor is an interest rate factor, the short rate of interest, and the other is a volatility factor, the variance of changes in the short rate of interest.

[2]This is somewhat like a two-factor version of Hull and White [1990].

[3]The first forward rate shown in the graph corresponds to the one-year spot rate.

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Bollerslev, Tim. "Conditional Autoregressive Conditional Heteroscedasticity." *Journal of Econometrics*, 31 (1986), pp. 307-327.

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