Quantifying Liquidity and Default Risks of Corporate Bonds over the Business Cycle*

Hui Chen
MIT Sloan and NBER

Rui Cui
Chicago Booth

Zhiguo He
Chicago Booth and NBER

Konstantin Milbradt†
MIT Sloan and NBER

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Abstract

This paper introduces time-varying liquidity frictions into a structural model of corporate bond pricing. We feature a combination of procyclical liquidity conditions and countercyclical macroeconomic fundamentals in characterizing the risks of corporate bonds over the business cycle. When calibrated to the historical moments of default probabilities and empirical measures of secondary market liquidity, our model matches the observed credit spreads of corporate bonds across high-grade to high-yield ratings, as well as measures of non-default components including Bond-CDS spreads and bid-ask spreads. In addition, we propose a novel structural decomposition scheme that captures the interaction between liquidity frictions and corporate default decisions via the rollover channel. We use this framework to quantitatively evaluate the effects of liquidity-provision policies during crisis time. Our structural approach identifies important economic forces that were previously overlooked by empirical researches in corporate bonds.

Keywords: Macroeconomic Conditions, Time-Varying Liquidity, Rollover Risk, Over-The-Counter Market, Search Friction, Structural Models, Bid-Ask Spread

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†Chen: MIT Sloan School of Management and NBER; e-mail: huichen@mit.edu. Cui: Booth School of Business, University of Chicago; e-mail: rcui@chicagobooth.edu. He: Booth School of Business, University of Chicago and NBER; e-mail: zhiguo.he@chicagobooth.edu. Milbradt: MIT Sloan School of Management; e-mail: milbradt@mit.edu.

1. Introduction

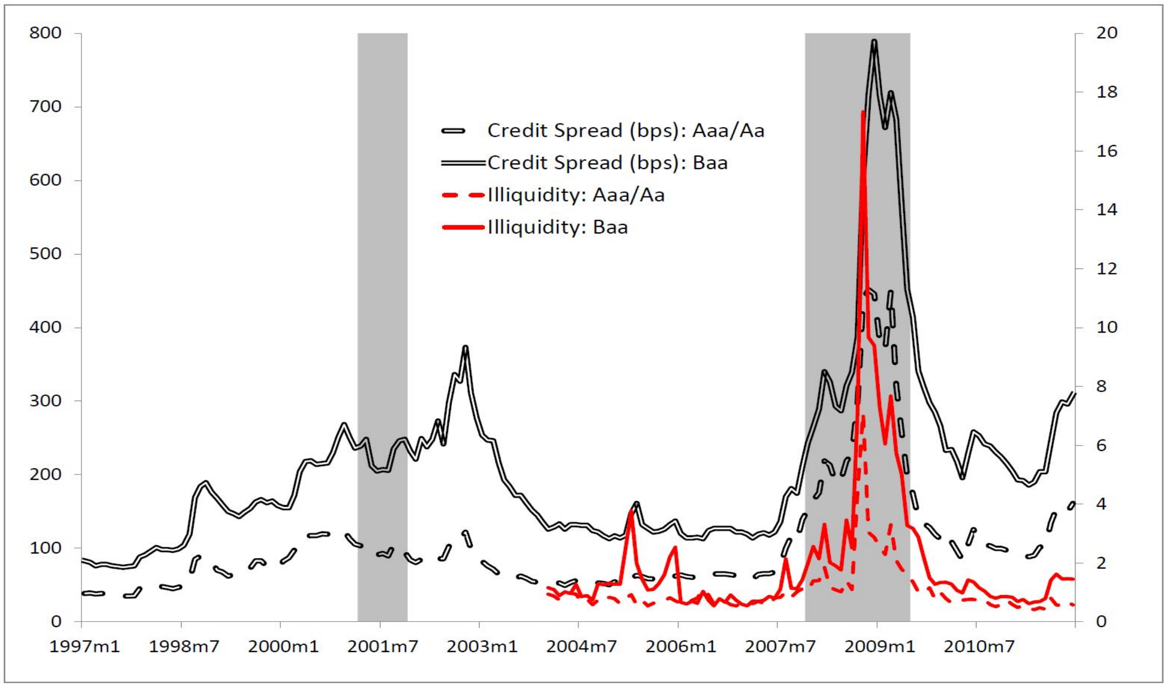
It is well known that default risk only accounts for part of the pricing of corporate bonds. For example, Longstaff et al. [2005] estimate that the default component explains about 50% of the spread between the yields of Aaa/Aa-rated bonds and Treasury bonds, and about 70% of the spreads of Baa-rated bonds. Furthermore, Longstaff et al. [2005] find that the non-default component of credit spreads is only weakly related to the differential state tax treatment on corporate bonds and Treasury bonds. Rather, consistent with the fact that the secondary corporate bond market being illiquid (e.g., Edwards et al. [2007], Bao et al. [2011]), the non-default component is strongly related to measures of bond liquidity.

Until recently, the literature on credit risk modeling has mostly focused on understanding the default component of credit spreads. The "credit spread puzzle," first discussed by Huang and Huang [2012], refers to the finding that, when calibrated to match the observed default rates and recovery rates, traditional structural models have difficulty explaining the credit spreads for bonds rated investment grade and above. By introducing time-varying macroeconomic risks into the structural models, Chen et al. [2009], Bhamra et al. [2010] and Chen [2010] are able to explain the default components of the credit spreads for investment-grade corporate bonds.[1] However, the significant non-default components in credit spreads still remain to be explained.

This paper attempts to provide a full resolution of the credit spread puzzle by quantitatively explaining both the default and non-default components of the credit spreads. It is commonly accepted that the non-default component of credit spreads is a liquidity premium to compensate investors for the liquidity risk when holding corporate bonds. The general empirical pattern of liquidity for corporate bonds, both in the cross-section and in the time-series, is shown in Figure 1, where we plot the credit spreads (left scale) and the "Roll's measure" of bond illiquidity (following Bao et al. [2011], right scale) over 1997-2011

[1] Chen [2010] relies on the estimates of Longstaff et al. [2005] to obtain the default component of the credit spread for Baa rated bonds, while Bhamra et al. [2010] focus on the difference between Baa and Aaa rated bonds. The difference of spreads between Baa and Aaa rated bonds presumably takes out the common liquidity component, which is a widely used practice in the literature. This treatment is accurate only if the liquidity components for both bonds are the same, which is at odds with Figure 1 shown below.

Figure 1: **Time Series Behavior of Credit Spread and Liquidity of Corporate Bonds across Rating Classes.** Credit spreads (double lines, left scale) and Roll's illiquidity measure (single line, right scale) for Aaa/Aa (dashed) and Baa (solid) rated bonds. Credit spread data is from St. Louis Fed at http://research.stlouisfed.org/fred2, and Roll's illiquidity measure follows from Bao et al. [2011] using TRACE data (available after 2004). Grey areas are NBER recessions.



for both Aaa/Aa and Baa rating classes that are studied extensively in the literature. Consistent with Dick-Nielsen et al. [2011] and Friewald et al. [2012], both credit spread and bond illiquidity exhibit strong counter-cyclical patterns, suggesting that recessions come with a soaring price of risk and worsening secondary market liquidity. Further, riskier bonds are coupled worse liquidity in the secondary market, and more so when the economy encounters a recession. This latter cross-sectional pattern implies the importance of *endogenous liquidity* in modeling the non-default component of corporate bonds.

We follow He and Milbradt [2012] by introducing a secondary over-the-counter market search friction (a la Duffie et al. [2005]) into a structural credit models with aggregate macroeconomic fluctuations (e.g., Chen [2010]). In our model, bond investors face the risk of idiosyncratic liquidity shocks that drive up their costs for holding the bonds. Market illiquidity arises endogenously because to sell their bonds, investors have to search for dealers to intermediate transactions with other investors not yet hit by liquidity shocks. The dealers set bid-ask spreads to capture part of trading surplus, and default risk affects the liquidity discount of corporate bonds by influencing the outside option of the illiquid bond investors in the ensuing bargaining.

The endogenous liquidity is further amplified by the *endogenous default* decision of the equity holders, as shown in Leland and Toft [1996] and emphasized recently by He and Xiong [2012] and He and Milbradt [2012]. As illustrated in He and Milbradt [2012], a default-liquidity spiral arises: when secondary market liquidity deteriorates, equity holders suffer heavier rollover losses in refinancing their maturing bonds and will consequently default earlier. This earlier default in turn worsens secondary bond market liquidity even further, and so on so forth. In contrast to He and Milbradt [2012], where primitive parameters associated with secondary market liquidity are assumed to be constant over time, in this paper we explicitly model the cyclical variation in market liquidity, which interacts with the cyclical variation in the firm's cash flows and aggregate risk prices. As the goal of this structural model is to deliver quantitative results, allowing for time-varying macroeconomic risk is important in explaining the credit risk puzzle, as shown by Chen et al. [2009], Bhamra et al. [2010], and Chen [2010].

We follow the literature in calibrating the pricing kernel parameters over binary macroeconomic states (normal and recession) to fit key moments of asset prices. The parameters governing secondary market liquidity over macroeconomic states are calibrated based on bond turnover, dealers' bargaining power, and bid-ask spreads; all empirical moments come from either existing empirical studies or TRACE. In our model, the liquidity risk when holding corporate bonds requires compensation, either because investors face uninsurable idiosyncratic liquidity shocks on holding costs, or because the secondary market liquidity worsens (e.g., the meeting intensity with dealers goes down) in recession during which the pricing kernel is high.

We apply our model to corporate bonds across four credit rating classes (Aaa/Aa, A, Baa, and Ba) and two different time-to-maturity (both 5-year and 10-year bonds). In addition to the two common measures — cumulative default probabilities and credit spreads — that the previous literature on corporate bonds calibration (e.g., Huang and Huang

[2012]) has focused on, modeling bond market liquidity directly allows us to investigate the model's quantitative performance in matching two measures of non-default risk for corporate bonds. First, we investigate the model's performance in matching observed Bond-CDS spreads, defined as the bond's credit spread minus the Credit Derivative Swap (CDS) spread. This is motivated by Longstaff et al. [2005] who argue that CDS contracts, whose secondary market is much more liquid than that of corporate bonds, mostly price the default risk of bonds. The second measure is bid-ask spreads for bonds of different ratings, and we compare our model implied bid-ask spreads to those documented in Edwards et al. [2007] and Bao et al. [2011]. These two measures crucially rely on secondary market illiquidity: in a model with a perfectly liquid bond market, both the implied Bond-CDS spread and bid-ask spread will be zero.

Our model features a great parsimony thanks to endogenously linking liquidity to a firm's distance-to-default, and we only change the distance-to-default across different credit ratings to match the corresponding historical default rates. We are able to match the cross-sectional pattern of the total credit spreads for 10-year bonds, including both the default premium and liquidity premium. Moreover, the model is also successful in generating the cross-sectional pattern in two empirical measures of non-default risk for 10-year bonds, i.e., Bond-CDS spread and bid-ask spread. The matching on 5-year bonds is less satisfactory. Although most of model-implied unconditional moments fall into the 95% confidence interval of the corresponding sample mean, in general our model features a steeper term structure of credit spreads and Bond-CDS spreads for corporate bonds than the data suggests.[2]

From an asset pricing perspective, the state-dependent liquidity risk contributes significantly to the overall risk exposure of a defaultable bond and thus goes a long way in explaining its total credit spread. The fact that the economy spends considerably longer time in the good state than in the bad state, and therefore most bond transactions happen in a good state with a fairly liquid secondary bond market, does not necessarily imply a low liquidity risk of holding such bonds. An investor is most likely to get stuck with the illiquid

[2]That is to say, given that we match the moments for 10-year bonds, the model-implied moments for credit spreads and Bond-CDS spreads for 5-year bonds are lower than corresponding empirical moments.

bond *precisely* in a recession, when prices of risk are high, low recovery values prevail and it takes a long time before the bond holdings can be sold off.

Our model has important implications in understanding the role of default and liquidity in determining a firm's borrowing cost. A common practice in the empirical literature is to decompose credit spreads into a liquidity and a default component, which naturally leads to the interpretation that these components are independent of each other. Our model suggests that both liquidity and default are endogenously linked, and thus there can be economically significant interaction terms. These dynamic interactions are difficult to capture using reduced-form models with *exogenously* imposed liquidity premia.

We propose a structural decomposition that nests the common additive default-liquidity decomposition to quantify the interaction between default and liquidity for corporate bonds. Similar to Longstaff et al. [2005] using CDS spread to proxy for default risk, we identify the "default" part by pricing a bond in a counterfactually perfectly liquid market but with the model implied default threshold. We identify the remaining credit spread after subtracting this "default" part as the "liquidity" part. We then further decompose the "default" ("liquidity") part into a "pure default" ("pure liquidity") component and a "liquidity-driven-default" ("default-driven liquidity") component, where the "pure default" or "pure liquidity" part is the spread implied by a counterfactual model where either the bond market is perfectly liquid as in Leland and Toft [1996] hence equity holders default later, or only the over-the-counter search friction for risk free bonds is at work as in Duffie et al. [2005], respectively. The two interaction terms that emerge, i.e., the "liquidity-driven default" and the "default-driven liquidity" components, capture the endogenous positive spiral between default and liquidity. For instance, "liquidity-driven-default" is driven by the rollover risk mechanism in that firms relying on finite-maturity debt financing will default earlier when facing worsening secondary market liquidity.

Besides giving a more complete picture of how the default and liquidity forces affect credit spreads, this model-based decomposition also offers important insight on evaluating hypothetical government policies, such as providing subsidized term loans to financial intermediaries active in the secondary bond market in order to improve the market liquidity. To evaluate the effectiveness of such policies, it is important to fully take into account of how a firm's default policy responds to liquidity conditions and how liquidity conditions respond to default risks. Imagine a policy that makes the secondary market in recession as liquid as in normal times, which lowers the credit spread of Ba rated bonds in recession by about 57 bps (about 15% of the spread). The liquidity-driven default part, which captures lower default risk from firms with mitigated rollover losses, can explain 18% of this drop; the default-driven liquidity part, which captures the endogenous reduction of liquidity premium for safer bonds, can explain about 32.84%. The prevailing view in the literature masks this interdependence between default and liquidity components and thus tends to miss these interaction terms.

The paper is structured as follows. Section 2 introduces the model. Section 3 gives the solutions to debt, equity valuations and default boundaries. Section 4 presents the main calibration. Section 5 discusses the model-based default-liquidity decomposition, and analyzes the effectiveness of a policy geared towards liquidity provision from the perspective of our decomposition. Section 7 concludes. The appendix provides proofs and a more general formulation of the model.

2. The Model

2.1 Aggregate States and the Firm

The following model elements are similar to Chen [2010] and Bhamra et al. [2010], except that we study the case in which firms issue bonds with an average finite maturity a la Leland (1998) so that rollover risk is present.

2.1.1 Aggregate states and stochastic discount factor

The aggregate state of the economy is described by a continuous time Markov chain, with the current Markov state denoted by s_t and the physical transition density between state i and state j denoted by $\zeta_{ij}^{\mathcal{P}}$. We assume an exogenous stochastic discount factor (SDF):

$$\frac{d\Lambda_t}{\Lambda_t} = -rdt - \eta\left(s_t\right) dZ_t^m + \sum_{s_t \neq s_t'} \left(e^{\kappa(s_{t-}, s_t)} - 1\right) dM_t^{(s_{t-}, s_t)}, \tag{1}$$

where $\eta(\cdot)$ is the state-dependent price of risk for Brownian shocks, $dM_t^{(j,k)}$ is a compensated Poison process capturing switches between states, and $\kappa(i,j)$ embeds the jump risk premia such that in the risk neutral measure, the distorted jump intensity between states is $\zeta_{ij}^{\mathcal{Q}} = e^{\kappa(i,j)}\zeta_{ij}^{\mathcal{P}}$.

Note that in Eq. (1), motivated by the US economy we have assumed a constant (i.e., state-independent) risk-free rate $r_f(s) = r$. In this paper we focus on the case with binary aggregate states, i.e., $s_t \in \{G, B\}$. In the Appendix we provide the general setup for the case with $n > 2$ aggregate states.

2.1.2 Firm cash flows and risk neutral measure

A firm has assets in place that generate cash flows at the rate of Y_t which follow, under the physical measure $\mathcal{P}$,

$$\frac{dY_t}{Y_t} = \mu_{\mathcal{P}}(s)\,dt + \sigma_m(s)\,dZ_t^m + \sigma_f dZ_t^f, \tag{2}$$

where s is the aggregate state that (possibly) influences the cash-flow process. Here, dZ_t^m captures aggregate Brownian risk, while dZ_t^f captures idiosyncratic Brownian risk. Given the stochastic discount factor m_t, risk neutral cash flow dynamics under the risk neutral measure $\mathcal{Q}$ follow

$$\begin{aligned}
\frac{dY_t}{Y_t} &= \mu_{\mathcal{Q}}(s)\,dt + \sigma(s)\,dZ_t^{\mathcal{Q}}, \\
dZ_t^{\mathcal{Q}} &= \frac{\sigma_m(s)}{\sqrt{\sigma_m^2(s_t) + \sigma_f^2}} dZ_t^m + \sqrt{1 - \frac{\sigma_m^2(s)}{\sigma_m^2(s) + \sigma_f^2}} dZ_t^f + \frac{\sigma_m(s)}{\sigma(s)} \eta(s)\,dt,
\end{aligned}$$

where $Z_t^{\mathcal{Q}}$ is a Brownian Motion under the risk-neutral measure $\mathcal{Q}$. The risk-neutral cash-flow drift is given by

$$\mu_{\mathcal{Q}}^s \equiv \mu_{\mathcal{P}}(s) - \sigma_m(s)\,\eta(s), \text{ and } \sigma_s \equiv \sqrt{\sigma_m^2(s) + \sigma_f^2}.$$

For ease of notation, we work with log cash flows $y \equiv \log(Y)$ throughout. Define

$$\mu_s \equiv \mu_{\mathcal{Q}}^s - \frac{1}{2}\sigma_s^2 = \mu_{\mathcal{P}}(s) - \sigma_m(s)\,\eta(s) - \frac{1}{2}\left(\sigma_m^2(s) + \sigma_f^2\right)$$

so that we have

$$dy_t = \mu_s dt + \sigma_s dZ_t^{\mathcal{Q}}. \tag{3}$$

From now on we work under measure $\mathcal{Q}$ unless otherwise stated, so we drop the superscript $\mathcal{Q}$ in $dZ_t^{\mathcal{Q}}$ and $\zeta_{ij}^{\mathcal{Q}}$ to simply write dZ_t and ζ_{ij} where no confusion can arise.

As standard in the asset pricing literature, we can obtain valuations for any asset as the expected discounted cash flows under the risk neutral measure $\mathcal{Q}$. The unlevered firm value, given the aggregate state s, is given by

$$\mathbf{v}_U(y) \equiv \begin{bmatrix} r - \mu_G + \zeta_G & -\zeta_G \\ -\zeta_B & r - \mu_B + \zeta_B \end{bmatrix}^{-1} \mathbf{1} \exp(y). \tag{4}$$

We will use v_U^s to denote the element of $\mathbf{v}_U$ in state s.

There is one caveat in applying the risk neutral pricing to bond valuations, as later we will introduce undiversifiable idiosyncratic liquidity shocks to bond investors. Because we model liquidity shocks as holding costs which can be interpreted as negative dividends, the risk neutral pricing for bonds with holding-cost adjusted cash flows is still valid provided that the bond holding is infinitesimal in the representative investor's portfolio.[3]

2.1.3 Firm's debt maturity structure and rollover frequency

The firm has bonds in place of measure 1 which are identical except for their time to maturity, and thus the aggregate and individual bond coupon (face value) is c (p). As in Leland (1998), equity holders commits to holding the aggregate coupon and outstanding face-value constant outside default, and thus issues new bonds of the same average maturity as the bonds maturing.

Each bond matures with intensity m, and the maturity event is i.i.d. across individual bonds. Thus, by (an appropriately chosen) law of large numbers over $[t, t + dt)$ the firm retires a fraction $m \cdot dt$ of its bonds. This implies an expected average debt maturity of

[3]Intuitively, if the representative agent's consumption pattern is not affected by the idiosyncratic shock brought on by the bond holdings (which is true if the bond holding is infinitesimal relative to the rest of the portfolio), then the representative agent's pricing kernel is independent of idiosyncratic undiversified shocks.

$1/m$.[4] The deeper implication of this assumption is that the firm adopts a "smooth" debt maturity structure with a uniform distribution, and the firm's average refinancing/rollover frequency is m. As shown later, the rollover frequency (at the firm level) is important for secondary market liquidity to affect a firm's endogenous default decisions. Later we will calibrate this number to the actual rollover frequency of US firms.

2.2 Liquidity in Secondary Over-the-Counter Corporate Bond Market

We follow He and Milbradt [2012] in modeling the over-the-counter corporate bond market. The setting builds on Duffie et al. [2005], in that it seamlessly integrates a search-based dealer intermediated OTC market into the Leland-type structural credit risk framework described above. Individual bond holders are subject to liquidity shocks that entail a positive holding cost. Bond holders hit by liquidity shocks will try to sell by searching for dealers in the over-the-counter secondary market, and transaction prices are determined by bargaining with a dealer once a contact is established.

More specifically, investors can either not hold the bond or hold one unit of the bond. Individual bond holders start in the H state without any holding cost when purchasing corporate bonds in the primary market. As time passes by, H type bond holders are hit independently by liquidity shocks with intensity ξ_s, which leads them to become L types who bear a positive holding cost χ_s per unit of time. We will assume that there is sufficient (to be made precise in the appendix) H investors without the bond waiting on the sideline, so it will be optimal for L investors to try to sell the bond to H investors. However, there

[4] An alternative interpretation is the exponential retiring interpretation of Leland 1998. This however requires additional assumptions. First, we have to relax the holding restriction as only at the time of primary market purchase is a full unit of the bond being held, so that asset holdings can be $q \in [0, 1]$. Second, we need to assume that regardless of what quantity $q \in [0, 1]$ of the bond is attempted to be sold by an individual bondholder, the contact intensity is λ. Third, holding costs are now scaled by the quantity q held by the investor. The fact that we assumed a *seller's market* now becomes central to the solvability and not simply an assumption for tractability — as buyers do not receive any surplus and are indifferent between buying and staying out of the market, the usual difficulty regarding inventory management stemming from holding restrictions does not apply. The value functions on the debt side should then be understood as value functions per unit of face-value. The equity value function does not change as in aggregate the same amount of debt is retired every instant as in the random maturity interpretation.

is a trading friction in moving the bond holdings from inefficient L type bond investors (sellers) to efficient H-type investors (buyers), in that trades have to be intermediated by dealers in the over-the-counter market.

Sellers meet dealers with intensity λ_s, which we interpret as the intermediation intensity of the financial sector. For simplicity, we assume that after L-type investors sell their holdings, they exit the market forever. The H-type buyers on the sideline currently not holding the bond also contact dealers with some intensity λ_s. As no dealer meets a buyer and a seller at the same time, dealers use the competitive (and instantaneous) interdealer market to lay off or buy a position in bonds. For simplicity, we assume that the flow of H-type buyers contacting dealers is greater than the flows of L-type sellers contacting dealers, so that the secondary market is a *seller's market*, a term that will become made clear below.

Fixing any aggregate state s, denote by D_l^s the individual bond valuation for the investor with type $l \in \{H, L\}$. We follow [duffie2007b] and assume Nash-bargaining weights β for the investor and $1-\beta$ for the dealer across all dealer-investor pairs. When a contact between a type L investor and a dealer occurs, the dealer can instantaneously sell a bond at a price M to another dealer who is in contact with an H investor via the interdealer market. If he does so, the bond travels from an L investor to an H investor via the help of the two dealers who are connected in the inter-dealer market. Denote by B^s the bid price at which the L type is selling his bond, by A^s the ask price at which the H type is purchasing this bond, and by M^s the inter-dealer market price. Similar to Duffie et al. [2005] and He and Milbradt [2012], we have the following proposition. Essentially, Bertrand competition, the holding restriction and the surplus of buyer-dealer pairs in the interdealer market drives the surplus of buyer-dealer pairs to zero.[5]

Proposition 1 *Fix valuations* D_H^s *and* D_L^s*, and denote the surplus from trade by* $\Pi^s = D_H^s - D_L^s > 0$*. The ask price* A^s *and inter-dealer market price* M^s *are equal to* D_H^s*, and*

[5]This further implies that the value function of investors not holding the asset is identically zero, which makes the model very tractable. Introducing for example direct bilateral trades or assuming a *buyer's market* would both entail tracking the value functions of investors on the sideline but would not add additional economic insights pertaining to credit risk in particular.

the bid price is given by $B^s = \beta D_H^s + (1-\beta) D_L^s$. *The dollar bid ask spread is* $A^s - B^s = (1-\beta)(D_H^s - D_L^s) = (1-\beta)\Pi^s$.

Empirical studies focus on the proportional bid-ask spread which is defined as the dollar bid-ask spread divided by the mid price, i.e.,

$$\Delta^s(y,\tau) = \frac{2(1-\beta)(D_H^s - D_L^s)}{(1+\beta) D_H^s + (1-\beta) D_L^s} = \frac{(1-\beta)\Pi^s}{D_H^s - \frac{1-\beta}{2}\Pi^2}. \tag{5}$$

2.3 State Transition

As notational conventions, we use capitalized bold-faced letters (e.g., $\mathbf{X}$) to denote matrices, lower case bold face letters (e.g. $\mathbf{x}$) to denote vectors (the only exceptions are the value functions for debt and equity, $\mathbf{D}, \mathbf{E}$ respectively, which will be vectors, and the (diagonal) matrix of drifts, $\boldsymbol{\mu}$), and non-bold face letters denote scalars (e.g. x). Dimensions for most objects are given underneath the expression. While we focus on 2-aggregate-state case where $s \in \{G, B\}$, the Appendix presents general results for arbitrary number of (Markov) aggregate states.

Denote by $\mathbf{Q}$ the Markov-transition matrix of aggregate *and* individual states, where each entry $q_{ls \to l's'}$ is the intensity of transitioning from (individual) liquidity state l to l' where $l, l' \in \{H, L\}$ and from aggregate state s to s' where $s, s' \in \{G, B\}$.[6] Thus, the

[6]Our intensity-based modeling rules out the possibility of coinciding jumps in the aggregate and individual states (i.e., $q_{ls \to l's'} = 0$ if $l \neq l'$ and $s \neq s'$), an assumption that can potentially be relaxed. Economically, this implies that the aggregate shock can bring about more liquidity shocks to individual debt holders given any time interval (but these shocks are still i.i.d across individuals).

transition matrix $\mathbf{Q}$ is

$$\underbrace{\mathbf{Q}}_{4\times4} \equiv \begin{bmatrix} -\sum_{ls\neq HG} q_{HG\to ls} & q_{HG\to LG} & q_{HG\to HB} & 0 \\ q_{LG\to HG} & -\sum_{ls\neq LG} q_{LG\to ls} & 0 & q_{LG\to LB} \\ q_{HB\to HG} & 0 & -\sum_{ls\neq HB} q_{HB\to ls} & q_{HB\to LB} \\ 0 & q_{LB\to LG} & q_{LB\to HB} & -\sum_{ls\neq LB} q_{LB\to ls} \end{bmatrix}$$
$$= \begin{bmatrix} -\xi_G-\zeta_G & \xi_G & \zeta_G & 0 \\ \beta\lambda_G & -\beta\lambda_G-\zeta_G & 0 & \zeta_G \\ \zeta_B & 0 & -\xi_B-\zeta_B & \xi_B \\ 0 & \zeta_B & \beta\lambda_B & -\beta\lambda_B-\zeta_B \end{bmatrix} \tag{6}$$

Further, define $\mathbf{Q}^{(1)} = \begin{bmatrix} -\xi_G-\zeta_G & \xi_G \\ \beta\lambda_G & -\beta\lambda_G-\zeta_G \end{bmatrix}$ and $\tilde{\mathbf{Q}}^{(1)} = \begin{bmatrix} \zeta_G & 0 \\ 0 & \zeta_G \end{bmatrix}$.

The entry $q_{Ls\to Hs}$ in the above transition matrix requires further explanation. In our model, given the aggregate state s, the intensity of switching from H state to L state is $\xi(s)$, and the L-state is absorbing (those L-type investors leave the market forever). An L type bond holders meets a dealer with intensity λ_s and sells the bond for B^s that he himself values at D_L^s. An L-type's intensity-modulated surplus when meeting the dealer, as shown in Section 2.2, can be rewritten as

$$\lambda_s\left(B^s - D_L^s\right) = \lambda_s\beta\left(D_H^s - D_L^s\right).$$

Thus, for the purpose of pricing, the "effective" transition intensity from L-type to H-type is $q_{Ls\to Hs} = \lambda_s\beta$ where λ_s is the state-dependent intermediation intensity and β is the investor's bargaining power.

2.4 Delayed Bankruptcy Payouts and Effective Recovery Rates

In Leland-type frameworks, when the firm's cash flow deteriorates, equity holders are willing to repay the maturing debt holders only when the equity value is still positive, i.e. the option value of keeping the firm alive justifies absorbing rollover losses and coupon payments. The firm defaults when its equity value drops to zero at some default threshold y_{def}, which is endogenously chosen by equity holders. As in Chen [2010], we will impose bankruptcy costs as a fraction $1-\hat{\alpha}_s$ of the value from unlevered assets $v_U^s\left(y_{def}\right)$ given in (4), where the debt holder's bankruptcy recovery $\hat{\alpha}_s$ may depend on the aggregate state s.

As emphasized in He and Milbradt [2012], because the driving force of liquidity in our model is that agents value receiving cash early, our bankruptcy treatment has to be careful in this regard (and different from typical Leland models). If bankruptcy leads investors to receive the bankruptcy proceeds immediately, then bankruptcy confers a "liquidity" benefit similar to a bond maturing. This "expedited payment" benefit runs counter to the fact that in practice bankruptcy leads to the freezing of assets within the company and a delay in the payout of any cash depending on court proceeding.[7] Moreover, bond investors with defaulted bonds may face a much more illiquid secondary market, and potentially a much higher holding cost once liquidity shocks hit due to regulatory or charter restrictions which prohibit institutions to hold defaulted bonds.

To capture above features, we assume that a bankruptcy court delay leads the bankruptcy cash payout $\hat{\alpha}_s v_U^s$ to occur at a Poisson arrival time with intensity θ.[8] The holding cost for L-type investors is $\chi_{def} v_U^s$ where $\chi_{def} > 0$, and the secondary market for defaulted bonds is illiquid with contact intensity λ_{def}. Given aggregate state s and default boundary y_{def} , denote the value of defaulted bonds by $D_H^{s,def}$ and $D_L^{s,def}$. Their valuation equations are

$$
\begin{aligned}
rD_H^{s,def} &= \theta\left[\hat{\alpha}_s v_U^s - D_H^{s,def}\right] + \xi_s\left[D_L^{s,def} - D_H^{s,def}\right] + \zeta_s\left[D_H^{s',def} - D_H^{s,def}\right] \\
rD_L^{s,def} &= -\chi_{def}^s v_U^s + \theta\left[\hat{\alpha}_s v_U^s - D_L^{s,def}\right] + \lambda_{def}^s \beta\left[D_H^{s,def} - D_L^{s,def}\right] + \zeta_s\left[D_L^{s',def} - D_L^{s,def}\right]
\end{aligned} \tag{7}
$$

Take $D_L^{s,def}$ for example: the first term is the illiquidity holding cost, the second term captures the bankruptcy payout, the third term captures trading the defaulted bonds with dealers, and the last term captures the jump of the aggregate state. In Eq. (7) we have assumed that the cash flow rate y remains constant at y_{def} during bankruptcy procedures, a simplify-

[7] The Lehman Brothers bankruptcy in September 2008 is a good case in point. After much legal uncertainty, payouts to the debt holders only started trickling out after about three and a half years.

[8] We could allow for a state-dependent bankruptcy court delay, i.e., $\theta(s)$; but the Moody's Ultimate Recovery Dataset reveals that there is little difference between the recovery time in good time versus bad time.

ing assumption that can be easily relaxed.[9] Defining $\mathbf{D}^{def} \equiv \left[D_H^{G,def}, D_L^{G,def}, D_H^{B,def}, D_L^{B,def}\right]^\top$, it is easy to show that

$$\underbrace{\mathbf{D}^{def}(y)}_{4\times 1} = diag\left(\left[\begin{array}{cccc} v_u^G(y) & v_u^G(y) & v_u^B(y) & v_u^B(y)\end{array}\right]\right)\underbrace{\left(\mathbf{R}-\mathbf{Q}_{def}+\theta\mathbf{I}\right)^{-1}\left(\theta\hat{\boldsymbol{\alpha}}-\boldsymbol{\chi}_{def}\right)}_{\equiv\boldsymbol{\alpha}}, \quad (8)$$

where $\boldsymbol{\chi}_{def} \equiv [0, \chi_{def}(G), 0, \chi_{def}(B)]^\top$ and where $\mathbf{Q}_{def}$ is the post-default counterpart of $\mathbf{Q}$ in (6). Throughout, $diag(\cdot)$ is the diagonalization operator mapping any row or column vector into a diagonal matrix (in which all off-diagonal elements are identically zero).

In (8), for easier comparison to existing Leland-type models where debt recovery at bankruptcy is simply $\hat{\alpha}v_U$, we denote the (bold face) vector $\boldsymbol{\alpha} \equiv \left[\alpha_H^G, \alpha_L^G, \alpha_H^B, \alpha_L^B\right]^\top$ as the *effective* bankruptcy recovery rates at the time of default. We will have $\alpha_H^s > \alpha_L^s$ to capture the fact that default is more hurtful for bond holders in the illiquid state. Since we mainly focus on bond spreads before firm default, for the rest of the paper we take $\boldsymbol{\alpha}$ as primitive parameters, because they are determined by post-default market structures. The important thing is that whatever assumptions on post-default secondary-market assumptions are made, any two post-default models are observationally equivalent pre-default as long as they imply the same $\boldsymbol{\alpha}$.

However, as emphasized in He and Milbradt [2012], because $\mathbf{v}_U(y_{def})$ depends on the endogenously determined bankruptcy boundary y_{def}, the dollar bid-ask spread of defaulted bonds is higher if the firm defaults earlier. Thus, the illiquidity of defaulted bonds, relative to that of default-free bonds whose dollar bid-ask spreads are proportional to principal p and coupon c, depends on the firm's pre-default parameters through the channel of endogenous default.

These effective bankruptcy recovery factors are the only critical ingredients for us to solve for the pre-default bond valuations, as well as their the market liquidity. In calibration, we will not rely on deeper structural parameters (say, post-default holding cost χ_{def}).

[9] And we assume that $\alpha V_{FB}^s(y_{def}) < p$, i.e., the total debt face value exceeds the payout. The result will be identical if we assume that y evolves as in (3), and debt holders receive the entire payout (net bankruptcy cost) of αV_{FB} eventually. The defaulted bonds values will be slightly lower if we take into account that equity holders receive some payouts in the event of $\alpha V_{FB} > p$, but one can derive the formula of $D_H^{s,def}$ and $D_L^{s,def}$ easily.

Instead, we choose these effective recovery rates $\boldsymbol{\alpha}$ to target both the market price of defaulted bonds observed immediately after default (which are close to L-type valuations) and the associated bid-ask spreads.

3. Debt Valuation and Default Boundaries

Denote by $D_l^{(s)}$ the l-type debt value in aggregate state s, $E_l^{(s)}$ the equity value in aggregate state s, and $\boldsymbol{y}_{def} = \left[y_{def}^G, y_{def}^B\right]^\top$ the vector of endogenous default boundaries. We derive the closed-form solution for debt and equity valuations in this section as a function of a given $\boldsymbol{y}_{def}$, along with the characterization of endogenous default boundaries $\boldsymbol{y}_{def}$.

3.1 Debt Valuations

Because equity holders will default at an earlier point in the bad state, i.e., $y_{def}^G < y_{def}^B$, the domains of debt valuations change when the aggregate state switches. We deal with this issue by the following treatment; see the Appendix for the generalization of this analysis.

Define two intervals $I_1 = \left[y_{def}^G, y_{def}^B\right]$ and $I_2 = \left[y_{def}^B, \infty\right)$, and denote by $D_l^{s,i}$ the restriction of D_l^s to the interval I_i, i.e., $D_l^{s,i}(y) = D_l^s(y)$ for $y \in I_i$. Clearly, $D_l^{B,1}(y) = \alpha_l V_U^B(y)$ is in the "dead" state, so that the firm immediately defaults in interval I_1 when switching into state B (from state G). In light of this observation, on interval $I_2 = \left[y_{def}^B, \infty\right)$ all bond valuations denoted by $\mathbf{D}^{(2)} = \left[D_H^{G,2}, D_L^{G,2}, D_H^{B,2}, D_L^{B,2}\right]^\top$ are "alive." We have the following system of ODEs for $\mathbf{D}^{(2)}$ when $y \in I_2 = [y_{def}(B), \infty)$:

$$[(r+m)\mathbf{I}_4 - \mathbf{Q}]\mathbf{D}^{(2)} = \left(c\mathbf{1}_4 - \boldsymbol{\chi}^{(2)}\right) + \boldsymbol{\mu}^{(2)}\left(\mathbf{D}^{(2)}\right)' + \frac{1}{2}\boldsymbol{\Sigma}^{(2)}\left(\mathbf{D}^{(2)}\right)'' + m \cdot p\mathbf{1}_4, \quad (9)$$

where $\mathbf{Q}^{(2)} = \mathbf{Q}$,

$$\boldsymbol{\mu}^{(2)} = diag\left([\mu_G, \mu_G, \mu_B, \mu_B]\right), \boldsymbol{\Sigma}^{(2)} = diag\left(\left[\sigma_G^2, \sigma_G^2, \sigma_B^2, \sigma_B^2\right]\right).$$

In contrast, on interval $I_1 = [y_{def}(G), y_{def}(B)]$, the bond is "dead" in state B, and the alive bonds $\mathbf{D}^{(1)} = \left[D_H^{(G,1)}, D_L^{(G,1)}\right]^\top$ solve

$$\left[(r+m)\,\mathbf{I}_2 - \mathbf{Q}^{(1)}\right]\mathbf{D}^{(1)} = \left(c\mathbf{1}_2 - \boldsymbol{\chi}^{(1)}\right) + \boldsymbol{\mu}^{(1)}\left(\mathbf{D}^{(1)}\right)' + \frac{1}{2}\boldsymbol{\Sigma}^{(1)}\left(\mathbf{D}^{(1)}\right)'' + m\cdot p\mathbf{1}_2 + \zeta_G \begin{bmatrix} \alpha_H^B \\ \alpha_L^B \end{bmatrix} v_U^B(y) \tag{10}$$

for

$$y \in I_1 = \left[y_{def}(G), y_{def}(B)\right],$$

where the last term is the recovery value in case of a jump to default brought about by a state jump.

Note that in the above valuation equations we simply treat holding costs given liquidity shocks as negative dividends, which effectively lower the coupon flows that investors are receiving. Moreover, we directly apply the pricing kernel (pricing under risk neutral measure $\mathcal{Q}$) given in (1) and make no risk adjustments on the liquidity shocks, which is justified by the assumption that the illiquid bond holding is infinitesimal in the representative investor's portfolio. For further discussions, see footnote 3 and the end of Section 2.1.2.

As shown in the Appendix, the general solution on interval i is given by

$$\underbrace{\mathbf{D}^{(i)}}_{2i\times 1} = \underbrace{\mathbf{G}^{(i)}}_{2i\times 4i} \cdot \underbrace{\exp\left(\boldsymbol{\Gamma}^{(i)} y\right)}_{4i\times 4i} \cdot \underbrace{\mathbf{b}^{(i)}}_{4i\times 1} + \underbrace{\mathbf{k}_0^{(i)}}_{2i\times 1} + \underbrace{\mathbf{k}_1^{(i)}}_{2i\times 1}\exp(y) \tag{11}$$

where the constants vector $\mathbf{b}^{(i)}$ will be determined via appropriate boundary conditions. The boundary conditions at $y = \infty$ and $y = y_{def}(G)$ are standard:

$$\lim_{y\to\infty}\left|\mathbf{D}^{(2)}(y)\right| < \infty, \text{ and } \mathbf{D}^{(1)}\left(y_{def}^G\right) = \begin{bmatrix} \alpha_H^G \\ \alpha_L^G \end{bmatrix} v_U^G\left(y_{def}^G\right) \tag{12}$$

For the boundary y_{def}^B, we must have value matching conditions for all functions across $y_{def}(B)$:

$$\mathbf{D}^{(2)}\left(y_{def}^B\right) = \begin{bmatrix} \mathbf{D}^{(1)}\left(y_{def}^B\right) \\ \begin{bmatrix} \alpha_H^B \\ \alpha_L^B \end{bmatrix} v_U^B\left(y_{def}^B\right) \end{bmatrix} \tag{13}$$

and smooth pasting conditions for functions that are alive across y^B_{def} ($\mathbf{x}_{[1,2]}$ selects the first 2 rows of vector $\mathbf{x}$):

$$\left(\mathbf{D}^{(2)}\right)'\left(y^B_{def}\right)_{[1,2]} = \left(\mathbf{D}^{(1)}\right)'\left(y^B_{def}\right). \tag{14}$$

3.2 Equity Valuations and Default Boundaries

When the firm refinances the maturing bonds, we assume that it can place newly issued bonds with H investors in a competitive primary market. This implies that there is a rollover cash flow (inflow or outflow) of $m\left[\mathbf{S}^{(i)}\cdot\mathbf{D}^{(i)}(y) - p\mathbf{1}_i\right]$ at each instant as a mass $m\cdot dt$ of debt holders matures on $[t, t+dt]$, where $\mathbf{S}^{(i)}$ is a $i\times 2i$ matrix that selects the appropriate D_H as we assumed the firm issues to H type investors in the primary market. For instance, for $y\in I_2 = [y_{def}(B), \infty)$, we have $\mathbf{D}^{(2)} = \left[D^{G,2}_H, D^{G,2}_L, D^{B,2}_H, D^{B,2}_L\right]^\top$ and $\mathbf{S}^{(2)} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}$.[10]

For ease of exposition, we will denote by double letters (e.g. $\mathbf{xx}$) a constant for equity that takes a similar place as a single letter (i.e. $\mathbf{x}$) constant for debt. We can write down the HJB equation for equity on interval I_i similar to (9) and (10). For instance, on interval I_2 we have

$$\begin{aligned}\underbrace{\left(r\mathbf{I}_2 - \mathbf{QQ}^{(2)}\right)}_{2\times 2}\underbrace{\mathbf{E}^{(2)}(y)}_{2\times 1} &= \underbrace{\boldsymbol{\mu\mu}^{(2)}}_{2\times 2}\underbrace{\left(\mathbf{E}^{(2)}\right)'(y)}_{2\times 1} + \frac{1}{2}\underbrace{\mathbf{\Sigma\Sigma}^{(2)}}_{2\times 2}\underbrace{\left(\mathbf{E}^{(2)}\right)''(y)}_{2\times 1} \\ &\quad + \underbrace{\mathbf{1}_2\exp(y)}_{Cashflow,2\times 1} - \underbrace{(1-\pi)\,c\mathbf{1}_2}_{Coupon,2\times 1} + \underbrace{m\left[\mathbf{S}^{(2)}\cdot\mathbf{D}^{(2)}(y) - p\mathbf{1}_2\right]}_{Rollover,2\times 1}\end{aligned} \tag{15}$$

here

$$\boldsymbol{\mu\mu}^{(2)} = diag\left([\mu_G, \mu_B]\right), \mathbf{\Sigma\Sigma}^{(2)} = diag\left(\left[\sigma^2_G, \sigma^2_B\right]\right), \mathbf{QQ}^{(2)} = \begin{bmatrix} -\zeta_G & \zeta_G \\ \zeta_B & -\zeta_B \end{bmatrix} \tag{16}$$

We refer the reader to the appendix for the full formulation of the equity HJBs.

[10] This formulation can also accommodate the situation where some maturing bonds are rolled over to L investors by adjusting $\mathbf{S}^{(i)}$, or any proportional issuance costs of say κ by redefining $\tilde{\mathbf{S}}^{(i)} = (1-\kappa)\,\mathbf{S}^{(i)}$ as the appropriate issuance matrix.

The general solution to equity value is

$$\underbrace{\mathbf{E}^{(i)}(y)}_{i\times 1} = \underbrace{\mathbf{GG}^{(i)}}_{i\times 2i} \cdot \underbrace{\exp\left(\mathbf{\Gamma\Gamma}^{(i)}y\right)}_{2i\times 2i} \cdot \underbrace{\mathbf{bb}^{(i)}}_{2i\times 1} + \underbrace{\mathbf{KK}^{(i)}}_{i\times 4i} \underbrace{\exp\left(\mathbf{\Gamma}^{(i)}y\right)}_{4i\times 4i} \underbrace{\mathbf{b}^{(i)}}_{4i\times i} + \underbrace{\mathbf{kk}_0^{(i)}}_{i\times 1} + \underbrace{\mathbf{kk}_1^{(i)}}_{i\times 1} \exp(y) \ \text{ for } y \in I_i$$

and the particular solution is

$$\begin{aligned}
\underbrace{\mathbf{E}^{(2)}(y)}_{2\times 1} &= \underbrace{\mathbf{GG}^{(2)}}_{2\times 4} \cdot \underbrace{\exp\left(\mathbf{\Gamma\Gamma}^{(2)}y\right)}_{4\times 4} \cdot \underbrace{\mathbf{bb}^{(2)}}_{4\times 1} + \underbrace{\mathbf{KK}^{(2)}}_{2\times 8} \underbrace{\exp\left(\mathbf{\Gamma}^{(2)}y\right)}_{8\times 8} \underbrace{\mathbf{b}^{(2)}}_{4\times 2} + \underbrace{\mathbf{kk}_0^{(2)}}_{2\times 1} + \underbrace{\mathbf{kk}_1^{(2)}}_{2\times 1} \exp(y) \ \text{ for } y \in I_2 \\
\underbrace{\mathbf{E}^{(1)}(y)}_{1\times 1} &= \underbrace{\mathbf{GG}^{(1)}}_{1\times 2} \cdot \underbrace{\exp\left(\mathbf{\Gamma\Gamma}^{(1)}y\right)}_{2\times 2} \cdot \underbrace{\mathbf{bb}^{(1)}}_{2\times 1} + \underbrace{\mathbf{KK}^{(1)}}_{1\times 4} \underbrace{\exp\left(\mathbf{\Gamma}^{(1)}y\right)}_{4\times 4} \underbrace{\mathbf{b}^{(1)}}_{4\times 1} + \underbrace{\mathbf{kk}_0^{(1)}}_{1\times 1} + \underbrace{\mathbf{kk}_1^{(1)}}_{1\times 1} \exp(y) \ \text{ for } y \in I_1
\end{aligned}$$

where $\mathbf{GG}^{(i)}$,$\mathbf{\Gamma\Gamma}^{(i)}$, $\mathbf{bb}^{(i)}$,$\mathbf{KK}^{(i)}$, $\mathbf{kk}_0^{(i)}$ and $\mathbf{kk}_1^{(i)}$ for $i \in \{1,2\}$ are given in the Appendix. In particular, the constant vector $\mathbf{bb}^{(i)}$ is determined by boundary conditions similar to those in Section 3.1.

Finally, the endogenous bankruptcy boundaries $\boldsymbol{y}_{def} = \left[y_{def}^G, y_{def}^B\right]^\top$ are given by the standard optimization / smooth pasting condition:

$$\left(\mathbf{E}^{(1)}\right)'\left(y_{def}^G\right)_{[1]} = 0, \text{ and } \left(\mathbf{E}^{(2)}\right)'\left(y_{def}^B\right)_{[2]} = 0. \tag{17}$$

3.3 Credit Default Swaps

One of our empirical measures for corporate bonds' non-default risk is the Bond/CDS spread, i.e., the bond spread minus its corresponding CDS spread. We compute the model implied CDS spread under the assumption that the CDS market is perfectly liquid, consistent with Longstaff et al. [2005]. In practice, the CDS market is much more liquid than the secondary corporate bond market.

Let τ (in years from today which is normalized to zero) be the time of default, $\tau \equiv \inf\{t : y_t \leq y_{def}^{s_t}\}$, which can be the first time the cash flow y_t reaches the default boundary y_{def}^s in state s, or when $y_{def}^G < y_t < y_{def}^B$ and a change of state from G to B triggers the default. The required flow payment f corresponding to a T-year CDS is the solution to the following equation:

$$\mathbb{E}^{\mathcal{Q}}\left[\int_0^{\min[\tau,T]}\exp\left(-rt\right)f dt\right]=\mathbb{E}^{\mathcal{Q}}\left[\exp\left(-r\tau 1_{\{\tau\leq T\}}\right)LGD_{\tau}\right] \tag{18}$$

where LGD_{τ} is the loss-given-default when the default occurs at time τ. If there is no default, no loss-given-default is paid out by the CDS seller. The loss-given-default LGD is defined as the bond face value p minus its recovery value, and we follow the practice to define the recovery value as the transaction price right after default (with the mid price when the firm defaults at y_b^s). We calculate the required flow payment f that solves (18) using a simulation method. Then the CDS spread f/p is defined as the ratio between the flow payment f and the bond's face value p.

4. Calibration

We calibrate the parameters governing firm fundamentals and pricing kernels to the key moments of the aggregate economy and asset pricing. Parameters governing time-varying liquidity conditions are calibrated to their empirical counterparts on bond turnover, dealer's bargaining power, and observed bid-ask spreads. Since the credit spreads of the randomly-maturing bonds in the model are not directly compared to the data, we use simulation methods to compute the spreads for fixed maturity bond whose holders are subject to the same liquidity shocks as modeled before.

4.1 Benchmark Parameters

[TABLE 1 ABOUT HERE]

4.1.1 SDF, cash flows, and liquidity parameters

We follow Chen et al. [2012] in calibrating firm fundamentals and investors' pricing kernel. Table 1 reports the benchmark parameters we use, which are standard in the literature. Transition intensities give the duration of the business cycle (10 years for expansions and 2 years for recessions). Jump risk premium $\exp(\kappa) = 2$ in state G (and the state B jump risk premium is the reciprocal of that of state G) is consistent with a long-run risk model with Markov-switching consumption conditional moments and calibrated to match the equity premium (Chen, 2010). The risk price η is the product of relative risk aversion γ and consumption volatility σ_c: $\eta = 0.2$ (0.3) in state G (state B) requires $\gamma = 10$ and $\sigma_c = 2\%$ ($\sigma_c = 3\%$). Cash flow growth rate is matched to the average real growth rate of aggregate corporate profits. Systematic volatility σ_m and idiosyncratic volatility σ_i are chosen to match the equity return volatilities and default probabilities of Baa firms.

Chen et al. [2009] argue that generating a reasonable equity Sharpe ratio is important criterion for a model that tries to simultaneously match the default rates and credit spreads, for otherwise one can simply raise credit spreads by imposing unrealistically high systematic volatility and prices of risk. Based on our calibration (especially the choices of σ_m, σ_i, κ, and η), we obtain the equity Sharpe ratio of 0.11 in state G and 0.20 in state B for the unlevered firm, which is close to what Chen et al. [2009] estimate for the mean firm-level Sharpe ratio for the whole universe of the CRSP firms (0.17).

For average maturity, we consider a firm with a continuum of bonds that matures (randomly) with intensity $m = 0.2$ so that the average debt maturity is about $1/m = 5$ years. This is close to the empirical median debt maturity (including bank loans and public bonds) reported in Chen et al. [2012].

For liquidity parameters, we assume a bondholder will be hit by liquidity shock with intensity $\xi = 0.7$. In our model, the liquidity shock intensity ξ mainly determines the turnover rate of corporate bonds in secondary markets. Because in TRACE the turnover of corporate bonds varies little over the business cycle, we set ξ to be state independent. When hit by a liquidity shock, it takes a bond holder on average a week ($\lambda_G = 50$) in the good state and 2.6 weeks ($\lambda_B = 20$) in the bad state to find an intermediary to sell all bond holdings. Taken together, the model implies an average annual turnover of about 70%, which is close to the value-weighted turnover in TRACE. Procyclicality of λ captures time-varying liquidity conditions in the secondary market, and is strongly supported in the data. We interpret lower λ as a weakening of the financial system and its ability to intermediate markets. In our model, adverse macroeconomic conditions (prices of risk) coincide and interact with weaker firm fundamentals and worsened secondary market liquidity to generate quantitatively important implications for the pricing of defaultable bonds. Finally, we set bond holders bargaining power to be fixed at $\beta = 0.05$ independent of the aggregate state. This number is taken from empirical work that estimates search frictions in the bond market (Feldhütter [2012]).

4.1.2 Effective recovery rates

The parameters that are specific to our model are the type and state-dependent recovery rates α_l^s for $l \in \{L, H\}$ and $s \in \{G, B\}$. We first borrow from the existing literature (say, Chen [2010]) who treats the traded prices right after default as recovery rates, and estimates recovery rates of $57.55\% \cdot v_u^G$ in normal times and $30.60\% \cdot v_u^B$ in recessions. Assuming that post-default prices are bid prices that investors are selling, then we have from Proposition 1 that

$$0.5755 \;=\; \alpha_L^G + \beta(\alpha_H^G - \alpha_L^G), \text{ and } 0.3060 = \alpha_L^B + \beta(\alpha_H^B - \alpha_L^B). \tag{19}$$

We need two more pieces of bid-ask information for defaulted bonds to pin down the α_l^s's. Edwards et al. [2007] report that in normal times, the transaction cost for defaulted bonds for median-sized trades is about $200bps$. To gauge the bid-ask spread for defaulted bonds during recessions, we take the following approach. Using TRACE data, we first follow Bao et al. [2011] to calculate the implied bid-ask spreads for low rated bonds (C and below) for both normal and recession times. We find that relative to normal times, during recessions the implied bid-ask spread is around 2.6 times higher. Given a bid-ask spread of $200bps$ for defaulted bonds, this multiplier implies that the bid-ask spread for defaulted bonds during recessions is about $520bps$. Hence we have

$$2\% \;=\; \frac{2\,(1-\beta)\left(\alpha_H^G - \alpha_L^G\right)}{\alpha_L^G + \beta(\alpha_H^G - \alpha_L^G) + \alpha_H^G}, \text{ and } 5.2\% = \frac{2\,(1-\beta)\left(\alpha_H^B - \alpha_L^B\right)}{\alpha_L^B + \beta(\alpha_H^B - \alpha_L^B) + \alpha_H^B}. \tag{20}$$

Solving (19) and (20) gives us the estimates of $\boldsymbol{\alpha} = \left[\alpha_H^G, \alpha_L^G, \alpha_H^B, \alpha_L^B\right]$.

[TABLE 2 ABOUT HERE]

4.1.3 Ultimate recovery rates

This subsection gives an estimation for the ultimate recovery rate $\hat{\alpha}_s$, which is the debt holders' final payout (as a fraction of the unlevered firm value) from bankruptcy settlement. As emphasized, our calibration exercise per se does not depend on these estimates of the ultimate recovery rates. However, later our structural default-liquidity decomposition relies on the hypothetical Leland and Toft [1996] situation without liquidity frictions in the the secondary market for corporate bonds. The hypothetical bond prices (and thus credit spreads) in the perfectly liquid Leland model then need to be derived from the ultimate recovery rates $\hat{\alpha}_s$'s.

To extract information on these ultimate recovery rates, we use *Moody's default and recovery database* that covers defaulted corporate bonds between 1987 and 2011. We track the price path for each defaulted bond from the default date to the settlement (or emergence) date, and follow Moody's preferred method when choosing the emergence price.[11]

We borrow from the empirical literature on venture capital / private equity (e.g. Kaplan and Schoar [2005]) to adjust for risk by discounting the return for each defaulted bond by a public market reference return over the same horizon (from default date to emergence date). We use the SP500 total return (including dividend) as the relevant benchmark, and the resulting excess returns are called "Public Market Equivalent" (PME).[12] To account for state dependence in risk premium, we sort our sample into two groups based whether the default month is classified as recession by NBER.[13] Table 8 in the Appendix provides summary statistics on our excess return matric and figure 2 plots its empirical distribution.

The average time from credit event to ultimate resolution is 501 days, implying a bankruptcy payout intensity of $\theta = 0.73$. And, this duration varies little across recession and non-recession periods. We also find that the average risk-adjusted buy-and-hold return when default occurs in recession is about 212%, and when default occurs in non-

[11] For each bond, Moody's calculates the emergence price using three methods: trading price, settlement price or liquidity price and indicates which one is preferred.

[12] We use this approach because it is difficult to estimate *beta* for this investment strategy due to unbalanced panels and unknown interim returns before emergence date, a well-known problem in the VC/PE literature. And, based on the unbalanced panel we find that the market *beta* of this strategy is about 0.7, which potentially would make the excess return even higher.

[13] Out of our full sample of 642 bonds, 130 defaulted in recession months.

recession time is about 153%. Since at aggregate state s the trading price right after default is $\left[\alpha_L^s + \beta(\alpha_H^s - \alpha_L^s)\right] v_U^s$ while the ultimate recovery is $\hat{\alpha}_s v_U^s$, we reach the estimates for $\hat{\alpha}_s$'s as (recall (19)):[14]

$$\hat{\alpha}_G = 1.53 \times 0.5749 = 87.96\%, \text{ and } \hat{\alpha}_B = 2.12 \times 0.3060 = 64.68\%.$$

4.2 Empirical Moments

Our calibration focuses on the following four rating classes: Aaa/Aa , A, Baa, and Ba; the first three rating classes are investment grade, while Ba is speculative grade. We combine Aaa and Aa together because there are few observations for Aaa firms. We emphasize that previous calibration studies on corporate bonds focus on the difference between Baa and Aaa only, while we are aiming to explain the credit spreads across a wide range of rating classes.

Our data of bond spreads is obtained using Mergent Fixed Income Securities Database (FISD) trading prices from January 1994 to December 2004, and TRACE data from January 2005 to June 2012. We exclude utility and financial firms. For each transaction, we calculate the bond spread by taking the difference between the bond's yield and the treasury yield with corresponding maturity. Within each rating class, we average these observations in each month to form a monthly time series of bond spreads for that rating. We then calculate the time-series average for each rating, and provide the standard deviation for the sample mean. To account for the autocorrelation of these monthly series, we calculate the standard deviation using Newey-West procedure with 15 lags.[15]

[TABLE 3 ABOUT HERE]

We report the time-series means for each rating category and their corresponding standard deviations for both 5-year and 10-year bonds in Table 3 column "Data." Compared

[14]This calculation implicitly assumes that there is no transitioning of aggregate states when waiting for ultimate recoveries, as we only have average buy-and-hold return for bonds that defaulted at a given aggregate state. One can potentially calculate the adjusted buy-and-hold return not only conditional on the state in which the firm defaults (which is our treatment), but also conditional on the state in which the ultimate recovery occurs. This will significantly complicate the analysis, and it is unclear which direction of bias that this treatment introduces.

[15]Using higher order of lags increases the standard deviation only slightly.

to the existing literature, e.g., Huang and Huang [2012] who mainly cover the period from the 1970's to the 1990's and report a credit spread of 63 bps for Aaa rated 10-year bonds, 91 bps for Aa, 123 for A, 194 for Baa, and 320 for Ba, our mean estimates based on FISD data from 1994 to 2010 are fairly close. The spread for 10-year Baa bonds based on our mean estimate is 166.4 bps, which is about 28 bps lower than 194 bps in Huang and Huang [2012]. However, credit spreads in the data have significant variations over time, and the average spreads are estimated with sizable errors. The spread of 194 bps falls into the 95% confidence interval, as the standard deviation of our mean estimate is 21.71 bps.

Our empirical moments for Bond-CDS spreads are obtained by matching FISD bond transaction data with CDS prices from Markit. We follow the same procedure as above, with two caveats. First, the data sample period only starts from 2005 when CDS data become available. Second, to address the selection issue, we follow Chen et al. [2012] and focus on firms that have both 5-year and 10-year bonds outstanding.

The default probabilities for 5-year and 10-year bonds are taken from Exhibit 33 of Moody's annual report on corporate default and recovery rates (2012), which gives the cumulative default probabilities over the period of 1920-2011. Finally, as we will be explained in detail later, we combine both Edwards et al. [2007] and Bao et al. [2011] to obtain the estimates of bid-ask spreads for corporate bonds across different ratings and over the business cycle.

4.3 Calibration Results on Credit Spreads and Default Probabilities

Table 3 presents our calibration results on aggregate default probability and total credit spread for bonds of four rating classes. Given a firm's default boundary, we compute the default probability and total credit spread of bonds at 5 and 10 year maturity using Monte-Carlo methods.[16] As typical in structural corporate bond pricing models, we find that the

[16]Specifically, we simulate the cash flow of the firm and aggregate state for 50,000 times for a fixed duration of 5 or 10 years and count the times where the cash flow cross the state dependent default boundary and also record the cash flow received by bond holders of either H or L type. Following the literature, we adjust the principle of the bond to make it issue at par.

model implied default probability and total credit spread are highly nonlinear in market leverage (see Figure 4). As in David [2008], the non-linearity inherent in the model implies that the average credit spreads are higher than the spreads at average market leverage. We thus follow David [2008] in computing model implied aggregate moments. Specifically, we compute the quasi market leverage (i.e., book debt over the sum of market equity and book debt) of all Compustat firms (excluding financial and utility firms and other standard filters) for which we have ratings data between 1994 and 2012, and classify each quarterly observation as either in "G State" or "B State" based on whether the specific quarter is classified as NBER recession.[17] We then match each firm-quarter observed in Compustat to its model counterpart based on quasi market leverage, and compute the average across aggregate states, and repeat the procedure for each rating class and each maturity (5 or 10 years).

The resulting summary statistics are presented in Table 3. As shown in Bhamra et al. [2010] and Chen [2010], allowing for state-dependent risk prices is the key to the success of structural models in explaining observed default probabilities and credit spread jointly. Since most of the existing corporate bond calibration exercises focus on 10-year bonds, to be more in line with the existing literature we have chosen state-dependent risk prices to deliver an overall good match for the default probabilities and credit spreads for 10-year bonds. Finally, recall that we calculate the model implied credit spreads based on the empirical leverage distribution observed in Compustat; thus, our model exactly matches the data counterpart on the dimension of leverage.

Table 3 shows that our quantitative model is able to deliver a satisfactory cross-sectional pattern in default probabilities and credit spreads across four ratings that we are considering. One number worth mentioning is the average spread for Baa bonds, which is 192.6 bps under our calibration. This is close to the Baa spread of 194 bps reported in Huang and Huang [2012], and falls into the 95% confidence interval $[124, 209]$ based on the FISD data covering 1994-2010.

Relative to the existing literature, our calibration aims at explaining the total credit

[17] For empirical distribution of market leverage for each rating, see Figure 3. Market leverage is defined as the ratio of book debt over book debt plus market equity (sometimes it is called quasi market leverage).

spread across ratings, rather than differences between ratings. For instance, Bhamra et al. [2010] and Chen [2010] only focus on explaining the difference between Baa and Aaa rated bonds, which is considered as the default component of Baa rated bonds under the assumption that the observed spreads for Aaa rated bonds are mostly driven by liquidity premium. Because our framework endogenously models bond liquidity, we are able to match the *total credit spread* that we observe in the data across the superior ratings (Aaa/Aa) and the high end of speculative rating bonds (say Ba). For example, for Aaa/Aa bonds, we generate a 10-year credit spread of 70 bps (68 bps in data) with a default probability of 2.09% (2.06 in data), and for Ba the 10-year credit spread is 349 bps (325 bps in data) with a default probability of 16.66% (19.01% in data).

While our model is able to quantitatively match the cross-sectional pattern for the credit spreads of 10-year bonds, the matching for 5-year bonds is less satisfactory. The general pattern is that although the default probability matches the data counterpart reasonably well, the model implied mean credit spread for 5-year bonds undershoots the data counterpart, suggesting that the model implied term structure of default probability and credit spread is steeper than the data suggests. In fact, consistent with the finding in Bhamra et al. [2010], the method of David [2008] which addresses the nonlinearity in the data (caused by the diverse distribution in leverage) has helped our model greatly to deliver a flatter term structure; but this treatment is not strong enough to get the term structure right. Some obvious extension of our model (e.g., introducing jumps in cash flows that are more likely to occur in state B) should help in this dimension, and we leave future research to address this issue.

4.4 Model Performance for Non-Default Risk

Our model features an illiquid secondary market for corporate bonds, which implies that the equilibrium credit spread must compensate the bond investors for bearing not only default risk but also liquidity risk. This new element allows us to investigate the model's quantitative performance on dimensions specific to bond market liquidity, in addition to the two common measures — cumulative default probabilities and credit spreads — that the previous literature has focused on (e.g., Huang and Huang [2012]).

Our model, which is an extension of He and Milbradt [2012], provides a coherent economic explanation for why bond liquidity goes down with rating classes documented in the data. The secondary market for post-default bonds is less illiquid than the corresponding pre-default secondary market, which leads to a greater wedge in the H and L investors' recovery values at default. As a bond moves closer to default, the valuation wedge between H and L type endogenously widens, giving rise to a larger bid-ask spread. This implies that investors will ask for a higher compensation for nondefault risk in holding bonds that are closer to default. The same logic applies when the economy switches to the B state since bonds are riskier in the B state. Lower intermediary intensity in the B state further reduces the outside option of L investors, driving up bid-ask spreads further. This worsened liquidity in turn leads to earlier default by equity holders – a positive liquidity-default spiral arises.

4.4.1 Bond-CDS Spread

Longstaff et al. [2005] argue that because the market for CDS contracts is much more liquid than secondary market for corporate bonds, the CDS spreads should mainly reflect the default risk of a bond, while the total credit spreads for bonds also includes liquidity premium to compensate for the illiquidity in the bond market. This is exactly what our model is trying to capture.

[TABLE 4 ABOUT HERE]

Similar to the above procedure in Section 4.3, following David [2008] we obtain our model-implied aggregated moments by first calculate the Bond-CDS spread for each firm-quarter observation in Compustat based on its quasi market leverage, from 1994 to 2012. Table 4 reports the model-implied Bond-CDS spreads, together with data counterparts (which are only available from 2005 onwards). Because our relatively short sample includes financial crisis, to ensure robustness we also report the results by excluding the crisis period (October 2008 to March 2009); note that this sample still covers most of the recession period (December 2007 to June 2009).

Overall, similar to the finding in Section 4.3, the quantitative matching of cross-sectional pattern of Bond-CDS spreads is reasonably good for 10-year bonds. For Aaa/Aa bonds our model implies a 10 year Bond-CDS spread of 42.7 bps (the average spread is 35 bps in data), while in the sample excluding the crisis, the model-implied spread is 42.4 bps (31.1 bps in the data). For lower rated Ba bonds, the implied 10-year Bond-CDS spread 93.7 bps slightly undershoots the data (108 bps), but overshoots the data a bit for the sample period excluding crisis (92.7 bps in model while 83 bps in data.) Finally, as there is a significant variations over time for the estimate of Bond-CDS spreads, the average spreads are estimated with sizable errors. In fact, all our model-implied Bond-CDS spreads are within one standard deviation of the estimated mean spreads from the data.

One area our model fails is to replicate the downward sloping Bond-CDS term structure in the data for highly rated bonds. In our data, the 5-year Bond-CDS spread is higher than or close to 10 years across rating categories, although because of sizable standard deviations their differences are not statistically significant. In the model, the Bond-CDS spreads are upward sloping for investment grade bonds. This is because bonds with shorter maturity are more liquid due to a better outside option of bond sellers who can sit out waiting for the principal payment (He and Milbradt, 2012). For bonds that are close to default, the bond's stated maturity matters little, thus 5-year and 10-year bonds face similar illiquidity. Thus, the illiquidity discount per year (which is stated maturity) is higher for 5-year bonds, leading to downward sloping curve for Bond-CDS spreads for risky bonds. It is also worth noting that, inconsistent with our sample, Longstaff et al. [2005] document a positive relation between Bond-CDS spread and maturity in their sample.

One possible explanation for the downward sloping Bond-CDS spreads is that the CDS spreads at different maturities are affected by liquidity differently, a dimension that our model abstracts from. Although the CDS market is quite liquid compared to the corporate bond market, it is well recognized that CDS contracts are the most liquid at the 5-year horizon when measured by the number of dealers offering quotes. If dealers are mainly selling CDS protections to regular investors and they possess market power, then the price of 10-year CDS contracts that are only offered by a small number of dealers tend to be higher than the price of 5-year CDS contracts with more competitive dealers, which may contribute to the relatively lower 10-year Bond-CDS spreads.

4.4.2 Bid-Ask Spread

The second non-default measure that we study is (proportional) bid-ask spreads in the secondary market for corporate bonds, whose model counterpart is given in (5). Unlike credit spreads, there is little study on the unconditional moments for the time series of bid-ask spreads. But previous empirical studies have uncovered rich patterns of bid-ask spreads across aggregate states and rating classes, and we investigate whether our model is able to match these patterns quantitatively.

We combine Edwards et al. [2007] and Bao et al. [2011] to construct the data counterparts for the bid-ask spread, because Edwards et al. [2007] only report the average bid-ask spread across ratings in normal times (2003-2005). The ratings considered in Edwards et al. [2007] are superior grade (Aaa/Aa) with an bid-ask spread of 40 bps, investment grade (A/Baa) with an bid-ask spread of 50 bps, and junk grade (below Ba) with a bid-ask spread of 70 bps.[18] For each grade, we then compute the Roll's measure of liquidity as in Bao et al. [2011] and used them to back out the ratio of B state bid-ask spread to the G-state bid-ask spreads. We multiply this ratio by the level of bid-ask spread estimated by Edwards et al. [2007] to arrive at bid-ask spread in B state. These empirical estimates are reported in Table 5.

On the model side, again we rely on the empirical leverage distribution in Compustat firms across ratings and aggregate states to calculate the average of model implied bid-ask spreads. We calibrate two state-dependent holding cost parameters (χ_G and χ_B) to match bid-ask spread of investment grade bonds across two aggregate states. Since the average maturity in TRACE data is around 8.3 years, the model implied bid-ask spread is calculated as the weighted average between the bid-ask spread of a 5-year bond and a 10-year bond.

[TABLE 5 ABOUT HERE]

[18] We take the median size trade around 240K. Edwards et al. [2007] show that trade size is an important determinants for transaction costs of corporate bonds. But, for tractability reasons, we have abstracted away from the trade size.

The model implied bid-ask spreads are reported in Table 5, together with their empirical counterparts. The model is able to generate several patterns that quantitatively matches what we observe in the data. First, in normal times, the average bid-ask spread is 37.23 bps for superior grade bonds, 47.99 bps for investment grade bonds and 75.94 bps for junk grade bonds, which are close to those estimated by Edwards et al. [2007]. Second, these bid-ask spreads doubles when the economy switches from state G to state B. Finally, although not reported here, the model-implied bid-ask spread of longer-maturity bonds are higher than shorter-maturity bonds and this is also consistent with previous empirical studies (eg. Edwards et al. [2007]; Bao et al. [2011]).

5. Structural Default-Liquidity Decomposition

Our structural model of corporate bonds features a full interaction between default and liquidity in determining the credit spreads of corporate bonds. It has been common practice in the empirical literature to decompose the credit spread into liquidity and default components in an additive way, such as in Longstaff et al. [2005]. From the perspective of our model, this decomposition — though intuitively appealing — over-simplifies the role of liquidity in determining the credit spread. More importantly, the additive structure often leads to a somewhat misguided interpretation that liquidity or default is the cause of the corresponding component, and each component would be the resulting credit spreads if we were to shut down the other channel.

We emphasize that this interpretation may give rise to misleading answers in certain policy related questions. For instance, as our decomposition indicates, part of the default risk comes from the illiquidity in the secondary market. Thus, when the government is considering providing liquidity to the market, it is not creating a direct effect on the credit spread by improving liquidity, but also an indirect effect in lowering the default risk via the rollover channel and the endogenous default decision of the equity holders. This additional indirect effect can be easily missed by the traditional perspective with an additive structure, as it imposes the assumption that default risk will not be affected by the improved bond market liquidity.

5.1 Decomposition Scheme

We propose a more detailed structural decomposition, which nests the additive default-liquidity decomposition that is common in the literature. Specifically, we further decompose the default part into pure-default part and liquidity-driven-default part, and similarly decompose the liquidity part into pure-liquidity and default-driven-liquidity parts:

$$\hat{cs} = \overbrace{\hat{cs}_{pureDEF} + \hat{cs}_{LIQ \to DEF}}^{\text{Default Component } \hat{cs}_{DEF}} + \overbrace{\hat{cs}_{pureLIQ} + \hat{cs}_{DEF \to LIQ}}^{\text{Liquidity Component } \hat{cs}_{LIQ}} \tag{21}$$

In this way, we separate *causes* and from *consequences*, and emphasize that lower liquidity (higher default risk) can lead to a rise in the credit spread through the default risk (liquidity) channel. Recognizing and further quantifying this endogenous interaction between liquidity and default is important in evaluating the economic consequence of improving market liquidity (e.g., Term Auction Facilities or discount window loans) or alleviating default issues (e.g, direct bailouts).

Let us start with the default component, and imagine a hypothetical investor who is not subject to liquidity problems (both pre and post default) and consider the spread that this investor demands over a Treasury bond for holding the corporate bond. The resulting spread, which denote by $\hat{cs}_{DEF}$, only prices the default event of hitting y_b, in line with Longstaff et al. [2005] who use information from the relatively liquid CDS market to back out the default premium. Importantly, the default boundary y_b in calculating $\hat{cs}_{DEF}$ is under the assumption that the other bond investors are still facing liquidity frictions as modeled above. In contrast, we define the "Pure-Default" component $\hat{cs}_{pureDEF}$ as the spread implied by the benchmark Leland model without liquidity frictions in the secondary market (e.g., setting $\xi = 0$ or $\lambda = \infty$ for both pre- and post-default) for all bond investors. Because the liquidity of the bond market leads to less rollover losses, equity holders will default later relative to our model with secondary market illiquidity, i.e., $y_{def}^{Leland,s} < y_{def}^{s}$ where $y_{def}^{Leland,s}$ denotes the endogenous default boundary in Leland and Toft [1996] with time-varying aggregate states.[19] Importantly, this distinction implies that

[19] Because of delayed bankruptcy payout, the recovery rates for Leland and Toft [1996] model where

the pure-default component $\hat{cs}_{pureDEF}$ is smaller than the default component $\hat{cs}_{DEF}$. The difference $\hat{cs}_{DEF} - \hat{cs}_{pureDEF}$ gives the novel "Liquidity-Driven Default" component, which quantifies the effect that bond illiquidity makes default more likely.

The liquidity component is defined as the difference between the credit spreads required by a representative H investor who is subject to liquidity frictions, i.e., the spread $\widehat{cs}$ implied by our model, and that required by a hypothetical investor without liquidity frictions, i.e., the spread $\hat{cs}_{DEF}$. Thus $\hat{cs}_{LIQ} \equiv \hat{cs} - \hat{cs}_{DEF}$, which is in line with Longstaff et al. [2005]. Following a similar treatment to the default component, we further decompose $\hat{cs}_{LIQ}$ into a "Pure-Liquidity" component and a "Default-driven Liquidity" component. We first calculate $\hat{cs}_{pureLIQ}$ as the spread of a bond that is subject to search/liquidity friction as in Duffie et al. [2005] but does not feature any default risk; this is the spread implied by our model as $y \to \infty$ so that the bond becomes default free. The residual $\hat{cs}_{LIQ} - \hat{cs}_{pureLIQ}$ is what we term the default-driven liquidity part of our credit spread. Economically, the default-driven liquidity part arises because default leads to a more illiquid post-default secondary market, which endogenously worsens the secondary market liquidity even before default as default becomes more likely.

5.2 Default-Liquidity Decomposition across Ratings over Business Cycle

We performed the above decomposition for typical 10-year bonds in the four rating classes considered above. Since we want to compare the decomposition across different aggregate states, as Table 3 only reports mean credit spreads, we calculate the implied state G credit spreads via the model implied (unconditional) state G probability of 83%. These state G

investors are not subject to liquidity problems are different from the ultimate recovery rates in Table 2. To calculate state-dependent Leland and Toft [1996] recovery rates, which are denoted by α^G_{Leland} and α^B_{Leland}, we first define state-dependent Price/Dividend ratios as $[x_G, x_B]^\top \equiv \begin{bmatrix} r-\mu_G+\zeta_G & -\zeta_G \\ -\zeta_B & r-\mu_B+\zeta_B \end{bmatrix}^{-1} \mathbf{1}$. Then taking into account the bankruptcy payout intensity θ and state switching intensities ζ_G and ζ_B, we have

$$\begin{bmatrix} \alpha^G_{Leland} \\ \alpha^B_{Leland} \end{bmatrix} = \begin{bmatrix} r+\theta+\zeta_G & -\zeta_G \frac{x_B}{x_G} \\ -\zeta_B \frac{x_G}{x_B} & r+\theta+\zeta_B \end{bmatrix}^{-1} \begin{bmatrix} \theta\hat{\alpha}_G \\ \theta\hat{\alpha}_B \end{bmatrix}.$$

credit spreads map into certain cash flows for each rating class, which we then use as a base for our state B calculations — we fix the cash flow levels at state G levels and simply investigate the credit spread if the state were B.

5.2.1 Level of credit spreads

We present our decomposition results for both aggregate states in Panel I in Table 6. As expected, the "pure liquidity" component accounts for a greater fraction of credit spread for higher rated bonds. For instance, for Aaa/Aa rated bonds, given the aggregate state G (B), 57% (54%) of the observed credit spread comes from the "pure liquidity" component. In contrast, for Ba rated bonds, only 12% (12%) of the credit spread is accounted for by the "pure liquidity" component. A similar intuitive pattern holds for the "pure default" component across credit ratings. The fraction of credit spreads that can be explained by the "pure default" component starts from about 27~30% for Aaa/Aa rated bonds, and monotonically increases to about 56% for Ba rated bonds.

The remaining part of the observed credit spreads, which is around 15%~25% depending on the credit rating, can be attributed to the novel interaction terms, either "liquidity-driven default" or "default-driven liquidity." The "liquidity-driven default" part captures how corporate optimal default decisions are affected by secondary market liquidity frictions via the rollover channel, which is non-negligible even for the highest rating firms: it accounts for 9% (8%) of the observed credit spreads in state G (B) for Aaa/Aa rated bonds. As expected, its quantitative importance rises for low rating bonds: for Ba rated bonds, the liquidity-driven default accounts for about 12% of observed credit spreads.

The second interaction term, i.e., the "default-driven liquidity" component, captures how secondary market liquidity endogenously worsens when a bond is closer to default. Given a more illiquid secondary market for defaulted bonds, a lower distance-to-default leads to a worse secondary market liquidity before default because of the reduced outside option of L investors when bargaining with dealers. Similar to "liquidity-driven default," the "default-driven liquidity" component becomes larger for bonds of lower rating classes. In our calibration, this component is slightly smaller than the "liquidity-driven default" part, but remains non-negligible even for high rated bonds (about 7~8% of the credit spread for Aaa/Aa rated bonds).

Finally, to illustrate the difference between our novel default-liquidity decomposition and the traditional approach, we also report the model implied CDS spreads for each rating class in the last column in Table 6. The common industry practice — motivated by the traditional additive perspective — typically views the CDS spread as the default component,[20] with the interpretation that if the secondary corporate bond market becomes as liquid as the CDS market then the resulting bond spread will equal the CDS spread. Our decomposition, by acknowledging endogenous default-liquidity interactions, gives a different answer to this counterfactual thought experiment. For instance, for Ba rated bonds, if the secondary market is perfectly liquid (as in Leland and Toft [1996]), then the resulting state G credit spread will go down to 175 bps. This is much lower than the CDS spread 241 bps which fails to take into account that improving secondary bond market liquidity helps alleviate the default risk via the rollover channel.

5.2.2 The change of credit spreads over aggregate states

This subsection focuses on a long-standing question that has interested empirical researchers, e.g., Dick-Nielsen et al. [2011] and Friewald et al. [2012]: How much of the soaring credit spread when the economy switches from boom to recession is due to increased credit risk, and how much is due to worsened secondary market liquidity? Our framework, which embeds aggregate states into the endogenous default-liquidity spiral proposed by He and Milbradt [2012], offers a fresh perspective on this question. The novel default-liquidity decomposition in (21) acknowledges that both liquidity and default risks for corporate bonds are endogenous and may affect each other. Given this feature, structural answers that rely on well-accepted economic structures seem more appropriate than reduced-form approaches.

As suggested by Panel II in Table 6, increased default risks constitute a large fraction of

[20] This result is subject to the caveat that the credit default swap premium equals the spread between corporate and riskless floating-rate notes (Duffie, 1999), while typically we use treasury fixed-coupon rate as riskless benchmark. See Longstaff et al. [2005] for more details.

the jump in credit spreads. The pure liquidity component is also quantitatively significant in explaining the rise of credit spreads: even for Ba rated bonds, about 14.5% of the rise when entering recessions is due to the lower secondary market illiquidity in state B.

When the economy encounters a recession, the higher default risk lowers secondary market liquidity further, giving rise to a greater "default-driven liquidity" part. Since worse liquidity in state B also pushes equity holders to default earlier through the rollover channel, bond spread rise because of a larger "liquidity-driven default" part. For low rated (say Ba) bonds, the quantitative importance of the "default-driven liquidity" channel (16.3%) dominates that of "liquidity-driven default" (11.7%). In this situation, separating *causes* from *consequences* — which is missing from the traditional additive decomposition — is important, because the effect of "default-driven liquidity" is reflected in the traditional "liquidity" component (i.e., the sum of "pure liquidity" and "default-driven liquidity") but ultimately caused by the default risk. Here, although the "liquidity component" constitutes 32% of the spread rise for Ba rated bonds, worsening liquidity in state B in fact only drives about 26% of this rise.

[TABLE 6 ABOUT HERE]

5.3 Implications on Evaluating Liquidity Provision Policy

Our decomposition and its quantitative results are highly informative for evaluating the effect of policies that target lowering the borrowing cost of corporations in recession times by injecting liquidity into the secondary market. As argued before, a full analysis of the effectiveness of such a policy must take account of how firms' default policies respond to liquidity conditions and how liquidity conditions respond to the default risks. These endogenous forces are exactly what our model is aiming to capture.

Suppose that the government is committed to launching certain liquidity enhancing programs (e.g., Term Auction Facilities or discount window loans) whenever the economy falls into a recession, envisioning that the improved funding environment for financial intermediaries alleviates the worsening liquidity in the secondary bond market. Suppose that the policy is effective in making the secondary market in state B as liquid as that of state

G. More precisely, the policy helps increase the meeting intensity between L investors and dealers in state B, so that λ_B rises from 50 to $\lambda_G = 100$; reduce the state B holding cost χ_B from 1.8 to $\chi_G = 1.5$; and make the post-default secondary market in state B to be as liquid as state G.[21]

In Table 7 we take the same cash flow levels derived above for the representative bond in each rating class, and calculate the credit spreads with and without the state-B liquidity provision policy. We find that a state-B liquidity provision policy lowers state B credit spreads by about 22 (76) bps for Aaa/Aa (Ba) rated bonds, which is about 26% (24%) of the corresponding credit spreads. Moreover, given the dynamic nature of our model, the state-B-only liquidity provision also affects firms' borrowing costs in state G: the state G credit spreads for Aaa/Aa (Ba) rated bonds go down by 14 (44) bps, or about 21% (20%) of the corresponding credit spreads.

Our structural decomposition further allows us to investigate the underlying driving force for the effectiveness of this liquidity provision policy, and Table 7 gives the percentage contribution of each of the following channels: "pure liquidity," "liquidity-driven default," and "default-driven liquidity." By definition, the "pure default" component remains unchanged given any policy that only affects the secondary market liquidity.[22] In Table 7, we observe that the pure-liquidity component accounts for about 80% of the drop in spread for Aaa/Aa rated bonds. However, the quantitative importance of the pure-liquidity component goes down significantly when we walk down the rating spectrum: for Ba rated bonds, it only accounts for about 46% of the decrease in the credit spread.

The market-wide liquidity provision not only reduces investors' required compensation for bearing liquidity risk, but also alleviates some default risk faced by bond investors. A better functioning financial market helps mitigate a firm's rollover risk and thus its default risk, and this force is captured by the "liquidity-driven default" part. The importance of this mechanism goes up for lower rated bonds (30%), but it remains quantitatively

[21] This implies a set of type/state-dependent recovery rates α_l^s that are different from Table 2. We take the ultimate recovery rates in Table 2 as given, and then apply the state-G buy-and-hold return (152%) to back out the hypothetical bond prices at default for both states. Finally, we obtain the hypothetical α_l^s's by imposing the state-G bid-ask spread at default (2%) for both states.

[22] The "pure default" component is defined by Leland and Toft [1996] which is independent of the secondary market liquidity.

significant even for Aaa/Aa rated bonds (10%).

Given that the hypothetical policy was limited to only improving secondary market liquidity, the channel of "default-driven liquidity" is more intriguing. Somewhat surprisingly, the contribution through "default-driven liquidity" is quantitatively significant across all ratings, with Ba rated bonds having the most significant effect (about 33% in state B). This interesting result only exists in models with a positive feedback loop between corporate default and secondary market liquidity. For instance, in He and Xiong (2012), a higher distance-to-default cannot affect the exogenously given secondary bond market liquidity. In contrast, in our model, when the market-wide liquidity provision improves the firm's financing environment and thus pushes the firm away from bankruptcy, the lower default risk endogenously enhances the outside option of L investors when bargaining with dealers, leading to further improved secondary market liquidity. This, in turn, improves default risk through the rollover channel, and so on so forth. The "default-driven liquidity" component is a result of this positive default-liquidity spiral given the initial liquidity provision policy.

[TABLE 7 ABOUT HERE]

6. Concluding Remarks

We build over-the-counter search frictions into a structural model of corporate bonds. In the model, firms default decisions interact with time varying macroeconomic and secondary market liquidity conditions. We calibrate the model to historical moments of default probability and empirical measures of liquidity. The model is able to match the observed total credit spread for corporate bonds with different rating classes, as well as various measures of non-default component studied in previous literature. We propose a structural decomposition that captures the interaction of liquidity and default risks of corporate bonds over the business cycle and use this framework to evaluate the effects of liquidity provision policies during crisis time. Our results identifies quantitatively important economic forces that were previously overlooked in empirical researches on corporate bonds.

References

J. Bao, J. Pan, and J. Wang. The illiquidity of corporate bonds. *The Journal of Finance*, 66(3):911–946, 2011.

H.S. Bhamra, L.A. Kuehn, and I.A. Strebulaev. The levered equity risk premium and credit spreads: A unified framework. *Review of Financial Studies*, 23(2):645–703, 2010.

H. Chen. Macroeconomic conditions and the puzzles of credit spreads and capital structure. *The Journal of Finance*, 65(6):2171–2212, 2010.

H. Chen, Y. Xu, and J. Yang. Systematic risk, debt maturity, and the term structure of credit spreads. Technical report, National Bureau of Economic Research, 2012.

Long Chen, Pierre Collin-Dufresne, and Robert S. Goldstein. On the relation between the credit spread puzzle and the equity premium puzzle. *Review of Financial Studies*, 22: 3367–3409, 2009.

A. David. Inflation uncertainty, asset valuations, and the credit spreads puzzle. *Review of Financial Studies*, 21(6):2487–2534, 2008.

J. Dick-Nielsen, P. Feldhütter, and D. Lando. Corporate bond liquidity before and after the onset of the subprime crisis. *Journal of Financial Economics*, 2011.

D. Duffie, N. Gârleanu, and L.H. Pedersen. Over-the-counter markets. *Econometrica*, 73 (6):1815–1847, 2005.

A.K. Edwards, L.E. Harris, and M.S. Piwowar. Corporate bond market transaction costs and transparency. *The Journal of Finance*, 62(3):1421–1451, 2007.

P. Feldhütter. The same bond at different prices: identifying search frictions and selling pressures. *Review of Financial Studies*, 25(4):1155–1206, 2012.

N. Friewald, R. Jankowitsch, and M.G. Subrahmanyam. Illiquidity or credit deterioration: A study of liquidity in the us corporate bond market during financial crises. *Journal of Financial Economics*, 2012.

Z. He and K. Milbradt. Endogenous liquidity and defaultable bonds. Technical report, National Bureau of Economic Research, 2012.

Z. He and W. Xiong. Rollover risk and credit risk. *The Journal of Finance*, 67(2):391–430, 2012.

Jing-Zhi Huang and Ming Huang. How much of the corporate-treasury yield spread is due to credit risk. *Review of Asset Pricing Studies*, 2(2):153–202, 2012.

S.N. Kaplan and A. Schoar. Private equity performance: Returns, persistence, and capital flows. *The Journal of Finance*, 60(4):1791–1823, 2005.

H.E. Leland and K.B. Toft. Optimal capital structure, endogenous bankruptcy, and the term structure of credit spreads. *The Journal of Finance*, 51(3):987–1019, 1996.

F.A. Longstaff, S. Mithal, and E. Neis. Corporate yield spreads: Default risk or liquidity? new evidence from the credit default swap market. *The Journal of Finance*, 60(5):2213–2253, 2005.

A Appendix

1.1 Appendix for Section 4.1.3 in Estimating Ultimate Recovery

[TABLE 8 ABOUT HERE]

[FIGURE 2 ABOUT HERE]

1.2 Appendix for Section 4.3: Empirical Leverage Distribution across Ratings

[FIGURE 3 ABOUT HERE]

[FIGURE 4 ABOUT HERE]

1.3 Generalization to n aggregate states

We follow the Markov-modulated dynamics approach of Jobert and Rogers (2006).

We note that there are multiple possible bankruptcy boundaries, $y_b(s)$, for each aggregate state s one boundary. Order states s such that $s > s'$ implies that $y_b(s) > y_b(s')$ and denote the intervals $I_s = [y_b(s), y_b(s+1)]$ where $y_b(n+1) = \infty$, so that $I_s \cap I_{s+1} = y_b(s+1)$. Finally, let $\boldsymbol{y}_b = [y_b(1), ..., y_b(n)]^\top$ be the vector of bankruptcy boundaries.

It is important to have a clean notational arrangement to handle the proliferation of states. Let $D_l^{(s)}$ denote the value of debt for an creditor in individual liquidity state l and with aggregate state s. We will use the following notation: $D_l^{(s,i)} \equiv D_l^{(s)}, y \in I_i$, that is $D_l^{(s,i)}$ is the restriction of $D_l^{(s)}$ to the interval I_i. It is now clear that $D_l^{(s,i)} = 0$ for any $i < s$, as it would imply that the company immediately defaults in interval I_i for state s. Let us, for future reference, call debt in states $i < s$ dead and in states $i \geq s$ alive. Finally, let us stack the alive functions along states s but still restricted to interval i so that $\mathbf{D}^{(i)} = \left[D_H^{(1,i)}, D_L^{(1,i)}, ..., D_H^{(i,i)}, D_L^{(i,i)}\right]^\top$ where $D_l^{(s,i)}$ has s denoting the state, i denotes the interval and l denotes the individual liquidity state. The separation of s and i will clarify the pasting arguments that apply when y crosses from one interval to the next. Let

$$\underbrace{\mathbf{I}_i}_{i\times i} = \begin{bmatrix} 1 & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & 1 \end{bmatrix} \tag{22}$$

i.e. a 2x2 diagonal identity matrix, and let

$$\underbrace{\mathbf{1}_i}_{i\times 1} = [1, ..., 1]^\top \tag{23}$$

be a column vector of just ones.

Fundamental parameters. For a 2x2 case, we have a transition matrix $\mathbf{Q}$ that looks like

$$\underbrace{\mathbf{Q}}_{2n\times 2n} = \begin{bmatrix} -\sum_{ls\neq H1}\xi_{H1\to ls} & \xi_{H1\to L1} & \xi_{H1\to H2} & \xi_{H1\to L2} \\ \xi_{L1\to H1} & -\sum_{ls\neq L1}\xi_{L1\to ls} & \xi_{L1\to H2} & \xi_{L1\to L2} \\ \xi_{H2\to H1} & \xi_{H2\to L1} & -\sum_{ls\neq H2}\xi_{H2\to ls} & \xi_{H2\to L2} \\ \xi_{L2\to H1} & \xi_{L2\to L1} & \xi_{L2\to H2} & -\sum_{ls\neq L2}\xi_{L2\to ls} \end{bmatrix} \tag{24}$$

Further, define the possibly state-dependent discount rates

$$\underbrace{\mathbf{R}}_{2n\times 2n} = \begin{bmatrix} \operatorname{diag}\left(\begin{bmatrix} r_H(1) \\ r_L(1) \end{bmatrix}\right) & \cdots & \mathbf{0}_2 \\ \vdots & \ddots & \vdots \\ \mathbf{0}_2 & \cdots & \operatorname{diag}\left(\begin{bmatrix} r_H(n) \\ r_L(n) \end{bmatrix}\right) \end{bmatrix} + m\mathbf{I}_{2n} \tag{25}$$

where we are including the intensity of the random maturity in the definition of $\mathbf{R}$ for notational convenience and brevity.

Building blocks for interval I_i. We now decompose the matrix $\mathbf{Q}$. Let $\mathbf{Q}^{(i)}$ be the transition matrix of jumping into an alive state $s' \leq i$ when currently in interval i and in an alive state $s \leq i$. Let $\tilde{\mathbf{Q}}^{(i)}$ be the transition matrix of jumping into a default state $s' > i$ when currently in interval i and in an alive state $s \leq i$.

Let $\mathbf{v}^{(i)}$ be the recovery or salvage value of the firm when default is declared in states $s > i$ when currently in interval i, where $v_l^{(s,i)} \exp(y) = \alpha_{(s,l)} \frac{\exp(y)}{r_H}$. Thus, $\mathbf{v}^{(i)}$ is a vector containing recovery values for states $(i+1, ..., n) \times (H, L)$ (i.e., it is of dimension $2(n-i) \times 1$).

Let $\boldsymbol{\chi}^{(i)}$ be a vector of holding costs in states $(1, ..., i) \times (H, L)$ (i.e, it is of dimension $2i \times 1$). The holding costs are all positive, and are deducted from the coupon payment. Higher holding costs indicate more severe liquidity states L for the agent.

First, let us start with the interval $i = n$. On this interval, all debt $D_l^{(s,n)}$ is alive. Let

$$\underbrace{\boldsymbol{\mu}^{(n)}}_{2n\times 2n} = \begin{bmatrix} \mu(1)\mathbf{I}_2 & \cdots & \mathbf{0} \\ \vdots & \ddots & \vdots \\ \mathbf{0} & \cdots & \mu(n)\mathbf{I}_2 \end{bmatrix} \tag{26}$$

and similarly let

$$\underbrace{\boldsymbol{\Sigma}^{(n)}}_{2n\times 2n} = \begin{bmatrix} \sigma^2(1)\mathbf{I}_2 & \cdots & \mathbf{0} \\ \vdots & \ddots & \vdots \\ \mathbf{0} & \cdots & \sigma^2(n)\mathbf{I}_2 \end{bmatrix} \tag{27}$$

and let

$$\mathbf{Q}^{(n)} = \mathbf{Q} \tag{28}$$
$$\mathbf{R}^{(n)} = \mathbf{R} \tag{29}$$
$$\tilde{\mathbf{Q}}^{(n)} = 0 \tag{30}$$

Next, for the interval $i = n-1$ we drop the last two rows and columns (i.e. rows and columns $2n$ and $2n-1$) (because they account for different liquidity states) of $\boldsymbol{\mu}^{(n)}, \boldsymbol{\Sigma}^{(n)}, \mathbf{Q}^{(n)}, \mathbf{R}^{(n)}$ to form $\boldsymbol{\mu}^{(n-1)}, \boldsymbol{\Sigma}^{(n-1)}, \mathbf{Q}^{(n-1)}, \mathbf{R}^{(n-1)}$ which are all $2(n-1) \times 2(n-1)$ matrices. In contrast, we form $\tilde{\mathbf{Q}}^{(n-1)}$ by dropping the last two rows and the first $2(n-1)$ columns of $\mathbf{Q}^{(n)}$ to form a $2(n-1) \times 2$ matrix.

We repeat this procedure, dropping rows and columns and thus shrinking the matrices, step by step all the all the way down to $i = 1$.

Debt valuation within an interval I_i. Debt valuation follows the following differential equation on interval I_i:

$$\left(\mathbf{R}^{(i)} - \mathbf{Q}^{(i)}\right)\mathbf{D}^{(i)} = \left(c\mathbf{1}_{2i} - \boldsymbol{\chi}^{(i)}\right) + \boldsymbol{\mu}^{(i)}\left(\mathbf{D}^{(i)}\right)' + \frac{1}{2}\boldsymbol{\Sigma}^{(i)}\left(\mathbf{D}^{(i)}\right)'' + \tilde{\mathbf{Q}}^{(i)}\mathbf{v}^{(i)}\exp(y) + m \cdot p\mathbf{1}_{2i} \tag{31}$$

where $\tilde{\mathbf{Q}}^{(i)}\mathbf{v}^{(i)}\exp(y)$ represents the intensity of jumping into default times the recovery in the default state and $m \cdot p\mathbf{1}_{2i}$ represents the intensity of randomly maturing times the payoff in the maturity state. Next, let us conjecture a solution of the kind $\mathbf{g}\exp() + \mathbf{k}_0^{(i)} + \mathbf{k}_1^{(i)}\exp(y)$ where $\mathbf{g}$ is a vector and γ is a scalar. The particular part stemming from $\mathbf{c}^{(i)}$ is solved by a term $\mathbf{k}_0^{(i)}$ with

$$\underbrace{\mathbf{k}_0^{(i)}}_{2i\times 1} = \underbrace{\left(\mathbf{R}^{(i)} - \mathbf{Q}^{(i)}\right)^{-1}}_{2i\times 2i}\underbrace{(c + m \cdot p)\,\mathbf{1}_{2i} - \boldsymbol{\chi}^{(i)}}_{2i\times 1} \tag{32}$$

and the particular part stemming from $\tilde{\mathbf{Q}}^{(i)}\mathbf{v}^{(i)}$ is solved by a term $\mathbf{k}_1^{(i)}\exp(y)$ with

$$\underbrace{\mathbf{k}_1^{(i)}}_{2i\times 1} = \underbrace{\left(\mathbf{R}^{(i)} - \mathbf{Q}^{(i)} - \boldsymbol{\mu}^{(i)} - \frac{1}{2}\boldsymbol{\Sigma}^{(i)}\right)^{-1}}_{2i\times 2i}\underbrace{\tilde{\mathbf{Q}}^{(i)}}_{2i\times 2(n-i)}\underbrace{\mathbf{v}^{(i)}}_{2(n-i)\times 1} \tag{33}$$

It should be clear that $\mathbf{k}_1^{(n)} = \mathbf{0}$ as on I_n there is no jump in the aggregate state that would result in immediate default. Plugging in, dropping the $\mathbf{c}^{(i)}$ and $\tilde{\mathbf{Q}}^{(i)}\mathbf{v}^{(i)}\exp(y)$ terms, canceling out $\exp() > 0$, we have

$$\mathbf{0}_{2i} = \left(\mathbf{Q}^{(i)} - \mathbf{R}^{(i)}\right)\mathbf{g} + \boldsymbol{\mu}^{(i)}\gamma\mathbf{g} + \frac{1}{2}\boldsymbol{\Sigma}^{(i)}\gamma^2\mathbf{g} \tag{34}$$

Following JR06, we premultiply by $2\left(\boldsymbol{\Sigma}^{(i)}\right)^{-1}$ and define $\mathbf{h} = \gamma\mathbf{g}$ to get

$$\gamma\mathbf{g} = \mathbf{h} \tag{35}$$

$$\gamma\mathbf{h} = -2\left(\boldsymbol{\Sigma}^{(i)}\right)^{-1}\boldsymbol{\mu}^{(i)}\mathbf{h} + 2\left(\boldsymbol{\Sigma}^{(i)}\right)^{-1}\left(\mathbf{R}^{(i)} - \mathbf{Q}^{(i)}\right)\mathbf{g} \tag{36}$$

Stacking the vectors $\mathbf{j} = \begin{bmatrix}\mathbf{g}\\ \mathbf{h}\end{bmatrix}$ we have

$$\gamma\mathbf{j} = \begin{bmatrix}\mathbf{0}_{2i} & \mathbf{I}_{2i}\\ 2\left(\boldsymbol{\Sigma}^{(i)}\right)^{-1}\left(\mathbf{R}^{(i)} - \mathbf{Q}^{(i)}\right) & -2\left(\boldsymbol{\Sigma}^{(i)}\right)^{-1}\boldsymbol{\mu}^{(i)}\end{bmatrix}\mathbf{j} = \underbrace{\mathbf{A}^{(i)}}_{4i\times 4i}\mathbf{j} \tag{37}$$

where $\mathbf{I}$ is of appropriate dimensions. The problem is now a simple eigenvalue-eigenvector problem and each solution j is a pair $\left(\underbrace{\gamma_j^{(i)}}_{1\times 1}, \underbrace{\mathbf{j}_j^{(i)}}_{4i\times 1}\right)$ (or rather $\left(\underbrace{\gamma_j^{(i)}}_{1\times 1}, \underbrace{\mathbf{g}_j^{(i)}}_{2i\times 1}\right)$, as the vector $\mathbf{j}_j^{(i)}$ contains the same information as $\mathbf{g}_j^{(i)}$ when we know $\gamma_j^{(i)}$, so we discard the lower half of $\mathbf{j}_j^{(i)}$). The number of solutions j to this eigenvector-eigenvalue problem is $4i$. Let

$$\mathbf{G}^{(i)} \equiv \left[\mathbf{g}_1^{(i)}, ..., \mathbf{g}_{2\times 2\times i}^{(i)}\right] \tag{38}$$

be the matrix of eigenvectors, and let

$$\boldsymbol{\gamma}^{(i)} \equiv \left[\gamma_1^{(i)}, ..., \gamma_{2\times 2\times i}^{(i)}\right]' \tag{39}$$

$$\boldsymbol{\Gamma}^{(i)} \equiv \operatorname{diag}\left[\boldsymbol{\gamma}^{(i)}\right] \tag{40}$$

be the corresponding vector and diagonal matrix, respectively, of eigenvalues.

The general solution on interval i is thus

$$\underbrace{\mathbf{D}^{(i)}}_{2i\times 1} = \underbrace{\mathbf{G}^{(i)}}_{2i\times 4i} \cdot \underbrace{\exp\left(\boldsymbol{\Gamma}^{(i)}y\right)}_{4i\times 4i} \cdot \underbrace{\mathbf{c}^{(i)}}_{4i\times 1} + \underbrace{\mathbf{k}_0^{(i)}}_{2i\times 1} + \underbrace{\mathbf{k}_1^{(i)}}_{2i\times 1}\exp(y) \tag{41}$$

where the constants $\mathbf{c}^{(i)} = \left[c_1^{(i)}, ..., c_{4i}^{(i)}\right]^\top$ will have to be determined via conditions at the boundaries of interval I_i (**NOTE:** $c_j^{(i)} \neq c$ where c is the coupon payment).

Boundary conditions. The different value functions $\mathbf{D}^{(i)}$ for $i \in \{1, ..., n\}$ are linked at the boundaries of their domains I_i. Note that $I_i \cap I_{i+1} = \{y_B(i+1)\}$ for $i < n$.

For $i = n$, we can immediately rule out all positive solutions to γ as debt has to be finite and bounded as $y \to \infty$, so that the entries of $\mathbf{C}^{(n)}$ corresponding to positive eigenvalues will be zero:[23]

$$\lim_{y\to\infty} \left|\mathbf{D}^{(n)}(y)\right| < \infty \tag{42}$$

For $i < n$, we must have value matching of the value functions that are alive across the boundary, and we must have value matching of the value functions that die across the boundary:

$$\mathbf{D}^{(i+1)}(y_B(i+1)) = \begin{bmatrix} \mathbf{D}^{(i)}(y_B(i+1)) \\ \begin{bmatrix} v_H^{i+1} \\ v_L^{i+1} \end{bmatrix} \exp(y_B(i+1)) \end{bmatrix} \tag{43}$$

For $i < n$, we must have mechanical (i.e. non-optimal) smooth pasting of the value functions that are alive across the boundary:

$$\left(\mathbf{D}^{(i+1)}\right)'(y_B(i+1))_{[1...2i]} = \left(\mathbf{D}^{(i)}\right)'(y_B(i+1)) \tag{44}$$

where $\mathbf{x}_{[1...2i]}$ selects the first $2i$ rows of vector $\mathbf{x}$.

Lastly, for $i = 1$, we must have

$$\mathbf{D}^{(1)}(y_B(1)) = \begin{bmatrix} v_H^1 \\ v_L^1 \end{bmatrix} \exp(y_B(1)) \tag{45}$$

Full solution. We can now state the full solution to the debt valuation given cut-off strategies:

Proposition 2 *The debt value functions* $\mathbf{D}$ *for a given default vector* $\boldsymbol{y}_B$ *are*

$$\mathbf{D}(y) = \begin{cases} \underbrace{\mathbf{D}^{(n)}(y)}_{2n\times 1} = \mathbf{G}^{(n)} \cdot \exp\left(\mathbf{\Gamma}^{(n)} y\right) \cdot \mathbf{c}^{(n)} + \mathbf{k}_0^{(n)} & y \in I_n \\ \vdots & \vdots \\ \underbrace{\mathbf{D}^{(i)}(y)}_{2i\times 1} = \mathbf{G}^{(i)} \cdot \exp\left(\mathbf{\Gamma}^{(i)} y\right) \cdot \mathbf{c}^{(i)} + \mathbf{k}_0^{(i)} + \mathbf{k}_1^{(i)} \exp(y) & y \in I_i \\ \vdots & \vdots \\ \underbrace{\mathbf{D}^{(1)}(y)}_{2\times 1} = \mathbf{G}^{(1)} \cdot \exp\left(\mathbf{\Gamma}^{(1)} y\right) \cdot \mathbf{c}^{(1)} + \mathbf{k}_0^{(1)} + \mathbf{k}_1^{(1)} \exp(y) & y \in I_1 \end{cases}$$

[23] According to JR06, there are exactly $2 \times |S| = 2n$ eigenvalues of $\mathbf{A}$ in the left open half plane (i.e. negative) and $2n$ eigenvalues in the right open half plane (i.e. positive) (actually, they only argue that this holds if $\boldsymbol{\mu} = \mathbf{R} - \frac{1}{2}\mathbf{\Sigma}$, but maybe not for general $\boldsymbol{\mu}$).

with the following boundary conditions to pin down vectors $\mathbf{c}^{(i)}$*:*

$$\lim_{y\to\infty}\left|\underbrace{\mathbf{D}^{(n)}(y)}_{2n\times1}\right| < \infty \tag{46}$$

$$\underbrace{\mathbf{D}^{(i+1)}(y_B(i+1))}_{2(i+1)\times1} = \underbrace{\begin{bmatrix}\mathbf{D}^{(i)}(y_B(i+1)) \\ \begin{bmatrix} v_H^{i+1} \\ v_L^{i+1}\end{bmatrix}\exp(y_B(i+1))\end{bmatrix}}_{2(i+1)\times1} \tag{47}$$

$$\underbrace{\left(\mathbf{D}^{(i+1)}\right)'(y_B(i+1))_{[1\ldots2i]}}_{2i\times1} = \underbrace{\left(\mathbf{D}^{(i)}\right)'(y_B(i+1))}_{2i\times1} \tag{48}$$

$$\underbrace{\mathbf{D}^{(1)}(y_B(1))}_{2\times1} = \underbrace{\begin{bmatrix} v_H^1 \\ v_L^1\end{bmatrix}\exp(y_B(1))}_{2\times1} \tag{49}$$

where $\mathbf{x}_{[1..2i]}$ *selects the first* $2i$ *rows of vector* $\mathbf{x}$*.*

.

Note that the derivative of the debt value vector is

$$\underbrace{\left(\mathbf{D}^{(i)}\right)'(y)}_{2i\times1} = \mathbf{G}^{(i)}\mathbf{\Gamma}^{(i)}\cdot\exp\left(\mathbf{\Gamma}^{(i)}y\right)\cdot\mathbf{c}^{(i)} + \mathbf{k}_1^{(i)}\exp(y) \tag{50}$$

where we note that $\mathbf{\Gamma}^{(i)}\cdot\exp\left(\mathbf{\Gamma}^{(i)}y\right) = \exp\left(\mathbf{\Gamma}^{(i)}y\right)\cdot\mathbf{\Gamma}^{(i)}$ as both are diagonal matrices (although this interchangeability only is important when $s=1$ as it then helps collapse some equations).

The first boundary condition (46) essentially implies that we can discard any positive entries of $\boldsymbol{\gamma}^{(n)}$ by setting the appropriate coefficients of $\mathbf{C}^{(n)}$ to 0. The second boundary condition (47) implies that we have value matching at any boundary $y_B(i+1)$ for $i<n$, be it to a continuation state or a bankruptcy state . The third boundary condition (48) implies that we also have smooth pasting at the boundary $y_B(i+1)$ for those states in which the firm stays alive on both sides of the boundary. Finally, the fourth boundary condition (49) implies value matching at the boundary $y_B(1)$, but of course only for those states in which the firm is still alive.

Thus, let us summarize the solution steps:

1. Order states so that the most restrictive/illiquid states are with the highest indices, such that $y_B(i) < y_B(j)$ implies $i<j$ (i.e. they appear in the lowest rows/columns in the following matrices).
2. Define the suitable matrices $\mathbf{R}, \mathbf{Q}$ for the transitions, and of course $\boldsymbol{\mu}, \mathbf{\Sigma}$ for drift and variance. These apply on the highest interval I_n.
3. Set up the eigenvalue-eigenvector problem and solve for (the matrix of) eigenvectors $\mathbf{G}^{(n)}$ and (the vector of) eigenvalues $\boldsymbol{\gamma}^{(n)}$. Solve for the constant $\mathbf{k}_0^{(n)}$ on this interval.
4. For intervals I_{n-i} we drop for each increment i the last pair of rows and columns of the appropriate matrices, with the following exception. We define $\mathbf{Q}^{(n-i)}$ as the matrix that arises out of $\mathbf{Q}$ when we drop the last i pair of rows and columns, i.e. rows 1-2 and columns 1-2 survive in the 4x4 case. We similarly define $\mathbf{R}^{(n-i)}, \boldsymbol{\mu}^{(n-i)}, \mathbf{\Sigma}^{(n-i)}$. We define $\tilde{\mathbf{Q}}^{(n-i)}$ as the matrix that arises out of $\mathbf{Q}$ when we drop the last i pair of rows and the first $n-i$ pairs of columns, i.e. rows 1-2 and columns 3-4 survive in the 4x4 case.
5. Set up the eigenvalue-eigenvector problem for interval I_{n-i} and solve for (the matrix of) eigenvectors $\mathbf{G}^{(n-1)}$ and (the vector of) eigenvalues $\boldsymbol{\gamma}^{(n-1)}$. Solve for the constant $\mathbf{k}_0^{(n-1)}$ on this interval and also for the particular part $\mathbf{k}_1^{(n-1)}\exp(y)$.

6. Build the system of boundary conditions via the matrix definitions of the debt to solve for the linear coefficients $\mathbf{c}^{(i)}$. To impose boundary condition (46), it is probably easiest to just use those entries of $\boldsymbol{\gamma}^{(n)}$ that are negative. Thus, the appropriate $\mathbf{C}^{(n)}$ for I_n is only a $2n \times 1$ vector, and not a $4n \times 1$ vector.

1.4 Equity

The equity holders are unaffected by the individual liquidity shocks the debt holders are exposed to. The only shocks the equity holders are directly exposed to are the shifts in $\mu(s)$ and $\sigma(s)$, i.e. shifts to the cash-flow process.

However, as debt has maturity and is rolled over, equity holders are indirectly affected by liquidity shocks in the market through the effect it has on debt prices. Thus, when debt matures, it is either rolled over if the debt holders are of type H, or it is reissued to different debt holders in the case that the former debt holder is of type L. Either way, there is a cash flow (inflow or outflow) of $m\left[\mathbf{S}^{(i)} \cdot \mathbf{D}^{(i)}(y) - p\mathbf{1}_i\right]$ at each instant as a mass $m \cdot dt$ of debt holders matures on $[t, t+dt]$.

For notational ease, we will denote by double letters (e.g. $\mathbf{xx}$) a constant for equity that takes a similar place as a single letter (i.e. $\mathbf{x}$) constant for debt. Then, the HJB for equity on interval I_i is given by

$$\begin{aligned}\left(\mathbf{RR}^{(i)} - \mathbf{QQ}^{(i)}\right)\mathbf{E}^{(i)}(y) &= \boldsymbol{\mu\mu}^{(i)}\left(\mathbf{E}^{(i)}\right)'(y) + \frac{1}{2}\boldsymbol{\Sigma\Sigma}^{(i)}\left(\mathbf{E}^{(i)}\right)''(y) \\ &\quad + \underbrace{\mathbf{1}_i\exp(y)}_{Cashflow} - \underbrace{(1-\pi)\,c\mathbf{1}_i}_{Coupon} + \underbrace{m\left[\mathbf{S}^{(i)} \cdot \mathbf{D}^{(i)}(y) - p\mathbf{1}_i\right]}_{Rollover}\end{aligned} \tag{51}$$

where

$$\begin{aligned}\mathbf{RR}^{(i)} &= \operatorname{diag}\left([r_H(1), ..., r_H(i)]\right) &(52)\\ \boldsymbol{\mu\mu}^{(i)} &= \operatorname{diag}\left([\mu(1), ..., \mu(i)]\right) &(53)\\ \boldsymbol{\Sigma\Sigma}^{(i)} &= \operatorname{diag}\left(\left[\sigma^2(1), ..., \sigma^2(i)\right]\right) &(54)\end{aligned}$$

are $i \times i$ square matrices, $\mathbf{QQ}^{(i)}$ is the transition matrix only between aggregate states that is also an $i \times i$ square matrix, and $\mathbf{S}^{(i)}$ is a $i \times 2i$ matrix that selects which debt values the firm is able to issue (each row has to sum to 1), and m is a scalar (**NOTE**: In contrast to $\mathbf{R}$, the matrix $\mathbf{RR}$ does not contain the maturity intensity m). For example, for $i = 2$, if the company is able to place debt only to H types, then $\mathbf{S}^{(2)} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}$. It is important that for reach row i only entries $2i-1$ and $2i$ are possibly nonzero, whereas all other entries are identically zero (otherwise, one would issue bonds belonging to a different state).

[TABLE 1.4 ABOUT HERE]

Writing out $\mathbf{D}^{(i)}(y) = \mathbf{G}^{(i)}\exp\left(\boldsymbol{\Gamma}^{(i)}y\right)\mathbf{c}^{(i)}$ and conjecturing a solution to the particular, non-constant part $\underbrace{\mathbf{KK}^{(i)}}_{i\times 4i}\underbrace{\exp\left(\boldsymbol{\Gamma}^{(i)}y\right)}_{4i\times 4i}\underbrace{\mathbf{c}^{(i)}}_{4i\times 1}$, we have

$$\begin{aligned}&\left(\mathbf{RR}^{(i)} - \mathbf{QQ}^{(i)}\right)\mathbf{KK}^{(i)}\exp\left(\boldsymbol{\Gamma}^{(i)}y\right)\mathbf{c}^{(i)} \\ = \;&\left[\boldsymbol{\mu\mu}^{(i)} \cdot \mathbf{KK}^{(i)} \cdot \boldsymbol{\Gamma}^{(i)} + \frac{1}{2}\boldsymbol{\Sigma\Sigma}^{(i)}\mathbf{KK}^{(i)} \cdot \left(\boldsymbol{\Gamma}^{(i)}\right)^2 + m \cdot \mathbf{S}^{(i)} \cdot \mathbf{G}^{(i)}\right]\exp\left(\boldsymbol{\Gamma}^{(i)}y\right)\mathbf{c}^{(i)}\end{aligned} \tag{55}$$

We can solve this by considering each $\gamma_j^{(i)}$ separately — recall that $\mathbf{c}^{(i)}$ is a vector and $\exp\left(\boldsymbol{\Gamma}^{(i)}y\right)$ is a *diagonal* matrix and in total there are $4i$ different roots. Consider the part of the particular part

$\mathbf{S}^{(i)} \cdot \mathbf{g}_j^{(i)} \exp\left(\gamma_j^{(i)} y\right) \cdot c_j^{(i)}$ and our conjecture gives $\underbrace{\mathbf{KK}_j^{(i)}}_{i\times 1} \underbrace{\exp\left(\gamma_j^{(i)} y\right)}_{1\times 1} \cdot \underbrace{c_j^{(i)}}_{1\times 1}$ for each root $j \in [1, ..., 4i]$. Plugging in and multiplying out the scalar $\exp\left(\gamma_j^{(i)} y\right) c_j^{(i)}$, we find that

$$\left(\mathbf{RR}^{(i)} - \mathbf{QQ}^{(i)}\right) \mathbf{KK}_j^{(i)} = \boldsymbol{\mu\mu}^{(i)} \cdot \mathbf{KK}_j^{(i)} \cdot \gamma_j^{(i)} + \frac{1}{2} \boldsymbol{\Sigma\Sigma}^{(i)} \mathbf{KK}_j^{(i)} \cdot \left(\gamma_j^{(i)}\right)^2 + m \cdot \mathbf{S}^{(i)} \cdot \mathbf{g}_j^{(i)} \tag{56}$$

Solving for $\mathbf{KK}_j^{(i)}$, we have

$$\underbrace{\mathbf{KK}_j^{(i)}}_{i\times 1} = \underbrace{\left[\mathbf{RR}^{(i)} - \mathbf{QQ}^{(i)} - \boldsymbol{\mu\mu}^{(i)} \cdot \gamma_j^{(i)} - \frac{1}{2} \boldsymbol{\Sigma\Sigma}^{(i)} \cdot \left(\gamma_j^{(i)}\right)^2\right]^{-1}}_{i\times i} m \cdot \underbrace{\mathbf{S}^{(i)}}_{i\times 2i} \underbrace{\mathbf{g}_j^{(i)}}_{2i\times 1} \tag{57}$$

Finally, for the homogenous part we use the same approach as above, but now we have less states as the individual liquidity state drops out. Thus, we conjecture $\mathbf{gg} \exp(\gamma)$ to get

$$\mathbf{0}_i = \left(\mathbf{QQ}^{(i)} - \mathbf{RR}^{(i)}\right) \mathbf{gg} + \boldsymbol{\mu\mu}^{(i)} \gamma\gamma \mathbf{gg} + \frac{1}{2} \boldsymbol{\Sigma\Sigma}^{(i)} \gamma\gamma \mathbf{gg} \tag{58}$$

so that, again, we have the following eigenvector eigenvalue problem

$$\gamma\gamma \mathbf{jj} = \begin{bmatrix} \mathbf{0}_i & \mathbf{I}_i \\ 2\left(\boldsymbol{\Sigma\Sigma}^{(i)}\right)^{-1} \left(\mathbf{RR}^{(i)} - \mathbf{QQ}^{(i)}\right) & -2\left(\boldsymbol{\Sigma\Sigma}^{(i)}\right)^{-1} \boldsymbol{\mu\mu}^{(i)} \end{bmatrix} \mathbf{jj} = \underbrace{\mathbf{AA}^{(i)}}_{2i\times 2i} \mathbf{jj} \tag{59}$$

which gives $\left(\gamma\gamma_j^{(i)}, \mathbf{gg}_j^{(i)}\right)$ for $j \in [1, ..., 2i]$ solutions. We stack these into a matrix of eigenvectors $\mathbf{GG}^{(i)}$ and a vector of eigenvalues $\boldsymbol{\gamma\gamma}^{(i)}$, from which we define the diagonal matrix of eigenvalues $\boldsymbol{\Gamma\Gamma}^{(i)} \equiv \operatorname{diag}\left(\boldsymbol{\gamma\gamma}^{(i)}\right)$. What remains is to solve for $\mathbf{kk}_0^{(i)}$ and $\mathbf{kk}_1^{(i)}$. We have

$$\mathbf{kk}_0^{(i)} = \left[\mathbf{RR}^{(i)} - \mathbf{QQ}^{(i)}\right]^{-1} \left[-(1-\pi) c \mathbf{1}_i + m\left(\mathbf{S}^{(i)} \mathbf{k}_0^{(i)} - p \mathbf{1}_i\right)\right] \tag{60}$$

and

$$\mathbf{kk}_1^{(i)} = \left[\mathbf{RR}^{(i)} - \mathbf{QQ}^{(i)} - \boldsymbol{\mu\mu}^{(i)} - \frac{1}{2} \boldsymbol{\Sigma\Sigma}^{(i)}\right]^{-1} \left(\mathbf{1}_i + m \cdot \mathbf{S}^{(i)} \mathbf{k}_1^{(i)}\right) \tag{61}$$

with $\mathbf{k}_1^{(n)} = \mathbf{0}$.

We are left with the following proposition.

Proposition 3 *The equity value functions* $\mathbf{E}$ *for a given default vector* $\boldsymbol{y}_B$ *are*

$$\mathbf{E}(y) = \begin{cases} \underbrace{\mathbf{E}^{(n)}(y)}_{n\times 1} = \mathbf{GG}^{(n)} \cdot \exp\left(\boldsymbol{\Gamma\Gamma}^{(n)} y\right) \cdot \mathbf{cc}^{(n)} + \mathbf{KK}^{(n)} \exp\left(\boldsymbol{\Gamma}^{(n)} y\right) \mathbf{c}^{(n)} + \mathbf{kk}_0^{(n)} + \mathbf{kk}_1^{(n)} \exp(y) & y \in I_n \\ \vdots & \vdots \\ \underbrace{\mathbf{E}^{(i)}(y)}_{i\times 1} = \mathbf{GG}^{(i)} \cdot \exp\left(\boldsymbol{\Gamma\Gamma}^{(i)} y\right) \cdot \mathbf{cc}^{(i)} + \mathbf{KK}^{(i)} \exp\left(\boldsymbol{\Gamma}^{(i)} y\right) \mathbf{c}^{(i)} + \mathbf{kk}_0^{(i)} + \mathbf{kk}_1^{(i)} \exp(y) & y \in I_i \\ \vdots & \vdots \\ \underbrace{\mathbf{E}^{(1)}(y)}_{1\times 1} = \mathbf{GG}^{(1)} \cdot \exp\left(\boldsymbol{\Gamma\Gamma}^{(1)} y\right) \cdot \mathbf{cc}^{(1)} + \mathbf{KK}^{(1)} \exp\left(\boldsymbol{\Gamma}^{(1)} y\right) \mathbf{c}^{(1)} + \mathbf{kk}_0^{(1)} + \mathbf{kk}_1^{(1)} \exp(y) & y \in I_1 \end{cases}$$

with the following boundary conditions to pin down the vector $\mathbf{cc}^{(i)}$*:*

$$\lim_{y\to\infty}\left|\underbrace{\mathbf{E}^{(n)}(y)\exp(-y)}_{n\times 1}\right| < \infty \tag{62}$$

$$\underbrace{\mathbf{E}^{(i+1)}(y_B(i+1))}_{(i+1)\times 1} = \underbrace{\begin{bmatrix}\mathbf{E}^{(i)}(y_B(i+1))\\ 0\end{bmatrix}}_{(i+1)\times 1} \tag{63}$$

$$\underbrace{\left(\mathbf{E}^{(i+1)}\right)'(y_B(i+1))_{[1\ldots i]}}_{i\times 1} = \underbrace{\left(\mathbf{E}^{(i)}\right)'(y_B(i+1))}_{i\times 1} \tag{64}$$

$$\underbrace{\mathbf{E}^{(i)}(y_B(1))}_{i\times 1} = 0 \tag{65}$$

where $\mathbf{x}_{[1\ldots i]}$ *selects the first* i *rows of vector* $\mathbf{x}$*.*

Note first the dimensionalities: $\underbrace{\mathbf{\Gamma\Gamma}^{(i)}}_{2i\times 2i}, \underbrace{\mathbf{GG}^{(i)}}_{i\times 2i}$ and $\underbrace{\mathbf{\Gamma}^{(i)}}_{4i\times 4i}, \underbrace{\boldsymbol{G}^{(i)}}_{2i\times 4i}$. Note second the derivative of the equity value vector is

$$\underbrace{\left(\mathbf{E}^{(i)}\right)'(y)}_{i\times 1} = \mathbf{GG}^{(i)}\mathbf{\Gamma\Gamma}^{(i)}\cdot\exp\left(\mathbf{\Gamma\Gamma}^{(i)}y\right)\cdot\mathbf{cc}^{(i)} + \mathbf{KK}^{(i)}\mathbf{\Gamma}^{(i)}\exp\left(\mathbf{\Gamma}^{(i)}y\right)\mathbf{c}^{(i)} + \mathbf{kk}_1^{(i)}\exp(y) \tag{66}$$

where we note that $\mathbf{\Gamma}^{(i)}\cdot\exp\left(\mathbf{\Gamma}^{(i)}y\right) = \exp\left(\mathbf{\Gamma}^{(i)}y\right)\cdot\mathbf{\Gamma}^{(i)}$ and $\mathbf{\Gamma\Gamma}^{(i)}\cdot\exp\left(\mathbf{\Gamma\Gamma}^{(i)}y\right) = \exp\left(\mathbf{\Gamma\Gamma}^{(i)}y\right)\cdot\mathbf{\Gamma\Gamma}^{(i)}$ as both are diagonal matrices (although this interchangeability only is important when $s = 1$ as it then helps collapse some equations).

The optimality conditions for bankruptcy boundaries $\{y_B(i)\}_i$ are given by

$$\left(\mathbf{E}^{(i)}\right)'(y_B(i))_{[i]} = 0 \tag{67}$$

i.e., a smooth pasting condition at the boundaries at which default is declared.

Table 1: **Baseline Parameters used in calibration.** Unreported parameters are tax rate $\pi = 0.35$, bond holders' bargaining power $\beta = 5\%$, and risk free rate $r = 0.02$.

Symbol	Interpretation	State G	State B
$\zeta^{\mathbb{P}}$	Transition Density	0.100	0.500
$\exp(\kappa)$	Jump Risk Premium	2.000	0.500
$\mu_{\mathbb{P}}$	Cash Flow Growth	0.045	0.015
η	Risk Price	0.175	0.245
σ_m	Systematic Vol	0.100	0.110
σ_i	Idiosyncratic Vol	0.225	0.225
m	Average Maturity Intensity	0.200	0.200
χ	Holding Cost	1.500	1.800
λ	Meeting Intensity	50.000	20.000
ξ	Liquidity Shock Intensity	0.700	0.700

Table 2: **Type-dependent Recovery Rates and Ultimate Recovery Rates.** The recovery values are calculated using data on risk-adjusted holding period returns of post-default corporate bonds in Moody's Default and Recovery Databases from 1984-2010. Details are in Appendix.

Symbol	Interpretation	State G	State B
α_H	Recovery Rate of H Type	58.71%	32.23%
α_L	Recovery Rate of L Type	57.49%	30.51%
$\hat{\alpha}$	Ultimate Recovery Rate	87.96%	64.68%

Table 3: **Default probabilities and credit spreads across credit ratings.** Default probabilities are cumulative default probabilities over 1920-2011 from Moody's investors service (2012), and credit spreads are from FISD transaction data over 1994-2010. We report the time series mean, with the standard deviation (reported underneath) being calculated using Newey-West procedure with 15 lags. The standard deviation of default probabilities are calculated based on the sample post 1970's due to data availability issue. On model part, we first calculate the quasi market leverage for Compustat firms (excluding financial and utility firms) for each rating over 1994-2010, then match observed quasi market leverage by locating the corresponding cashflow level δ. We then calculate the time series average of model-implied credit spreads and Bond-CDS spreads across these firm-quarter observations. This procedure implies that our model-implied leverages exact match the empirical counterpart.

Rating	Default Prob. %		Credit Spread, bps	
	Data (std.)	Model	Data (std.)	Model
	10 year			
Aaa/Aa	2.06 (0.26)	1.50	68.02 (7.20)	69.84
A	3.44 (0.29)	3.99	99.48 (11.60)	115.73
Baa	7.03 (0.41)	8.35	166.40 (21.71)	192.60
Ba	19.01 (2.99)	16.80	325.05 (35.34)	347.10
	5 year			
Aaa/Aa	0.66 (0.09)	0.27	65.42 (7.20)	44.48
A	1.31 (0.09)	1.04	99.53 (12.99)	72.12
Baa	3.09 (0.24)	2.96	166.88 (24.30)	133.25
Ba	9.81 (1.91)	8.64	348.89 (48.07)	301.83

Table 4: **Bond-CDS spreads across credit ratings.** The sample to construct Bond-CDS spreads are firms with both 5-year and 10-year bonds, over the sample period from 2005 to 2012. We report the time series mean for both including and excluding 08/09 crisis, with the standard deviation (reported underneath) being calculated using Newey-West procedure with 15 lags. Crisis period is defined as from 10/2008 to 03/2009. On model part, we calculate the quasi market leverage for Compustat firms (excluding financial and utility firms) for each rating classes. We match the observed quasi market leverage by locating the corresponding cash flow level δ, and calculate the time series average of model-implied credit spreads and Bond-CDS spreads across these firm-quarter observations. This procedure implies that our model-implied leverages exactly match the empirical counterpart.

Rating	Data (std.), bps		Model, bps
	Including crisis	Excluding crisis	
	10 year		
Aaa/Aa	34.98 (12.08)	31.12 (11.23)	43.73
A	53.29 (14.37)	45.61 (10.60)	50.42
Baa	85.41 (24.29)	71.59 (16.72)	65.71
Ba	107.72 (41.08)	83.23 (27.61)	93.03
	5 year		
Aaa/Aa	36.68 (12.45)	30.81 (10.29)	35.34
A	61.82 (16.98)	52.20 (13.98)	41.42
Baa	102.49 (26.01)	86.39 (16.73)	56.16
Ba	136.14 (43.00)	109.52 (28.22)	95.35

Table 5: **Bid-Ask Spreads across credit ratings.** The normal time bid-ask spread are taken from Edward et. al. (2007) for a median size trade. The recession time numbers are normal time numbers multiplied by the ratio of bid-ask spread implied by Roll's measure of illiquidity (following Bao, Pan, and Wang (2010)) in recession time to normal time. The model counterpart is computed for a bond with time to maturity of 8.3 years, which is the mean time-to-maturity of frequently traded bonds (where we can compute a Roll's measure) in the TRACE sample.

Rating Classes	State G		State B	
	Data	Model	Data	Model
Superior Grade (bps)	40.00	40.83	71.86	84.36
Investment Grade (bps)	50.00	53.82	108.33	119.39
Junk Grade (bps)	70.00	78.61	144.20	171.45

Table 6: **Structural Liquidity-Default Decomposition for Ten Year Bonds Across Ratings.** For each rating, we locate the cash flow y that corresponds to the historical total credit spread of a ten year bond at this rating. We perform the structural liquidity-default decomposition following the procedure discussed in the text across aggregate states. We quantitatively evaluate the channels that give rise to the observed level of credit spreads and their changes when the economy shifts from normal time to recession. As a comparison to previous literature (e.g. Longstaff. et. al (2005)) , we also report the CDS spread implied by the model across ratings and aggregate states.

Rating	State	Credit Spread	Default-Liquidity Decomposition *Pure Def*	*Pure Liq*	*Liq* → *Def*	*Def* → *Liq*	CDS
		Panel I: Explaining Credit Spread Levels					
Aaa/Aa	G (bps)	65.97	17.70	37.89	5.88	4.50	28.34
	(%)	100.00%	26.83%	57.43%	8.92%	6.82%	
	B (bps)	82.71	24.79	44.73	6.69	6.50	38.03
	(%)	100.00%	29.97%	54.09%	8.09%	7.86%	
A	G (bps)	94.27	37.44	40.54	10.42	5.88	53.96
	(%)	100.00%	39.71%	43.01%	11.05%	6.24%	
	B (bps)	119.72	47.58	47.88	14.36	9.91	71.81
	(%)	100.00%	39.74%	39.99%	11.99%	8.27%	
Baa	G (bps)	163.91	83.57	46.58	18.90	14.87	116.52
	(%)	100.00%	50.98%	28.42%	11.53%	9.07%	
	B (bps)	205.97	103.79	55.03	24.64	22.50	149.08
	(%)	100.00%	50.39%	26.72%	11.96%	10.93%	
Ba	G (bps)	310.89	174.90	59.97	38.07	37.94	240.85
	(%)	100.00%	56.26%	19.29%	12.25%	12.20%	
	B (bps)	386.38	218.33	70.93	46.88	50.24	305.10
	(%)	100.00%	56.51%	18.36%	12.13%	13.00%	
		Panel II: Explaining Credit Spread Changes					
Aaa/Aa	$G \to B$ (bps)	16.74	7.09	6.84	0.81	2.00	9.69
	(%)	100.00%	42.35%	40.89%	4.81%	11.94%	
A	$G \to B$ (bps)	25.45	10.14	7.33	3.95	4.03	17.85
	(%)	100.00%	39.86%	28.82%	15.50%	15.82%	
Baa	$G \to B$ (bps)	42.06	20.22	8.45	5.75	7.64	32.56
	(%)	100.00%	48.08%	20.10%	13.66%	18.15%	
Ba	$G \to B$ (bps)	75.49	43.43	10.96	8.81	12.30	64.25
	(%)	100.00%	57.52%	14.52%	11.66%	16.29%	

Table 7: **Effect of Liquidity Provision Policy on 10-Year Bonds Across Ratings.** We consider a policy experiment that improves the liquidity condition (χ, λ) in the B state to be as good as G state. We compute the credit spread under the policy for both G and B state, and perform the structural liquidity-default decomposition to examine the channels that are responsible for the reduced borrowing cost.

Rating	State	Credit Spread (bps)		Contribution of Each Component		
		w/o. policy	w. policy	*pure LIQ* (%)	*LIQ→DEF* (%)	*DEF→LIQ* (%)
Aaa/Aa	G	65.97	52.33	79.23	10.80	9.97
	B	82.71	60.95	81.12	5.88	13.00
A	G	94.27	77.88	70.61	18.49	10.90
	B	119.72	93.55	72.24	14.78	12.98
Baa	G	163.91	138.93	53.32	16.16	18.52
	B	205.97	169.10	59.06	13.07	27.88
Ba	G	310.89	273.50	46.07	27.54	26.39
	B	386.38	329.04	49.16	18.00	32.84

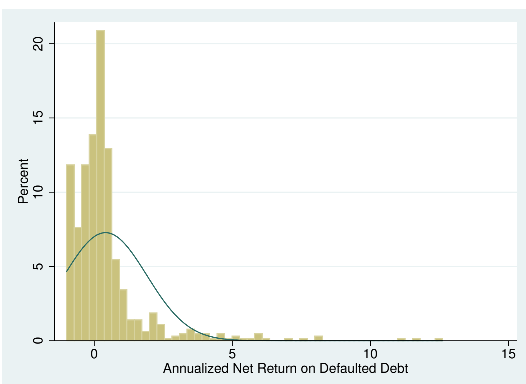
Table 8: **Summary Statistics for Annualized Net PME on Defaulted Bond by Default Time** Data on holding period return of post-default bonds are from Moody's Default and Recovery Database 1984-2010. We adjust for risk by discounting the return of holding defaulted bonds by a public market benchmark over the same investment horizon. The resulting measure is called "Public Market Equivalent" as reported below.

Default Time	# of Def. Bond	Mean Annual Net PME	Mean Annual Net Return
Non-Recession	512	0.3126	0.3922
Recession	130	0.5537	0.4672
Full Sample	642	0.3613	0.4074

Table 9: Matrix & Vector Dimensions.

Debt Parameters			Equity Parameters		
Symbol	Interpretation	Dimension	Symbol	Interpretation	Dimension
$\mathbf{D}^{(i)}(y)$	Debt Value Function	$2i \times 1$	$\mathbf{E}^{(i)}(y)$	Equity Value Function	$i \times 1$
$\boldsymbol{\mu}^{(i)}$	(Log)-Drifts	$2i \times 2i$	$\boldsymbol{\mu\mu}^{(i)}$	(Log-)Drifts	$i \times i$
$\boldsymbol{\Sigma}^{(i)}$	Volatilities	$2i \times 2i$	$\boldsymbol{\Sigma\Sigma}^{(i)}$	Volatilities	$i \times i$
$\mathbf{R}^{(i)}$	Discount rates and maturity	$2i \times 2i$	$\mathbf{RR}^{(i)}$	Discount rates	$i \times i$
$\boldsymbol{\chi}^{(i)}$	Holding costs	$2i \times 1$	c	Coupon	1×1
$\mathbf{Q}^{(i)}$	Transition to cont. states	$2i \times 2i$	$\mathbf{QQ}^{(i)}$	Transition to cont. states	$i \times i$
$\tilde{\mathbf{Q}}^{(i)}$	Transition to default states	$2i \times 2(n-i)$	$\mathbf{AA}^{(i)}$	Matrix to be decomposed	$2i \times 2i$
$\mathbf{v}^{(i)}$	Vector of recovery values	$2(n-i) \times 1$	$\boldsymbol{\Gamma\Gamma}^{(i)}$	Diag matrix of eigenvalues	$2i \times 2i$
$\mathbf{A}^{(i)}$	Matrix to be decomposed	$4i \times 4i$	$\mathbf{GG}^{(i)}$	Matrix of eigenvectors	$i \times 2i$
$\boldsymbol{\Gamma}^{(i)}$	Diag matrix of eigenvalues	$4i \times 4i$	$\mathbf{kk}_0^{(i)}, \mathbf{kk}_1^{(i)}$	Coeff. of particular sol.	$i \times 1$
$\mathbf{G}^{(i)}$	Matrix of eigenvectors	$2i \times 4i$	$\mathbf{S}^{(i)}$	Issuance matrix	$i \times 2i$
$\mathbf{k}_0^{(i)}, \mathbf{k}_1^{(i)}$	Coeff. of particular sol.	$2i \times 1$	$\mathbf{KK}^{(i)}$	Coeff. of particular sol.	$i \times 4i$
$\mathbf{c}^{(i)}$	Vector of constants	$4i \times 1$	$\mathbf{cc}^{(i)}$	Vector of constants	$2i \times 1$

Figure 2: Distribution of Annualized Net Return (left) and Public Market-Adjusted Return (right) of Defaulted Bonds



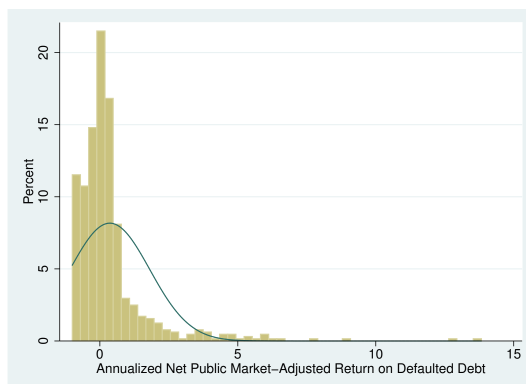


Figure 3: **Empirical Distribution of Market Leverage for Compustat Firms by Aggregate State and Rating classes.** We compute quasi-market leverage for each firm-quarter observation in the Compustat database from 1997-2012. The B state is defined as quarters for which at least two months are classified as NBER recession month. The remaining quarters are G state. We drop financial and utility firms in our sample. We also exclude firms with zero leverage.

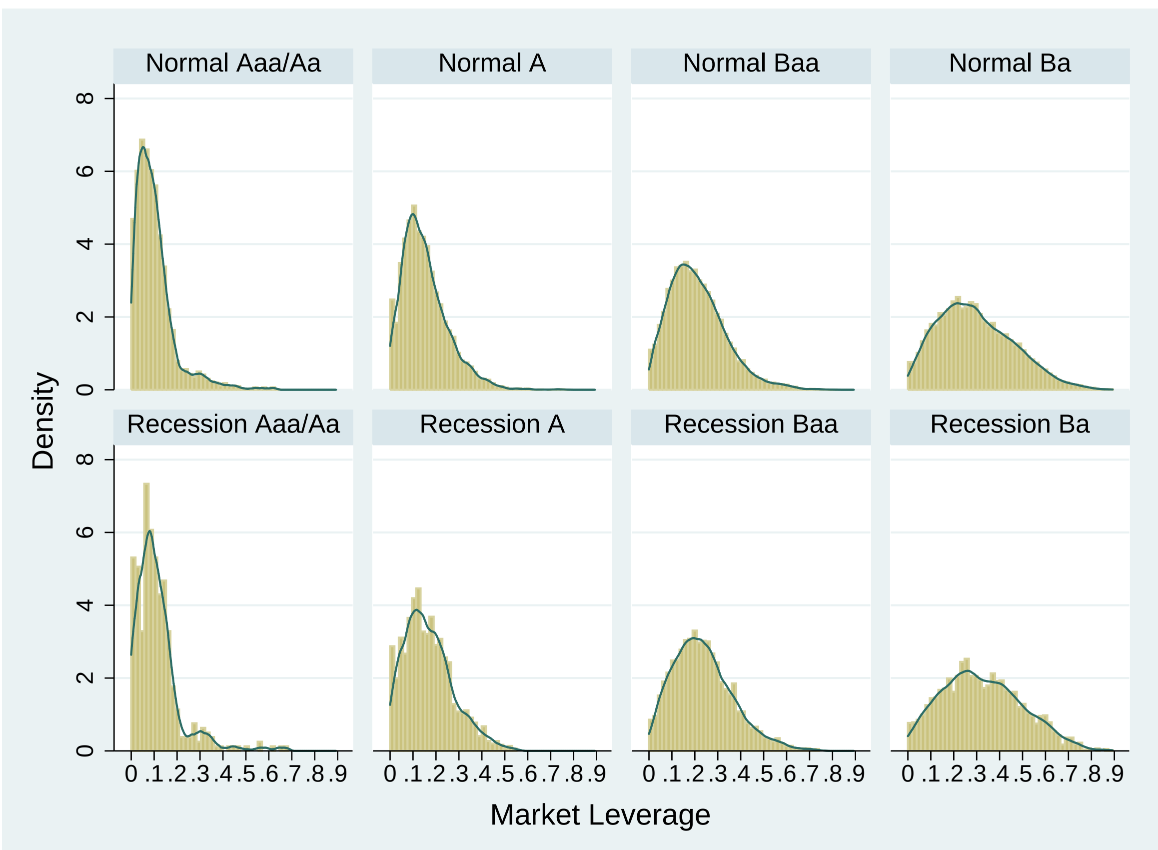


Figure 4: Model Implied Nonlinearity between Market Leverage, Default Rates and Total Credit Spread