Chapter 2
Multiple Demands on the Stockpile

2.1 Modification of Assumption (6)

The model, as previously defined, contains the implicit hypothesis that there is only one demand source withdrawing items from the stockpile. Except for Zehna [11], Chapter 4, all of the previous work done on the deterministic inventory depletion model necessarily requires this single demand source assumption. Zehna, however, proved that when \( L(S) = aS + b \ (b > 0 > a > -1) \) then FIFO is optimal for one or more demand sources. In addition, he showed that if \( L(S) \) is either a convex or a concave differentiable function with \( L'(S) < -1 \), then LIFO is optimal for one or more demand sources.

One of the objectives of the present work, is to remove the assumption of a single demand source. We will denote the number of demand sources requesting items from the stockpile by the letter "\( v \)". \( v \) is an integer and is bounded by \( 1 \leq v \leq n \) where \( n \) is the number of items initially in the stockpile. We do not consider \( v > n \) since the policy of issuing the \( n \) items to the \( n \) demand sources cannot be improved upon in terms of maximizing the total field life of the stockpile. The demand sources will be denoted by \( M_1, M_2, \ldots, M_v \).
Since assumption (6) contains the implicit assumption of a single
demand source we will modify assumption (6) as follows:

(6)' An item is issued from the stockpile whenever any demand source
has consumed the entire useful field life of the item previously
issued to it. If two or more demand sources request a
new item at the same time, the new items will be issued to
them in the same order as they received their last previously
issued items.

A policy is said to be feasible if a demand on the stockpile is
always satisfied, provided (i) the stockpile is not empty and (ii) the
remaining items in the stockpile have positive field life. In seeking
the optimal policy we will only be concerned with the optimal policy
which belongs to the class of feasible policies.

Before proceeding further, it will be useful to define the notation
which is used to describe a policy. An issuing policy for \( v \) demand
sources:

(1) List the items assigned to a particular demand source in their
order of use from the first item used until the last item used,
and

(2) separate the items for different demand sources by a semicolon.
For example, a policy \( A \) can be described as follows:

\[
A = [S_{11}, S_{12}, \ldots , S_{11}; S_{21}, \ldots , S_{21}; \ldots ; S_{v1}, \ldots , S_{v1}]
\]
Thus $A$ is the issuing policy which assigns

items $S_{11}, S_{12}, \ldots, S_{1i_1}$ to demand source $M_1$ in that order,
items $S_{21}, S_{22}, \ldots, S_{2i_2}$ to demand source $M_2$ in that order,
items $S_{v1}, S_{v2}, \ldots, S_{vi_v}$ to demand source $M_v$ in that order.

Note that $\sum_{j=1}^{i} i_j = n$ if all items are assigned. It is obvious that
the choice of $M_1, M_2, \ldots, M_v$ for the particular assignment of items
above was arbitrary. Hence the $v!$ policies obtained by permuting the
$M_i$'s are equivalent policies in the sense that the total field life
obtained from the $n$ items is unchanged regardless of how the demand
sources are labelled.

It is assumed that the process begins by issuing $v$ items, one to
each $M_1, M_2, \ldots, M_v$.

2.2 General Relationships

Among the items which have a deteriorating field life function
there are several interesting relationships which will be useful at
various times throughout the subsequent chapters. For this reason these
relationships have been gathered together and stated as lemmas in this
section.

Lemma 2.1: Let $L(S)$ be a continuous nonincreasing function with
$L^{-}(S) \geq -1$ for $0 < S \leq S_0$. Let $v = 1$. If the items in the stock-
pile are issued according to FIFO, the field life of any item at the
time of issue is strictly positive.
Proof of Lemma 2.1: If $S_o = +\infty$, the lemma is trivially true. Hence assume $S_o < +\infty$. By FIFO $S_n$ is the first item issued and by assumption (7) of the model $L(S_n) > 0$. Now assume the lemma is true for the first $k$ items issued and it will be proved true for the first $k + 1$ items issued.

Since items are withdrawn from the stockpile in decreasing order of their index numbers (under FIFO), let the $k^{th}$ item issued be denoted by $S_j$ and the $k + 1^{st}$ item issued by $S_{j+1}$. Let $x$ denote the total field life of the first $k - 1$ items (under FIFO). Then the inductive hypothesis is:

$$L(S_j + x) > 0 \quad (2.2.1)$$

which implies that $S_j + x < S_o$. Now it must be proved that

$$L(S_{j-1} + x + L(S_j + x)) > 0 \quad (2.2.2)$$

or in other words that

$$S_{j-1} + x + L(S_j + x) < S_o.$$ 

Now by hypothesis $L(S) \geq -1$ for all $S$ with $0 < S \leq S_o$ and since $L(\cdot)$ is continuous and since $S_j + x < S_o$ by (2.2.1) we can form

$$\frac{L(S_j + x) - L(S_o)}{S_j + x - S_o} \geq -1 \quad (2.2.3)$$
hence
\[ L(S_j + x) - L(S_o) \leq S_o - S_j - x \]

and since \( L(S_o) = 0 \) we obtain
\[ L(S_j + x) + x + S_j \leq S_o . \tag{2.2.4} \]

But \( S_{j-1} + x < S_j + x \), hence in (2.2.4)
\[ S_{j-1} + x + L(S_j + x) < S_j + x + L(S_j + x) \leq S_o \]

which proves (2.2.2). Therefore by induction the lemma is proved.
\[ \text{q.e.d.} \]

The next lemma is concerned with the effect on total field life when \( M \) items of arbitrary ages are combined with the inventory of \( n = N \) items and the process then starts. We assume a FIFO issuing policy is used.

**Lemma 2.2:** Let \( L(S) \) be a continuous nonincreasing function with \( L(S) \geq -1 \) for all \( S \) with \( 0 < S \leq S_o \). Let \( \nu = 1 \). Denote by \( Q_{TN}^N \), the total field life obtained by issuing the \( n = N \) items according to FIFO. Let \( M \geq 1 \) additional items of initial ages \( S_1^* < S_2^* < \cdots < S_M^* < S_o \) be combined with the original \( N \) items. Let the \( N + M \) items be issued by FIFO and denote the total field life by \( Q_{TN+M}^N \). Let \( S_i^* \neq S_j^* \) for all \( i, j \).

Then \( Q_{TN+M}^N \geq Q_T^N \) for any finite \( N, M \).
Proof of Lemma 2.2: The proof will be by induction. Let $M = 1$ and $N \geq 1$. Three cases are possible.

**Case 1** \[ S_1^* < S_1 \]

\[
Q_N^F \leq Q_N^F + L(S_1^* + S_N^*) = Q_{N+1}^F
\]

**Case 2** \[ S_N < S_1^* \]

For this case we will use induction on $N$ to show $Q_{N+1}^F \geq Q_N^F$. Let $N = 1$. We have since $L(S)$ is nonincreasing $L(S_1^*) \geq L(S_1)$ and since $L(\cdot)$ is continuous and $L^{-1}(S) \geq -1$ for $0 < S < S_0$, then

\[
\frac{L(S_1) - L(S_1^* + L(S_1^*))}{-L(S_1^*)} \geq -1
\]

\[
L(S_1) - L(S_1^* + L(S_1^*)) \leq L(S_1^*)
\]

and \[
Q_{F_1} = L(S_1) \leq L(S_1^*) + L(S_1 + L(S_1^*)) = Q_{F_2}.
\]

Now assume true for $N = j$ and prove true for $N = j + 1$. Let $x$ be the total field life from issuing items $[S_1^*, S_{j+1}^*, S_j, \ldots, S_2]$ by FIFO and let $y$ be the total field life from issuing $[S_{j+1}, S_j, \ldots, S_2]$ by FIFO. Then the inductive assumption states

\[ x \geq y. \] \hspace{1cm} (2.2.5)
We must show

\[ Q_{T_{N+1}} = x + L(S_1 + x) \geq y + L(S_1 + y) = Q_{T_N} \quad (2.2.6) \]

(i) if \( x = y \), then (2.2.6) holds with equality

(ii) if \( x > y \), then by lemma 2.1, \( S_1 + x < S_0 \) and \( S_1 + y < S_0 \) and since \( L(\cdot) \) is continuous and \( L^{-}(S) \geq -1 \) for \( S \leq S_0 \) we have

\[
\frac{L(S_1 + x) - L(S_1 + y)}{x - y} \geq -1
\]

\[ L(S_1 + x) - L(S_1 + y) \geq -x + y \]

and

\[ L(S_1 + x) + x \geq L(S_1 + y) + y \]

which proves (2.2.6); hence by induction the lemma is true for this case.

**Case 3** \( S_1 < \cdots < S_i < S^*_1 < S_{i+1} < \cdots < S_N \)

Let \( x \) denote the total field life of the \( N^{th}, N-1^{st}, \ldots, 1^{st} \) items issued by FIFO (i.e., of items \( S_N, S_{N-1}, \ldots, S_{i+1} \)). Then \( S^*_1 + x > S_1 + x > S_1 \) and since \( L(\cdot) \) is nonincreasing

\[ L(S_1^* + x) \leq L(S_1 + x) \quad (2.2.7) \]

Now let

\[ S_1^* + x = T_1^* \]
\[ S_j + x = T_j \quad \text{for all} \quad j = 1, \ldots, i. \]

Then (2.2.7) becomes \( L(T^*_1) \leq L(T_1) \) and \( 0 < S_1 + x < S_2 + x < \cdots < S_i + x < S^*_i + x \) is rewritten as

\[ 0 < T_1 < T_2 < \cdots < T_i < T^*_i \quad (2.2.8) \]

but (2.2.8) shows that we now have case 2 above with \( N = 1 \) hence

\[ x + Q_{F_{i+1}} \geq x + Q_{F_1} \quad \text{by case 2. But} \]

\[ Q_{F_{N+1}} = x + Q_{F_{i+1}} \geq x + Q_{F_1} = Q_{F_N}. \]

Therefore \( Q_{F_{N+1}} \geq Q_{F_N} \) in all three cases and since the three cases exhaust all possibilities the lemma is proved for \( M = 1 \) and \( N \geq 1 \).

Let \( M > 1 \) and \( N \geq 1 \).

Assume the lemma is true for \( M > 1 \) and consider adding \( M + 1 \) items of initial ages \( (S^*_i)_{i=1}^{M+1}, (S^*_1 < S^*_i) \). Ignoring \( S^*_{M+1} \) temporarily, the total field life of the remaining items \( Q_{F_{N+M}} \) satisfies

\[ Q_{F_{N+M}} \geq Q_{F_N} \quad \text{by the inductive assumption. Then adding} \ S^*_{M+1} \text{ can only increase the total field life by the case} \ M = 1 \ i.e.,} \ Q_{F_{N+M+1}} \geq Q_{F_N} \]

and by induction the lemma is proved.

q.e.d.

It is very important to know the ordering of a FIFO assignment to \( v > 1 \) demand sources. Lemma 2.3 below gives such an assignment under a deteriorating field life function with slope \( \geq -1 \).
Lemma 2.3: Let $L(S)$ be a continuous nonincreasing function with $L'(S) \geq -1$ for all $S$ such that $0 < S \leq S_0$. Let $\nu \geq 1$. Then starting from the oldest item $S_n$, FIFO assigns every $\nu^{th}$ item to the same demand source, i.e., without loss of generality we can arbitrarily let $M_1$ receive $S_n$, $M_2$ receive $S_{n-1}$ etc. to start, then

- demand source $M_1$ receives items indexed by $n - k\nu$
- demand source $M_2$ receives items indexed by $n - k\nu - 1$
- $\vdots$
- $\vdots$
- demand source $M_j$ receives items indexed by $n - k\nu - j + 1$
- $\vdots$
- $\vdots$
- demand source $M_\nu$ receives items indexed by $n - k\nu - \nu + 1$

for $k = 0, 1, 2, \ldots$ until all items have been assigned. Conversely if the assignment of items is as given above, then the assignment is FIFO.

Proof of Lemma 2.3: The proof proceeds by the use of induction in several parts.

Let $k = 1$.

Now we have arbitrarily assigned
\[ S_n \rightarrow M_1 \]
\[ S_{n-1} \rightarrow M_2 \]
\[ \ldots \]
\[ S_{n-v+1} \rightarrow M_v \]

and since \( L(\cdot) \) is nonincreasing

\[ L(S_n) \leq L(S_{n-1}) \leq \cdots \leq L(S_{n-v+1}) \] (2.2.9)

hence the next assignment is \( S_{n-v} \rightarrow M_1 \) since \( L(S_n) \) is the smallest field life. Now assume the lemma is true for assigning \( S_{n-v-i+1} \rightarrow M_i \) for \( i \in \{1, 2, \ldots, v-1\} \) we must prove

\[ S_{n-v-i} \text{ is assigned to } M_{i+1}. \] (2.2.10)

To prove (2.2.10) it is useful to show for all \( j \) such that \( 2 \leq j \leq i \) that

\[ L(S_{n-v-j+2} + L(S_{n-j+2})) + L(S_{n-j+2}) \leq L(S_{n-v-j+1} + L(S_{n-j+1})) + L(S_{n-j+1}). \] (2.2.11)

Inequality (2.2.11) states that the field life obtained from the two items assigned to \( M_{j-1} \) is less than the field life obtained from the two items assigned to \( M_j \). For simplicity let \( x = L(S_{n-j+2}) \) and \( y = L(S_{n-j+1}) \) and since \( L(\cdot) \) is nonincreasing \( x \leq y \).

(i) If \( x = y \) then (2.2.11) is obviously true.

(11) If \( x < y \), then since \( L(\cdot) \) is continuous, since \( L^{-}(S) \geq -1 \) for \( S \leq S_0 \) and since by lemma 2.1 we have \( S_{n-v-j+2} + y < S_0 \) then
\[
\frac{L(S_{n-v-j+2} + x) - L(S_{n-v-j+2} + y)}{x - y} \geq -1
\]

hence

\[
L(S_{n-v-j+2} + x) + x \leq L(S_{n-v-j+2} + y) + y
\]

\[
\leq L(S_{n-v-j+1} + y) + y
\]

since

\[
L(S_{n-v-j+2} + y) \leq L(S_{n-v-j+1} + y)
\]

and (2.2.11) is satisfied, for all \( j = 2, \ldots, i \).

Now we can telescope (2.2.11) into the inequality

\[
L(S_{n-v} + L(S_n)) + L(S_n) \leq L(S_{n-v-j+1} + L(S_{n-j+1})) + L(S_{n-j+1})
\]

for all \( j = 1, \ldots, i \)

and we will show

\[
L(S_{n-1}) \leq L(S_{n-v} + L(S_n)) + L(S_n).
\]  (2.2.12)

Then, if (2.2.12) holds we have by (2.2.12) and (2.2.9) that demand source \( M_{i+1} \) (who received \( S_{n-i} \)) is the next source to demand an item from the stockpile which by FIFO and the inductive assumption is item \( S_{n-v-i} \). We now prove (2.2.12). By lemma 2.1, \( S_{n-i} + L(S_n) < S_o \) and since \( L(\cdot) \) is continuous and \( L^{-1}(S) \geq -1 \) for \( S \leq S_o \) we have
\[
\frac{L(S_{n-i} + L(S_n)) - L(S_{n-i})}{L(S_n)} \geq -1
\]

\[
L(S_{n-i}) \leq L(S_n) + L(S_{n-i} + L(S_n)) \leq L(S_n) + L(S_{n-v} + L(S_n))
\]

since \(L(S_{n-i} + L(S_n)) \leq L(S_{n-v} + L(S_n))\).

Hence (2.2.12) holds and \(S_{n-v-i}\) is assigned to \(M_{i+1}\). Therefore by induction the lemma is true for \(k = 1\).

Assume the lemma is true for \(k = t\) and it will be proved true for \(k = t + 1\). For all \(j = 1, \ldots, v\) let \(x_j\) be the total field life of all the items assigned to \(M_j\) up through cycle \(t - 1\), i.e., of items \(S_{n-j+1}, S_{n-v-j+1}, \ldots, S_{n-(t-1)v-j+1}\). Then the inductive assumption states

\[
x_1 \leq x_2 \leq \cdots \leq x_j \leq \cdots \leq x_v. \tag{2.2.13}
\]

In order to assert that \(M_1\) receives item \(S_{n-(t+1)v}\), it is necessary to prove

\[
L(S_{n-tv} + x_1) + x_1 \leq L(S_{n-tv-j} + x_{j+1}) + x_{j+1} \tag{2.2.14}
\]

for all \(j = 1, \ldots, v - 1\).

If \(x_{j+1} = x_1\), then (2.2.14) obviously holds since \(L(\cdot)\) is nonincreasing.

If \(x_{j+1} > x_1\), then since \(L(\cdot)\) is continuous and nonincreasing and by lemma 2.1, \(S_{n-tv} + x_{j+1} < S_o\), thus
\[
\frac{L(S_{n-tv} + x_{j+1}) - L(S_{n-tv} + x_1)}{x_{j+1} - x_1} \geq -1
\]

implies

\[
L(S_{n-tv} + x_1) + x_1 \leq L(S_{n-tv} + x_{j+1}) + x_{j+1} 
\]

\[
\leq L(S_{n-tv} - j + x_{j+1}) + x_{j+1}.
\]

And (2.2.14) holds for all \( j = 1, \ldots, v - 1 \) since \( j \) was arbitrary.

Therefore item \( S_{n-(t+1)v} \) is issued to \( M_1 \) and the lemma has been proved for the first assignment in the \( t + 1^{\text{st}} \) cycle.

Now assume the lemma is true for the \( j^{\text{th}} \) assignment in the \( t + 1^{\text{st}} \) cycle and it will be proved true for the \( j + 1^{\text{st}} \) assignment in the \( t + 1^{\text{st}} \) cycle \( (j + 1 \leq v) \).

Let

\( y_1 \) be the total field life of items issued to \( M_1 \) up through cycle \( t \)

\( y_2 \) be the total field life of items issued to \( M_2 \) up through cycle \( t \)

\[ \vdots \]

\( y_j \) be the total field life of items issued to \( M_j \) up through cycle \( t \).

Then by the inductive assumption on \( t \) and on \( j \)

\[
x_{j+1} \leq x_{j+2} \leq \cdots \leq x_v \leq y_1 \leq \cdots \leq y_j
\]

(2.2.15)

where \( y_i = x_i + L(S_{n-tv-i+1} + x_1) \) for \( i = 1, \ldots, j \).
It must be shown that
\[ x_{j+1} + L(S_{n\cdot tv\cdot j} + x_{j+1}) \leq x_k + L(S_{n\cdot tv\cdot k+1} + x_k) \]
\[ \leq y_1 + L(S_{n\cdot (t+1)\cdot v-i+1} + y_1) \]  
(2.2.16)

for all \( k = j + 2, \ldots, v \) and \( v = 1, \ldots, j \).

But (2.2.16) follows immediately by the same reasoning as used above.

If \( x_{j+1} = x_k \), then the first inequality in (2.2.16) holds since \( L(\cdot) \) is nonincreasing. If \( x_{j+1} < x_k \) then since \( S_{n\cdot tv\cdot j} + x_k < S_0 \) by lemma 2.1 then
\[ \frac{L(S_{n\cdot tv\cdot j} + x_k) - L(S_{n\cdot tv\cdot j} + x_{j+1})}{x_k - x_{j+1}} \geq -1 \]
implies
\[ L(S_{n\cdot tv\cdot j} + x_{j+1}) + x_{j+1} \leq L(S_{n\cdot tv\cdot j} + x_k) + x_k \]
\[ \leq L(S_{n\cdot tv\cdot k+1} + x_k) + x_k , \]
for \( k = j + 2, \ldots, v \) since \( k \) was arbitrary. Similarly if \( x_k = y_1 \) then the second inequality in (2.2.16) is obviously satisfied, and if \( x_k < y_1 \) then since \( S_{n\cdot tv\cdot k+1} + y_1 < S_0 \) by lemma 2.1, then
\[ L(S_{n\cdot tv\cdot k+1} + x_k) + x_k \leq L(S_{n\cdot tv\cdot k+1} + y_1) + y_1 \]
\[ \leq L(S_{n\cdot (t+1)\cdot v-i+1} + y_1) + y_1 \]
for all \( i = 1, \ldots, j \) since \( i \) was arbitrary. Thus both inequalities in (2.2.16) hold.
But (2.2.16) implies that $M_{j+1}$ is in need of an item before $M_{j+2}, \ldots, M_v, M_1, \ldots, M_j$ in that order. Therefore the next item to be assigned must be assigned to $M_{j+1}$. However the last item assigned (by the inductive assumption) was item $S_{n-(t+1)v-j+1}$ and since we are following FIFO then $S_{n-(t+1)v-j}$ is the next item and is assigned to $M_{j+1}$. But this last assignment is precisely what this lemma states it should be. Hence by induction on $k = t$ and on $j$ the lemma is proved. The converse is obviously true since we assign the oldest item each time an assignment is made.

q.e.d.

There is an interesting corollary to this lemma which states exactly how many items each demand source receives under FIFO issuance when the field life function is as given in lemma 2.3.

**Corollary 2.3.1:** Let $L(S)$ be a continuous nonincreasing function with $L''(S) \geq -1$ for all $S$ such that $0 < S \leq S_0$. If FIFO is used to assign the $n$ items to $v \geq 1$ demand sources, then demand source $M_j$ receives exactly $\left\lceil \frac{n-j}{v} \right\rceil + 1$ items ($j = 1, \ldots, v$) where $[x]$ denotes the greatest integer $\leq x$.

**Proof of Corollary 2.3.1:** By lemma 2.3 any demand source $M_j$ receives items indexed by $n - kv - j + 1$ for $k = 0, 1, \ldots, t$ where $t$ is the largest integer such that

$$n - tv - j + 1 \geq 1.$$
Thus \( n - tv - j \geq 0 \) and \( t \leq \frac{n-j}{v} \). But \( t \) is the largest integer satisfying this condition, hence \( t = \left\lfloor \frac{n-j}{v} \right\rfloor \). Now since \( k = 0, 1, \ldots, t \) items then \( M_j \) receives exactly
\[
\sum_{k=0}^{\left\lfloor \frac{n-j}{v} \right\rfloor} 1 = 1 + \left\lfloor \frac{n-j}{v} \right\rfloor \text{ items.}
\]
q.e.d.

For example, let \( n = 11, \ v = 3 \) then

\[
M_1 \text{ receives } 1 + \left\lfloor \frac{11 - 1}{3} \right\rfloor = 1 + \left\lfloor \frac{10}{3} \right\rfloor = 4 \text{ items}
\]

\[
M_2 \text{ receives } 1 + \left\lfloor \frac{11 - 2}{3} \right\rfloor = 1 + \left\lfloor \frac{9}{3} \right\rfloor = 4 \text{ items}
\]

\[
M_3 \text{ receives } 1 + \left\lfloor \frac{11 - 3}{3} \right\rfloor = 1 + \left\lfloor \frac{8}{3} \right\rfloor = 3 \text{ items}
\]

total 11 items

The next two lemmas are very useful in the proofs of theorems in subsequent chapters. In the case \( v = 1 \) demand source, lemma 2.4 states that the FIFO issuance of a set of \( n \) items has a greater total field life than another set of \( n \) items also issued by FIFO whenever the initial ages of each member of the second set of items is at least as great as its corresponding member in the first set. This result holds under fairly general \( L(S) \). Lemma 2.5 generalizes lemma 2.4 to the case of more than one demand source. We have stated the \( v = 1 \) and \( v \geq 1 \) cases separately since the proof of lemma 2.5 becomes quite simple once lemma 2.4 has been proved.
Lemma 2.4: Let \( L(S) \) be a continuous nonincreasing function with \( L^{-}(S) \geq -1 \) for all \( S \) such that \( 0 < S \leq S_{o} \). Let \( v = 1 \). Consider two sets of \( n \) items which the following characteristics:

\[
I = \{ S_{1}, \ldots, S_{n} | S_{i} < S_{i+1} < S_{o} \text{ for all } i = 1, \ldots, n - 1 \}
\]

\[
II = \{ \hat{S}_{1}, \ldots, \hat{S}_{n} | \hat{S}_{i} < \hat{S}_{i+1} < S_{o} \text{ for all } i = 1, \ldots, n - 1 \}
\]

and \( S_{i} \leq \hat{S}_{i} \) for all \( i = 1, \ldots, n \). Denote by \( Q_{T} \) and \( \hat{Q}_{T} \) the total field life by FIFO issuance of the items in sets \( I \) and \( II \) respectively. Then \( Q_{T} \geq \hat{Q}_{T} \).

Proof of Lemma 2.4: The proof is by induction.

Let \( n = 1 \). Since \( L(\cdot) \) is nonincreasing then

\[
Q_{T} = L(S_{1}) \geq L(\hat{S}_{1}) = \hat{Q}_{T}.
\]

Now assume the lemma is true for \( n = k \) items and it will be proved true for \( n = k + 1 \) items. Let \( x \) and \( \hat{x} \) denote the total field life obtained by the FIFO issuance of the first \( k \) items issued (i.e., the \( k \) oldest items) of sets \( I \) and \( II \) respectively. Thus by the inductive assumption \( x \geq \hat{x} \). If \( x = \hat{x} \), then

\[
Q_{T} = L(S_{1} + x) + x \geq L(\hat{S}_{1} + \hat{x}) + \hat{x} = \hat{Q}_{T}
\]

since \( L(\cdot) \) is nonincreasing. If \( x > \hat{x} \), then since \( L(\cdot) \) is continuous nonincreasing and \( L^{-}(S) \geq -1 \) for \( S \leq S_{o} \) and by lemma 2.1, \( S_{1} + x < S_{o} \) and \( \hat{S}_{1} + \hat{x} < S_{o} \) then

\[
\frac{L(S_{1} + x) - L(S_{1} + \hat{x})}{x - \hat{x}} \geq -1
\]
implies
\[ Q_F = L(S_1 + x) + x \geq L(S_1 + \hat{x}) + \hat{x} \geq L(\hat{S}_1 + \hat{x}) + \hat{x} = \hat{Q}_F. \]

And by induction the lemma is proved. \(\Box\)

**Lemma 2.5**: Let \(L(S)\) and sets I and II have the same properties as in lemma 2.4. Let \(v \geq 1\). Denote by \(Q^v_{F,n}\) and \(\hat{Q}^v_{F,n}\) the total field life by FIFO issuance of the \(n\) items to the \(v\) demand sources with the items from sets I and II respectively. Then
\[
Q^v_{F,n} \geq \hat{Q}^v_{F,n}.
\]

**Proof of Lemma 2.5**: Using lemma 2.3 the following table for the assignment of the \(n\) items to the \(v\) demand sources can be constructed.

<table>
<thead>
<tr>
<th>Demand Source</th>
<th>Set I</th>
<th>Set II</th>
<th>Total field life contributed by (M^I_1)</th>
</tr>
</thead>
<tbody>
<tr>
<td>(M_1)</td>
<td>(S_n, \ldots, S_{n-kv})</td>
<td>(\hat{S}<em>n, \ldots, \hat{S}</em>{n-kv})</td>
<td>(x_1)</td>
</tr>
<tr>
<td>(M_2)</td>
<td>(S_{n-1}, \ldots, )</td>
<td>(\hat{S}_{n-1}, \ldots, )</td>
<td>(x_2)</td>
</tr>
<tr>
<td></td>
<td>(S_{n-kv-1})</td>
<td>(\hat{S}_{n-kv-1})</td>
<td></td>
</tr>
<tr>
<td>(\vdots)</td>
<td>(\vdots)</td>
<td>(\vdots)</td>
<td></td>
</tr>
<tr>
<td>(M_v)</td>
<td>(S_{n-v+1}, \ldots, )</td>
<td>(\hat{S}_{n-v+1}, \ldots, )</td>
<td>(x_v)</td>
</tr>
<tr>
<td></td>
<td>(S_{n-(k+1)v+1})</td>
<td>(\hat{S}_{n-(k+1)v+1})</td>
<td></td>
</tr>
</tbody>
</table>

where the subscripts on the \(S\)'s are such that \(n - kv - i \geq 1\) for all \(k = 0, 1, \ldots\) and \(i = 0, 1, \ldots, v - 1\), i.e., the inventory is
exhausted. Note that lemma 2.3 tells us that the subscripts on the items for a particular demand source are the same for both sets I and II. Hence the items assigned to $M_j$ from sets I and II obey the conditions

(i) $S_{n-k} + 1 \leq \hat{S}_{n-k} + 1$ for all $k = 0, 1, 2, \ldots$

(ii) there are the same number of items assigned to $M_j$ from set I as there is from set II.

But these conditions hold for all $M_j, j = 1, \ldots, v$. Hence by lemma 2.4 $x_j \geq \hat{x}_j$ for all $j = 1, \ldots, v$ and

$$Q_{n,v} = \sum_{j=1}^{v} x_j \geq \sum_{j=1}^{v} \hat{x}_j = q_{n,v}.$$  

q.e.d.

Zehna [11] stated that for $L(S)$ concave and differentiable with $0 \geq L'(S) \geq -1$ for $0 \leq S \leq S_o$ when there are $v = 2$ demand sources and either $n = 3$ or $n = 4$ items in the stockpile, then FIFO is the optimal issuing policy. He did not present the proof of this statement and since these results are essential to the proof of Theorem 2.6, the proof is presented here.

Lemma 2.6: Let $L(S)$ be a concave function with $L'(S) \geq -1$ for $0 < S \leq S_o$. Let $v = 2$ and $n = 3$ or $n = 4$. Then FIFO is the optimal issue policy.
Proof of Lemma 2.6: The proof will proceed by the
elimination of all non-FIFO policies. By Lieberman [9] Theorem 3, it is
only necessary to consider allocations to each demand source, \( M_1 \) and
\( M_2 \), which are FIFO within the allocation. Hence the possible policies
are:

\[ n = 4 \]
1 = \([S_1, S_2, S_3, S_4]\) \hspace{2cm} 2 = \([S_1, S_2, S_3, S_4]\)
3 = \([S_1, S_2, S_3, S_4]\) \hspace{2cm} 4 = \([S_1, S_2, S_3, S_4]\)
5 = \([S_1, S_2, S_3, S_4]\) \hspace{2cm} 6 = \([S_1, S_2, S_3, S_4]\)
7 = \([S_1, S_2, S_3, S_4]\)

\[ n = 3 \]
A = \([S_1, S_2, S_3]\) \hspace{2cm} B = \([S_1, S_2, S_3]\)
C = \([S_1, S_2, S_3]\) \hspace{2cm} D = \([S_1, S_2, S_3]\)

Policies 1, 5, and A may be eliminated immediately since they are not
feasible (they contradict assumption (6')). A general result is now
proved which eliminates policies 4, 7, and C. Consider the two policies
\( I = [A, S_2, B, S_1] \) and \( II = [A, S_1, B, S_2] \) where \( A \) and \( B \) are any
items in FIFO order. Let \( y \) and \( x \) denote the total field life of the
items represented by \( A \) and \( B \) respectively. It will be shown that for
\( x \geq y \geq 0 \)

\[ Q_I = L(S_2 + y) + y + L(S_1 + x) \geq L(S_1 + y) + y + L(S_2 + x) \]

\[ Q_{II} - L(S_2 + y) + y + L(S_1 + x) + x = Q_{II} \cdot \]

(2.2.17)
If \( x = y \) then (2.2.17) holds with equality. If \( x > y \) then since 
\[ S_2 + y < S_0 \] 
by lemma 2.1 and since \( L(\cdot) \) is concave for \( S \leq S_0 \) then 
\[
\frac{L(S_2 + y) - L(S_1 + y)}{S_2 - S_1} \geq \frac{L(S_2 + x) - L(S_1 + x)}{S_2 - S_1}.
\]

With \( S_2 > S_1 \) we then have 
\[ L(S_2 + y) + L(S_1 + x) \geq L(S_2 + x) + L(S_1 + y) \]
and (2.2.17) holds for all \( x \geq y \).

Now in policy \( 4 \) \( A = S_4, \ B = S_3 \) and \( L(S_4) \leq L(S_3) \)
in policy \( 7 \) \( A = \emptyset, \ B = S_4, S_3 \) and \( 0 < L(S_4) + L(S_3 + L(S_4)) \)
in policy \( C \) \( A = \emptyset, \ B = S_3 \) and \( 0 < L(S_3) \).

We apply the above result for \( Q_I \geq Q_{II} \) and see that policy \( 4 \) is
dominated by policy \( 3 \), policy \( 7 \) is dominated by policy \( 6 \), and policy \( C \) is
 dominated by policy \( B \).

Thus for \( n = 3 \), \( A \) and \( C \) have been eliminated, hence \( B \) is
optimal but by lemma 2.3, policy \( B \) is FIFO.

For \( n = 4 \) by lemma 2.3, policy \( 3 \) is FIFO. It is necessary to show
that policy \( 3 \) dominates policies \( 2 \) and \( 6 \). That is, show

\[
L(S_4) + L(S_2 + L(S_4)) + L(S_3) + L(S_1 + L(S_3))
\]
\[
\geq L(S_4) + L(S_3 + L(S_4)) + L(S_2) + L(S_1 + L(S_2))
\]

(2.2.18)

and

\[
L(S_4) + L(S_2 + L(S_4)) + L(S_3) + L(S_1 + L(S_3))
\]
\[
\geq L(S_4) + L(S_3 + L(S_4)) + L(S_1 + L(S_4)) + L(S_2 + L(S_4)) + L(S_2).
\]

(2.2.19)
For (2.2.18) since \( S_3 + L(S_4) < S_o \) and \( S_1 + L(S_2) < S_o \) by lemma 2.1 and since \( L(\cdot) \) is concave for \( S \leq S_o \) then
\[
\frac{L(S_3 + L(S_4) - L(S_2 + L(S_4)))}{S_3 - S_2} \leq \frac{L(S_1) - L(S_2)}{S_3 - S_2}
\]
implies
\[
L(S_3 + L(S_4)) + L(S_2) \leq L(S_2 + L(S_4)) + L(S_3) . \tag{2.2.20}
\]

Furthermore since \( L(\cdot) \) is nonincreasing
\[
L(S_4) + L(S_1 + L(S_2)) \leq L(S_4) + L(S_1 + L(S_3)) . \tag{2.2.21}
\]

Combining (2.2.20) and (2.2.21) we obtain (2.2.18). For (2.2.19) since \( L(\cdot) \) is nonincreasing and by lemma 2.2
\[
L(S_3) \leq L(S_4) + L(S_3 + L(S_4))
\]
implies
\[
L(S_1 + L(S_2)) \geq L(S_1 + L(S_4) + L(S_3 + L(S_4))) . \tag{2.2.22}
\]

Combining (2.2.22) and (2.2.20) we obtain (2.2.19). Hence policy 3, which is FIFO, dominates policies 2 and 6. FIFO is optimal for \( n = 4 \). q.e.d.

In the next section we will use some of the foregoing lemmas and corollaries to prove some interesting results on optimal inventory depletion policies when \( v > 1 \).
2.3 Bounds on the Optimal Policy

As Zezma [11] points out, the extension of the results for \( v = 1 \) to the case \( v \geq 1 \) when \( L(S) \) is concave nonincreasing is not a simple matter. He gives a counterexample to show that such an extension is not possible in general. However, for the particular case \( L(S) = aS + b \), \( b > 0 > a > -1 \), for \( 0 \leq S \leq S_o \), the results for \( v = 1 \) and \( v \geq 1 \) coincide, viz. FIFO is optimal in both cases.

Presented below are a set of theorems which provide upper and lower bounds on the optimal policy when \( v > 1 \) and \( L(S) \) is concave nonincreasing for \( S \leq S_o \). These bounds for the optimal policy coincide with the bounds for the FIFO policy for the same \( n \) items and \( v > 1 \). And since not all policies are included in these bounds, the optimal policy and the FIFO policy are "close" in the sense that the difference between the optimal policy and the FIFO policy cannot exceed the difference between their common upper and lower bounds.

Since \( L(S) \) takes the same form for all of the theorems and lemmas of this section we will say:

\( L(S) \) has property \( \Omega \) if \( L(S) \) is a concave nonincreasing function for all \( S \) such that \( 0 \leq S \leq S_o \) and \( L'(S) \geq -1 \) for \( 0 < S \leq S_o \).

Theorem 2.1: Let \( L(S) \) have property \( \Omega \). Let \( v \geq 1 \). Denote by \( Q^*_{n,v} \), the total field life obtained from the \( n \) items in the stockpile when the number of demand sources is \( v \) and when an optimal issuing policy is followed. Then

\[
Q^*_{n,v} \leq Q^*_{n,v+1} \quad \text{for any } v = 1, \ldots, n - 1.
\]
Proof of Theorem 2.1: Let the optimal policy which achieves \( Q^*_{n,v} \) be denoted by \( A = \left[ \begin{array}{cccc} S_{i_{11}}, & S_{i_{12}}, & \ldots, & S_{i_{1j_1}}; \ S_{i_{21}}, & \ldots, & S_{i_{2j_2}}; \ \vdots & \vdots & \ddots & \vdots; \ S_{i_{v1}}, & \ldots, & S_{i_{vj_v}} \end{array} \right] \). Now since \( n > v \) then at least one of the subscripts \( j_1, j_2, \ldots, j_v \) is an integer strictly greater than \( 1 \) (i.e., at least one demand source must have two or more items assigned to it). Let us say \( j_k > 1 \). Then the total field life contributed by demand source \( M_k \) to the total field life \( Q^*_{n,v} \) is given by

\[
Q^*_{M_k} = L(S_{i_{k1}}) + L(S_{i_{k2}}) + L(S_{i_{k3}}) + \cdots + L(S_{i_{kj_k}}) + \cdots.
\]

(2.3.1)

Now consider the following issuing policy \( B_{v+1} \) for the case of \( \nu + 1 \) demand sources:

Issue the same items in the same order to all demand sources \( M_i \) for all \( i \neq k \) as are issued to them when policy \( A \) is followed.

Issue item \( S_{i_{k_j}} \) to demand source \( M_{v+1} \) and issue the remaining \( j_k - 1 \) items to demand source \( M_k \) in the same order as under policy \( A \).

Let \( Q^*_{B_{n,v+1}} \) denote the total field life obtained from policy \( B_{v+1} \).

We will show \( Q^*_{B_{n,v+1}} > Q^*_{n,v} \).

Now the total field life contributed by demand sources \( M_i \) for all \( i \neq k \) is the same for both policy \( A \) and policy \( B_{v+1} \). Hence we only need to examine the field life contributed by \( M_k \) and \( M_{v+1} \). Let

\[
x = L(S_{i_{k1}}) + L(S_{i_{k2}}) + L(S_{i_{k3}}) + \cdots + L(S_{i_{kj_k} - 1}) + L(S_{i_{k1}}) + \cdots
\]

(2.3.2)
then

\[ Q_{M_k} = x + L(S_{i_{k,j_k}}) \]  \hspace{1cm} (2.3.3)

by using (2.3.1). We must show

\[ x + L(S_{i_{k,j_k}}) \geq Q_{M_k} \]  \hspace{1cm} (2.3.4)

but \( L(\cdot) \) is nonincreasing hence \( L(S_{i_{k,j_k}}) \geq L(S_{i_{k,j_k}} + x) \) since \( x > 0 \). Therefore (2.3.4) holds. But \( x \) is the field life contributed by \( M_k \) and \( L(S_{i_{k,j_k}}) \) is the field life contributed by \( M_{v+1} \) under policy \( B_{v+1} \).

Therefore

\[
Q_{B_{n,v+1}} = \sum_{i=1, i \neq k}^{v} Q_{M_i} + x + L(S_{i_{k,j_k}}) \\
\geq \sum_{i=1}^{v} Q_{M_i} = Q^*_{n,v}.
\]

Now \( Q^*_{n,v+1} \) is the optimal policy for \( v + 1 \) demand sources, hence

\[
Q^*_{n,v+1} \geq Q_{B_{n,v+1}} \geq Q^*_{n,v}.
\]

q.e.d.
Theorem 2.2: Let \( L(S) \) have property \( \Omega \). Let \( \nu \geq 1 \). Then when the FIFO issuing policy is followed

\[
q_{\nu}^{\frac{n}{n}} \leq q_{\nu}^{\frac{n+1}{n+1}} \quad \text{for any } \nu = 1, \ldots, n - 1.
\]

Proof of Theorem 2.2: By lemma 2.1 (applied to each demand source separately) the FIFO issuance of the \( n \) items in the stockpile results in each item having positive field life on issuance under either \( F_{\nu, n} \) or \( F_{\nu, n+1} \). Furthermore in any FIFO ordering of the \( n \) items for any \( \nu \leq n \) there are then exactly \( n \) terms \( L(S_{i} + \cdots) \) for \( i = 1, \ldots, n \). Hence there is a one-one correspondence between the terms in \( q_{\nu}^{\frac{n}{n}} \) and \( q_{\nu}^{\frac{n+1}{n+1}} \) where this correspondence is established on the basis of the index letter \( i \) for \( L(S_{i} + \cdots) \) and \( i = 1, \ldots, n \). Now using lemma 2.3

\[
q_{\nu}^{\frac{n}{n}} = L(S_{n}) + L(S_{n-1} + L(S_{n})) + \cdots \quad M_{1}
\]

\[
+ L(S_{n-1}) + L(S_{n-1} + L(S_{n-1})) + \cdots \quad M_{2}
\]

\[
+ \cdots \quad M_{\nu}
\]

\[
+ L(S_{n-\nu+1}) + L(S_{n-2\nu+1} + L(S_{n-\nu+1})) + \cdots \quad M_{v}
\]

(2.3.5)
\(Q_{n, v+1}^F = L(S_n) + L(S_{n-v-1} + L(S_n)) + \cdots \) \(M_1\)

\( + L(S_{n-1}) + L(S_{n-v-2} + L(S_n)) + \cdots \) \(M_2\)

\[+ \cdots \]

\(+ L(S_{n-v+1}) + L(S_{n-2v} + L(S_{n-v+1})) + \cdots \) \(M_v\)

\(+ L(S_{n-v}) + L(S_{n-2v-1} + L(S_{n-v})) + \cdots \) \(M_{v+1}\)

\((2.3.6)\)

Now choose any \(L(S_i + x_i)\) for \(i = 1, \ldots, n\) belonging to \(Q_{n, v}^F\) and the corresponding \(L(S_i + y_i)\) for \(i = 1, \ldots, n\) belonging to \(Q_{n, v+1}^F\). We will show

\[L(S_i + y_i) \geq L(S_i + x_i) \text{ for all } i = 1, \ldots, n;\]

but since \(L(\cdot)\) is nonincreasing, it is only necessary to show \(x_i \geq y_i\) for all \(i = 1, \ldots, n\).

**Case 1:** \(i \in \{n - v, n - v + 1, \ldots, n\}\)

Then \(y_i = 0\) and since \(x_i \geq 0\) we have \(x_i \geq y_i\) \((2.3.7)\)

**Case 2:** \(1 \leq i \leq n - v - 1\)

Then

\[x_i = L(S_{i+t_v}) + L(S_{i+(t-1)v} + L(S_{i+t_v})) + \cdots + L(S_{i+v} + \cdots)\]
\[ y_i = L(S_{i+s(v+1)}) + L(S_{i+(s-1)(v+1)} + L(S_{i+s(v+1)}) + \cdots + L(S_{i+v+1} + \cdots) \]

where these equations follow from lemma 2.3. Now \( s \leq t \) since by lemma 2.3 every \( v^{th} \) item is assigned to the \( j^{th} \) demand source (say \( M_j \) receives \( S_1 \)) under \( Q_{n,v} \) and every \((v+1)^{st}\) item is assigned under \( Q_{n,v+1} \). Hence when the \( F_{n,v} \) policy is followed, the demand source which receives \( S_1 \) will have already received more (or equal) items than the demand source which receives \( S_1 \) under \( F_{n,v+1} \). Hence \( x_1 \) and \( y_1 \) have the following policies

\[
F_{x_1} = [S_{i+tv}, S_{i+(t-1)v}, \ldots, S_{i+v}]
\]

\[
F_{y_1} = [S_{i+s(v+1)}, S_{i+(s-1)(v+1)}, \ldots, S_{i+v+1}].
\]

But

\[
i + v < i + v + 1 \quad \Rightarrow \quad S_{i+v} < S_{i+v+1}
\]

\[
i + 2v < i + 2(v + 1) \quad \Rightarrow \quad S_{i+2v} < S_{i+2(v+1)}
\]

\[
\vdots
\]

\[
i + sv < i + s(v + 1) \quad \Rightarrow \quad S_{i+sv} < S_{i+s(v+1)}. \quad (2.3.8)
\]

Now consider the FIFO policy of issuing the \( s \) items \( S_{i+v}, \ldots, S_{i+sv} \) and denote this policy by \( A \) i.e.,

\[
A = [S_{i+sv}, S_{i+(s-1)v}, \ldots, S_{i+v}].
\]
Now by (2.3.8) and lemma 2.4
\[ Q_A \geq y_1 \]

where \( Q_A \) is the field life from policy A. Furthermore, since \( s \leq t \) then by lemma 2.2
\[ Q_A \leq x_1 \]

Thus \( x_1 \geq y_1 \). And since the choice of \( L(S_i + x_i) \) was arbitrary for \( 1 \leq i \leq n - v - 1 \)
\[ x_i \geq y_i \quad \text{for all } i \text{ with } 1 \leq i \leq n - v - 1. \]
(2.3.9)

Combining (2.3.7) and (2.3.9) we have
\[ x_1 \geq y_1 \quad \text{for all } i = 1, \ldots, n, \]

therefore
\[ L(S_i + x_i) \leq L(S_i + y_i) \quad \text{for all } i = 1, \ldots, n \]

and
\[ Q_{n,v} = \sum_{i=1}^{n} L(S_i + x_i) \leq \sum_{i=1}^{n} L(S_i + y_i) = Q_{n,v+1}. \]

q.e.d.
Theorem 2.3: Let \( L(S) \) have property \( \Omega \). Let \( v \geq 1 \). If one item is added to the initial stockpile of \( n \) items prior to the issuance of any of the items, then

\[
Q_{n,v}^{F} \leq Q_{n+1,v}^{F}
\]

when the FIFO issuing policy is followed.

Proof of Theorem 2.3: Before beginning the proof it should be noted that for \( v = 1 \), this theorem reduces to lemma 2.2.

Let \( S_{n+1} \) denote the initial age of the new item. We consider three cases:

**Case 1** \( S_{n+1} < S_1 \) and say \( S_1 \) is assigned to \( M_j \) for some \( j \in \{1, \ldots, v\} \). Then by lemma 2.3

\[
Q_{n,v}^{F} = L(S_n) + L(S_{n-v} + L(S_n)) + \cdots
\]

\[
+ L(S_{n-1}) + L(S_{n-v-1} + L(S_{n-1})) + \cdots
\]

\[
+ L(S_{n-2}) + L(S_{n-v-2} + L(S_{n-2})) + \cdots
\]

\[
+ \cdots + L(S_{n-j+1}) + \cdots + L(S_1 + L(S_{n-j+1}) + \cdots)
\]

\[
+ \cdots + L(S_{n-v+1}) + \cdots
\]

\[
(2.3.12)
\]
\[ Q_{T_{n+1}, v} = L(S_n) + \cdots + L(S_{n-1}) + \cdots + L(S_{n-j+1}) + \cdots + L(S_{n-j}) + \cdots \]

\[ + L(S_{n-j+1}) + \cdots + L(S_{n-j}) + \cdots + L(S_{n+1} + L(S_{n-j}) + \cdots) \]

\[ + \cdots + L(S_{n-v+1}) + \cdots \]

(2.3.13)

and

\[ Q_{T_{n+1}, v} - Q_{T_{n}, v} = L(S_{n+1} + L(S_{n-j}) + \cdots) > 0 \]

by lemma 2.1. Therefore \( Q_{T_{n+1}, v} > Q_{T_{n}, v} \) for this case.

**Case 2**

\( S_n < S_{n+1} < S_0 \)

\( Q_{T_{T_{n+1}, v}} \) is still given by (2.3.12); however \( Q_{T_{n+1}, v} \) now becomes

\[ Q_{T_{n+1}, v} = L(S_{n+1}) + L(S_{n-v+1} + L(S_{n+1})) + \cdots \]

\[ + L(S_n) + L(S_n - v + L(S_n)) + \cdots \]

\[ + L(S_{n-1}) + L(S_{n-v-1} + L(S_{n-1})) + \cdots \]

\[ + \cdots + L(S_{n-j+1}) + L(S_{n-v-j+1} + L(S_{n-j+1})) + \cdots \]

\[ + \cdots + L(S_{n-v+2}) + L(S_{n-2v+2} + L(S_{n-v+2})) + \cdots \]
\[ Q_{n+1, \nu} - Q_{n, \nu} = L(S_{n+1}) + L(S_{n-\nu +1} + L(S_{n+1})) + \ldots \]
\[ - [L(S_{n-\nu+1}) + L(S_{n-2\nu+1} + L(S_{n-\nu+1})) + \ldots ] \geq 0 \]

by lemma 2.2 since \( Q_{n+1, \nu} - Q_{n, \nu} \) represents the difference of the two policies

\[ A = [S_{n+1}, S_{n-\nu+1}, S_{n-2\nu+1}, \ldots, S_{n-k\nu+1}, \ldots] \]
\[ B = [S_{n-\nu+1}, S_{n-2\nu+1}, \ldots, S_{n-k\nu+1}, \ldots] \]

where \( B \) has the same items as \( A \) after \( A \) has issued its first item \( S_{n+1} \).

Therefore

\[ Q_{n+1, \nu} \geq Q_{n, \nu} \] .

**Case 3**  \( S_i < S_{n+1} < S_{i+1} \) for any \( i = 1, \ldots, n-1 \)

Then for items \( S_n \) down through \( S_{i+1} \) the two total field life functions are identical. Let \( x_j \) denote the total field life contributed by \( M_j \) (\( j = 1, \ldots, \nu \)) down through item \( S_{i+1} \). Without loss of generality let \( S_{n+1} \) be assigned to \( M_j \). Now we can rewrite (2.3.12) in the following manner:
\[ Q_{F, n, v} = x_1 + L(S_{n-t} + x_1) + \cdots \]
\[ + x_2 + L(S_{n-t} + x_2) + \cdots \]
\[ + \cdots + x_j + L(S_{n-(t-1)v+j} + x_j) + \cdots \]
\[ + x_{j+1} + L(S_{n-(t-1)v+j+1} + x_{j+1}) + \cdots \]
\[ + \cdots + x_v + L(S_{n-t} + x_v) + \cdots \]

where \( n - tv < i + 1 \), and \( S_{n-(t-1)v+j+1} = S_1 \) when numbering from above.

And
\[ Q_{F, n+1, v} = x_1 + L(S_{n-t} + x_1) + \cdots \]
\[ + x_2 + L(S_{n-t} + x_2) + \cdots \]
\[ + \cdots + x_j + L(S_{n+1} + x_j) + \cdots \]
\[ + x_{j+1} + L(S_{n-(t-1)v+j} + x_{j+1}) + \cdots \]
\[ + x_{j+2} + L(S_{n-(t-1)v+j+2} + x_{j+2}) + \cdots \]
\[ + \cdots + x_v + L(S_{n-(t-1)v-2} + x_v) + \cdots \]

By induction we will prove \( Q_{F, n+1, v} > Q_{F, n, v} \) for this case. Let \( S_1 = S_1 \) then \( S_{n-(t-1)v+j+1} = S_1 \) and
\[
Q_{F_{n+1}, \nu} - Q_{F_{n}, \nu} = L(S_{n+1} + x_j) + L(S_1 + x_{j+1}) - L(S_1 + x_j).
\] (2.3.14)

Now by the definition of the \( x_i \)'s and since FIFO is being followed then by lemma 2.3

\[
x_j \leq x_{j+1} \leq x_{j+2} \leq \cdots \leq x_\nu \leq x_1 \leq \cdots \leq x_{j-1}.
\] (2.3.15)

If \( x_j = x_{j+1} \) then (2.3.14) has

\[
Q_{F_{n+1}, \nu} - Q_{F_{n}, \nu} = L(S_{n+1} + x_j) > 0 \text{ and } Q_{F_{n+1}, \nu} > Q_{F_{n}, \nu}.
\]

If \( x_j < x_{j+1} \), since \( S_{n+1} + x_{j+1} < S_0 \) by lemma 2.1 and since \( L(\cdot) \) is concave for \( S \leq S_0 \) then

\[
\frac{L(S_{n+1} + x_{j+1}) - L(S_{n+1} + x_j)}{x_{j+1} - x_j} \leq \frac{L(S_1 + x_{j+1}) - L(S_1 + x_j)}{x_{j+1} - x_j}
\]

implies

\[
L(S_{n+1} + x_j) + L(S_1 + x_{j+1}) \geq L(S_{n+1} + x_{j+1}) + L(S_1 + x_j)
\]

\[
> L(S_1 + x_j).
\]

Therefore in (2.3.14) we have

\[
Q_{F_{n+1}, \nu} > Q_{F_{n}, \nu}.
\]
Now assume $Q_{F_{n+1},v} - Q_{F_n,v} \geq 0$ for $S_i = S_k$ (i.e., $S_k < S_{n+1} < S_{k+1}$) and it will be proved true for $S_i = S_{k+1}$ (i.e., $S_{k+1} < S_{n+1} < S_{k+2}$) for $k = 1, 2, \ldots, n - 2$.

Now for $S_i = S_{k+1}$ we have

$$Q_{F_{n+1},v} - Q_{F_n,v} = L(S_{n+1} + x_j) + L(S_{k+1} + x_{j+1}) + L(S_k + x_{j+2})$$

$$+ \cdots + L(S_1 + x_{m+1} + \cdots)$$

$$- [L(S_{k+1} + x_j) + L(S_k + x_{j+1}) + \cdots + L(S_1 + x_m + \cdots)]$$

(2.3.16)

where we assume that $S_1$ is assigned to $M_{n+1}$ under $F_{n+1,v}$.

Now using (2.3.15) and since $S_{n+1} + x_{j+1} < S_o$ by lemma 2.1 and $L(\cdot)$ is concave for $S \leq S_o$ then if $x_{j+1} > x_j$, then

$$\frac{L(S_{n+1} + x_{j+1}) - L(S_{n+1} + x_j)}{x_{j+1} - x_j} \leq \frac{L(S_{k+1} + x_{j+1}) - L(S_{k+1} + x_j)}{x_{j+1} - x_j}.$$ 

This implies

$$L(S_{n+1} + x_j) + L(S_{k+1} + x_{j+1}) \geq L(S_{k+1} + x_j) + L(S_{n+1} + x_{j+1}).$$

(2.3.17)

If $x_{j+1} = x_j$, then (2.3.17) holds with equality. Now adding the same quantities to both sides of (2.3.17) we obtain
\[ L(S_{n+1} + x_j) + L(S_{k+1} + x_{j+1}) + L(S_k + x_{j+2}) + \cdots + L(S_1 + x_{m+1} + \cdots) \]
\[ \geq L(S_{k+1} + x_j) + L(S_{n+1} + x_{j+1}) + L(S_k + x_{j+2}) + \cdots + L(S_1 + x_{m+1} + \cdots). \]
\[(2.3.18)\]

But by the inductive assumption
\[ L(S_{n+1} + x_{j+1}) + L(S_k + x_{j+2}) + L(S_{k-1} + x_{j+3}) + \cdots + L(S_1 + x_{m+1} + \cdots) \]
\[ \geq L(S_k + x_{j+1}) + L(S_{k-1} + x_{j+2}) + \cdots + L(S_1 + x_m + \cdots) \]
\[(2.3.19)\]

where the left side of (2.3.19) is just the right side of (2.3.18) after omitting \( L(S_{k+1} + x_j) \). Hence we can write the new inequality using (2.3.18) and (2.3.19)
\[ L(S_{n+1} + x_j) + L(S_{k+1} + x_{j+1}) + L(S_k + x_{j+2}) + \cdots + L(S_1 + x_{m+1} + \cdots) \]
\[ \geq L(S_{k+1} + x_j) + L(S_k + x_{j+1}) + L(S_{k-1} + x_{j+2}) + \cdots + L(S_1 + x_m + \cdots). \]
\[(2.3.20)\]

However, (2.3.20) is precisely what we want to show for (2.3.16). Therefore \( Q_{n+1, \nu} - Q_{n, \nu} \geq 0 \). And by induction case 3 has been proved.

Combining cases 1, 2, and 3 the theorem is proved.

q.e.d.

**Theorem 2.4**: Let \( L(S) \) have property \( \Omega \). Let \( \nu \geq 1 \). If one item is added to the initial stockpile of \( n \) items prior to the issuance of any of the items, then
\[ Q_{n, \nu}^* \leq Q_{n+1, \nu}^* \]

when an optimal issuing policy is followed.
Proof of Theorem 2.4: Let $\psi_n$ denote the optimal policy which yields total field life $Q^*_{n,\nu}$. Let $A$ be the policy where the original $n$ items are issued according to $\psi_n$ and the new item, $S_{n+1}$, is issued last, to the demand source which first finishes its consumption of its items under $\psi_n$. Then if the field life of item $S_{n+1}$ at the time of issuance is denoted by $x$ we have $x \geq 0$, and
\[
Q^*_{n,\nu} \leq Q^*_{n+1,\nu} + x = Q^*_{A_{n+1},\nu} \leq Q^*_{n+1,\nu}.
\]
q.e.d.

Before presenting the next theorem, we should point out an interesting extension of Theorems 2.3 and 2.4.

Corollary 2.4.1: Let $L(S)$ have property $\Omega$. Let $\nu \geq 1$. If $M \geq 1$ items are added to the initial stockpile of $n$ items prior to the issuance of any of the items, then

(i) $Q_{F, n+M,\nu} \geq Q_{F, n,\nu}$ if the FIFO issuing policy is followed

(ii) $Q^*_{n+M,\nu} \geq Q^*_{n,\nu}$ if an optimal issuing policy is followed.

Proof of Corollary 2.4.1: We will just prove (i) since the proof for (ii) follows mutatis mutandis.

In Theorem 2.3 we have already proved the corollary true for $M = 1$. Assume the corollary is true for $M > 1$ and consider adding $M + 1$ items to the stockpile. Ignoring item $S_{M+1}$ temporarily, the total field life of the remaining items satisfies $Q_{F, n+M,\nu} \geq Q_{F, n,\nu}$ by the
inductive hypothesis. Then adding $S_{M+1}$ can only increase the total
field life by $M = 1$ hence by Theorem 2.3

$$Q_{F_{n+M+1}, \nu} \geq Q_{F_{n+M}, \nu} \geq Q_{F_{n}, \nu}$$

q.e.d.

**Theorem 2.5:** Let $I(S)$ have property $\Omega$. Let $\nu \geq 1$. If
\[\frac{1}{2}(n + 1) \leq \nu \leq n,\]
then any feasible policy which assigns more than two items to any demand source has a lower total field life than some policy which assigns at most two items to each demand source.

Before beginning the proof of this theorem two things should be
pointed out. First, although the theorem doesn't explicitly state the
improved policy, the proof does state it. Second, if we call the set
"$G$" the set of all policies which issue at most two items to each
demand source, the theorem states that for any feasible policy for
issuing the $n$ items, there is a member of $G$ which dominates it. The
theorem does not state that this member of $G$ is a feasible policy.
Indeed, this may not be the case at all. At this point though, it should
be noted that FIFO $\in G$ and by lemma 2.3 FIFO is feasible. Theorem 2.6
will show that of all the policies in $G$, FIFO maximizes the total field
life for the $n$ items; hence FIFO is the optimal policy.

**Proof of Theorem 2.5:** Since $\nu$ is an integer which is greater than or
equal to $\frac{1}{2}$ the number of items in the stockpile then if $i$ demand
sources have $k_i > 2$ items assigned to them, there are at least
$$\sum_{i} (k_i - 2)$$
demand sources which have only one item assigned to them
(since all demand sources must have at least one item by the initial
assignment).
We only need to consider one demand source with \( k_1 > 2 \) items and \( k_1 - 2 \) demand sources with only one item each since the same procedure (following) applies to all other demand sources with \( k_j > 2 \) items assigned to them.

Let \( 1 > 2 \) items be assigned to \( M_k \). In particular let these items be denoted by \( S_{k_1} < S_{k_2} < \cdots < S_{k_i} \). Let \( M_j \) be a demand source with only one item assigned to it.

Let \( \psi = [S_{t_1}, \ldots, S_{t_2}, S_{t_1} ; S_{\ell_1}] \) be the part of any arbitrary feasible policy which assigns \( S_{t_1}, \ldots, S_{t_2}, S_{t_1} \) to \( M_k \) and \( S_{\ell_1} \) to \( M_j \) where \( S_{t_1}, \ldots, S_{t_2}, S_{t_1} \) is any permutation of the items \( S_{k_1}, \ldots, S_{k_i} \).

We will now show \( S_{\ell_1} < S_{k_2} \). Assume to the contrary that \( S_{\ell_1} > S_{k_2} \). Then since \( L(\cdot) \) is nonincreasing \( L(S_{\ell_1}) \leq L(S_{k_2}) \). We have two cases:

**Case (i):** \( S_{k_2} \neq S_{t_1} \) then \( S_{k_2} \) is issued before item \( S_{t_1} \).

Let \( x \) be the total field life up to but not including the issuance of item \( S_{k_2} \). If \( x = 0 \) then \( L(S_{\ell_1}) \leq L(S_{k_2}) \) above. If \( x > 0 \) then

\[
\frac{L(S_{k_2} + x) - L(S_{k_2})}{x} \geq -1
\]

implies

\[
L(S_{\ell_1}) \leq L(S_{k_2}) \leq L(S_{k_2} + x) + x .
\]

\((2.3.21)\)
But (2.3.21) says that policy \( \psi \) is infeasible since item \( S_{j_1} \) is consumed prior to \( S_{k_2} \) hence some \( S_{t_j} \) should be assigned to \( M_{j_1} \) rather than \( M_k \). This result contradicts the hypothesis that \( \psi \) is feasible. Therefore in this case \( S_{j_1} < S_{k_2} \).

Case (ii): \( S_{k_2} = S_{t_1} \) then \( S_{k_1} \neq S_{t_1} \) and item \( S_{k_1} \) is issued before item \( S_{t_1} \). Since we are assuming \( S_{j_1} > S_{k_2} \) then \( S_{j_1} > S_{k_1} \) and \( L(S_{j_1}) \leq L(S_{k_1}) \). By the same argument as in case (i) above we obtain

\[
L(S_{j_1}) \leq L(S_{k_1}) \leq L(S_{k_1} + x) + x
\]

and we obtain the contradiction that some \( S_{t_j} \) should be issued to \( M_{j_1} \) rather than \( M_k \). Hence in this case also \( S_{j_1} < S_{k_1} < S_{k_2} \). Thus

\[
S_{j_1} < S_{k_2}.
\]

(2.3.22)

Now from Lieberman [9] Theorem 3 we have that the FIFO issuance of
\( S_{k_1}, \ldots, S_{k_l} \) yields a greater total field life than any other permutation such as given by \( S_{t_1}, \ldots, S_{t_l} \) in policy \( \psi \). Therefore if we let policy \( A \) be

\[
A = [S_{k_1}, \ldots, S_{k_2}, S_{k_1}, S_{j_1}]
\]

then

\[
Q_A > Q_\psi.
\]
Now policy A may not be a feasible policy; however since we only wish to show that there exists a policy which belongs to \( G \) which is better than \( \psi \), in the sense of greater total field life, we do not need feasibility for \( A \). It will be shown that the policy which belongs to \( G \) has field life \( Q \) and \( Q \geq Q_A \). Thus \( Q \geq Q_\psi \).

Now since \( S_{j_1} < S_{k_2} \) and \( S_{k_1} < S_{j_2} \) we consider two cases.

**Case 1**

\[ S_{j_1} < S_{k_1} \]

Then policy \( B = [S_{k_1}, S_{k_1-1}, \ldots, S_{k_2}, S_{j_1}, S_{j_1}] \) results in a greater total field life than policy A. The proof of this statement follows:

Let \( Q_A \) and \( Q_B \) be the total field life from policy A and policy B respectively. Let \( x = L(S_{k_1}) + \cdots + L(S_{k_2}) + L(S_{j_1}) + \cdots \) then

\[
Q_A = x + L(S_{k_1} + x) + L(S_{j_1})
\]

\[
Q_B = x + L(S_{j_1} + x) + L(S_{k_1});
\]

we must show \( Q_B \geq Q_A \). Now \( x > 0 \) by lemma 2.1 and \( S_{k_1} - S_{j_1} > 0 \).

Furthermore by lemma 2.1 \( x + S_{k_1} < S_0 \) and \( L(\cdot) \) is concave for \( S \leq S_0 \) by hypothesis, thus

\[
\frac{L(S_{k_1} + x) - L(S_{j_1} + x)}{S_{k_1} - S_{j_1}} \leq \frac{L(S_{k_1}) - L(S_{j_1})}{S_{k_1} - S_{j_1}}
\]

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which implies
\[ L(S_{k_1} + x) + L(S_{j_1}) \leq L(S_{j_1} + x) + L(S_{k_1}) \]
hence \( Q_B \geq Q_A \).

**Case 2**

\[ S_{k_1} < S_{j_1} \]

Then policy \( C = [S_{k_1}, S_{k_1-1}, \ldots, S_{k_2}, S_{j_1}; S_{k_2}, S_{k_1}] \) results in a greater total field life than policy \( A \).

Let \( y = L(S_{k_1}) + \cdots + L(S_{k_2}) \) then

\[
Q_A = y + L(S_{k_2}) + y + L(S_{k_1} + y) + L(S_{j_1})
\]

\[
Q_C = y + L(S_{j_1} + y) + L(S_{k_2}) + L(S_{k_1} + L(S_{k_2})) \cdot
\]

We must show \( Q_C \geq Q_A \).

By lemma 2.2 since \( y > 0, y + L(S_{k_2} + y) \geq L(S_{k_2}) \). Now since \( L(\cdot) \) is nonincreasing

\[
L(S_{k_1} + y + L(S_{k_2} + y)) \leq L(S_{k_1} + L(S_{k_2})) \cdot \quad (2.3.23)
\]

By lemma 2.1, \( S_{k_2} + y < S_0 \) and since \( L(\cdot) \) is concave for \( S \leq S_0 \) and since \( S_{k_2} - S_{j_1} > 0 \) then
\[
\frac{L(S_{k_2} + y) - L(S_{j_1} + y)}{S_{k_2} - S_{j_1}} \leq \frac{L(S_{k_2}) - L(S_{j_1})}{S_{k_2} - S_{j_1}}
\]

which implies

\[
L(S_{k_2} + y) + L(S_{j_1}) \leq L(S_{j_1} + y) + L(S_{k_2}). \tag{2.3.24}
\]

Upon combining (2.3.23) and (2.3.24) we have proved

\[
Q_C \geq Q_A.
\]

Note that policy C has reduced the problem by assigning only 1 - 1 items to \( M_k \) and 2 items to \( M_{j_1} \). We will now show that there exists a policy D which is better than policy B where D assigns 1 - 1 items to \( M_k \) and 2 items to \( M_{j_1} \).

Let \( D = [S_{k_1}, S_{k_1-1}, \ldots, S_{k_2}, S_{k_1}; S_{j_1}, S_{j_2}] \) where \( S_{j_1} < S_{k_1} \)

But the same argument as in case 2 above applies since policy B presents the same situation relative to policy D that policy A presents to policy C. Thus \( Q_D \geq Q_B \). Hence in either case \( S_{j_1} < S_{k_1} \) or \( S_{k_1} < S_{j_1} \) we have a policy \( Q_D \geq Q_A \) or \( Q_C \geq Q_A \) which is better than policy A and which assigns 1 - 1 items to \( M_k \) and 2 items to \( M_{j_1} \).

This reduction process continues. If \( i - 1 > 2 \), we consider demand source \( M_{j_2} \) with only one item, \( S_{j_2} \) then

\[
A^* = [S_{k_1}, S_{k_1-1}, \ldots, S_{k_3}, S_{k_1}; S_{j_1}, S_{j_2}]
\]
where $S^*$ is either $S_{11}$ or $S_j$ of the first iteration. In the same manner that it was shown that $S_{j_1} < S_{k_2}$, it is also the case that $S_{j_2} < S_{k_3}$ (actually $S_{j_2} < S_{k_2}$ also but this is not necessary at this point). Then policy $A^*$ is dominated by

$$C^* = [S_{k_i}, S_{k_{i-1}}, \ldots, S_{k_4}, S_{j_2}; S_{k_3}, S_{k_1}, S^*] \text{ if } S_{j_2} > S^*$$

or

$$D^* = [S_{k_i}, S_{k_{i-1}}, \ldots, S_{k_4}, S^*; S_{k_3}, S_{j_2}] \text{ if } S_{j_2} < S^*$$

which reduces the problem to 1 - 2 items assigned to $M_k$ and 2 items assigned to $M_{j_2}$.

We must now show that it is better to go from an $i = 3$ problem to an $i = 2$ problem and then by reduction the theorem has been proved.

Let $S_{k_1} < S_{k_2} < S_{k_3}$ be the $i = 3$ items assigned to $M_k$ and let $S_{j_1}$ be the single item assigned to $M_j$. Now $S_{j_1} < S_{k_2}$ by the same reasoning as given before.

Case 1a: $S_{j_1} > S_{k_1}$ and let $A = [S_{k_3}, S_{k_2}, S_{k_1}; S_{j_1}]$;

$$B = [S_{k_3}, S_{j_1}; S_{k_2}, S_{k_1}]$$

$$Q_A = L(S_{k_3}) + L(S_{k_2} + L(S_{k_3})) + L(S_{k_2} + L(S_{k_3}) + L(S_{k_2} + L(S_{k_3}))) + L(S_{j_1})$$

$$Q_B = L(S_{k_3}) + L(S_{j_1} + L(S_{k_3})) + L(S_{k_2}) + L(S_{k_1} + L(S_{k_2})) .$$
We must show $Q_B \geq Q_A$. Now by lemma 2.2

$$L(s_{k_3}) + L(s_{k_2} + L(s_{k_3})) \geq L(s_{k_2});$$

therefore since $L(\cdot)$ is nonincreasing

$$L(s_{k_2} + L(s_{k_3})) + L(s_{j_1} + L(s_{k_3})) \leq L(s_{k_1} + L(s_{k_2})). \quad (2.3.25)$$

By lemma 2.1 $s_{k_2} + L(s_{k_3}) < s_o$ and since $L(\cdot)$ is concave for $s \leq s_o$

$$\frac{L(s_{k_2} + L(s_{k_3})) - L(s_{j_1} + L(s_{k_3}))}{s_{k_2} - s_{j_1}} \leq \frac{L(s_{k_2}) - L(s_{j_1})}{s_{k_2} - s_{j_1}}.$$

Thus

$$L(s_{k_2} + L(s_{k_3})) + L(s_{j_1}) \leq L(s_{j_1} + L(s_{k_3})) + L(s_{k_2}) \quad (2.3.26)$$

and combining (2.3.25) and (2.3.26) we have $Q_B \geq Q_A$ as desired.

**Case 2a**

$s_{j_1} < s_{k_1}$

Let $C = [s_{k_3}, s_{k_2}, s_{j_1}, s_{k_1}]$ then

$$Q_C = L(s_{k_3}) + L(s_{k_2} + L(s_{k_3})) + L(s_{j_1} + L(s_{k_3})) + L(s_{k_2} + L(s_{k_3})) + L(s_{k_1}).$$
We must show $Q_C \geq Q_A$. By lemma 2.1 $S_{k_1} + L(S_{k_1}) + L(S_{k_2} + L(S_{k_3})) < S_o$

and since $L(\cdot)$ is concave $S \leq S_o$

$$\frac{L(S_{k_1} + L(S_{k_1}) + L(S_{k_2} + L(S_{k_3}))) - L(S_{k_1} + L(S_{k_2} + L(S_{k_3})))}{S_{k_1} - S_{j_1}} \leq \frac{L(S_{k_1}) - L(S_{j_1})}{S_{k_1} - S_{j_1}}$$

and

$$L(S_{k_1} + L(S_{k_1}) + L(S_{k_2} + L(S_{k_3}))) + L(S_{j_1}) \leq L(S_{j_1} + L(S_{k_1}) + L(S_{k_2} + L(S_{k_3}))) + L(S_{k_1})$$

(2.3.27)

Hence by (2.3.27), $Q_C \geq Q_A$. Now let

$$D = [S_{k_1}, S_{k_2}, S_{j_1} \quad S_{k_3}, S_{j_1}]$$

the proof that $Q_D \geq Q_C$ is the same as given in case 1a only with the proper subscripts interchanged. Hence $Q_D \geq Q_A$, and we have shown a better policy exists where 2 items are assigned to $M_k$ and 2 items are assigned to $M_j$.

By reduction, the theorem is proved.

q.e.d.
Theorem 2.6: Let \( L(S) \) have property \( \Omega \). If \( \frac{1}{2}(n + 1) \leq v \leq n \), then FIFO is the optimal issuing policy.

Proof of Theorem 2.6: Note that not only does FIFO \( \in G \) and FIFO is feasible but also that FIFO issues all \( n \) of the items, i.e., none of the items deteriorate to zero in the stockpile.

We will now show that an optimal policy for the conditions given in this theorem also must issue all of the items. This last statement is proved by contradiction. Assume that the optimal policy allows at least one item, say \( S_j \), to expire in the stockpile. Then since \( \frac{1}{2}(n + 1) \leq v \leq n \) there is at least one demand source which receives only one item, say \( S_i \). In addition \( S_i < S_j \) or else by lemma 2.1, \( S_j \) would have positive field life upon the consumption of \( S_i \), i.e., \( S_j + L(S_i) < S_0 \), and \( S_j \) would then be issued. Thus assume \( S_i < S_j \). Now by Lieberman [9] Theorem 3 we have

\[
L(S_j) + L(S_i + L(S_j)) \geq L(S_i)
\]

where equality holds only if \( L'(S) = -1 \) over the range of \( S_i \) and \( S_j \), and strict inequality holds at all other times. Therefore letting \( S_j \) deteriorate to zero in the stockpile can't be optimal. And we obtain a contradiction to the assumption of optimality. But \( S_j \) was a general item which deteriorated in the stockpile, thus the contradiction obtained applies to all \( S_j \), and the optimal policy must issue all \( n \) items.

Thus the optimal policy as well as the FIFO policy issues all items in the stockpile. Now in looking at all policies in \( G \) we can restrict
our attention to looking at only those policies which issue all \( n \) items. Let \( A \in G \) be one of these policies and consider any two demand sources \( M_i \) and \( M_j \) under policy \( A \).

**Case 1** \( M_i \) receives \( S_{i_1}, S_{i_2} \) with \( S_{i_1} < S_{i_2} \)

\( M_j \) receives \( S_{j_1}, S_{j_2} \) with \( S_{j_1} < S_{j_2} \)

Then if the four items \( S_{i_1}, S_{i_2}, S_{j_1}, S_{j_2} \) are not assigned to \( M_i \) and \( M_j \) according to FIFO, then the total field life can be increased by a FIFO assignment since by lemma 2.6 FIFO is optimal for \( n = 4, \nu = 2 \).

**Case 2** \( M_i \) receives \( S_{i_1} \)

\( M_j \) receives \( S_{j_1}, S_{j_2} \) with \( S_{j_1} < S_{j_2} \)

Again, if the three items \( S_{i_1}, S_{j_1}, S_{j_2} \) are not assigned to \( M_i \) and \( M_j \) according to FIFO then the total field life can be increased by a FIFO assignment. By lemma 2.6 FIFO is optimal for \( n = 3, \nu = 2 \).

**Case 3** \( M_i \) receives \( S_{i_1} \) \( M_j \) receives \( S_{j_1} \). Then FIFO is obviously optimal (there is only one policy).

Thus the total field life from all demand sources can be improved until every demand source has a FIFO ordering of its items relative to every other demand source. We will call such an ordering a pairwise-FIFO ordering. It should be noted that any other ordering results in a lower total field life hence pairwise-FIFO is optimal.
We must now show that pairwise-FIFO is the same as FIFO for the total assignment of the n items to the v demand sources. Assume the items are in pairwise-FIFO order. Now relabel the demand sources such that

\[
\begin{align*}
M_v & \text{ has item } S_n \text{ assigned to it } \\
M_{v-1} & \text{ has item } S_{n-1} \text{ assigned to it } \\
& \ddots \\
M_p & \text{ has item } S_{n-p+1} \text{ assigned to it } \\
M_1 & \text{ has item } S_{n-v+1} \text{ assigned to it } .
\end{align*}
\]

(2.3.26)

This relabelling is possible since no two of the items \( S_n, \ldots, S_{n-v+1} \) can be assigned to the same demand source under pairwise-FIFO. Now consider the demand source \( M_p \) which has the two items \( S_{n-p+1} \) and \( S_{i_1} \) assigned to it, for any \( p = 1, \ldots, v \). We must show that \( S_{i_1} = S_{n-v+1} \) then by lemma 2.3 we have a FIFO ordering for the total assignment (since \( p \) was arbitrary). The proof of \( S_{i_1} = S_{n-v+1} \) is by contradiction. Assume \( S_{i_1} \neq S_{n-v+1} \).

**Case 1**

\[ S_{i_1} > S_{n-v+1} \]

Now from above we know \( S_{i_1} < S_{n-v+1} \) and since \( S_{i_1} > S_{n-v+1} \) there are at most \( (n - v) - (n - v - 1 + l) - 1 = i - 2 \) items, with initial life greater than \( S_{i_1} \), which are available for assignment to demand.
sources $M_v, \ldots, M_{p+1}, M_v, \ldots, M_{p+1}$ are the first i - 1 demand sources to consume their initial items. Hence some item $S_i < S_i$ must be assigned to one of these i - 1 demand sources, say demand source $M_{p+j}$ ($j \geq 1$). Then the pairwise ordering for $M_p$ and $M_{p+j}$ is $[S_{n-i+1}, S_i, S_{n-i+1+j}, S_{i+j}]$; but $S_{n-i+1} < S_{n-i+1+j}$ and $S_i > S_{i+j}$ is not a FIFO ordering, hence we obtain a contradiction to the assumption of pairwise-FIFO. Therefore $S_i \not< S_{n-v-i+1}$.

Case 2

$S_i < S_{n-v-i+1}$

As shown above (2.3.28) there are i - 1 demand sources with items whose initial ages are greater than $S_{n-i+1}$. And since $S_i < S_{n-v-i+1}$ there are at least $(n - v + 1) - (n - v - i + 1) = i$ items such that $S_{n-v-i+1+j} > S_{i}$ for $j = 0, 1, \ldots, i - 1$ and these items are available for issuance to the first i - 1 demand sources requesting items viz. $M_v, \ldots, M_{p+1}$. Hence there is at least one $S_{n-v-i+1+j}$ which must be issued to one of the $M_t$ where $t = 1, 2, \ldots, i - 1$. But then the issue policy for $M_p$ and $M_t$ is $[S_{n-i+1}, S_i, S_{n-i+1+k}, S_{n-v-i+1+j}]$ where $k = p - t$. But $S_{n-i+1} > S_{n-i+1-k}$ and $S_i < S_{n-v-i+1+j}$ is not a FIFO ordering; hence we obtain a contradiction to pairwise-FIFO. Therefore $S_i \not< S_{n-v-i+1}$.

Combining cases 1 and 2 we have $S_i = S_{n-v-i+1}$ and by lemma 2.3 since p was arbitrary, FIFO is optimal.

q.e.d.

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