An Auction Model Arising from an Internet Search Service Provider

Wei Shi Lim and Christopher S. Tang

September 16, 2004

Abstract

Most search service providers such as Lycos and Google either produce irrelevant search results or unstructured company listings to the consumers. To overcome these two shortcomings, search service providers such as GoTo.com have developed mechanisms for firms to advertise their services and for consumers to search for the right services. To provide relevant search results, each firm who wishes to advertise at the GoTo site must specify a set of keywords. To develop structured company listings, each firm bids for priority listing in the search results that appear on the GoTo site. Since the search results appear in descending order of bid price, each firm has some control over the order in which the firm appears on the list resulting from the search. In this paper, we present a one-stage game for two firms that captures the advertising mechanism of a search service provider (such as GoTo). This model enables us to examine the firm’s optimal bidding strategy and evaluate the impact of various parameters on the firm’s strategy. Moreover, we analyze the conditions under which all firms would increase their bids at the equilibrium. These conditions could be helpful to the service provider when developing mechanisms to entice firms to submit higher bids.

Keywords: On-line Advertising, Search Service Provider, Game Theory, Auction Theory, Bidding, Bayesian Nash Equilibrium.

1Address all correspondence to Lim, weishi@nus.edu.sg: The NUS Business School, 1 Business Link, National University of Singapore, Singapore 117592. Tang, ctang@anderson.ucla.edu: The Anderson School, UCLA, Los Angeles, CA 90095. The model developed in this paper is motivated by the business model offered by GoTo.com. No implication of the actual practice is intended. This research is partially supported by UCLA James Peters Research Fellowship.
1 Introduction

As Internet becomes the common place for billions of companies to market their products and services, many companies find it difficult to reach the target consumers and consumers find it challenging to search for the companies with the right products or services. Specifically, most Internet search engines such as Lycos and Google have the fundamental problems of irrelevant search results and unstructured company listings. Consequently, many firms (or advertisers) find advertising on the Internet to be a weak revenue model.

To overcome these shortcomings, we witnessed the formation of electronic Yellow Pages that allow firms to place their advertisements. However, the electronic Yellow Pages charges the firm (or advertiser) a fixed charge, and lists the companies in alphabetical order. This Internet advertising mechanism has two drawbacks. Firstly, the firm has to pay a fixed charge regardless of the effectiveness of the advertisement (measured in terms of the number of clicks to the advertising firm’s homepage). Secondly, a firm with a name that starts with ‘Z’ will be listed last on the list, getting little exposure to consumers.

As a way to respond to the above challenges, GoTo.com (renamed as Overture in October, 2001\footnote{Internet giant Yahoo announced on July 14 2003 that it was acquiring Overture for about $1.63 billion in cash. Yahoo said in a statement that commercial search is ‘the most dynamic and fastest growing segment’ of the Internet, with revenues in the sector expected to grow to about $5 billion by 2006 (Business Times, July 14, 2003).}) has created an innovative search/advertising service that works as follows. First, firms are grouped according to a specific set of keywords. Second, each group of firms that corresponds to a specific keyword can list their firms on GoTo’s search site. Third,
each firm bids for priority listing in the search results that appear on the GoTo site. Specifically, the search results appear in descending order of bid price. Fourth, when a consumer clicks on the name of a firm listed in the search results, the consumer will be linked automatically to the homepage of this firm and this firm has to pay GoTo an amount equal to the bid price regardless of whether the consumer makes a purchase or not.

To illustrate, we have used the GoTo search service to find a group of firms associated with the keyword ‘Palm Pilot.’ GoTo provided a list of 240 firms with bids varying from $0.42 to $0.01. Specifically, the top three bidders were Office Depot ($0.42 bid), Camera Club ($0.41 bid) and HandSpring ($0.38 bid). Since the search results appeared in descending order of the bid, we observed that Office Depot was listed first, Camera Club second, and HandSpring third. In this case, Office Depot would have to pay GoTo $0.42 had we clicked on the hyperlink associated with Office Depot that appeared on this list.

The GoTo search service offers firms two major benefits over the electronic Yellow Pages and other Internet search engines. First, since the firm pays GoTo the amount of its bid only when a consumer clicks on the firm’s listing, GoTo offers pay-for-performance service (i.e., variable cost of advertising instead of fixed cost). Second, since the search results appear in descending order of the bid and since the bids are revealed to all firms, each firm can submit a new bid anytime so as to change the order at which it appears on the list. However, before a firm submits a higher (or a lower) bid, the firm needs to evaluate the financial impact associated with the uncertain rank-based response (i.e., number of clicks the firm will receive if the firm is ranked first, second, third, etc.). The uncertain
rank-based response motivates us to develop a model to examine the following questions:

1. From the perspective of the service provider, how can it induce participating firms to bid aggressively?

2. Given the bids of other firms, how should a firm determine its optimal bid?

3. How would the uncertain rank-based response affect a firm’s bidding strategy?

4. How would a firm’s current market share affect its bidding strategy?

5. Is it beneficial for the service provider to disclose more information regarding the rank-based response to the firms?

Our model enables us to develop various insights, including the following results:

1. A firm is more likely to bid more aggressively when the rank-based response is highly ‘rank sensitive’ (i.e., more clicks for being ranked first) or when the marginal profit resulting from each click is high. In particular, a small loyal market or a large disloyal market encourages aggressive bidding. As a result, all firms will bid the highest when they perceive the rank-based response to be highly rank sensitive and when their rank-based response are positively correlated. On the other hand, firms bid the lowest as a dominant strategy if they perceive the rank-based response to be insufficiently rank sensitive.

2. From the perspective of the service provider, offering more information about the effectiveness of the advertising platform encourages firms to bid aggressively if the
perceived mean of the rank-based response is sufficiently low.

In this paper, we construct a new bidding model that captures the advertising mechanism of the search service provider (such as GoTo). Although the auction literature is rich [7], our model differs from traditional auction models in the following aspects: Firstly, the ‘items’ that are put up for bidding here are the rankings in the listing of firms associated with the same keyword. As such, the value of each item is captured by the uncertain rank-based response, which depends on factors that are common to all firms as well as factors (such as market share, profit margin, etc.) that are private to individual firms. As such, unlike most auction models that deal with either common value auction or private value auction, our model captures the mixture of the common and private values of the items being auctioned. Secondly, although our auction mechanism resembles that of an English auction with multiple items, the ‘items’ up for grabs are not identical in our case: the value of being ranked first is significantly different from being ranked second, and a firm which fails to ‘win’ a higher ranking automatically qualifies to ‘win’ a lower one.

This paper is organized as follows. Section 2 presents a model that captures the advertising and bidding mechanisms. We also determine the payoff function associated with different bidding strategies. In Section 3, we analyze the best response function (i.e., the optimal bidding strategy of a firm given the bids of other firms and one’s private forecast). Section 4 evaluates the conditions under which the pure-strategy Bayesian Nash equilibrium would be the bidding strategy that the service provider desires the most. The paper ends with some concluding remarks in Section 5.
2 The Model

To simplify the exposition of our model, we shall consider a one-stage game in which two firms are planning to list their names under the same keyword by using the search service provider that operates in the same manner as GoTo. The notion of loyal customers is commonly used in the marketing literature to capture the market segment that is dedicated to one brand or one firm only. Among the consumers who use this particular keyword to conduct a search, let $a$ be the total number of loyal customers (customers who have a clear preference of a firm, will not switch to another firm, and thus will not be interested in another firm at all) and let $b$ be the total number of disloyal customers (those who do not have a clear firm preference). For those customers who visit more than one site, they are considered as ‘disloyal’ customers in our paper. We assume that both $a$ and $b$ are common knowledge to both firms. This assumption is reasonable as a fairly good assessment of the values $a$ and $b$ can be obtained through market research. Also, by including two segments of customers, our model captures the possibility that loyal customers visit the search site as well. Let $b_i$ be the bid submitted by Firm $i$ ($i = 1, 2$). In this case, Firm $i$ will appear first on the list and Firm $j$ will appear second if $b_i > b_j$, ($j \neq i$). The reverse holds if $b_i < b_j$.

Let $l_i$ denote the market share of Firm $i$; i.e., $l_i$ is the proportion of clicks that Firm $i$ receives from the loyal customers. Since the loyal customers have to be loyal to either Firm 1 or Firm 2 but not both, we must have $l_1 + l_2 = 1$. Let $p_t$ ($t = 1, 2$) denote the rank-based response (i.e., the proportion of clicks that a firm receives from the disloyal
customers when a firm is ranked \(t\). To simplify our exposition, we shall refer to \(p_1\) as the first rank-based response that represents the proportion of clicks that a firm receives for being ranked first. We shall assume that \(p_t\) is a random variable that depends solely on the ranking. After receiving the bids, the service provider announces the bid and the ranking of each firm to both firms. Firm \(i\) that is ranked \(t\) will receive the total number of clicks, \(N_{i,t}\), where the random variable \(N_{i,t}\) can be expressed as \(N_{i,t} = a_i + b p_t\). It is reasonable to assume that a disloyal customer may click on only one or both firms upon receiving the search results, we shall assume that \(p_1 + p_2 = k\), where \(k \in [0, 2]\) and \(k\) represents the average number of clicks committed by a disloyal customers upon receiving the search results.\(^2\) For simplicity, we write the first rank-based response \(p_1\) as \(p\) and \(p_2\) as \(k - p\). To model the uncertain nature of the rank-based response \(p_t\), we assume that \(p\) is a random variable such that \(p = \bar{p} + e\), where \(e\) is assumed to be normally distributed with mean zero and variance \(V\), i.e., \(E[(p - \bar{p})^2] = V\). The normality assumption has limitations because it allows for values of \(p\) outside the range \([0, 1]\). However, it has been used extensively in past literature because it simplifies the analysis considerably (c.f., [1], [3], [6], [8], [9]).

We consider the case in which Firm \(i\) obtains private forecast \(f_i\) (\(i = 1, 2\)) about the first rank-based response \(p\).\(^3\) Clearly, Firm \(i\)’s forecast of \(k - p\) is \(k - f_i\). Let \(f_i = p + \epsilon_i\),

\(^2\)As the service provider can easily announce the value of \(k\) to all firms based on historical data that it has captured over time, we shall assume that \(k\) is common knowledge to all firms.

\(^3\)If a firm has prior experience with the service provider, one can also regard the forecast as an outcome of Bayesian updating. Here, we shall simply assume that the forecast is obtained by the firm using whatever market information gathering techniques at its disposal.
\( i = 1, 2, \) where \( \epsilon_i \) is normally distributed, independent of \( p \), with mean zero and variance \( s_i \).

The variance \( s_i \) measures Firm \( i \)'s forecast accuracy of \( p \). In this paper, we shall examine the impact of Firm \( i \)'s forecasting accuracy \( s_i \) on its bidding strategy. In addition, since the service provider has more information regarding \( p \), the service provider can help the firm to reduce the magnitude of \( s_i \) by disclosing more information on \( p \) to the firms. This observation motivates us to examine the conditions under which it is beneficial for the search provider to disclose more information on \( p \) to the firms so as to entice them to bid more aggressively.

Since \( p = \bar{p} + e \) and since \( f_i = p + \epsilon_i \), the deviation of the forecast \( f_i \) from the mean \( \bar{p} \) is given by \( e + \epsilon_i \). We assume that the error terms \( (\epsilon_1, \epsilon_2) \) follow a bivariate normal distribution, independent of \( p \), with mean zero and covariance matrix \( \begin{pmatrix} s_1 & \sigma_{12} \\ \sigma_{12} & s_2 \end{pmatrix} \), where \( s_i \geq \sigma_{12} \geq 0, \ i = 1, 2 \).\(^4\) Since \( p = \bar{p} + e \) and since \( f_i = p + \epsilon_i \), it is easy to show that the posterior distribution of \( p \) given \( f_i \) is normally distributed (c.f., [4], [5], [8], [9]), and

\[
E(p|f_i) = (1 - t_i)\bar{p} + t_i f_i, \quad i = 1, 2, \quad \text{where} \quad t_i = \frac{V}{V + s_i}. \tag{1}
\]

Observe from (1) that \( E(p|f_i) = f_i \) when \( s_i = 0 \), and \( E(p|f_i) = \bar{p} \) when \( s_i = \infty \).

\(^4\)If the firms pool their resources and allow each of their private information to be put in a common pool available to both firms, \( s_1 = s_2 = \sigma_{12} \). On the other hand, if there is no sharing of information, the error terms of the forecasts are independent and \( \sigma_{12} = 0 \).
2.1 Expected Payoffs

We now determine the expected profit of Firm \( i \), conditional on the forecast \( f_i \). Let \( \theta_i \) denote the average margin that Firm \( i \) gets from each click to its homepage.\(^5\) For simplicity, we shall assume that each Firm \( i \) (\( i = 1, 2 \)) chooses its bid \( b_i \) from the set \( \{ \alpha, \beta, \gamma \} \), where \( \alpha = 3\gamma, \beta = 2\gamma \).\(^6\) We assume that the net profit per click is more than the maximum bid, i.e., \( \theta_i > 3\gamma \); otherwise, the firm would not bid ‘high’ when it uses this search service provider to advertise, thereby defeating our main objective in this paper, which is to derive the conditions under which a firm bids aggressively. When Firm \( i \) bids \( b_i \) and Firm \( j \) bids \( b_j \), the expected payoff of Firm \( i \), conditional on the forecast \( f_i \) of Firm

---

\(^5\)One can estimate \( \theta_i \) by using the information on the percentage of clicks that leads to a purchase and the profit margin of each purchase. Specifically, all firms belong to the same category selling similar products or services, which is why they use similar keywords to market themselves. Given the fact that they are in the same market selling similar products, each firm can deduce the margins of other firms. We assume that \( \theta_i \) is common knowledge as the purpose of the paper is to study the bidding behavior of the firms and how the search service provider can influence these bids and \( \theta_i \) is not a parameter that can be manipulated by the search service provider.

\(^6\)In reality, the firm has many bidding options; however, we limit our study to this special case for the following reasons. First, if we consider more than 3 choices of bids, then our analysis would become intractable. Second, we aim to understand the optimal bidding strategy of each firm and hope to obtain some insights regarding how different parameters affect the firm’s bidding behavior. We hope the analysis presented in this paper can be viewed as a building block for future studies.
The above equation can be explained as follows. When $b_i > b_j$, Firm $i$ has a higher bid and is ranked first; hence, Firm $i$ will obtain $al_i$ clicks from the loyal customers and $bE(p|f_i)$ clicks from the disloyal customers. It follows from the fact that the ‘effective’ margin per click is equal to $(\theta_i - b_i)$, we obtain the first expression. Using the same reasoning as before, we obtain the second and the third expressions. In the case of a tie (when $b_i = b_j$), we adopt the usual convention in auction studies that the ‘winner’ is randomly selected from the two firms. Substituting (1) into (3), we have:

$$E(\Pi_i(b_i, b_j)|f_i) = \begin{cases} 
(\theta_i - b_i)(al_i + bE(p|f_i)) & \text{if } b_i > b_j, \\
(\theta_i - b_i)(al_i + b(k - E(p|f_i))) & \text{if } b_i < b_j, \\
(\theta_i - b_i)(al_i + bE(p|f_i)) & \text{if } b_i = b_j.
\end{cases}$$

(3)

In order for Firm $i$ to determine a good bidding strategy, Firm $i$ has to anticipate the strategy of Firm $j$, which entails Firm $i$ to estimate the expected profit of Firm $j$. To do so, we first estimate Firm $j$’s private forecast $f_j$ conditional on Firm $i$’s private forecast $f_i$, and then we estimate Firm $j$’s expected profit. First, given the private forecast $f_i$, Firm $i$ can also deduce the forecast of Firm $j$ ($j \neq i$). As shown in [8] (Page 1078) and [9] (Page 76), the expected value of $f_j$ given $f_i$ is given as:

$$E(f_j|f_i) = (1 - d_i)\bar{p} + d_if_i, \quad i = 1, 2, j \neq i, \quad \text{where}$$

(5)
\[ d_i = \frac{V + \sigma_{12}}{V + s_i}. \]

Observe that when \( f_i \) and \( f_j \) are independent, i.e., \( \sigma_{12} = 0 \), \( d_i = t_i \) and \( E(f_j|f_i) \) is simply \( E(p|f_i) \). In this case, Firm \( i \)'s estimate of \( f_j \) given \( f_i \) is simply its estimate of \( p \). On the other hand, if \( f_i \) and \( f_j \) are perfectly correlated with \( \sigma_{12} = s_i \), \( d_i = 1 \) and \( E(f_j|f_i) = f_i \). Firm \( i \)'s estimate of \( f_j \) is simply \( f_i \) itself.

Firm \( i \) can utilize (5) to determine the conditional expectation of Firm \( j \)'s expected payoff \( E(\Pi_j(b_i, b_j)|f_i) \) as follows:

\[
E(\Pi_j(b_i, b_j)|f_i) = \begin{cases} 
(\theta_j - b_j)(al_j + b\bar{p} + bt_jd_i(f_i - \bar{p})) & \text{if } b_j > b_i, \\
(\theta_j - b_j)(al_j + bk - b\bar{p} - btjd_i(f_i - \bar{p}) & \text{if } b_j < b_i, \\
(\theta_j - b_j)(al_j + bk^2) & \text{if } b_i = b_j.
\end{cases}
\]

Equation (6) can be explained as follows. For instance, when \( b_j > b_i \), Firm \( j \) has a higher bid and is ranked first. In this case, Firm \( j \) obtains \( al_j \) clicks from the loyal customers and \( b(E(\Pi_j(b_i, b_j)|f_i)) \) clicks from the disloyal customers. From (1) and (5), it is easy to show that Firm \( j \) will obtain a total of \( (al_j + b\bar{p} + btjd_i(f_i - \bar{p})) \) clicks from the customers. Combining this with the fact that the 'effective' margin for Firm \( j \) is equal to \( \theta_j - b_j \), we obtain the first expression. We can use the same reasoning for the other two expressions. In this case, Firm 1 can utilize (4) and (6) to construct the expected payoff matrix of Firm 1 and Firm 2, given Firm 1's private forecast \( f_1 \). This payoff matrix, as envisioned by Firm 1, is displayed in Table 1. In Table 1, the strategy choices of Firm 1 are given in the rows while the strategic choices of Firm 2 are given in the columns. Also, in each cell, the expression on the top corresponds to the expected payoff of Firm 1 (i.e., \( E(\Pi_1(b_1, b_2)|f_1) \)), while the
expression at the bottom corresponds to Firm 1’s estimate of the expected payoff of Firm 2; i.e., $E(\Pi_2(b_1, b_2)|f_1)$. By symmetry, given it’s private forecast of $f_2$, Firm 2 has similar expressions for its expected payoffs and that of Firm 1. The expected payoff matrix, as envisioned by Firm 2, is omitted here.

$$
\begin{array}{|c|c|c|}
\hline
\text{Firm 1 / Firm 2} & \alpha(=3\gamma) & \beta(=2\gamma) \\
\hline
\alpha(=3\gamma) & (\theta_1 - 3\gamma)(a_1 + b_k + b_2) & (\theta_1 - 3\gamma)(a_1 + b_k + b_2) \\
 & (\theta_2 - 3\gamma)(a_2 + b_k + b_2) & (\theta_2 - 3\gamma)(a_2 + b_k + b_2) \\
\hline
\beta(=2\gamma) & (\theta_1 - 2\gamma)(a_1 + b_k - b_p - b_2) & (\theta_1 - 2\gamma)(a_1 + b_k - b_p - b_2) \\
 & (\theta_2 - 3\gamma)(a_2 + b_k + b_2 d_1) & (\theta_2 - 3\gamma)(a_2 + b_k + b_2 d_1) \\
\hline
\gamma & (\theta_1 - \gamma)(a_1 + b_k - b_p - b_2) & (\theta_1 - \gamma)(a_1 + b_k - b_p - b_2) \\
 & (\theta_2 - 3\gamma)(a_2 + b_k + b_2 d_1) & (\theta_2 - 3\gamma)(a_2 + b_k + b_2 d_1) \\
\hline
\end{array}
$$

Table 1: Expected Payoffs of Firm 1 and Firm 2 As Envisioned by Firm 1

Given the payoff matrix, we shall analyze the bidding strategy of each firm in Section 3. Essentially, a bidding strategy for Firm $i$ ($i = 1, 2$) is a function that specifies a bid $b_i$ for any given forecast $f_i$. In Section 4, we find the conditions under which aggressive bidding is the only Bayesian Nash equilibrium. The Bayesian Nash equilibrium is used here as we are concerned with the question of how firms end up bidding high even before they observe the bids of the opposing firms. A Bayesian Nash equilibrium is a pair of strategies and
a pair of beliefs such that (i) each strategy is the best response to the firm’s belief about the behavior of the other firm, and (ii) the beliefs are correct. In our case, given Firm \(i\)’s forecast \(f_i\), his correct conjecture regarding Firm \(j\)’s forecast is given by (5). In our analysis, we assume that both firms are risk neutral.

3 Best Response Function

In this section, we analyze the best response function of a firm, given the strategy choice of the other firm.

3.1 Critical Points

Let us first define the notation \(\phi_{ki} (i = 1, 2, k = 1, 2, 3, 4)\), which will be shown to be critical points that signify changes in the bidding strategies of the firms as we derive the best response functions later. For \(i = 1, 2\), let \(\phi_{1i} = \left(\frac{2\gamma a_i}{b(\theta_i - \gamma)} + \frac{\theta_i + \gamma k}{\theta_i - \gamma} - \bar{p}\right)(1 + \frac{s_i}{V})\), \(\phi_{2i} = \left(\frac{\gamma a_i}{b(\theta_i - \gamma)} + \frac{\theta_i - \gamma}{\theta_i - 2\gamma} - \bar{p}\right)(1 + \frac{s_i}{V})\), \(\phi_{3i} = \left(\frac{\gamma a_i}{b(\theta_i - 2\gamma)} + \frac{\theta_i - \gamma}{\theta_i - 2\gamma} - \bar{p}\right)(1 + \frac{s_i}{V})\), \(\phi_{4i} = \left(\frac{\gamma a_i}{b(\theta_i - \gamma)} + \frac{\theta_i - \gamma}{\theta_i - 2\gamma} - \bar{p}\right)(1 + \frac{s_i}{V})\). These values \(\phi_{ki} (k = 1, 2, 3, 4)\) change as the magnitude of \(\theta_i\) varies. It is straightforward to compare these values to determine their relative order. For instance, since \(\theta_i\) is greater than the minimum bid \(\gamma\), we have \(\phi_{1i} > \phi_{4i}\) because \(\phi_{1i} - \phi_{4i} = \left(\frac{a_i}{b} + \frac{k}{2}\right)\frac{\gamma}{\theta_i - \gamma}(1 + \frac{s_i}{V}) > 0\).

By using similar arguments, it is easy to show that:

- **Observation 1** For \(\theta_i \in (3\gamma, 5\gamma)\), \(\phi_{4i} < \phi_{3i} < \phi_{1i} < \phi_{2i}\).

- **Observation 2** For \(\theta_i \in (5\gamma, \infty)\), \(\phi_{4i} < \phi_{3i} < \phi_{2i} < \phi_{1i}\).
The critical points $\phi_{ki}$ have the following properties that will prove useful in our subsequent analysis. (All proofs are given in the Appendix.)

**Lemma 3.1** Suppose $\theta_i > \gamma$. Then $\phi_{1i}$ and $\phi_{4i}$ are increasing in $l_i, a, k$ and $\gamma$ but decreasing in $b, \bar{p}$ and $\theta_i$. Furthermore, $\phi_{ki}$ ($k = 1, 4$) is increasing in $s_i$ if and only if $\phi_{ki}$ is positive.

**Lemma 3.2** Suppose $\theta_i > 3\gamma$. Then $\phi_{2i}$ is increasing in $\gamma, l_i, a, k$ but decreasing in $\theta_i, b, \bar{p}$. Furthermore, $\phi_{2i}$ is increasing in $s_i$ if and only if $\phi_{2i}$ is positive.

In the remainder of this section, we shall deduce the best response function of Firm $i$, given the strategic choices of Firm $j$ ($j \neq i$). We use $R_i(x, f_i)$ to denote Firm $i$’s best response function when Firm $j$ bids $x$ and the forecast of Firm $i$ is $f_i$. We shall proceed by considering three cases, namely, when Firm $j$ bids $\alpha, \beta$ and $\gamma$.

### 3.2 Best Response Function: when Firm $j$ bids $\alpha$

When Firm $j$ bids $\alpha = 3\gamma$, it is clear that Firm $i$ should never bid $\beta (= 2\gamma)$ because Firm $i$ can preserve its ranking by submitting a lower bid $\gamma$ instead of $\beta$. By comparing Firm $i$’s expected payoffs associated with bids $\alpha$ and $\gamma$, we obtain the following lemma.

**Lemma 3.3** Suppose Firm $j$ bids $\alpha$. For $\theta_i > \alpha (= 3\gamma)$, the best response function of Firm $i$ given its forecast $f_i$ is given by:

$$R_i(\alpha, f_i) = \begin{cases} \alpha & \text{if } f_i \geq \bar{p} + \phi_{1i} \\ \gamma & \text{otherwise.} \end{cases}$$
The above lemma implies that for Firm $i$ to match a bid of $\alpha$ by Firm $j$ with a bid of $\alpha$, the forecast of the first rank-based response $p$ must be large enough, i.e., $f_i \geq \bar{p} + \phi_{1i}$. Therefore, as this critical value $\bar{p} + \phi_{1i}$ decreases, the chance of Firm $i$ bidding $\alpha$ increases. By examining the properties of $\phi_{1i}$ as stated in Lemma 3.1, we can develop the following corollary by using the result stated in Lemma 3.3.

**Corollary 3.1** Suppose Firm $j$ bids $\alpha$. Then Firm $i$ is more likely to bid $\alpha$ (rather than $\gamma$) if and only if $\phi_{1i}$ is smaller. In other words, aggressive response by Firm $i$ is encouraged as $\gamma, l_i, a, k$ decreases, $\theta_i, b, \bar{p}$ increases.

The above corollary has four interpretations. Firstly, as the number of disloyal customers $b$ becomes larger or when the number of loyal customers $a l_i$ becomes smaller, it is easy to check that $\phi_{1i}$ becomes smaller. As such, Firm $i$ has more incentive to submit a higher bid in order to rank first, which will result in more clicks from the disloyal customers. More specifically, this is also true when no loyal customers search using the site (i.e., $a = 0$). Secondly, recall that $k$ represents the average number of clicks from a disloyal customer. Therefore, as $k$ decreases, the number of clicks expected from being ranked second ($= k - p$) decreases. This in turn enhances the incentive for Firm $i$ to bid aggressively as $k$ decreases.

We say that the first ranked-based response is more rank sensitive if the pool of disloyal customers is large, the current market share is small and $k$ is small. This is because the benefits for a firm to be ranked first is increased. In other words, a firm is more likely to respond aggressively if the first rank-based response is highly ‘rank sensitive’. Thirdly, as the ‘effective’ margin $\theta_i$ increases, the critical point $\bar{p} + \phi_{1i}$ becomes smaller (Lemma 3.1). 
Therefore, Firm \( i \) has more incentive to bid higher in order to increase the total number of clicks. Finally, as the average value of the first rank-based response \((\bar{p})\) increases, Firm \( i \) will be more inclined to bid more aggressively.

The following corollary examines the impact of information on Firm \( i \)'s bidding behavior when Firm \( j \) bids \( \alpha \).

**Corollary 3.2** Suppose Firm \( j \) bids \( \alpha \). Given its forecast \( f_i \), Firm \( i \) is more likely to respond aggressively with a bid of \( \alpha \) as \( s_i \) decreases if and only if \[
\frac{2\gamma a_i}{b(\theta_i-\gamma)} + \frac{\theta_i + \gamma k}{\theta_i - \gamma} - \bar{p} > 0,
\]
or equivalently, if \( a_i, k \) are large and \( \theta_i - \gamma, b, \bar{p} \) are small.

The implication of the above corollary is very interesting as it is not entirely intuitive. In general, the conditions for \( \phi_{1i} > 0 \) (i.e., large loyal market or small disloyal market for Firm \( i \), low profit margin \((\theta_i - \gamma)\), large \( k \) or \( \bar{p} \), etc.) would discourage Firm \( i \) from bidding aggressively. However, the effects of these disincentives are reduced when the uncertainty level of the rank-based response \( s_i \) is reduced. Specifically, as \( s_i \) decreases, the critical value \( \bar{p} + \phi_{1i} \) decreases, and hence, Firm \( i \) is more likely to bid aggressively. This is because as uncertainty reduces, Firm \( i \) exhibits more trust in its forecast \( f_i \) and as a result is less stringent on the cut off point from which it will bid \( \alpha \). The opposite holds when \( \phi_{1i} < 0 \).

The implication of the above corollary for the management of information of the service provider is significant. Firstly, we note that \( \phi_{1i} = \frac{2\gamma a_i}{b(\theta_i-\gamma)} + \frac{\theta_i + \gamma k}{\theta_i - \gamma} - \bar{p} \) is common knowledge to all. When \( \phi_{1i} > 0 \), information can lead a firm to be more aggressive in his bidding behavior. In this case, the search service provider may provide more information on \( p \) so as to help Firm \( i \) to reduce the uncertainty of the rank-based response \( s_i \). By doing so,
the service provider would entice the firm to bid higher so as to increase the search service provider’s revenue. However, when \( \phi_{1i} < 0 \), information can lead a firm to be conservative in his bidding behavior. In this case, the search service provider should provide little information on \( p \) so as to get the firm to bid higher. Therefore, our analysis provides the underlying conditions under which it is beneficial for the search service provider to provide more information on \( p \) to the firms.

3.3 Best Response Function: when Firm \( j \) bids \( \beta \)

When Firm \( j \) bids \( \beta \), Firm \( i \) may choose to bid \( \gamma \), \( \beta \), or \( \alpha \) when \( (\alpha =)3\gamma < \theta_i \). By a direct comparison of these three potential bids and applying Observations 1 and 2, we observe the following:

**Lemma 3.4** Suppose Firm \( j \) bids \( \beta \). For \( \theta_i > 3\gamma \), the best response function of Firm \( i \), given its forecast \( f_i \), is given by

\[
R_i(\beta, f_i) = \begin{cases} 
\alpha & \text{if } f_i \geq \bar{p} + \phi_{2i}, \\
\beta & \text{if } \bar{p} + \phi_{4i} \leq f_i \leq \bar{p} + \phi_{2i}, \\
\gamma & \text{if } f_i \leq \bar{p} + \phi_{4i}.
\end{cases}
\]

When \( \theta_i \geq 3\gamma (= \alpha) \), the best response of Firm \( i \), given Firm \( j \)'s bid of \( \beta \) is non-decreasing in Firm \( i \)'s forecast \( f_i \). Specifically, when Firm \( i \)'s forecast is sufficiently high (i.e., when \( f_i \geq \bar{p} + \phi_{2i} \)), Firm \( i \) responds with an aggressive bid of \( \alpha \). If its forecast is relatively low (i.e., when \( f_i < \bar{p} + \phi_{4i} \)), the best response of Firm \( i \) is to bid at the lowest possible bid of \( \gamma \). For moderate values of the forecast, Firm \( i \) merely matches Firm \( j \)'s bid of \( \beta \).
Like Corollaries 3.1 and 3.2, the following corollaries examine the conditions under which Firm $i$ is more likely to respond with a higher bid when Firm $j$ bids $\beta$ and the impact of information on Firm $i$’s bidding strategy.

**Corollary 3.3** Suppose Firm $j$ bids $\beta$. Then Firm $i$ is more likely to bid aggressively if and only if $\gamma, l_i, a, k$ decreases and $\theta_i, b, \bar{p}$ increases.

**Corollary 3.4** Suppose Firm $j$ bids $\beta$. Given its forecast $f_i$, Firm $i$ is more likely to respond aggressively as $s_i$ decreases if and only if $\frac{\gamma a_i}{b(\theta_i - 3\gamma)} + \frac{\theta_i - 2\gamma k}{\theta_i - 3\gamma 2} - \bar{p} > 0$ for $\theta_i \in (\gamma, 3\gamma)$, or $\frac{\gamma a_i}{b(\theta_i - 3\gamma)} + \frac{\theta_i - 2\gamma k}{\theta_i - 3\gamma 2} - \bar{p} > 0$ for $\theta_i > 3\gamma$.

Both Corollaries 3.3 and 3.4 are analogous to Corollaries 3.1 and 3.2. Generally, we find that firms are more likely to respond aggressively if the incentives to do so increase. These incentives include a ‘rank sensitive’ first ranked-based response, a high profit margin and a high average first rank-based response $\bar{p}$. Moreover, as the uncertainty level of the rank-based response reduces (i.e., as $s_i$ decreases), firms will bid more aggressively (conservatively) when the corresponding critical points $\phi_{4i}$ and $\phi_{2i}$ are positive (negative).

### 3.4 Best Response Function: when Firm $j$ bids $\gamma$

Given that Firm $j$ bids $\gamma$, there is no need for Firm $i$ to bid $\alpha$. This is because Firm $i$ can always secure the first position by bidding slightly lower at $\beta$. Therefore, it suffices for Firm $i$ to consider bidding either $\beta$ or $\gamma$. By comparing Firm $i$’s expected payoff when bidding $\beta$ and $\gamma$, we established the following result:
Lemma 3.5 Suppose Firm \( j \) bids \( \gamma \). For \( \theta_i > 3\gamma \), the best response function of Firm \( i \), given its forecast \( f_i \) is

\[
R_i(\gamma, f_i) = \begin{cases} 
\beta & \text{if } f_i \geq \bar{p} + \phi_{3i}, \\
\gamma & \text{otherwise.}
\end{cases}
\]

In the following, we derive results regarding how the parameters affect the bidding behavior of Firm \( i \) when Firm \( j \) bids \( \gamma \). Not unexpectedly, these results corroborate with Corollaries 3.1, 3.2 (when Firm \( j \) bids \( \alpha \)), and Corollaries 3.3, 3.4 (when Firm \( j \) bids \( \beta \)).

Corollary 3.5 Suppose Firm \( j \) bids \( \gamma \). For \( \theta_i > 3\gamma \), Firm \( i \) is more likely to bid aggressively (i.e., bid \( \beta \) rather than \( \gamma \)) if and only if \( \phi_{3i} \) is smaller. Equivalently, aggressive response by Firm \( i \) is encouraged as \( \gamma, l_i, a, k \) decreases, \( \theta_i, b, \bar{p} \) increases.

Corollary 3.6 Suppose Firm \( j \) bids \( \gamma \). Given its forecast \( f_i \), Firm \( i \) is more likely to respond aggressively as \( s_i \) decreases if and only if \( \frac{\gamma a_i}{b(\theta_i - 2\gamma)} \frac{b_i - \gamma}{\theta_i - 2\gamma} - \bar{p} > 0 \).

Corollaries 3.2, 3.4 and 3.6 enable us to deduce that, as the uncertainty level of the rank-based response reduces (i.e., as \( s_i \) decreases), firms will bid more aggressively (conservatively) when the corresponding critical points \( \phi_{ki} \) are positive (negative), or rather when \( \bar{p} \) is sufficiently small (large). Therefore, as the search provider ponders over whether it should provide firms with more information about the rank-based response \( p \) so as to reduce the uncertainty level \( s_i \), the search provider needs to evaluate the conditions (i.e., whether the critical points are positive) to see if it is beneficial to do so. Offering more information to the firms can benefit the search provider under some conditions and it can damage them under other conditions.
4 Bayesian Nash Equilibrium and Aggressive Bidding

In this section, we shall utilize the best response functions developed in the last section to analyze the equilibrium of the bidding game. It is clear that the game has multiple equilibria, depending on the parameters of the game. Since the search service provider receives the highest (lowest) revenue per click when a firm bids $\alpha$ ($\gamma$), the service provider would like to promote (avoid) the conditions under which the firm bids $\alpha$ ($\gamma$). Specifically, bidding the least at $\gamma$ is attainable as a dominant strategy equilibrium and the occurrence of this outcome is something that the search service provider should avoid. For this reason, we shall first establish the conditions under which both firms bid $\gamma$ as a dominant strategy at the equilibrium. We obtain the conditions under which both firms bid $\alpha$ as the unique Bayesian Nash equilibrium. Clearly, the best outcome from the perspective of the search service provider is when both firms choose to bid $\alpha$ at the equilibrium. We note, however, that bidding $\alpha$ can never be achieved as a dominant strategy equilibrium. This is because when the other firm bids $\gamma$, there is no reason for the firm to bid more than $\beta$, regardless of the firm’s belief on $p$. Nonetheless, we will derive the conditions under which $(\alpha, \alpha)$ can be achieved as a unique pure-strategy Bayesian Nash equilibrium of the game.

Lemma 4.1 For all $\theta_i > 3\gamma$, bidding $\gamma$ is a dominant strategy for both Firms 1 and 2 if and only if $f_i < \bar{p} + \phi_{4i}$ for $i = 1, 2$.

Lemma 4.1 above implies that if a firm’s forecast of $p$ (i.e., $f_i$) is sufficiently small, it will always bid $\gamma$, regardless of the strategy choices of the other firm. That is, when a firm
is pessimistic about the value of the first rank-based response \( p \), there is no incentive for
the firm to bid high at all. Therefore, to entice a firm to submit a higher bid, it is crucial
for the service provider to develop various marketing mechanisms (say, more eye catching
listing for the company which is ranked first) so that firms will not form a pessimistic view
about the value of \( p \).

In the following, we summarize the best response function for the case where \( \theta_i \) is
greater than \( \alpha \) and \( f_i \) is greater than \( \bar{p} + \phi_{1i} \). These results will be useful later, when we
establish the conditions under which \((\alpha, \alpha)\) is the unique Bayesian Nash equilibrium.

**Lemma 4.2** If \( f_i > p + \phi_{1i} \) and \( \theta_i \in \left(3\gamma, 5\gamma\right) \), then

\[
\{R_i(\alpha, f_i), R_i(\beta, f_i), R_i(\gamma, f_i)\} = \begin{cases} 
\{\alpha, \beta, \beta\}, & f_i \in (\bar{p} + \phi_{1i}, \bar{p} + \max_k \phi_{ki}), \\
\{\alpha, \alpha, \beta\}, & f_i \geq \bar{p} + \max_k \phi_{ki}.
\end{cases}
\]

**Lemma 4.3** If \( \theta_i \in [5\gamma, \infty) \) and \( f_i \geq \bar{p} + \max_k \phi_{ki} = \bar{p} + \phi_{1i} \), then \( \{R_i(\alpha, f_i), R_i(\beta, f_i), R_i(\gamma, f_i)\} = \{\alpha, \beta, \beta\} \).

Before we use these results to establish the necessary and sufficient condition for \((\alpha, \alpha)\) to
be the unique pure-strategy Bayesian Nash equilibrium, let us define the following terms
that would simplify the exposition. For \( i, j = 1, 2, j \neq i \), let \( \tilde{f}_i = \bar{p} + \phi_{1i}, f_i^* = \bar{p} + \max_k \phi_{ki} \).

We further define \( \tilde{f}_i = \bar{p} + \frac{\phi_{ij}}{d_i}, f_i^{**} = \bar{p} + \frac{\max_k \phi_{ki}}{d_i} \). It can be shown that from (5):

\[
E(f_j|\tilde{f}_i) = \tilde{f}_j, \quad \text{and} \quad \quad E(f_j|f_i^{**}) = f_j^*.
\]

20
By applying Lemmas 4.2 and 4.3, we have:

**Theorem 4.1** The necessary and sufficient condition for \((\alpha, \alpha)\) to be the unique Bayesian Nash equilibrium of the game is either

1. \(\theta_i > 3\gamma, f_i > \max(f_i^*, f_i^{**}); \theta_j \in (3\gamma, 5\gamma), f_j > \max(f_j^*, f_j^{**}), i \neq j, \) or

2. \(\theta_i \in (3\gamma, 5\gamma), f_i \in (\max(\tilde{f}_i, f_i^{**}), f_i^*); \theta_j \in (3\gamma, 5\gamma), f_j \in (\max(f_j^*, \tilde{f}_j), f_j^{**}), i \neq j.\)

Firstly, we note that both firms will bid high at the equilibrium if and only if \(f_i\) and \(f_j\) (i.e., their private forecast of the first rank-based response \(p\)) are sufficiently high. Furthermore, when the margins of the firms are not too different \((\theta_i \in (3\gamma, 5\gamma)\) for \(i = 1, 2\)) and when the forecasts of both firms are ‘within the same range’ (i.e., either both very high, as illustrated by Condition (1), or both moderately high, as given in Condition (2)), both firms will bid high at the equilibrium. This is because if the firms forecasts \(f_i\) and \(f_j\) are very different, then the firms will submit different bids. Such ‘mismatch’ in beliefs will not lead to an equilibrium with high bids from both firms.

The following corollary analyzes the impact of the correlation between the private forecasts on the equilibrium.

**Corollary 4.1** As the covariance \(\sigma_{12}\) between the private forecasts of the firms increases, both firms are more likely to bid high at the equilibrium.

The above corollary suggests that it is in the interest of the service provider to manage the information flow such that the firms’ private forecasts are highly correlated. This
is because when Firm $i$ perceives Firm $j$ to have a low forecast of the first rank-based response, Firm $i$ will anticipate Firm $j$ to bid low. This would deter Firm $i$ to bid high because Firm $i$ believes that it can rank first with a moderate bid.

5 Discussion and Future Research

In this paper, we have developed a model to analyze the bidding behavior of each firm when facing uncertain rank-based response. We have also analyzed the bidding behavior of both firms at the equilibrium. We found that aggressive bidding at the equilibrium is only possible when both firms’ private forecasts are sufficiently high. More importantly, the management of information regarding the effectiveness of the service by the service provider is pivotal to the bidding behavior of the firms. In addition, our analysis enables us to conclude that aggressive bidding will occur when the first ranked-based response is highly rank sensitive (the current market share $ad_i$ is small, the disloyal segment $b$ is large and the proportion of clicks from being ranked second is low), the ‘effective’ margin $\theta_i$ is high, the first rank-based response $p$ is high, the uncertainty level of the rank-based response $s_i$ is low (if and only if the critical point is positive), or when the covariance of $\sigma_{12}$ between the private forecasts of the firms is high.

As a first attempt to analyze bidding behavior arising from Internet advertising, our model has a number of limitations. Firstly, our analysis is limited to that of two firms, each with three bids. However, we believe that the dynamic interactions between the firms and the essence of our results are not compromised. In particular, the key focus of this paper
is on the conditions under which firms bid aggressively, and this, can be, and has been addressed in our model. Secondly, our model did not allow for multiple periods whereby other issues may be addressed. These include sabotage; i.e., a firm can distort the belief of the other firm by clicking on the competing firm’s name but makes no purchase. Finally, we did not consider any signaling effect, that is, a high ranking firm may be bidding higher because it has inferior quality or lower reputation. We view our analysis as the building block for further analysis.

References


6 Appendix

Proofs of Lemmas 3.1, 3.2

Since \( \theta_i > \gamma \) and all other parameters are positive, it is clear that \( \phi_{1i} \) is increasing in \( l_i, a \) and \( k \) but decreasing in \( b \) and \( \bar{p} \). To prove that \( \phi_{1i} \) is increasing in \( \gamma \) but decreasing in \( \theta_i \), rewrite \( \phi_{1i} \) given above as follows:

\[
\phi_{1i} = (\frac{\gamma}{\theta_i - \gamma} + \frac{2al_i}{b} + k)(\frac{k}{2} - \bar{p})(1 + \frac{s_i}{V}).
\]

Since \( \frac{2al_i}{b} \) and \( k \) are positive, it is clear that \( \phi_{1i} \) decreases as \( \theta_i \) increases. Furthermore, the derivative of \( \frac{\gamma}{\theta_i - \gamma} \) with respect to \( \gamma \) is \( \frac{\theta_i}{(\theta_i - \gamma)^2} \), which is positive. Hence, \( \phi_{1i} \) is increasing in \( \gamma \).

Similarly, we can prove the result for \( \phi_{4,i} \).

Lastly, it is clear that \( \phi_{ki}(k = 1, 4) \) is increasing in \( s_i \) if and only if \( \phi_{ki} \) is positive. \( \square \)

Using the same argument as above, we can prove the other lemma.

Proof of Lemma 3.3

Firm \( i \) prefers to bid \( \alpha \) over \( \gamma \) if and only if

\[
E(\Pi_i(\alpha, \alpha)|f_i) - E(\Pi_i(\gamma, \alpha)|f_i) = b(\theta_i - \gamma)((1 - t_i)\bar{p} + t_i f_i) - 2\gamma a l_i - (\theta_i + \gamma)\frac{k}{2}b \geq 0,
\]

or equivalently,

\[
f_i \geq \bar{p} + \frac{1}{t_i}(\frac{2\gamma a l_i}{b(\theta_i - \gamma)} + \frac{\theta_i + \gamma k}{a_i - \gamma^2} - \bar{p}) = \bar{p} + \phi_{1i}.
\]

\( \square \)

Proof of Lemma 3.4

Compare the respective payoffs when Firm \( i \) bids \( \alpha, \beta \) and \( \gamma \):

1. Firm \( i \) is better off bidding \( \beta \) as opposed to bidding \( \alpha \) if and only if

\[
E(\Pi_i(\beta, \beta)|f_i) - E(\Pi_i(\alpha, \alpha)|f_i) = -b(\theta_i - 3\gamma)((1 - t_i)\bar{p} + t_i f_i) + \gamma a l_i + (\theta_i - 2\gamma)\frac{k}{2}b \geq 0,
\]

or \( f_i \leq \bar{p} + \phi_{2i} \).
2. Firm $i$ prefers to bid $\beta$ over $\gamma$ if and only if $E(\Pi_i(\beta, \beta) | f_i) - E(\Pi_i(\gamma, \beta) | f_i) = b(\theta_i - \gamma)((1 - t_i)p + t_if_i) - \gamma al_i - \theta_i^\beta b \geq 0$, or equivalently, $f_i \geq \bar{p} + \phi_{3i}$.

3. Firm $i$ prefers to bid $\gamma$ over $\alpha$ if and only if $E(\Pi_i(\gamma, \beta) | f_i) - E(\Pi_i(\beta, f_i) | f_i) = -2b(\theta_i - 2\gamma)((1 - t_i)p + t_if_i) - \gamma al_i - b\bar{2}(\theta_i - \gamma) \geq 0$ if and only if $f_i \geq \bar{p} + \phi_{3i}$. 

Since $\alpha = 3\gamma < \theta_i$, we can apply Observations 1 and 2 to show that $\phi_{4i} < \phi_{3i} < \phi_{2i}$. Combining this with the above observations, we can obtain the result in the lemma. \hfill \Box

Proof of Lemma 3.5

It is easy to check that $E(\Pi_i(\beta, \gamma) | f_i) - E(\Pi_i(\gamma, \gamma) | f_i) = b(\theta_i - 2\gamma)((1 - t_i)p + t_if_i) - \gamma al_i - b\bar{2}(\theta_i - \gamma) \geq 0$ if and only if $f_i \geq \bar{p} + \phi_{3i}$. \hfill \Box

Proof of Lemma 4.1

For $\theta_i > 3\gamma$, Lemmas 3.3, 3.4 and 3.5 imply that $R_i(\alpha, f_i) = R_i(\beta, f_i) = R_i(\gamma, f_i) = \gamma$ if and only if $f_i < \bar{p} + \min_{k=1,3,4} \phi_{ki} = \phi_{4i}$ (Observations 1, 2). \hfill \Box

Proof of Lemmas 4.2 and 4.3

We obtain the result directly from Lemmas 3.3, 3.4, 3.5 and Observations 1 and 2. \hfill \Box

Proof of Theorem 4.1

Suppose Condition (1) holds. We deduce from (5) and (8) that $E(f_i | f_j) > E(f_i | f_j^\ast) = f_i^\ast = \bar{p} + \max_k \phi_{ki}$. Coupled with $f_i > f_i^\ast$, we have $\{R_i(\alpha, f_i), R_i(\beta, f_i), R_i(\gamma, f_i)\} = \{R_i(\alpha, E(f_i | f_j)), R_i(\beta, E(f_i | f_j)), R_i(\gamma, E(f_i | f_j))\} = \{\alpha, \alpha, \beta\}$, $j \neq i$ (Lemma 4.2), where the right-hand term denotes the best response function for Firm $i$ (and Firm $j$) as envisioned by Firm $j$ (Firm $i$). Furthermore, $\{R_j(\alpha, f_j), R_j(\beta, f_j), R_j(\gamma, f_j)\} = \{R_j(\alpha, E(f_j | f_i)), R_j(\beta, E(f_j | f_i)), R_j(\gamma, E(f_j | f_i))\} = \{\alpha, \alpha, \beta\}$ if $\theta_j \in (3\gamma, 5\gamma)$ or $\{R_j(\alpha, f_j), R_j(\beta, f_j), \ldots$
\( R_j(\gamma, f_j) = \{ R_j(\alpha, E(f_j|f_i)), R_j(\beta, E(f_j|f_i)), R_j(\gamma, E(f_j|f_i)) \} = \{ \alpha, \beta, \beta \} \) if \( \theta_j > 5\gamma \).

Similarly, we can check that with Condition (2), the best response functions are \{ \alpha, \beta, \beta \} and \{ \alpha, \alpha, \beta \} respectively (Lemma 4.2). Given the best response functions (given as a triplet) of \( (\alpha, \alpha, \beta) \) for Firm \( i \) and either \( (\alpha, \alpha, \beta) \) or \( (\alpha, \beta, \beta) \) for Firm \( j \), one can easily verify that there is only one Bayesian Nash equilibrium, i.e., only the pair \( (\alpha, \alpha) \) (where both Firm \( i \) and Firm \( j \) choose strategy \( \alpha \) is a pair of mutual best responses).

Next, we proceed to prove that Conditions (1) and (2) are indeed the only conditions under which \( (\alpha, \alpha) \) can be the unique pure-strategy Bayesian Nash equilibrium. To achieve this, we need to identify all possible pairs of best response functions \( R_i(\cdot, \cdot) \) and beliefs \( f_i \) \((i = 1, 2)\) such that \( (\alpha, \alpha) \) can be sustained as a Bayesian Nash equilibrium.

To simplify our analysis, we proceed in the following manner. Firstly, we note that for \( (\alpha, \alpha) \) to be an equilibrium, it must necessarily be true that Firm \( i \) bids \( \alpha \) in best response to Firm \( j \)'s bid of \( \alpha \). In addition, based on its private forecast \( f_i \), Firm \( i \) believes that Firm \( j \) will bid \( \alpha \), i.e.,

\[
R_i(\alpha, f_i) = \alpha = R_j(\alpha, E(f_j|f_i)).
\]  

Furthermore, we need to ensure that there does not exist other equilibria, that is,

\[
\exists (x, y) (\in \{ \alpha, \beta, \gamma \}) \text{ such that } R_i(y, f_i) = x, R_j(x, f_j) = y.
\]  

In the process of identifying such pairs of best response functions, we will simplify our analysis by not imposing any restriction on the beliefs \( f_i, f_j \) initially. Once pairs of such functions are identified, we then determine the firms’ beliefs \( f_i, f_j \) that are consistent with
these best response functions. In particular, given our choice of $R_i, R_j$, we choose $f_i, f_j$ such that given $f_i$ ($f_j$), Firm $i$ ($j$) not only chooses $R_i$ ($R_j$) but believes that Firm $j$ ($i$) will choose $R_j$ ($R_i$).

For $R_i(\cdot, \cdot)$ to satisfy (9), we must have $\theta_i > \alpha$ and $f_i > \bar{p} + \phi_i$ ($i = 1, 2$). From Lemmas 4.2 and 4.3, we note that we need only consider the following types of best response functions in our analysis. For easy exposition, we label these best response functions as $R_1$ and $R_2$, i.e., $R_1(\alpha, \cdot) = \alpha, R_1(\beta, \cdot) = \beta, R_1(\gamma, \cdot) = \beta$ (Lemmas 4.2, 4.3), $R_2(\alpha, \cdot) = \alpha, R_2(\beta, \cdot) = \alpha, R_2(\gamma, \cdot) = \beta$ (Lemma 4.2). We consider all pairwise combinations (one for each firm) of the two best response functions given above in order to identify the combinations that will give $(\alpha, \alpha)$ as the unique pure-strategy Bayesian Nash equilibrium. There are altogether three cases to consider. In each case, we either provide evidence for the existence of a Bayesian Nash equilibrium $(x, y), x, y \neq \alpha$, or we verify that $(\alpha, \alpha)$ is indeed the unique Bayesian Nash equilibrium.

**Case 1** $R_i = R_1, i = 1, 2$.

It is straightforward to verify that there exists $f_i$ ($i = 1, 2$) such that $R_i(\beta, f_i) = \beta = R_j(\beta, E(f_j|f_i))$. Therefore, $(\alpha, \alpha)$ cannot be a unique equilibrium here as we also have $(\beta, \beta)$ as an equilibrium.

**Case 2** $R_i = R_2, i = 1, 2$.

By direct verification, we check that (9) and (10) are satisfied. Thus, $(\alpha, \alpha)$ is the unique pure-strategy Bayesian Nash equilibrium. We deduce also from Lemmas 4.2 and

28
4.3 that $R^2$ is achieved as the best response function if

$$f_i \geq \max(\bar{p} + \max_k \phi_{ki}, \bar{p} + \frac{\max_k \phi_{kj}}{d_i}) = \max(f_i^*, f_i^{**}). \quad (11)$$

The rationale behind the choice of $f_i$ is as follows. Given the beliefs above, we would have $f_i > f_i^* = \bar{p} + \max_k \phi_{ki}$ and $E(f_i|f_j) > E(f_i|f_j^{**}) = \bar{p} + \max_k \phi_{ki} = f_i^*$. One can easily verify that these beliefs give rise to the best response function $R^2$ for the firms. Furthermore, these beliefs also imply that a firm’s assessment of the best response of the opposing firm given its private forecast coincides with the opposing firm’s actual best response function, based on the opposing firm’s own private forecast. This ensures that each firm not only finds bidding $\alpha$ to be a best response when the other firm bids $\alpha$, it also believes that the other firm has the same assessment and that each firm believes that there is no other equilibrium and believes that the other firm also holds the same belief.

**Case 3** $R_i = R^1, R_j = R^2, j \neq i$.

As in Case 2 above, we can easily verify from (9) and (10) that $(\alpha, \alpha)$ is the only pure-strategy Bayesian Nash equilibrium. From Lemma 4.2, it is clear that

$$\theta_j \in (3\gamma, 5\gamma), f_j \geq \bar{p} + \max_k \phi_{kj} = f_j^*.$$ \hfill (12)

However, there are two sets of forecasts for Firm $i$ (Lemmas 4.2, 4.3), namely,

$$\theta_i \in (5\gamma, \infty), f_i \geq \bar{p} + \phi_{1i} = \bar{p} + \max_k \phi_{ki} = f_i^*,$$ \hfill (13)

or

$$\theta_i \in (3\gamma, 5\gamma), f_i \in (\bar{p} + \phi_{1i}, \bar{p} + \max_k \phi_{ki}) = (\tilde{f}_i, f_i^*).$$ \hfill (14)

Combining (12) and (13), and taking into account the rationale for the choices of $f_i, f_j$ as given in the paragraph below (11), we have
\[ \theta_i \in (5\gamma, \infty), f_i \geq \max(f_i^*, f_i^{**}), \]
\[ \theta_j \in (3\gamma, 5\gamma), f_j \geq \max(f_j^*, f_j^{**}). \]  \hspace{1cm} (15)

Incorporating (15) with (11), we obtain Condition (1) in the statement of the theorem.

Similarly, combining (12) and (14), we have Condition (2). \(\Box\)

**Proof of Corollary 4.1**

We recall that the correlation between the firms’ forecasts is represented by \(\sigma_{12}\). A high correlation implies that \(d_i = \frac{V_i + \sigma_{12}}{V_i + s_i}\) approaches 1, while a low correlation means that \(\sigma_{12}\) approaches 0 and \(d_i < 1\). Hence, it is immediate that \(f_i^{**}, \tilde{f}_i\) decrease as the correlation increases, resulting in a higher likelihood that the conditions stated in Theorem 4.1 can be satisfied. \(\Box\)