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STRICTLY CONCAVE PARAMETRIC PROGRAMMING, PART I: BASIC THEORY*

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This paper, which is presented in two parts, develops a computational approach to strictly concave parametric programs of the form: Maximize $\alpha f_1(x) + (1 - \alpha) f_2(x)$ subject to concave inequality constraints for each fixed value of α in the unit interval, where f_1 and f_2 are strictly concave and certain additional regularity conditions are satisfied. This class of problems subsumes a corresponding class of vector maximum problems and also, by means of a simple device, provides a deformation method for ordinary concave programming. The approach is based on exploiting the continuity properties of the parametric program so as to efficiently maintain a solution to the associated Kuhn-Tucker conditions as α traverses the unit interval. The same approach can be adapted to much more general parametric programs than the one above.

In Part I, a Basic Parametric Procedure is derived and shown to be finite in a certain sense. It forms the basis of various parametric programming algorithms, depending on what special assumptions are made on the functions. In Part II, additional theory is developed that facilitates computational implementation, and one possible general-purpose algorithm for a digital computer is given. An illustrative graphical example is presented and several extensions are indicated.

1. Introduction

In Part I we present and justify a computational approach to concave parametric programming problems of the form

$$(P\alpha) \qquad \text{Maximize}_x \ \alpha f_1(x) + (1 - \alpha) f_2(x) \quad \text{subject to} \quad g(x) \geqq 0$$

for each $\alpha \varepsilon [0, 1]$, where x is a decision n-vector, the functions f_1, f_2 and $g = (g_1, \cdots, g_m)$ are concave and satisfy certain additional regularity conditions (see section 2.1), and a solution of $(P\alpha)$ is available for some value of α in the unit interval.

One motivation for studying $(P\alpha)$ derives from the fact that it subsumes a corresponding class of vector maximum problems of the form

$$(1) \qquad \text{“Maximize”}_x \ f_1(x), f_2(x) \quad \text{subject to} \quad g(x) \geqq 0.$$

The quotation marks in (1) signify that it is desired to find all efficient decisions,

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where a feasible decision x^0 is said to be efficient if there exists no other feasible decision x' such that $f_j(x') \geqq f_j(x^0)$, $j = 1, 2$, with strict inequality holding for at least one value of j. Under the conditions imposed in this paper, it can be shown [8, 11, 12] that $(P\alpha)$ and (1) are equivalent in that every efficient decision, for (1) is an optimal solution of $(P\alpha)$ for some $0 \leqq \alpha \leqq 1$, and conversely.

Another motive for studying $(P\alpha)$ is that by a simple device it subsumes the standard problem of concave programming. Suppose that it is desired to solve the concave programming problem

$$\text{(2)} \qquad \text{Maximize}_x\, F(x) \quad \text{subject to} \quad g(x) \geqq 0.$$

If x^0 is any feasible decision of (2), put $(P\alpha)$ equal to

$$(3\alpha) \quad \text{Maximize}_x\, \alpha F(x) + (1 - \alpha) \sum_{i=1}^{n} - (x_i - x_i^0)^2 \text{ subject to } g(x) \geqq 0.$$

Then x^0 clearly is the (known) optimal solution of (3_0), and (3_1) is identical to (2). Applying an algorithm for parametric concave programming to (3α) beginning with $\alpha = 0$ and increasing α until $\alpha = 1$, one obtains a "deformation" method of concave programming.[1] It is to be noted that this device renders much less restrictive the assumption stated in the opening paragraph that an optimal solution of $(P\alpha)$ is available for some value of α in the unit interval, for $(P\alpha)$ with α fixed is precisely of the form (2).

We turn now to a brief review of presently available methods for solving $(P\alpha)$ on the unit interval. It should be noted at the outset that since $(P\alpha)$ is a concave programming problem for each fixed value of α, then in principle the parametric problem can often be solved on an arbitrarily fine grid of parameter values by repeated applications of some known concave programming algorithm. This straightforward approach may be fairly practical for some problems, because the optimal solution for one parameter value can be expected to provide a nearly optimal solution at an adjacent parameter value on the grid. It is possible, however, to make more fundamental and efficient use of the basic continuity properties which arise from the parametric nature[2] of $(P\alpha)$. This has been done by various authors for a few simple classes of functions for f_1, f_2, and g. If the constraints are all linear, then several efficient parametric programming algorithms are available for certain special classes of criterion functions: when f_1 and f_2 are both linear functions, $(P\alpha)$ can be solved by parametric linear programming [7]; when one function is linear and the other is a quadratic polynomial, the algorithms of Houthakker [10], Markowitz [13], and Wolfe [15] are available;[3] when f_1 and f_2 are both quadratic polynomials, an algorithm of Zahl [16] essentially solves $(P\alpha)$. Little if anything appears to have been done

[1] If a feasible x^0 is unknown, then the successive application of the same device temporarily considering the constraints as objective functions may produce one.

[2] Indeed, it is noteworthy that essentially non-parametric problems have often been parameterized in order to imbue them with continuity properties which facilitate their solution [5, 6, 10, 15, 16].

[3] See also [4, 14].

to devise efficient algorithms for parametric problems involving more general classes of criterion functions or feasible regions other than convex polyhedra. This paper is intended as a contribution in this direction.

The present approach is not essentially a "simplicial" or "gradient" one, but is addressed to the aim of efficiently maintaining a solution to the Kuhn-Tucker conditions [12] as α traverses the unit interval. Under our assumptions, these conditions are necessary and sufficient for an optimal solution of $(P\alpha)$.

In subsection 1.1 we establish some notation and ideas and give what is for our purposes a particularly convenient version of the Kuhn-Tucker conditions. Section 2 is devoted to presenting and validating a Basic Parametric Procedure for solving $(P\alpha)$ on [0, 1]. The procedure can be adapted to more general parametric programs.

1.1 Preliminaries

For simplicity of notation, we sometimes write $f(x;\ \alpha)$ for $\alpha f_1(x) + (1 - \alpha)f_2(x)$. We also denote the first m positive integers by M, and use the minus sign to denote the relative complement of two sets. For example, if S is a set of integers then $M - S \equiv \{i \varepsilon M : i \not\varepsilon S\}$. The gradient of a function of several variables is denoted by the symbol ∇, and the matrix of second partial derivatives by ∇^2.

For α_0 fixed, $(P\alpha_0)$ is a standard concave programming problem for which fundamental theoretical results have been given by Kuhn and Tucker [12]. A convenient version of their Theorem 3 is recorded here without proof.

Theorem (Kuhn-Tucker): Consider $(P\alpha_0)$ with α_0 fixed. Let $f(x;\ \alpha_0)$ and the $g_i(x)$ be differentiable on the feasible region $\{x : g(x) \geqq 0\}$, let $f(x;\ \alpha_0)$ be concave on the feasible region, and let the $g_i(x)$ be concave on E^n. Assume that the constraint functions satisfy the Kuhn-Tucker constraint qualification (see the remark below).

Then x^0 is an optimal solution of $(P\alpha_0)$ if and only if there exist m real numbers u_i^0 and a subset S^0 of constraint indices such that $(x^0,\ u^0,\ S^0)$ satisfies the following conditions at $\alpha = \alpha_0$:

$$(=S)\alpha \begin{cases} \text{(KT-1)} & \nabla_x f(x;\ \alpha) + \sum_{i=1}^m u_i \nabla_x g_i(x) = 0 \\ \text{(KT-2)} & g_i(x) = 0, \quad i \varepsilon S; \qquad u_i = 0, \quad i \varepsilon M - S \end{cases}$$

$$\text{(KT-3)} \qquad g_i(x) \geqq 0, \quad i \varepsilon M - S$$

$$\text{(KT-4)} \qquad u_i \geqq 0, \quad i \varepsilon S.$$

Remark: For a statement and discussion of the Kuhn-Tucker constraint qualification, see [12, p. 483] or [2]. It has been shown, for example, that if all the constraints are linear, then this qualification is satisfied; and that the existence of a point for which all the constraint functions are positive is also sufficient for the qualification to be satisfied. The sufficient condition that will be of direct use in the sequel is: if $x^*(\alpha_0)$ is an optimal solution of $(P\alpha_0)$, then the matrix whose rows are $\nabla_x g_i(x^*(\alpha_0))$, i such that $g_i(x^*(\alpha_0)) = 0$, is of maximal rank [2].

Equations (KT-1) and (KT-2) appear together so often in the sequel that we introduce the special symbol $(=S)\alpha$ to denote them (in this notation, S and α may vary). The u_i will be referred to as *dual variables*. Dual variables with the property that there exists (x^0, S^0) such that (x^0, u^0, S^0) satisfies (KT-1, ..., 4) will be called *optimal* dual variables.

The concept of a *valid set* plays a central role in this work. A subset S^0 of constraint indices is said to be valid at α_0 if and only if there exists (x^0, u^0) such that (x^0, u^0, S^0) satisfies (KT-1) through (KT-4) at α_0. Under appropriate assumptions the optimal solution $x^*(\alpha)$ of $(P\alpha)$ and the associated optimal dual variables $u^*(\alpha)$ are unique and continuous. This fact, coupled with the observation that there is only a finite number of subsets of constraints, suggests that if S^0 is valid at α_0, then S^0 is likely to be valid in some interval including α_0. If this is the case, then it will be shown that one obtains $x^*(\alpha)$ and $u^*(\alpha)$ in that interval by solving $(=S^0)\alpha$ parametrically, and (KT-3) and (KT-4) are automatically satisfied. If this is not the case, then even though $(=S^0)\alpha$ has a solution near α_0, either (KT-3) or (KT-4) will be violated, and it is necessary to find a new valid set before being able to proceed. Because of continuity, moreover, a set which is valid near α_0 will usually differ by very few constraint indices from S^0. This approach leads to a decomposition of $(P\alpha)$ on [0, 1] into a chain of parametric subproblems. Each subproblem involves the parametric solution of the Lagrange multiplier equations $(=S)\alpha$ associated with the constraints specified by a set valid on a subinterval of [0, 1]. By continuity, the optimal terminal solution to one subproblem is the optimal initial solution to the next subproblem of the chain, and the valid sets of adjacent subproblems are both valid at the transition point between them.

2. A Basic Parametric Procedure

In this section we state and prove a Basic Parametric Procedure for solving $(P\alpha)$ for each value of α in the unit interval. It can be modified and implemented in various ways, as will be indicated in sections 2 and 4 of Part II.

2.1 Assumptions

We assume that an optimal solution of $(P\alpha)$ is available for some value of α in the unit interval, say $\alpha = 0$ (in view of our previous discussion, this assumption is not restrictive).

Throughout this work the following conditions will be imposed upon $(P\alpha)$. We denote the feasible region $\{x: g(x) \geqq 0\}$ by X.

Condition 1: The functions $f_i(x) (i = 1, 2)$ and $g_i(x) (i = 1, \cdots, m)$ are analytic[4] on some open region containing X, and the constraint functions are concave on E^n.

[4] A function of several real variables is *analytic* in a region R, if in some neighborhood of every point of R, the function is the sum of a convergent power series with real coefficients. The class of all analytic functions includes, for example, all polynomials, and seems amply

Condition 2: X is non-empty and bounded.

Condition 3: The hessian matrices $\nabla_x^2 f_i(x)\,(i = 1, 2)$ are negative definite for all $x \in X$.

Condition 4: If $\alpha_0 \in [0, 1]$ and $x^*(\alpha_0)$ is an optimal solution of $(P\alpha_0)$, then the matrix whose rows are the gradients $\nabla_x g_i(x^*(\alpha_0))$, i such that $g_i(x^*(\alpha_0)) = 0$, is of maximal rank.

Conditions 1 and 2 imply that X is convex and compact.

Condition 3 implies that f_1 and f_2 are "locally" strictly concave on X, which is slightly stronger than strict concavity. This, in turn, implies that $f(x; \alpha) = \alpha f_1(x) + (1 - \alpha) f_2(x)$ is locally strictly concave on X for each fixed value of $\alpha \in [0, 1]$. In the presence of Conditions 1 and 2, this last assertion remains true even on some open region containing X and on some open interval containing $[0, 1]$.

Condition 4 is equivalent to requiring that the gradients $\nabla_x g_i(x^*(\alpha_0))$, i such that $g_i(x^*(\alpha_0)) = 0$, must be linearly independent; hence at most n constraints can be satisfied with exact equality at an optimal solution of $(P\alpha_0)$. In the remark following the Kuhn-Tucker theorem, it was noted that this condition implies that the Kuhn-Tucker constraint qualification holds. Thus the hypotheses of the Kuhn-Tucker theorem are satisfied by $(P\alpha_0)$ for each fixed $\alpha_0 \in [0, 1]$ when Conditions 1, 3, and 4 hold.

2.2 Statement of the Basic Parametric Procedure

For convenience we view α as increasing from 0 toward 1.

Step 1: Solve $(P0)$ by any convenient method, so that $(x^*(0), u^*(0), S^*)$ satisfying (KT-1) through (KT-4) at $\alpha = 0$ is at hand. Put $\alpha^0 = 0$, $S^0 = S^*$, and $(x, u)^0 = (x^*(0), u^*(0))$.

Step 2: Solve equations $(=S^0)\alpha$ by any convenient method as α increases above α^0 for the unique continuous solution[5] $(x^{S^0}(\alpha), u^{S^0}(\alpha))$ satisfying the left end-point value $(x, u)^0$ so long as this solution satisfies (KT-3) and (KT-4); that is, until $\alpha = \alpha'$, where

$$\alpha' = \operatorname{Max}\{\alpha : \alpha^0 \leqq \alpha \leqq 1,\ g_i(x^{S^0}(\alpha)) \geqq 0,\ i \in M - S^0,\quad \text{and} \quad u_i^{S^0}(\alpha) \geqq 0,\ i \in S^0, \text{ on } [\alpha^0, \alpha]\}.$$

If $\alpha' = 1$, terminate. Otherwise put $(x, u)^0 = (x^{S^0}(\alpha'), u^{S^0}(\alpha'))$ and go to Step 3.

Step 3: Solve equations $(=S)\alpha$ by any convenient method as α increases above α' for the unique continuous solution $(x^S(\alpha), u^S(\alpha))$ satisfying the left end-point value $(x, u)^0$ for different sets S which satisfy

$$\{i \in M : u_i^{S^0}(\alpha') > 0\} \subset S \subset \{i \in M : g_i(x^{S^0}(\alpha')) = 0\} \tag{4}$$

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wide enough to include most continuous functions likely to be encountered in applications. Except for Theorem 3, the Basic Parametric Procedure requires only twice continuous differentiability rather than analyticity of all functions.

[5] Throughout this work we employ the symbol $(x^S(\alpha), u^S(\alpha))$ to denote a solution of the equations $(=S)\alpha$.

until for some S', $(x^{S'}(\alpha), u^{S'}(\alpha))$ satisfies (KT-3) and (KT-4) on $[\alpha', \alpha' + \epsilon]$ for some $\epsilon > 0$. Put $\alpha^0 = \alpha'$, $S^0 = S'$, and return to Step 2.

Complete justification requires proof of the following.

Theorem (Basic): Assume that Conditions 1 through 4 hold. Then the following assertions regarding the Basic Parametric Procedure hold:

(i) Step 2 is well-defined.

(ii) At each execution of Step 2, $(x^{S^0}(\alpha), u^{S^0}(\alpha)) = (x^*(\alpha), u^*(\alpha))$ on $[\alpha^0, \alpha']$.

(iii) Step 3 is well-defined.

(iv) Step 3 will be executed only a finite number of times before termination obtains.

2.3 Theoretical Development

Continuity plays a crucial role in parametric programming.

Theorem 1 (Continuity):

(i) Assume that Conditions 1 through 3 hold. Then $(P\alpha)$ has a unique optimal solution $x^*(\alpha)$, and $x^*(\alpha)$ is continuous on some open interval containing [0, 1].

(ii) Assume that Conditions 1 through 4 hold. Then $(P\alpha)$ has unique optimal dual variables $u_i^*(\alpha)(i = 1, \cdots, m)$ and $u^*(\alpha)$ is continuous, on some open interval containing [0, 1].

Proof: The continuity of $x^*(\alpha)$ can be obtained from standard continuity and compactness arguments. The existence of $u^*(\alpha)$ is obtained by showing that the Kuhn-Tucker theorem applies on some open interval containing [0, 1]; uniqueness and continuity by expressing $u^*(\alpha)$ in the neighborhood of any $\alpha_0 \varepsilon [0, 1]$ as a continuous function of $x^*(\alpha)$ [8, p. 68 ff].

It will prove convenient to introduce some special notations. Define $A\alpha$ to be the set of constraint indices corresponding to the constraints which are *active* at α in the sense that their dual variables are strictly positive:

$$A\alpha = \{i \varepsilon M: u_i^*(\alpha) > 0\}.$$

Define $B\alpha$ to be the set of constraint indices corresponding to the constraints which are *binding* at $x^*(\alpha)$:

$$B\alpha = \{i \varepsilon M: g_i(x^*(\alpha)) = 0\}.$$

The sets $A\alpha$ and $B\alpha$ are well-defined on some open interval containing [0, 1] because of the existence and uniqueness of $(x^*(\alpha), u^*(\alpha))$ on some such interval. We can now state two important corollaries of Theorem 1.

Corollary 1.1: Assume that Conditions 1 through 4 hold. Then for each $\alpha_0 \varepsilon [0, 1]$ there exists an open interval containing α_0 such that, on this interval,

$$A\alpha_0 \subset A\alpha \subset B\alpha \subset B\alpha_0 .$$

Proof: The outermost relations follow directly from the definitions of $A\alpha$ and $B\alpha$ and the continuity of $x^*(\alpha)$ and $u^*(\alpha)$. The middle relation follows from (KT-2) and (KT-4).

Corollary 1.2: Assume that Conditions 1 through 4 hold. Then there is an open interval containing [0, 1] such that, for each fixed value of α in this open interval, a subset S of constraint indices is valid at α if and only if $A\alpha \subset S \subset B\alpha$.

Proof: This assertion is a simple consequence of the uniqueness of $(x^*(\alpha), u^*(\alpha))$ and the definition of validity.

The significance of Corollaries 1.1 and 1.2 is that the totality of valid sets at $\alpha_0 \varepsilon [0, 1]$ contains the totality of valid sets for α sufficiently near α_0. Hence the optimal solution of $(P\alpha_0)$, which yields $A\alpha_0$ and $B\alpha_0$, gives a strong indication of the identity of a valid set for α near α_0.

The next theorem shows that equations $(=S)\alpha$ can be solved on some open interval about $\alpha_0 \varepsilon [0, 1]$ if S is valid at α_0.

Theorem 2: Let $\alpha_0 \varepsilon [0, 1]$ be fixed, let S be valid at α_0, and assume Conditions 1 through 4 hold.

Then there exists an open interval $I\alpha_0$ containing and symmetric about α_0, and an open neighborhood $N(x^*(\alpha_0), u^*(\alpha_0))$ containing $(x^*(\alpha_0), u^*(\alpha_0))$, such that on $I\alpha_0$ there is a unique function $(x^S(\alpha), u^S(\alpha))$ in $N(x^*(\alpha_0), u^*(\alpha_0))$ which satisfies $(=S)\alpha$. Furthermore, $(x^S(\alpha), u^S(\alpha))$ is analytic on $I\alpha_0$.

Proof: The proof [8, p. 74 ff.] is an application of the appropriate version of the Implicit Function Theorem [3, p. 39] to $(=S)\alpha$.

Corollary 2.1: Let $\alpha_0 \varepsilon [0, 1]$ be fixed, let S be valid at α_0, and assume that Conditions 1 through 4 hold.

Then there exists an open interval containing α_0 and contained in $I\alpha_0$ such that, for each fixed value of α in this interval, the following three assertions are equivalent:

(i) S is valid at α.

(ii) $(x^S(\alpha), u^S(\alpha)) = (x^*(\alpha), u^*(\alpha))$.

(iii) $g_i(x^S(\alpha)) \geqq 0, i \varepsilon M - S$ (i.e., (KT-3) is satisfied)
$u_i^S(\alpha) \geqq 0, i \varepsilon S$ (i.e., (KT-4) is satisfied).

Proof: (i) $\Rightarrow$ (ii). By continuity, $(x^*(\alpha), u^*(\alpha)) \varepsilon N(x^*(\alpha_0), u^*(\alpha_0))$ for all α sufficiently near α_0; by the validity of S at α and Corollary 1.2, one concludes that $(x^*(\alpha), u^*(\alpha))$ satisfies $(=S)\alpha$; since the solution of $(=S)\alpha$ is *unique* in $N(x^*(\alpha_0), u^*(\alpha_0))$ for $\alpha \varepsilon I\alpha_0$, assertion (ii) follows.

(ii) $\Rightarrow$ (iii). Obvious.

(iii) $\Rightarrow$ (i). Assertion (iii) and the fact that $(x^S(\alpha), u^S(\alpha))$ satisfies $(=S)\alpha$ imply by the definition of validity that S is valid at α.

One more result must be established before a complete proof of the Basic Theorem can be given.

Define a *point of change* of $B\alpha$ as a point α' with the property that there is no open interval containing α' such that $B\alpha = B\alpha'$ everywhere on that interval. A similar definition holds for a point of change of $A\alpha$. In the sequel, the phrase "point of change" is used to refer to either a point of change of $A\alpha$ or of $B\alpha$, or possibly of both.

Theorem 3 (Finiteness): Assume that Conditions 1 through 4 hold. Then $A\alpha$ and $B\alpha$ each have a finite number of points of change on [0, 1].

Proof: Suppose that $B\alpha$ has an infinite number of points of change on [0, 1]. Then there is a cluster point $\alpha^* \varepsilon [0, 1]$ of these points of change. Let

$\langle\alpha^\nu\rangle$, $\alpha^\nu \varepsilon [0, 1]$, be a sequence of distinct points of change of $B\alpha$ which converges to α^*. Applying Corollary 1.1 at α^ν, we see that there exists an open interval containing α^ν such that $A\alpha^\nu \subset A\alpha \subset B\alpha \subset B\alpha^\nu$ on this interval. By the definition of a point of change of $B\alpha$, for each α^ν there exists a number β^ν contained in this interval and in $(\alpha^\nu - 1/\nu, \alpha^\nu + 1/\nu)$ such that $A\alpha^\nu \subset A\beta^\nu \subset B\beta^\nu < B\alpha^\nu$ (note the use of $<$ to denote a proper subset of $B\alpha^\nu$). Clearly $\langle\beta^\nu\rangle \rightarrow \alpha^*$. From Corollary 1.1 applied at α^*, we see that we have demonstrated the existence of two sequences $\langle\alpha^\nu\rangle \rightarrow \alpha^*$, $\langle\beta^\nu\rangle \rightarrow \alpha^*$, such that $A\alpha^* \subset A\alpha^\nu \subset A\beta^\nu \subset B\beta^\nu < B\alpha^\nu \subset B\alpha^*$ for all ν sufficiently large. Since there is but a finite number (2^m) of possible sets which $B\beta^\nu$ or $B\alpha^\nu$ could possibly be, we may assume, taking a subsequence if necessary, that there exist sets B' and B'' such that $B\beta^\nu = B'' < B\alpha^\nu = B'$ for all ν.

Consider the function $x^{B''}(\alpha)$ defined as in Theorem 2 applied at α^*. Since B'' is valid at α^* and at all α^ν and β^ν, ν sufficiently large, $x^{B''}(\alpha) = x^*(\alpha)$ at these points. Take $i_0 \varepsilon B' - B''$. Then $g_{i_0}(x^{B''}(\alpha^\nu)) = 0$ and $g_{i_0}(x^{B''}(\beta^\nu)) > 0$, all ν sufficiently large, and $g_{i_0}(x^{B''}(\alpha^*)) = 0$. In other words, we have shown that α^* is a non-isolated zero of $g_{i_0}(x^{B''}(\alpha))$, and that this function is not identically zero on any open interval about α^*. But this leads to a contradiction of the well-known fact [1, p. 518] that the zeros of an analytic function which is not identically zero are isolated, for by Theorem 2 and Condition 1 we have that $g_{i_0}(x^{B''}(\alpha))$ is analytic on some open interval about α^*. Hence the supposition that $B\alpha$ has an infinite number of points of change on [0, 1] is false.

A similar argument shows that $A\alpha$ cannot have an infinite number of points of change on [0, 1].

Applying the result of Theorem 3 to a given $(P\alpha)$, define $0 \leqq \alpha'_1 < \alpha'_2 < \cdots < \alpha'_N \leqq 1$ to be the collection of all points of change of $A\alpha$ or $B\alpha$ or both. As a matter of convention, we take $\alpha'_0 = 0$ and $\alpha'_{N+1} = 1$. From Corollaries 1.1 and 1.2 we conclude that any set which is valid at α, $\alpha'_j < \alpha < \alpha'_{j+1}$, is also valid on the entire closed interval $[\alpha'_j, \alpha'_{j+1}]$. In addition, it may also be valid on other intervals, of course. Among the sets which are valid at α'_j there are all those which are valid on $[\alpha'_{j-1}, \alpha'_j]$ or on $[\alpha'_j, \alpha'_{j+1}]$.

We are now in a position to prove the Basic Theorem.

Proof (*Basic Theorem*): First we prove parts (i) and (ii). At the beginning of each Step 2, $(x, u)^0$ and S^0 satisfy (KT-1) through (KT-4) at α^0, so that S^0 is valid at α^0 and $(x, u)^0 = (x^*(\alpha^0), u^*(\alpha^0))$. Let J, $1 \leqq J \leqq N + 1$ be the largest integer such that S^0 is valid on $[\alpha^0, \alpha'_J]$. ($\alpha'_J = \alpha'_1 = \alpha^0 = 0$ is permissible the first time Step 2 is executed.) Applying Theorem 2 at each point of $[\alpha^0, \alpha'_J]$, it follows that $(= S^0)\alpha$ has a unique analytic solution $(x^{S^0}(\alpha), u^{S^0}(\alpha))$ satisfying the left end-point value $(x, u)^0$ on some interval containing $[\alpha^0, \alpha'_J]$. This solution satisfies (KT-3) and (KT-4) and equals $(x^*(\alpha), u^*(\alpha))$ on $[\alpha^0, \alpha'_J]$ by Corollary 2.1. If $\alpha'_J = 1$, the solution of $(P\alpha)$ on [0, 1] is complete. If $\alpha'_J < 1$, however, $(x^{S^0}(\alpha), u^{S^0}(\alpha))$ does *not* satisfy (KT-3) and (KT-4) for any $\alpha \varepsilon (\alpha'_J, \alpha'_{J+1})$, for otherwise, by Corollary 2.1 applied at α'_J, S^0 would be valid on $[\alpha'_J, \alpha'_{J+1}]$, which would violate the definition of J. Clearly the scalar α' defined in Step 2 is precisely α'_J, and (i) and (ii) hold.

Next we prove (iii). Any set S which satisfies (4) is valid at α', by Corollary 1.2 and the fact that $(x^{S^0}(\alpha'), u^{S^0}(\alpha')) = (x^*(\alpha'), u^*(\alpha'))$. Applying Theorem 2 at α', we see that if S satisfies (4) then $(=S)\alpha$ has a solution as stated on $[\alpha', \alpha' + \epsilon_1]$ for some $\epsilon_1 > 0$. By Corollary 1.1 we know that at least one such S, say S', is valid on $[\alpha', \alpha' + \epsilon_2]$ for some $0 < \epsilon_2 \leqq \epsilon_1$; by Corollary 2.1 applied at α', $(x^{S'}(\alpha), u^{S'}(\alpha))$ satisfies (KT-3) and (KT-4) on $[\alpha', \alpha' + \epsilon]$ for some $0 < \epsilon \leqq \epsilon_2$. Since there is only a finite number of sets satisfying (4), S' will be found after a finite number of trials.

Finally, we prove (iv). It was established in the proof of (i) that Step 3 is entered each time a point of change α' is encountered at Step 2 such that the current set S^0 being used at Step 2 is not valid immediately above α'. It was established in the proof of (iii) that Step 3 finds a set which *is* valid immediately above α' in a finite number of trials, and control is returned to Step 2 along with the new valid set. By convention, we have taken α increasing, and by Theorem 3 there is but a finite number of points of change on [0, 1]; it follows that Step 3 will only have to be executed a finite number of times before termination obtains. The proof is complete.[6]

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[6] At Step 2, α' need not be the *next* point of change above α^0, for S^0 may remain valid on an interval spanning several points of change. The procedure could be modified to require $S^0 = B\alpha$ at Step 2, so that α' would assume, in turn, the values of each point of change of $B\alpha$; or one could require $S^0 = A\alpha$ at Step 2, so that α' would assume, in turn, the values of each point of change of $A\alpha$. The minimum requirement (the one adopted here) is $A\alpha \subset S^0 \subset B\alpha$ at Step 2, and seems more symmetrical and less arbitrary than either of the extreme requirements just mentioned.

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