SYMMETRIC MATHEMATICAL PROGRAMS*

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Mathematical programs with S-convex and with symmetric objective and feasibility regions are investigated. Optimal solutions and algorithms are presented for both continuous and discrete optimization problems.

The principal purpose of this note is to show that certain symmetric nonlinear integer programs can be reduced to corresponding extremal problems in just one variable.[1]

We consider the mathematical program given by:

$$\text{Maximize } f(x): x \, \varepsilon \, X$$

where f is a real-valued function mapping $X \rightarrow E^1$, and X is a closed subset of E^n. We postulate that the above program (and its later modification to integer x) has a solution.

Our notation uses S to denote a doubly stochastic matrix and more specifically P to denote a permutation matrix.

We consider the following definitions:

Definition 1. X is a symmetric set if $x \, \varepsilon \, X$ implies

$$xP \, \varepsilon \, X \text{ for all } P.$$

Definition 2. f is a symmetric function on a symmetric set X if for any P

$$f(xP) = f(x) \text{ for all } x \, \varepsilon \, X.$$

Definition 3. X is S-convex if $x \, \varepsilon \, X$ implies $xS \, \varepsilon \, X$, for all S.

Defintiion 4. f is an S-concave function on an S-convex set X if for any S

$$f(xS) \geqq f(x) \text{ for all } x \, \varepsilon \, X.$$

Definition 5. A point $x = (x_i)$ is symmetric if $x_i = z$ for $i = 1, \cdots, n$.

Definition 6. A point $x = (x_i)$ is nearly symmetric if $|x_i - x_j| \leqq 1$ for $i, j = 1, \cdots, n$.

Berge [2] gives further discussion of S-concave functions and S-convex sets. Symmetric convex sets are S-convex, but not necessarily conversely. Further, symmetric, quasi-concave functions defined over a symmetric convex set X are S-concave, but not necessarily conversely. Every nonempty S-convex set contains a symmetric point.

The results of this note are three theorems and some discussion of applications.

THEOREM 1. *If X is a closed, S-convex set and f is S-concave on X, then the set X^* of points maximizing f over X is a closed S-convex set.*

Let I denote the set of integer points in E^n.

THEOREM 2. *If X is S-convex and f is S-concave on X, then for any point $x \, \varepsilon \, X \cap I$ there is a nearly symmetric point $y \, \varepsilon \, X \cap I$ such that $f(y) \geqq f(x)$.*

PROOF. Since f is S-concave, it is symmetric. We therefore need only consider the case where $x_1 \geq x_2 \geq \cdots \geq x_n$. Let y be the unique nearly symmetric vector in I satisfying $y_1 \geq y_2 \geq \cdots \geq y_n$, $\sum_{i=1}^{k} y_i \leq \sum_{i=1}^{k} x_i$, $k = 1, \cdots, n-1$, and $\sum_{i=1}^{n} y_i = \sum_{i=1}^{n} x_i$. Then, y is majorized by x so that $y = xS$ (see [5], p. 49). Thus $y \varepsilon X \cap I$ and $f(y) \geq f(x)$, which completes the proof.

We shall say that y rounds x if y is integer and $\mathrm{Max}_i |x_i - y_i| < 1$. There are 2^n ways of rounding a point, x, with nonintegral coordinates. Under some additional assumptions we may further specify the solution to the integer problem of Theorem 2. Let Y denote the set of symmetric points of X.

THEOREM 3. *Assume X is S-convex, Y is convex, f is S-concave on X and f is quasi-concave on Y. Then for every optimal symmetric point zl which maximizes f on X there exists a point obtained by rounding zl that maximizes f on $X \cap I$.*[2]

PROOF. It suffices to show that if $x \varepsilon X \cap I$ is not obtained by rounding zl, then there is a $y \varepsilon X \cap I$ which is obtained by rounding zl and for which $f(y) \geq f(x)$. In view of Theorem 2, we may assume x is nearly symmetric. Let $[z]$ and $\langle z \rangle$ denote the largest integer not exceeding z and the smallest integer not less that z, respectively. The mean

$$x^* = l\left[\frac{\sum_{i=1}^{n} x_i}{n}\right]$$

is in X and $f(x^*) \geq f(x)$ by S-concavity of f. Thus, we need only consider two cases:

(i) $\mathrm{Max}_i\, x_i \leq [z]$

and

(ii) $\mathrm{Min}_i\, x_i \geq \langle z \rangle$.

In the former case we have

$$[z]l = \alpha zl + (1 - \alpha)x^*$$

for some $\alpha \varepsilon [0, 1]$. Since f is quasi-concave, we have $f(x^*) \leq f([z]l)$. Thus, $y = [z]l$ has the desired property. In the latter case, $y = \langle z \rangle l$ has the desired property and hence the proof is complete.

We further remark that in applying Theorem 3 only $n + 1$ rounding points need be considered rather than 2^n. This follows if we define

$$h(m) = f(\underbrace{[z] + 1, \cdots, [z] + 1}_{m}, [z], \cdots, [z]).$$

By the symmetry of f, only the value of m matters in choosing a rounding point and $0 \leq m \leq n$.

It should be noted that Theorem 2 (as well as Theorem 3) reduces the search for an optimal integer point to a one-dimensional search problem. This is useful to know for the case when not all of the hypotheses of Theorem 3 are satisfied but when f is S-concave and X is S-convex. To see this observe that for each integer K, f is constant for all nearly symmetric points of $X \cap I$ such that $\sum_{i=1}^{n} x_i = K$. Let $g(K)$ denote the common value of f on this set. Then it is only necessary to maximize g in order to maximize f. Obviously this concept of maximizing g can be used when Theorem 3 applies also.

Now let us consider some applications. Bessler and Veinott [3] have applied a

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[2] z is a scalar and l is a vector with each coordinate equal to 1.

special case of these results to symmetric networks. Kielson [6], [7] has considered a random walk model. Berge [2] discusses further application to quasi-convex programming and to the theory of inequalities. One may note that $E_r(x)^{1/r}$ is concave [8] over $x \geqq 0$, where $E_r(x)$ is the rth elementary symmetric function given by

$$E_r(x) = \sum_{\text{all } r\text{-tuples}} x_{i_1} \cdots x_{i_r}.$$

That is, $E_r(x)$ is the sum of r-tuples. Thus, $E_r(x)^k$ is quasi-concave and symmetric for $k \geqq 0$. A symmetric polynomial given by

$$f(x) = \sum_{j=1}^{m} a_j \prod_{r=1}^{n} E_r(x)^{k_{rj}}$$

is S-concave for a_j, $k_{rj} \geqq 0$, and we may apply Theorem 1 in such cases.

Theorem 3 may be applied by noting if $E_r(x)$ is integer maximized by the integer vector x^*, then $E_t(x)$ is also for $1 \leqq t \leqq n$. Further, in general we have

$$\text{Max} f(x) \leqq \sum_{j=1}^{m} a_j \prod_{r=1}^{n} \text{Max} E_r(x)^{k_{rj}}.$$

Equality is obtained at x^*. For example, if $X = \{x : x \geqq 0 \text{ and } \sum_{i=1}^{n} x_i \leqq b\}$, then the integer maximum of f occurs at

$$x_i^* = [b/n] + 1 \qquad \text{for } i = 1, \cdots, b - [b/n]n$$
$$x_i^* = [b/n] \qquad \text{for } i = b - [b/n]n + 1, \cdots, n.$$

Another application is in the stirring tank problem described by Aris [1]. By a change of variables given by

$$x_i = \text{Ln} [(k + \theta_i)/k], \qquad i = 1, \cdots, n$$

where we assume $(b_e - b_0)/(b_e - \gamma) > 1$, we can solve the equivalent problem:

$$\text{Min} \sum_{i=1}^{n} e^{x_i}:$$

$$\sum_{i=1}^{n} x_i \geqq \text{Ln} \left(\frac{b_e - b_0}{b_e - \gamma} \right)$$

$$x \geqq 0.$$

By Theorem 1 we obtain the symmetric solution

$$\theta_i^* = k \left\{ \left(\frac{b_e - b_0}{b_e - \gamma} \right)^{1/n} - 1 \right\} \quad \text{for } i = 1, \cdots, n.$$

As a final example we may consider the problem of congressional redistricting. We define

n = number of districts
b = total number of representatives
p = population of each district
x_i = number of representatives in district i
$a = b/n$ (one man, one vote)

Let r be an increasing function defined on E^1 (e.g., $r_i(x) = x/p$) then $| r_i(x) - a |$ is quasi-convex. Consider the problem

$$\text{Min} [\max_i |r(x_i) - a|]: \sum_{i=1}^{n} x_i = b, \qquad x \geqq 0, \qquad x \varepsilon I.$$

The objective is symmetric and quasi-convex (and hence S-convex). By Theorem 3

we see that

$$x_i^* = [b/n] + 1 \quad \text{for } i = 1, \cdots, b - n[b/n]$$

$$x_i^* = [b/n] \quad \text{for } i = b - n[b/n] + 1, \cdots, n.$$

In closing, observe that Theorems 1, 2, and 3 may be extended. We define a program to be compatibly symmetric of degree m $(1 \leq m \leq n)$ if x can be partitioned into m subvectors, say $x_1, \cdots, x_m$, such that f and X are symmetric for each x_j. Degrees of S-concavity follow similarly. Then, the theorems may be appropriately applied to reduce the program to m dimensions.

References

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