

## SURROGATE MATHEMATICAL PROGRAMMING

Harvey J. Greenberg and William P. Pierskalla

*Southern Methodist University, Dallas, Texas*

(Received January 1969)

This paper presents an approach, similar to penalty functions, for solving arbitrary mathematical programs. The surrogate mathematical program is a lesser constrained problem that, in some cases, may be solved with dynamic programming. The paper deals with the theoretical development of this surrogate approach.

CONSIDER A mathematical program given by (the bar below  $x$  indicating a row vector, the bar above a column vector, so that the transpose of  $x$  is  $\bar{x}$ ):

$$P: \text{Maximize } f(\underline{x}) : \underline{g}(\underline{x}) \leq \underline{b}^0 \quad \text{and} \quad \underline{x} \in X,$$

where  $f$  maps  $E^n$  into the reals;  $\underline{g}$  maps  $E^n$  into  $E^m$ ;  $X$  is a closed subset of  $E^n$ ;  $\underline{x}$  is a decision vector; and  $\underline{b}^0$  is a given 'resource' vector in  $E^m$ . We postulate that  $P$  has a solution and that  $X$  is compact.

EVERETT<sup>[6]</sup> has developed a Lagrangian penalty-function approach. His generalized-Lagrange-multiplier (GLM) method solves a succession of problems given by:

$$E: \max f(\underline{x}) - \underline{\lambda} \bar{g}(\underline{x}) : \underline{x} \in X.$$

The  $m$ -dimensional vector,  $\underline{\lambda}$ , is selected from  $E_+^m = [\underline{\lambda} : \underline{\lambda} \geq 0]$ . A solution to  $P$  is at hand when the solution to  $E$  satisfies the complementary slackness condition given by:  $\underline{\lambda}^* \bar{g}(\underline{x}^*) = \underline{\lambda}^* \bar{b}^0$  and  $\underline{g}(\underline{x}^*) \leq \underline{b}^0$ , where  $\underline{x}^*$  solves  $E$  for  $\underline{\lambda} = \underline{\lambda}^*$ .

BROOKS AND GEOFFRION<sup>[5]</sup> have related this to the classical work of KUHN AND TUCKER'S<sup>[12]</sup> saddle-point equivalence by showing that, if  $\underline{\lambda}^*$  exists, then  $(\underline{x}^*, \underline{\lambda}^*)$  is a saddle point of the Lagrangian:

$$L(\underline{x}, \underline{\lambda}) = f(\underline{x}) - \underline{\lambda} \bar{g}(\underline{x}) + \underline{\lambda} \bar{b}^0.$$

It is possible for no value of  $\underline{\lambda}$  to exist to satisfy the complementary slackness required for termination. The values of  $\underline{b}$  for which no such  $\underline{\lambda}$  exists are termed 'gaps.' BELLMORE, GREENBERG, AND JARVIS<sup>[3]</sup> have elaborated upon this and proved some theorems regarding the gap problem. Moreover, the gap problem can be related to the conjugate function of the optimal response  $f^*(\underline{b})$ , which is defined by

$$f^*(b) = f(x_P^*), \quad \text{where } x_P^* \text{ solves } P.$$

This is discussed in GREENBERG,<sup>[11]</sup> and further analyzed geometrically by GOULD.<sup>[10]</sup>

In this paper we use the conceptual framework of GLM to construct a different approach having less of a gap problem. A surrogate problem is defined by

$$S: \max f(x) : \lambda \bar{g}(x) \leq \lambda \bar{b}^0 \quad \text{and } x \in X,$$

where  $\lambda$  is selected from  $E_+^m$ . A surrogate gap exists if there does not exist a  $\lambda \geq 0$  such that  $S$  solves  $P$ .

Without loss of generality, we consider  $\lambda \in \Lambda = \{\lambda : \lambda \geq 0, \sum_{i=1}^{i=m} \lambda_i = 1\}$ . The advantage of using  $\Lambda$  instead of  $E_+^m$  lies in the fact that  $\Lambda$  is a bounded as well as a closed convex set of points, and one of the multipliers is determined. Henceforth we shall consider the  $S$  problem to be defined for  $\lambda \in \Lambda$ .

GLOVER<sup>[8]</sup> first introduced this approach for 0-1 integer programming. The constraint given by  $\lambda \bar{g}(x) \leq \lambda \bar{b}$  is called a *surrogate constraint* to  $P$ . The use of surrogate constraints in 0-1 programming has been further analyzed by BALAS,<sup>[2]</sup> GEOFFRION,<sup>[7]</sup> and GLOVER.<sup>[9]</sup> Our approach will generalize their results and unify them with other approaches taken in mathematical programming (viz., Everett's GLM).

### BASIC THEORY

IN THIS SECTION we shall describe some of the many relations among  $E$ ,  $S$ , and  $P$  problems. Clearly  $f(x_S^*) \geq f(x_P^*)$  for all  $\lambda \in \Lambda$ , where  $x_S^*$  solves  $S$ , and  $x_P^*$  solves  $P$ . Hence, if  $x_S^*$  is feasible in  $P$ , it solves  $P$ . One question is existence: Does  $\Lambda$  contain a point, say  $\lambda^*$ , such that  $x_S^*$  solves  $P$ ? This is similar to the gap question in  $E$ , which asks: Does  $E_+^m$  contain a point,  $\lambda^*$ , such that  $x_E^*$  solves  $P$ ?

LEMMA 1. *If there exists  $\lambda^* \neq 0$  such that  $x_E^*$  solves  $P$ , then  $x_E^*$  solves  $S$  for  $\lambda = (\lambda^* / \sum_{i=1}^{i=m} \lambda_i^*)$ .*

Note that, when  $\lambda^* = 0$ , then  $x_E^*$  solves  $S$  for any  $\lambda \in \Lambda$ , since this means that the unconstrained maximum of  $f$  is feasible in  $P$  (and hence  $S$ ).

Lemma 1 says that, if Everett's GLM can solve  $P$ , then the surrogate approach can also solve  $P$ . The converse is not true, as we shall see shortly.

*Proof.* If  $x_E^*$  solves  $P$  for  $\lambda = \lambda^*$ , then  $f(x_E^*) - \lambda^* \bar{b}^0 \geq f(x) - \lambda^* \bar{g}(x)$  for all  $x \in X$ . Therefore,  $f(x_E^*) \geq f(x) + \lambda^* [\bar{b}^0 - \bar{g}(x)]$  for  $x \in X$ . Hence,  $f(x_E^*) \geq f(x)$  for all  $x \in X$  such that  $\lambda^* \bar{g}(x) \leq \lambda^* \bar{b}^0$  or, equivalently, such that  $(\lambda^* / \sum_{i=1}^{i=m} \lambda_i^*) \bar{g}(x) \leq (\lambda^* / \sum_{i=1}^{i=m} \lambda_i^*) \bar{b}^0$ . Therefore,  $x_E^*$  solves  $S$ , and the proof is complete.

It will be convenient to define the optimal Lagrangian for  $P$  (and  $S$ ) as

$$H(\gamma, \underline{\lambda}) = \sup_{\underline{x} \in X} \{f(\underline{x}) - \gamma \underline{\lambda} \bar{g}(\underline{x}) + \gamma \underline{\lambda} \bar{b}^0\}$$

over the set

$$\{(\gamma, \underline{\lambda}) : \gamma \geq 0, \underline{\lambda} \in \Lambda \text{ and } \sup_{\underline{x} \in X} \{f(\underline{x}) - \gamma \underline{\lambda} \bar{g}(\underline{x}) + \gamma \underline{\lambda} \bar{b}^0\} < +\infty\}.$$

It is well known that this set is convex. The scale factor  $\gamma$  in the Lagrangian is necessary, since the Lagrange multipliers must be chosen from the nonnegative orthant, whereas our  $\lambda$ 's belong to the compact set  $\Lambda$ . Thus, it is apparent that  $S$  has one less multiplier than  $E$ . Further, we define

$$F(\underline{\lambda}) = \sup_{\underline{x} \in X} f(\underline{x}) : \underline{\lambda} \bar{g}(\underline{x}) \leq \underline{\lambda} \bar{b}^0$$

over the set

$$\Lambda' = \{\underline{\lambda} : \underline{\lambda} \in \Lambda, \sup_{\underline{x} \in X} [f(\underline{x}) : \underline{\lambda} \bar{g}(\underline{x}) \leq \underline{\lambda} \bar{b}^0] < +\infty\}.$$

It is easy to prove that  $\Lambda'$  is convex. Moreover, for the results of concern here it is convenient to assume  $\Lambda' = \Lambda$ . Also, applications of this theory will assume that 'sup' can be replaced by 'max.' Of course, this may not be valid, but context dictates the implied assumptions.

It can be shown<sup>[6]</sup> that, every time the  $E$  model is solved for some  $\underline{\lambda}$ , a  $P$  problem is solved for  $\underline{b}$ :  $\underline{\lambda} \bar{b} = \underline{\lambda} \bar{g}(\underline{x}_S^*)$  and  $\bar{b} \geq \bar{g}(\underline{x}_S^*)$ . A similar result holds for the  $S$  problem.

**LEMMA 2.** *If  $\underline{x}_S^*$  solves  $S$ , then  $\underline{x}_S^*$  solves  $P$  for any  $\underline{b}^*$ :  $\underline{b}^* \geq \underline{g}(\underline{x}_S^*)$  and  $\underline{\lambda} \bar{b}^* \leq \underline{\lambda} \bar{b}^0$ .*

The proof follows from the fact that  $\underline{x}_S^*$  solves  $S$  for any  $\underline{b}^*$  given by the above relations and that  $\underline{x}_S^*$  is feasible in  $P$  for  $\underline{b}^* \geq \underline{g}(\underline{x}_S^*)$ .

This is more general than the above statement regarding the  $E$  model in that we do not require complementary slackness. Note that, if  $\underline{x}_S^*$  solves  $S$ , then Lemma 2 implies the analogue of the  $E$ -model property that  $\underline{x}_S^*$  solves  $P$  for any  $\underline{b}^*$ :  $\underline{b}^* \geq \underline{g}(\underline{x}_S^*)$  and  $\underline{\lambda} \bar{b}^* = \underline{\lambda} \bar{g}(\underline{x}_S^*)$ .

It is also possible to relate the surrogate multiplier to the Lagrange multiplier.

**LEMMA 3.** *If  $P$  is Lagrange regular (i.e., the Kuhn-Tucker conditions are necessary<sup>[12]</sup>), and if  $\underline{\lambda}^*$  exists such that  $S$  solves  $P$ , then  $\gamma \cdot \underline{\lambda}^*$  is a Lagrange multiplier for  $P$  for some  $\gamma \geq 0$ .*

*Proof.* If  $P$  is Lagrange regular, so is  $S$ . Hence,  $\underline{x}_S^*$  satisfies  $\nabla f - \gamma \sum_{i=1}^{i=m} \lambda_i \nabla g^i = \underline{0}$  for some  $\gamma \geq 0$ . This completes the proof.

Suppose  $\underline{\lambda}^*$  does exist such that  $S$  solves  $P$ . How do we find it? To develop a convergent search scheme, it is important to recognize properties of  $\underline{\lambda}^*$ . In Everett's model, we are utilizing saddle-point equivalence. That is, if  $\underline{\lambda}^*$  exists such that  $E$  solves  $P$ , then  $(\underline{x}^*, \underline{\lambda}^*)$  is a saddle-point of the Lagrangian. This fact leads to classical search schemes for minimization with respect to  $\underline{\lambda} \geq \underline{0}$ . It is also well known that  $H(\gamma, \underline{\lambda})$  is convex

over  $E_+^m$ . This is an important property, since the multipliers we seek in GLM minimize  $H(\gamma, \lambda)$ .

One of the results of this paper is to show a similar minimizing property for  $S$ .

**THEOREM 1.** *If  $S$  solves  $P$  for  $\underline{\lambda} = \underline{\lambda}^*$ , then  $\underline{\lambda}^*$  minimizes  $F(\underline{\lambda}) : \underline{\lambda} \in \Lambda$ .*

*Proof.* By definition, we have  $F(\underline{\lambda}) = \max f(\underline{x}) : \underline{\lambda} \bar{g}(\underline{x}) \leq \underline{\lambda} \bar{b}^0$ . Since  $S$  solves  $P$ , let their solution be denoted as  $\underline{x}^*$ . Now  $\underline{g}(\underline{x}^*) \leq \bar{b}^0$  implies  $\underline{\lambda} \bar{g}(\underline{x}^*) \leq \underline{\lambda} \bar{b}^0$  for all  $\underline{\lambda} \in \Lambda$ . Hence,  $F(\underline{\lambda}) \geq f(\underline{x}^*) = F(\underline{\lambda}^*)$ . This completes the proof.

Lemma 1 shows that  $\underline{\lambda}^*$  exists for  $S$  and provides a solution  $\underline{x}_S^*$  to  $P$  whenever  $E$  has no gap (e.g., a convex program). In that case, if  $P$  is Lagrange regular, then

$$\max_{\underline{x} \in X} [f(\underline{x}) : \underline{\lambda}^* \underline{g}(\underline{x}) \leq \underline{\lambda}^* \bar{b}^0] = \max_{\underline{x} \in X} [f(\underline{x}) : \underline{\lambda}^* \bar{g}(\underline{x}) = \underline{\lambda}^* \bar{b}^0],$$

since Lagrange regularity implies the complementary slackness condition:  $\underline{\lambda}^* \bar{g}(\underline{x}^*) = \underline{\lambda}^* \bar{b}^0$ , where  $\underline{x}^*$  is a solution to  $P$ . However, for the surrogate problem in general we have

$$\max_{\underline{x} \in X} [f(\underline{x}) : \underline{\lambda} \bar{g}(\underline{x}) \leq \underline{\lambda} \bar{b}^0] \geq \max_{\underline{x} \in X} [f(\underline{x}) : \underline{\lambda} \bar{g}(\underline{x}) = \underline{\lambda} \bar{b}^0].$$

We will now give an example to show that, when the  $E$  problem has a gap, the corresponding  $S$  problem may not have a gap, and, indeed, for the example, the solution to  $S$  solves  $P$ .

Consider the integer program  $P$ : Maximize  $x_1 + x_2$  subject to (1)  $x_1 + 2x_2 \leq 4 = b_1^0$ ; (2)  $2x_1 + x_2 \leq 3 = b_2^0$ ;  $x_1, x_2 \geq 0$  and integer.

First, consider two  $E$  problems:

A. *Place (1) in objective.* Then,  $E_A$ : Maximize  $x_1 + x_2 - \lambda(x_1 + 2x_2)$  subject to:  $2x_1 + x_2 \leq 3$ ;  $x_1, x_2 \geq 0$  and integer. For  $\lambda = \frac{1}{3}$ , we obtain two solutions as  $\underline{x}^1 = (1, 1)$ ,  $\underline{x}^2 = (0, 3)$ . The values of  $b_1$  obtained are  $b_1^1 = 3$  and  $b_1^2 = 6$ . Thus, the desired  $b_1 = b_1^0 = 4$  is in a gap.

B. *Place (2) in objective.* Then,  $E_B$ : Maximize  $x_1 + x_2 - \lambda(2x_1 + x_2)$  subject to:  $x_1 + 2x_2 \leq 4$ ;  $x_1, x_2 \geq 0$  and integer. For  $\lambda = \frac{1}{3}$ , the solutions to  $E_B$  are  $\underline{x}^1 = (4, 0)$ ,  $\underline{x}^2 = (0, 2)$ . The corresponding values of  $b_2$  obtained are  $b_2^1 = 8$ ,  $b_2^2 = 2$ . Thus,  $b_2 = 3$  is in a gap.

Since A and B yield gaps, placing both constraints in the objective must necessarily yield a gap. [This follows, since, if  $\underline{x}^*$  solves  $P$  and  $E$ , then  $\underline{x}^*$  solves  $\max [f(\underline{x}) - \lambda_1 \bar{g}_1(\underline{x}) : \bar{g}_2(\underline{x}) \leq \bar{b}_2^0]$  for any partition of  $\bar{g} = (\bar{g}_1, \bar{g}_2)$ .] We thus omit this case. Let us now consider the surrogate model with  $\Lambda = \{\lambda : 0 \leq \lambda \leq 1\}$  so that  $\lambda_2 = \lambda$  and  $\lambda_1 = 1 - \lambda$ :

$$S: \text{Maximize}_{\underline{x} \in X} x_1 + x_2 \text{ subject to } (1 + \lambda)x_1 + (2 - \lambda)x_2 \leq 4 - \lambda,$$

where  $0 \leq \lambda \leq 1$ . For  $\lambda = \frac{1}{2}$ , then, solutions to  $S$  are  $\underline{x}^1 = (2, 0)$ ,  $\underline{x}^2 = (1, 1)$ ,  $\underline{x}^3 = (0, 2)$ , and  $\underline{x}^2$  and  $\underline{x}^3$  are feasible in  $P$ . Thus,  $S$  solves  $P$  for  $\lambda = \frac{1}{2}$ .

We note that the source of the  $E$ -gap was lack of complementary slack-

ness due to integer constraints, and both models A and B actually obtained a solution (but could not deduce that it was optimal). The surrogate model successfully resolved this gap because complementary slackness did not need to hold.

Moreover, it is possible for  $S$  to have a solution, while  $E$  is unbounded.

For example, consider:  $\max \sum_{i=1}^{i=n} x_i^2$ , subject to:  $\sum_{i=1}^{i=n} x_i \leq b$ ;  $x_1 + x_n \leq a$ ;  $x \geq 0$ , where  $b > a > 0$ .  $E$  has no solution, but  $S$  has a solution for  $\underline{\lambda} = (1, 0)$  given by  $x_{i^*} = b$  for some  $i \neq 1, n$ , and  $x_{j^*} = 0$  for  $j \neq i$ . This solves  $P$ .

Heuristically, we are trading the gap problem and the number of multipliers for the extra computation of solving a one-constraint mathematical program rather than an unconstrained mathematical program. To carry this further,  $P$  represents a difficult problem insofar as multiple constraints tend to reduce the efficiency of methods such as dynamic programming or gradient projection. Problem  $E$  represents the other extreme, where an entirely different objective is constructed, and may fail to solve  $P$ . The surrogate problem,  $S$ , is somewhat between these extremes.

But what can be said about the gap problem in  $S$ ? The next result states how to detect a surrogate gap.

Let problem  $S$  have a collection  $\Omega^*$  of multiple solutions for a given  $\underline{\lambda}$ . That is,  $\Omega^*(\underline{\lambda}) = \{x^* : f(x^*) = \max f(x) : \underline{\lambda} \bar{g}(x) \leq \underline{\lambda} \bar{b}^0, x \in X\}$ . Define  $\Gamma(\underline{\lambda})$  as the index set of the elements of  $\Omega^*(\underline{\lambda})$  [ $\Gamma(\underline{\lambda})$  may be finite, denumerable or nondenumerable]. It may be convenient for the reader to consider the finite case to best understand Theorem 2.

For any solution  $x_{\gamma^*} \in \Omega^*(\underline{\lambda})$ ,  $\gamma \in \Gamma(\underline{\lambda})$ , let

$$B_{\gamma}(\underline{\lambda}) = \{ \underline{b} : \underline{b} \geq \underline{g}(x_{\gamma^*}) \text{ and } \underline{\lambda} \underline{b} \leq \underline{\lambda} \bar{b}^0 \},$$

where  $\bar{b}^0$  is the original  $\bar{b}$  given in  $P$ , i.e., by Lemma 2,  $x_{\gamma^*}$  solves  $P$  for any  $\underline{b} \in B_{\gamma}(\underline{\lambda})$ . Let

$$B(\underline{\lambda}) = \bigcup_{\gamma \in \Gamma(\underline{\lambda})} B_{\gamma}(\underline{\lambda}) \text{ and } B = \bigcup_{\underline{\lambda} \in \Lambda} B(\underline{\lambda}).$$

Thus, if  $\bar{b} \in B$ , there is a  $\underline{\lambda} \in \Lambda$  such that a solution to  $S$  exists that solves  $P$  for that  $\bar{b}$ . Of course, we would like  $\bar{b}^0$  to be in  $B$ , because if  $\bar{b}^0 \notin B$  there is no  $\underline{\lambda} \in \Lambda$  such that  $S$  solves  $P$  for  $\bar{b}^0$  and  $\bar{b}^0$  is in a surrogate gap.

NOTE.  $B$  contains all right-hand sides obtained by GLM, but may contain others failing the complementary slackness conditions. To see this, observe that, if  $\underline{\lambda} \bar{b}^0 = \underline{\lambda} \bar{g}(x_{\gamma^*})$  for some  $\gamma \in \Gamma(\underline{\lambda})$  and  $\underline{\lambda} \in \Lambda$ , then  $B_{\gamma}(\underline{\lambda})$  contains those  $\bar{b}$  with complementary slackness as in GLM. That is,  $g_i(x_{\gamma^*}) = b_i$  if  $\lambda_i > 0$ , and  $g_i(x_{\gamma^*}) \leq b_i$  if  $\lambda_i = 0$ , and  $x_{\gamma^*}$  is optimal for  $P$ .

Denote the convex hull of the points  $\underline{g}(x_{\gamma^*})$ ,  $\gamma \in \Gamma(\underline{\lambda})$  by  $C(\underline{\lambda})$ ; that is,

$\underline{b} \in C(\underline{\lambda})$  if, and only if,  $\underline{b}$  is a convex combination of the vectors  $g(\underline{x}_\gamma^*)$  that are said to span  $C(\underline{\lambda})$ .

**THEOREM 2.** *Suppose there exists  $\underline{\lambda} \in \Lambda$  such that  $\underline{b}^0 \in C(\underline{\lambda})$ . Then, either there exists  $\gamma \in \Gamma(\underline{\lambda})$  such that  $\underline{x}_\gamma^*$  solves  $P$ , or there exists no  $\underline{\lambda} \in \Lambda$  such that a solution to  $S$  is a solution to  $P$  (i.e.,  $\underline{b}^0$  is in a gap).*

*Proof.* Let  $\underline{x}^0$  be any optimal solution to  $P$ . Let  $\underline{\lambda} \in \Lambda$  be such that  $\underline{b}^0 \in C(\underline{\lambda})$ . If  $\underline{x}^0$  solves  $S$ , then  $\underline{x}^0 = \underline{x}_\gamma^*$  for some  $\gamma \in \Gamma(\underline{\lambda})$ , and, since  $\underline{x}^0$  is feasible in  $P$ ,  $\underline{b}^0 \in B(\underline{\lambda})$ . If  $\underline{x}^0$  does not solve  $S$ , then  $f(\underline{x}^0) < f(\underline{x}_\gamma^*)$  for all  $\gamma \in \Gamma(\underline{\lambda})$ , since  $\underline{x}^0$  is always feasible in  $S$ . Now assume, to the contrary, that  $\underline{b}^0 \in B$ ; that is, assume there is a multiplier  $\underline{\lambda}^0 \in \Lambda$  such that  $S$  solves  $P$ . By virtue of the inequality  $f(\underline{x}^0) < f(\underline{x}_\gamma^*)$  for all  $\gamma \in \Gamma(\underline{\lambda})$ ,  $\underline{x}_\gamma^*$  is not a solution to  $P$  for any  $\gamma \in \Gamma(\underline{\lambda})$ , and  $\underline{\lambda}^0 \bar{g}(\underline{x}_\gamma^*) > \underline{\lambda}^0 \bar{b}^0$  for all  $\gamma \in \Gamma(\underline{\lambda})$ . Therefore, for any point  $\bar{y} \in C(\underline{\lambda})$ ,  $\underline{\lambda}^0 \bar{y} > \underline{\lambda}^0 \bar{b}^0$ , since  $\bar{y}$  is just a convex combination of the vectors  $\bar{g}(\underline{x}_\gamma^*)$ ,  $\gamma \in \Gamma(\underline{\lambda})$ . But  $\bar{b}^0 \in C(\underline{\lambda})$ , and hence we obtain the contradiction. A more elaborate description of the argument for Everett gaps appears in reference 3.

For the case of two constraints ( $m=2$ ), Fig. 1 gives a geometrical interpretation of how a gap can be viewed in the resource plane. Since there are only two constraints, we use only one surrogate variable  $\lambda \in [0, 1]$ , i.e.,  $\lambda_1 = \lambda$  and  $\lambda_2 = 1 - \lambda$ . Assume that for some  $\lambda^* \in (0, 1)$  we obtain the two alternate optima  $\hat{x}$  and  $x'$  which correspondingly solve  $S$  for  $g(\hat{x}) = \hat{b}$  and  $g(x') = b'$ , where  $\hat{b} = (\hat{b}_1, \hat{b}_2)$  and  $b' = (b'_1, b'_2)$ . Also assume that  $b^0$  is a strict convex combination of  $\hat{b}$  and  $b'$  [thus  $b^0 = \alpha \hat{b} + (1 - \alpha)b'$  for some  $\alpha \in (0, 1)$ ]. Now, neither  $\hat{x}$  nor  $x'$  is feasible for  $P$  and  $f(\hat{x}) = f(x') > f(x^0)$  (recall  $x^0$  is the optimal for  $P$ ; thus  $x^0$  is feasible for  $S$ ).

In Fig. 1 the dark region represents the points  $[g_1(x), g_2(x)]$  such that  $g_i(x) \leq b_i^0$ ,  $i=1, 2$ ; thus, it is the image in the resource plane of the feasibility region for  $P$ . The vertically lined region represents the points  $[g_1(x), g_2(x)]$  such that  $\lambda g_1(x) + (1 - \lambda)g_2(x) \leq \lambda b_1^0 + (1 - \lambda)b_2^0$ ; thus, it is the image in the resource plane of the feasibility region for  $S$ . Note that the dark region is contained in the vertically lined region, since any  $x$  feasible in  $P$  is also feasible in  $S$ . The cross-lined region (in the upper right corner of the figure) represents the points  $[g_1(x), g_2(x)]$  such that  $g_i(x) > b_i^0$ ,  $i=1, 2$ .

Now, if  $\lambda^*$  is decreased, the point  $x'$  remains feasible, since  $b'$  is still in the vertically lined region, so  $F(\lambda^* - \epsilon) \geq F(\lambda^*) = f(x')$ . If  $\lambda^*$  is increased, the point  $\hat{x}$  remains feasible, since  $\hat{b}$  is still in the vertically lined region, so  $F(\lambda^* + \epsilon) \geq F(\lambda^*) = f(\hat{x})$ .

Thus, no matter whether  $\lambda^*$  is increased or decreased, either  $\hat{x}$  or  $x'$  remains feasible in the new  $S$  problem, the new value of  $f$  is at least as large as  $f(\hat{x}) = f(x')$ , and we can never find a  $\lambda$  that allows us to solve  $P$  by solving  $S$ .

We noted that Theorem 2 gives a sufficient condition for gaps to occur. With a few additional assumptions, it is possible to state sufficient conditions such that no such gap exists. In this regard, we know from refer-

ences 3, 10, and 11 that, in the  $E$  problem, no gap exists for all  $\underline{b}$  such that  $P$  has a strict interior [i.e., there exists an  $x \in X$  such that  $g(x) < \underline{b}$ ] if, and only if,  $f^*(\underline{b})$  is a concave function. For the  $S$  problem we can weaken the concavity of  $f^*(\underline{b})$  to say: if  $f^*(\underline{b})$  is a closed quasiconcave function then  $S$  has no gaps. [A function  $F$  defined over a convex set  $\Lambda$  is quasiconvex if  $F(\alpha\lambda_1 + (1-\alpha)\lambda_2) \leq \max[F(\lambda_1), F(\lambda_2)]$  for all  $\lambda_1, \lambda_2 \in \Lambda$  and  $\alpha \in [0, 1]$ . It is closed if its epigraph is a closed set.]

COROLLARY 2.1. If  $\bar{b}^0 \in C(\underline{\lambda}^*)$ , where  $\underline{\lambda}^*$  minimizes  $F(\underline{\lambda})$ , if  $f^*(\underline{b})$  is a closed

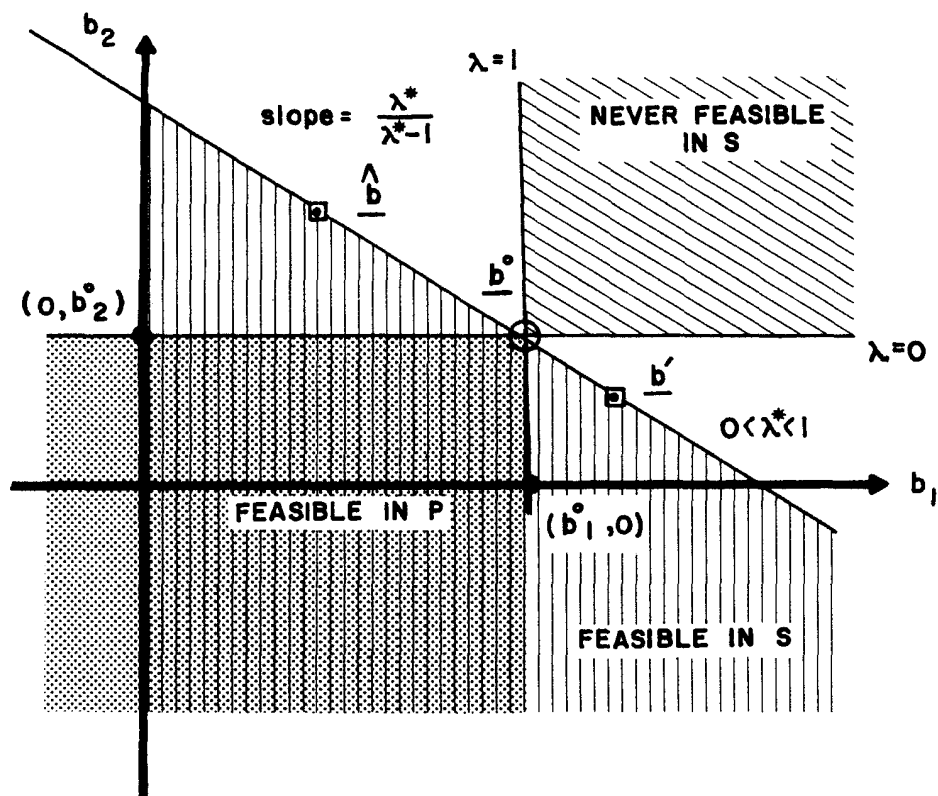


Fig. 1. Illustration of a surrogate gap in the resource plane ( $m=2$ ).

quasiconcave function, and if  $B_\gamma(\underline{\lambda})$  is a closed (convex) set for all  $\underline{\lambda} \in \Lambda$ , then  $\bar{b}^0 \in B_\gamma(\underline{\lambda}^*)$  (i.e.,  $S$  has no such gap).

Proof. Let

$$\psi(\underline{\lambda}) = \{ \underline{b}^* : f^*(\underline{b}^*) \geq f^*(\underline{b}) \text{ for all } \underline{b} \in B_\gamma(\underline{\lambda}) \} \cap B_\gamma(\underline{\lambda}).$$

Since  $f^*(\underline{b})$  is a closed quasiconcave function and  $B_\gamma(\underline{\lambda})$  is a closed convex set for all  $\underline{\lambda} \in \Lambda$ , then  $\psi(\underline{\lambda})$  is a closed convex set for all  $\underline{\lambda} \in \Lambda$ . In particular,  $\psi(\underline{\lambda}^*)$  is a closed convex set and  $g(x_\gamma^*) \in \psi(\underline{\lambda}^*)$  for  $\gamma \in \Gamma(\underline{\lambda}^*)$ . Since  $\bar{b}^0 \in C(\underline{\lambda}^*)$ ,

$\bar{b}^0 = \sum_{i=1}^{m+1} \alpha_i g(x_i^*)$  for  $\alpha_i \geq 0, \sum_{i=1}^{m+1} \alpha_i = 1$ . Hence,  $\bar{b}^0 \in \psi(\underline{\lambda}^*) \subset B_\gamma(\underline{\lambda}^*)$ .

The proof is complete.

This corollary may be further weakened if we define

$$F(\underline{\lambda}) = \sup\{f(x) : \underline{\lambda} \bar{g}(x) \leq \underline{\lambda} \bar{b}^0, x \in X\},$$

$$= \sup\{f^*(b) : \underline{\lambda} \bar{b} \leq \underline{\lambda} \bar{b}^0 \text{ and } \bar{g}(x) \leq \bar{b} \text{ for some } x \in X\}.$$

In addition, we need to assume that, for some  $\underline{\lambda}^0 \in \Lambda, F(\underline{\lambda}^0) < +\infty$  and since the sets of interest are closed and  $f^*(\bar{b})$  is closed, then, for this  $\underline{\lambda}^0$  [and all  $\underline{\lambda} : F(\underline{\lambda}) \leq F(\underline{\lambda}^0)$ ], we can replace sup by max. Then in the corollary we can replace the sets  $B_\gamma(\underline{\lambda})$  by

$$R(\underline{\lambda}) = \{\bar{b} : \underline{\lambda} \bar{b} \leq \underline{\lambda} \bar{b}^0 \text{ and } \bar{g}(x) \leq \bar{b} \text{ for some } x \in X\}.$$

In LUENBERGER's paper,<sup>[13]</sup> sufficient conditions for the quasiconcavity of  $f^*(\bar{b})$  were given, and the conclusions of Corollary 2.1 deduced for his more restrictive assumptions.

We now proceed to a theorem that yields some bounds on the optimal value to the objective function in  $P$ . The theorem essentially states that every solution to  $S$  yields an upper bound for  $P$ . Let  $\Omega^* = \bigcup_{\underline{\lambda} \in \Lambda} \Omega^*(\underline{\lambda})$ .

**THEOREM 3.** *If  $x^0$  is an optimal solution to  $P$ , then (i) if  $\bar{b}^0 \in B, f(x^0) \leq f(x^*)$  for all  $x^* \in \Omega^*$ , and (ii) if  $\bar{b}^0 \notin B, f(x^0) < f(x^*)$  for all  $x^* \in \Omega^*$ .*

*Proof.* Part (i) is obvious from the fact that  $x^0$  is feasible in  $S$  for any  $\underline{\lambda} \in \Lambda$  and hence  $f(x_s^0) \geq f(x^0)$ . Part (ii) follows directly by contradiction.

We also note, by (ii):

**COROLLARY 3.1.** *If  $\bar{b}^0 \notin B$  and if  $F(\underline{\lambda})$  is continuous, then  $f(x^0) < \min_{\underline{\lambda} \in \Lambda} F(\underline{\lambda})$ . Without the continuity assumption,  $F$  may not assume its minimum in  $\Lambda$ . All we can say then is  $f(x^0) \leq \inf_{\underline{\lambda} \in \Lambda} F(\underline{\lambda})$ . The inequality is no longer strict.*

A comparable result that relates the preceding theorem involving the  $S$  and  $P$  problems to the  $E$  problem can also be given.

**THEOREM 4.** *If  $x^0$  is an optimal solution to  $P$ , then for fixed  $\underline{\lambda}^* \in \Lambda$  and all  $x \in X$ :*

(i)

$$f(x^0) \begin{cases} \leq \max\{f(x) : \underline{\lambda}^* \bar{g}(x) \leq \underline{\lambda}^* \bar{b}^0\}, \\ \leq \max\{f(x) - \gamma \underline{\lambda}^* [\bar{g}(x) - \bar{b}^0]\} \text{ for all } \gamma \geq 0, \end{cases}$$

and

(ii) *if  $x_s^*$  is optimal for  $S$  for  $\underline{\lambda}^* \in \Lambda$  and if  $\underline{\lambda}^* \bar{g}(x_s^*) < \underline{\lambda}^* \bar{b}^0$ , then*

$$f(x^0) \leq f(x_s^*) < \max\{f(x) - \gamma \underline{\lambda}^* [\bar{g}(x) - \bar{b}^0]\} \text{ for all } \gamma > 0.$$

*Proof.* Let  $H(\gamma, \underline{\lambda}) = \max\{f(x) - \gamma \underline{\lambda} [\bar{g}(x) - \bar{b}^0]\}, x \in X$ . Then part (i) says that  $f(x^0) \leq F(\underline{\lambda}^*) \leq H(\gamma, \underline{\lambda}^*)$ , and part (ii) says that  $f(x^0) \leq F(\underline{\lambda}^*) < H(\gamma, \underline{\lambda}^*)$ . In both parts the first inequality follows from Theorem 3. To show  $F(\underline{\lambda}^*) \leq H(\gamma, \underline{\lambda}^*)$ , observe that  $-\underline{\lambda}^* [\bar{g}(x_s^*) - \bar{b}^0] \geq 0$  implies



$f(x_s^*) \leq f(x_s^*) - \gamma \lambda^* [\bar{g}(x_s^*) - \bar{b}] \leq \max\{f(x) - \gamma \lambda^* [\bar{g}(x) - \bar{b}^0]\}$ , and the inequality is strict when  $-\lambda^* [\bar{g}(x_s^*) - \bar{b}^0] > 0$  and  $\gamma > 0$ . The proof is complete.

The essence of the preceding theorem lies in the fact that the surrogate gap region is always contained in the GLM gap region, and under (ii) the surrogate gap region is a proper subset of the GLM gap region when complementary slackness  $\lambda^* \bar{g}(x_s^*) = \lambda^* \bar{b}^0$  does not hold in the surrogate model. That is, if we solve  $E$  and obtain a gap for  $\lambda = \gamma \lambda^*$ , where  $\lambda^* \in \Lambda$  and  $\gamma \geq 0$ , and, if we then solve  $S$  for  $\lambda^*$  and obtain  $\lambda^* \bar{g}(x_s^*) < \lambda^* \bar{b}^0$ , we may have been able to strictly reduce the gap region. The knapsack example presented earlier is of this type.

### SOME PROPERTIES OF $F(\lambda)$

IN THE PRECEDING section we noted that, once a minimizing  $\lambda \in \Lambda$  is found, we either have the optimal solution to  $P$  or else we have discovered an  $S$  gap. It is natural therefore to seek a minimizing  $\lambda \equiv \lambda^*$ . In attempting to find  $\lambda^*$ , it would be helpful to know whether  $F(\lambda)$  possesses certain properties. We now present three theorems pertaining to the structure of  $F(\lambda)$ .

**THEOREM 5.**  $F(\lambda)$  is quasiconvex in  $\lambda \in \Lambda$ .

*Proof.* Since  $\Lambda$  is convex for any  $\lambda^1, \lambda^2 \in \Lambda$  and  $0 \leq \alpha \leq 1$ , we have  $\lambda = \alpha \lambda^1 + (1 - \alpha) \lambda^2$  also belongs to  $\Lambda$ . Let  $x^*(\lambda)$ ,  $x^*(\lambda^1)$ , and  $x^*(\lambda^2)$  be any optimal solutions to  $F(\lambda)$ ,  $F(\lambda^1)$ , and  $F(\lambda^2)$  respectively. Now for any  $\alpha \in [0, 1]$ , either (i)  $\lambda^1 \bar{g}[x^*(\lambda)] \leq \lambda^1 \bar{b}^0$ , or (ii)  $\lambda^2 \bar{g}[x^*(\lambda)] \leq \lambda^2 \bar{b}^0$ , or both. (If neither (i) nor (ii) were true then we would have

$$\alpha \lambda^1 \bar{g}[x^*(\lambda)] + (1 - \alpha) \lambda^2 \bar{g}[x^*(\lambda)] > \alpha \lambda^1 \bar{b}^0 + (1 - \alpha) \lambda^2 \bar{b}^0,$$

so that  $\lambda \bar{g}[x^*(\lambda)] > \lambda \bar{b}^0$ , which is a contradiction.) Thus, for any  $\alpha \in [0, 1]$ ,  $x^*(\lambda)$  is feasible for either  $\lambda^1$  or  $\lambda^2$  or both, and

$$\begin{aligned} F(\lambda) &= f[x^*(\lambda)] \leq \max\{f[x^*(\lambda^1)], f[x^*(\lambda^2)]\} \\ &= \max[F(\lambda^1), F(\lambda^2)]. \end{aligned}$$

The proof is complete.

Algorithms for solving  $P$  with  $S$  should capitalize on this quasiconvexity property of  $F(\lambda)$ . (See MARTOS<sup>[14]</sup> for properties of quasiconvex functions.) The development of such algorithms and their testing will be left for future work; however, it should be mentioned that, when we have only two constraints ( $m=2$ ), then we can consider one multiplier, say  $\lambda \in [0, 1]$ , and we are searching  $F(\lambda)$  over the unit interval. Some efficient techniques for doing this are Fibonacci search and Fibonacci search modified by certain a priori knowledge concerning the location of  $\lambda^* \in [0, 1]$ . When  $F(\lambda)$  has

flat regions (so that Fibonacci search does not apply), then a bisection method always works. Of course, any scheme may be used that only requires quasiconvexity.

Before proceeding, it is interesting to compare the advantages and disadvantages of GLM to  $S$ . Table I presents five factors; the first two are in GLM's favor and the last three favor  $S$ . Moreover, suppose  $m-1$  of the constraints are placed in the objective and GLM is used with  $m-1$  multipliers and one constraint remains. Then factors (1) and (3) are the same in GLM and  $S$ . In general, we cannot compare the tradeoffs of (2), (4), and (5); however, if we have only two constraints (if  $m=2$ ), then (2) is insignificant (by the argument in the preceding paragraph) so that  $S$  is to be preferred for the case of one multiplier. In particular, for state-reduction dynamic programming, the gap problem is dominant, and  $S$

TABLE I  
COMPARISON OF GLM AND S

	GLM	S
(1)	Solves sequence of unconstrained problems	Solves sequence of $r$ constraint problems
(2)	$H(\gamma, \lambda)$ is convex	$F(\lambda)$ is quasiconvex
(3)	$m$ multipliers	$m - r$ multipliers
(4)	Closed, convex, but unbounded region for multipliers	Compact, convex region for multipliers
(5)	Gaps arise when $f^*(\underline{b})$ is not concave	Reduced gap region and if the conditions of Corollary 2.1 hold, no gap exists

should be used. The authors are presently conducting an empirical comparison of GLM and  $S$  for two- and three-dimensional knapsack-type problems.

It is interesting to note one of the generalizations in  $S$  over GLM with reference to factor (5), concerning gaps. For convex programs, where  $f(x)$  is concave,  $g(x)$  is convex and  $X$  is a closed, convex set, GLM has no gaps at those  $\underline{b}$  yielding a nonempty strict interior of the feasibility region of  $P$ . This is because of the saddle-point equivalence theorem (or an alternative proof from the gap theorem in reference 3). In  $S$  we may relax this to where  $f$  is quasiconcave and upper semicontinuous along lines.<sup>[13]</sup>

When  $\lambda$  is perturbed one component at a time, GLM has a monotonicity relation<sup>[6]</sup> between  $\lambda_i$  and  $g_i(x_E^*)$ . The principal value of that result is that since we are minimizing  $H(\gamma, \lambda)$  in GLM, we know not to decrease the multiplier for the infeasible resources. A similar result holds for  $S$ .

THEOREM 6. Let  $\epsilon > 0$  be given and let

$$J = \{j : g_j[x^*(\lambda)] > b_j^0, x^*(\lambda) \in \Omega^*(\lambda)\}.$$

Then

$$F(\lambda - \epsilon e_j + \epsilon e_k) \geq F(\lambda) \quad \text{for any } j \in J,$$

and  $k \notin J$  such that  $\lambda_j - \epsilon \geq 0$  and  $\lambda_k + \epsilon \leq 1$ .

In this theorem,  $e_j$  denotes a vector whose components are all 0 except the  $j$ th one, which is 1.

*Proof.* Let  $\lambda - \epsilon e_j + \epsilon e_k \equiv \hat{\lambda}$ . Now  $\hat{\lambda} \in \Lambda$ , and (a)  $\lambda \bar{g}[x^*(\lambda)] \leq \lambda \bar{b}^0$ , (b)  $-\epsilon g_j[x^*(\lambda)] < -\epsilon b_j^0$ , and (c)  $\epsilon g_k[x^*(\lambda)] \leq \epsilon b_k^0$ . Adding (a), (b), and (c), we obtain

$$(\lambda - \epsilon e_j + \epsilon e_k) \bar{g}[x^*(\lambda)] < (\lambda - \epsilon e_j + \epsilon e_k) \bar{b}^0,$$

and we note that  $x^*(\lambda)$  is a feasible solution for  $\hat{\lambda}$ . Thus

$$F(\hat{\lambda}) = \max\{f(x) : \lambda \bar{g}(x) \leq \lambda \bar{b}^0, x \in X\} \geq F(\lambda).$$

The proof is complete.

COROLLARY 6.1. Let  $\epsilon > 0$  be given and define  $\hat{\lambda} = \lambda + \sum_{j \in J} \epsilon_j e_j - \sum_{j \in J} \epsilon_j e_j$  such that  $\hat{\lambda} \in \Lambda$ . Then  $F(\hat{\lambda}) \geq F(\lambda)$ .

Theorem 6 and Corollary 6.1 imply that a consideration which might be useful in searching for the minimizing  $\lambda^*$  would be not to decrease the elements of  $\lambda$  that correspond to infeasible inequalities in  $P$  and not to increase the elements that correspond to feasible inequalities in  $P$ .

Another helpful result in searching  $\Lambda$  is the following. Suppose for two  $\lambda \in \Lambda$ , say  $\lambda_1, \lambda_2$ , there is a common optimal solution—i.e.,  $\Omega(\lambda_1) \cap \Omega(\lambda_2) \neq \emptyset$ . Then, for all  $\lambda = \alpha \lambda_1 + (1 - \alpha) \lambda_2$ ,  $\alpha \in [0, 1]$ , we have  $\Omega(\lambda) \cap \Omega(\lambda_1) \cap \Omega(\lambda_2) \neq \emptyset$ , and hence  $F(\lambda) = F(\lambda_1) = F(\lambda_2)$ . The proof is based on the fact that  $F(\lambda) \leq F(\lambda_1)$  by quasiconvexity of  $F$ , and  $F(\lambda) \geq F(\lambda_1)$  because  $x^*(\lambda_1)$  is feasible for  $\lambda$ .

A further result in this section concerning  $F(\lambda)$  is an  $\epsilon$  feasibility theorem. This theorem parallels Everett's  $\epsilon$  theorem and yields the same type of qualitative results. That is, we can answer the question of stability of the  $S$ -problem method for solving  $P$ . It will be shown that a solution  $x^*$  that nearly solves  $S$  also nearly solves the modified  $P$  problem  $P^*$ , where  $b^0$  is replaced by  $\underline{g}(x^*)$ .

THEOREM 7. If  $f(x^*) \geq F(\lambda) - \epsilon$ , where  $\epsilon > 0$  and  $x^*$  is feasible for  $S$ , then  $x^*$  is an  $\epsilon$  solution for  $P^*$  in the sense that for all  $x : \underline{g}(x) \leq \underline{g}(x^*)$ ,  $x \in X$ , we have  $f(x^*) \geq f(x) - \epsilon$ .

*Proof.* From  $S$  we have

$$f(x^*) \geq \max\{f(x) : \lambda \bar{g}(x) \leq \lambda \bar{b}^0, x \in X\} - \epsilon,$$

and  $\lambda \bar{g}(x^*) \leq \lambda \bar{b}^0$ . Let  $\hat{x}$  solve  $S$  and  $x^{**}$  solve  $P^*$ . Then  $f(x^*) \geq f(\hat{x}) - \epsilon \geq f(x^{**}) - \epsilon$ . Furthermore  $x^*$  is feasible for  $P^*$ . Hence, the proof is complete.

We are now interested in continuity properties of  $F(\lambda)$  over  $\Lambda$ .

**THEOREM 8.** *If (1)  $f$  is upper semicontinuous, (2)  $\underline{g}$  is continuous, and (3)  $X$  is compact, then  $F$  is upper semicontinuous.*

*Proof.* Let  $\Omega(\lambda) = \{x: x \text{ solves } S \text{ for } \lambda\}$ . If we consider a sequence  $\{\lambda^\epsilon\}$ , where  $\lambda^\epsilon \rightarrow \lambda^0 \in \Lambda$ , then there exists a subsequence of  $\{x^\epsilon\}$ , call it  $\{x^i\}$ , that converges to a point  $x^*$ , where  $x^\epsilon \in \Omega(\lambda^\epsilon)$ . Since  $\underline{g}(x)$  is continuous and since  $\lambda^i \bar{g}(x^i) \leq \lambda^i \bar{b}^0$  for all  $i$ , we have  $\lambda^0 \bar{g}(x^*) \leq \lambda^0 \bar{b}^0$ . Thus,  $x^*$  is feasible in  $S$  for  $\lambda = \lambda^0$ . Then  $F(\lambda^0) \geq f(x^*)$ . Since  $f$  is upper semicontinuous,  $f(x^*) \geq \overline{\lim}_{i \rightarrow \infty} f(x^i) = \overline{\lim}_{i \rightarrow \infty} F(\lambda^i)$ . Thus,  $F(\lambda^0) \geq \overline{\lim}_{i \rightarrow \infty} F(\lambda^i)$ , as desired.

An algorithm to search over  $\Lambda$ , particularly when  $m > 2$ , should take advantage of the 'large' discrete decrease in  $F(\lambda)$  that can occur when we are near a point of discontinuity.

Luenberger<sup>[13]</sup> proves that if  $f(x)$  is quasiconcave and lower semicontinuous along lines, if  $\bar{g}(x)$  are convex constraint functions, and if the constraint set has a nonempty interior, then  $F(\lambda)$  is quasiconvex and lower semicontinuous for all  $\lambda: F(\lambda) < +\infty$ . Our Theorem 5 shows that  $F(\lambda)$  is quasiconvex under no restrictions on  $f$  and  $\bar{g}$ , and Theorem 8 shows  $F(\lambda)$  is upper semicontinuous under reasonable assumptions on  $f$ ,  $\bar{g}$ , and  $X$ . We thus state:

**COROLLARY 8.1.** *If (1)  $f$  is a continuous quasiconcave function, (2)  $X$  is convex and compact, and (3)  $\underline{g}$  is convex and there exists  $x \in X$  such that  $\underline{g}(x) < \bar{b}^0$ , then  $F$  is continuous.*

The proof is merely an application of the Luenberger lemma and Theorem 8.

### GENERALIZATIONS

THERE ARE several aspects of the preceding results that are subject to generalization. For example, we can talk about combining the  $E$  and  $S$  problems by putting some of the constraints in the objective function and leaving some of them in the surrogate constraint. This modification might be useful in resolving an  $E$  gap or an  $S$  gap. For example, we may wish to put some or all of the infeasible constraints into the surrogate in an attempt to resolve the  $E$  gap. Clearly more work of both a theoretical and empirical nature will be required before it can be said definitively whether or not this type of modification is practical in the general context of solving  $P$ .

Another generalization of interest is to define a hierarchy of  $S$  problems say  $S_0, S_1, S_2, \dots, S_k, \dots, S_m$  where  $S_0$  is the  $E$  problem,  $S_m$  is the  $P$  prob-

lem,  $S_1$  is the previously stated  $S$  problem, and  $S_k$  is a class of problems given by:

$S_k$ : max  $f(x)$  subject to

$$\begin{aligned} (1) \quad & \underline{\lambda}^1 \bar{g}^1(x) \leq \underline{\lambda}^1 \bar{b}^1, \\ & \dots \quad \dots \\ (k) \quad & \underline{\lambda}^k \bar{g}^k(x) \leq \underline{\lambda}^k \bar{b}^k, \\ (k+1) \quad & x \in X, (\underline{\lambda}^1, \dots, \underline{\lambda}^k) = \underline{\lambda} \in \Lambda, \end{aligned}$$

where  $\underline{\lambda}$  is a partitioned  $m$ -vector,  $[\bar{g}^1(x), \dots, \bar{g}^k(x)]$  is a partition of  $\bar{g}(x)$ , and  $(\bar{b}^1, \dots, \bar{b}^k)$  is the same partition of  $\bar{b}^0$ .

We will occasionally represent the inequalities (1) through (k) by the mapping  $\lambda[g(x)] \leq \lambda(b)$ , where the mapping  $\lambda$  is from  $E^m$  to  $E^k$ .

In each problem  $s_k \in S_k$  there are  $k$  surrogate constraints. For example, for  $m=4$  and  $k=2$  there are seven problems for  $S_2$ , namely (since the objective function is the same for all seven problems we only state it once; also we suppress the argument  $x$  for conciseness):

$S_2$ : max  $_{x \in X} f$

$$\begin{aligned} s_{2_1}: \quad & \lambda_1 g_1 \leq \lambda_1 b_1, \quad \lambda_2 g_2 + \lambda_3 g_3 + \lambda_4 g_4 \leq \lambda_2 b_2 + \lambda_3 b_3 + \lambda_4 b_4. \\ s_{2_2}: \quad & \lambda_2 g_2 \leq \lambda_2 b_2, \quad \lambda_1 g_1 + \lambda_3 g_3 + \lambda_4 g_4 \leq \lambda_1 b_1 + \lambda_3 b_3 + \lambda_4 b_4. \\ s_{2_3}: \quad & \lambda_3 g_3 \leq \lambda_3 b_3, \quad \lambda_1 g_1 + \lambda_2 g_2 + \lambda_4 g_4 \leq \lambda_1 b_1 + \lambda_2 b_2 + \lambda_4 b_4. \\ s_{2_4}: \quad & \lambda_4 g_4 \leq \lambda_4 b_4, \quad \lambda_1 g_1 + \lambda_2 g_2 + \lambda_3 g_3 \leq \lambda_1 b_1 + \lambda_2 b_2 + \lambda_3 b_3. \\ s_{2_5}: \quad & \lambda_1 g_1 + \lambda_2 g_2 \leq \lambda_1 b_1 + \lambda_2 b_2, \quad \lambda_3 g_3 + \lambda_4 g_4 \leq \lambda_3 b_3 + \lambda_4 b_4. \\ s_{2_6}: \quad & \lambda_1 g_1 + \lambda_3 g_3 \leq \lambda_1 b_1 + \lambda_3 b_3, \quad \lambda_2 g_2 + \lambda_4 g_4 \leq \lambda_2 b_2 + \lambda_4 b_4. \\ s_{2_7}: \quad & \lambda_1 g_1 + \lambda_4 g_4 \leq \lambda_1 b_1 + \lambda_4 b_4, \quad \lambda_2 g_2 + \lambda_3 g_3 \leq \lambda_2 b_2 + \lambda_3 b_3. \end{aligned}$$

Similarly, for  $m=4, k=3$  we can write the six  $S_3$  problems:

$S_3$ : max  $_{x \in X} f$

$$\begin{aligned} s_{3_1}: \quad & \lambda_1 g_1 + \lambda_2 g_2 \leq \lambda_1 b_1 + \lambda_2 b_2; \quad \lambda_3 g_3 \leq \lambda_3 b_3; \quad \lambda_4 g_4 \leq \lambda_4 b_4. \\ s_{3_2}: \quad & \lambda_1 g_1 + \lambda_3 g_3 \leq \lambda_1 b_1 + \lambda_3 b_3; \quad \lambda_2 g_2 \leq \lambda_2 b_2; \quad \lambda_4 g_4 \leq \lambda_4 b_4. \\ s_{3_3}: \quad & \lambda_1 g_1 + \lambda_4 g_4 \leq \lambda_1 b_1 + \lambda_4 b_4; \quad \lambda_2 g_2 \leq \lambda_2 b_2; \quad \lambda_3 g_3 \leq \lambda_3 b_3. \\ s_{3_4}: \quad & \lambda_2 g_2 + \lambda_3 g_3 \leq \lambda_2 b_2 + \lambda_3 b_3; \quad \lambda_1 g_1 \leq \lambda_1 b_1; \quad \lambda_4 g_4 \leq \lambda_4 b_4. \\ s_{3_5}: \quad & \lambda_2 g_2 + \lambda_4 g_4 \leq \lambda_2 b_2 + \lambda_4 b_4; \quad \lambda_1 g_1 \leq \lambda_1 b_1; \quad \lambda_3 g_3 \leq \lambda_3 b_3. \\ s_{3_6}: \quad & \lambda_3 g_3 + \lambda_4 g_4 \leq \lambda_3 b_3 + \lambda_4 b_4, \quad \lambda_1 g_1 \leq \lambda_1 b_1; \quad \lambda_2 g_2 \leq \lambda_2 b_2. \end{aligned}$$

It will be convenient to define ancestor sets. Form a subset of the set of  $S_k$  problems by

$$A_k(\hat{s}_{k+1}) = (S_k | \hat{s}_{k+1}) = \left\{ \begin{array}{l} \text{set of all } S_k \text{ problems that can be formed} \\ \text{from a given } \hat{s}_{k+1} \text{ problem by adding together} \\ \text{any two of the } k+1 \text{ surrogate constraints to} \\ \text{form an } S_k \text{ problem} \end{array} \right\}.$$

We call  $A_k(\hat{s}_{k+1})$  the ancestor set of  $\hat{s}_{k+1}$ .

In like manner, we form a subset of the set of  $S_{k+1}$  problems by:

$$D_{k+1}(\hat{s}_k) = (S_{k+1}|\hat{s}_k) = \left. \begin{array}{l} \text{set of all } S_{k+1} \text{ problems that can be formed} \\ \text{from a given } \hat{s}_k \text{ problem by taking any one of} \\ \text{the } k \text{ constraints (with two or more } \lambda_i \text{'s) and} \\ \text{forming two new surrogate constraints from} \\ \text{this single constraint} \end{array} \right\}.$$

We call  $D_{k+1}(\hat{s}_k)$  the descendent set of  $\hat{s}_k$ . Using the previous  $S_2$  and  $S_3$  sets we illustrate these definitions:  $A_2(s_{3_6}) = \{s_{2_1}, s_{2_2}, s_{2_5}\}$ ,  $D_3(s_{2_1}) = \{s_{3_4}, s_{3_5}, s_{3_6}\}$ .

Furthermore, every ancestor and every descendent set have at least one member for  $k=2, \dots, m$  and  $k=1, \dots, m-1$ , respectively.

Many of the lemmas and theorems of the preceding sections can be generalized to the  $S_k$  class of problems. In order to accomplish these generalizations it is convenient to define:

$$F_{s_k}(\underline{\lambda}) = \max\{f(\underline{x}) : s_k \in S_k, \underline{\lambda} \in \Lambda\}, \quad k=0, 1, \dots, m,$$

and

$$F_k(\underline{\lambda}) = \min_{s_k \in S_k} F_{s_k}(\underline{\lambda}).$$

**THEOREM 4'.** *If  $\underline{x}^0$  is optimal for  $P$ , then for all  $k=0, 1, \dots, m-1$  and  $\underline{\lambda} \in \Lambda$ , (i)  $f(\underline{x}^0) \leq F_{s_k}(\underline{\lambda})$ , and (ii)  $F_{k+1}(\underline{\lambda}) \leq F_k(\underline{\lambda}) \leq F_{s_k}(\underline{\lambda})$ .*

*Proof.* Part (i) follows immediately, since  $\underline{x}^0$  is feasible in every  $s_k$  problem for all  $k$ . Part (ii): The case  $k=0$  is proved in Theorem 4. Hence, we only consider the case  $k>0$ . Take any problem  $s_k \in S_k$  and consider any  $\hat{s}_{k+1} \in D_{k+1}(s_k)$ . Let  $\hat{\underline{x}}(\underline{\lambda})$  be any optimal solution to  $\hat{s}_{k+1}$ . Now  $s_k \in A_k(\hat{s}_{k+1})$ , so  $\hat{\underline{x}}(\underline{\lambda})$  is feasible for  $s_k$ . Therefore,  $F_{s_k}(\underline{\lambda}) \geq f[\hat{\underline{x}}(\underline{\lambda})] = F_{\hat{s}_{k+1}}(\underline{\lambda})$ . Since every  $s_k$  has a nonempty descendent set, then

$$\begin{aligned} F_{s_k}(\underline{\lambda}) &\geq \min_{s_k \in S_k} F_{s_k}(\underline{\lambda}) \geq \min_{s_{k+1} \in D_{k+1}(s_k), s_k \in S_k} F_{s_{k+1}}(\underline{\lambda}) \\ &\geq \min_{s_{k+1} \in S_{k+1}} F_{s_{k+1}}(\underline{\lambda}) = F_{k+1}(\underline{\lambda}). \end{aligned}$$

The proof is complete.

This extension of Theorem 4 is important in that each class of problems successively reduces the gap problem encountered in the preceding class. Computationally, this reduction may not be useful beyond  $S_2$ , since the sets  $S_k$  become factorially large. However, we may be able to use  $S_2$  (which for small  $m$  is not large) to resolve gaps. More will be said about this in a later paper.

In addition to generalizing Theorem 4 as shown above, it is possible to generalize many of the lemmas and theorems in the preceding section.

**LEMMA 1'.** *Let  $\underline{x}_k^*$  solve  $s_k$  for  $\underline{\lambda}^*$ . If  $\underline{x}_k^*$  solves  $P$ , then  $\underline{x}_k^*$  solves  $F_{k+j}(\underline{\lambda}^*)$  for all  $j \geq 0$ .*

*Proof.* By Theorem 4',  $f(\underline{x}_k^*) \leq F_{k+1}(\underline{\lambda}^*) \leq F_{s_k}(\underline{\lambda}^*) = f(\underline{x}_k^*)$ .

LEMMA 2'. If  $x_k^*$  solves  $s_k$  for  $\underline{\lambda}^*$ , then  $x_k^*$  solves  $P$  for any  $\underline{b}^* : \underline{b}^* \geq \underline{g}(x_k^*)$  and the  $k$ -vector  $\lambda^*(\underline{b}^*) \leq \lambda^*(\underline{b}^0)$ .

Proof.  $x_k^*$  solves  $s_k$  for any  $\underline{b}^*$  such that  $\lambda^*(\underline{b}^*) \leq \lambda^*(\underline{b}^0)$ , since  $\lambda^*[g(x_k^*)] \leq \lambda^*(\underline{b}^*)$ . Furthermore,  $x_k^*$  is feasible in  $P$  for  $\underline{b}^*$ ; hence,  $x_k^*$  is optimal in  $P$  for  $\underline{b}^*$ .

LEMMA 3'. If  $P$  is Lagrange regular and if  $\underline{\lambda}^*$  exists such that  $s_k$  solves  $P$ , then  $\gamma \cdot \underline{\lambda}^*$  is a Lagrange multiplier for  $P$  for some  $\gamma \geq 0$ .

Proof. The proof is the same as for Lemma 3.

THEOREM 1'. If some problem  $s_k \in S_k$  solves  $P$  for  $\underline{\lambda}^*$  and any  $k = 1, \dots, m-1$ , then  $\underline{\lambda}^*$  minimizes  $F_{s_k}(\underline{\lambda})$  for all  $\underline{\lambda} \in \Lambda$ .

The proof of Theorem 1' is essentially the same as that of Theorem 1, and is omitted.

The final generalization to be considered in this section is the extension of the  $S$  problem to a nonlinear surrogate-constraint function  $\lambda$ . Gould<sup>[10]</sup> introduced the notion that  $\lambda \cdot \bar{g}(x)$  is a special case of considering the functional  $\lambda[g(x)]$ , where  $\lambda$  is selected from the class of separable monotone nondecreasing functions mapping  $E^m$  into the extended nonnegative real numbers with  $\lambda(\underline{b}^0) < +\infty$ . The modified Lagrangian is thus

$$L(x, \lambda) = f(x) - \sum_{j=1}^{j=m} \lambda_j [g_j(x)] + \sum_{j=1}^{j=m} \lambda_j (b_j^0),$$

where  $\lambda_j : E^1 \rightarrow E^1 \cup \{+\infty\}$ ,  $\lambda_j$  is monotone nondecreasing, and  $\lambda_j(b_j^0) < +\infty$ . Gould proved that when the linear form yields gaps, a more general functional still exists to resolve it. From a theoretical view, there are no gaps. Further, Gould showed that for the functional  $L$  that solves  $P$  there is a saddle-point corresponding to the solution in that  $L(x, \lambda^*) \leq L(x_P^*, \lambda^*) \leq L(x_P^*, \lambda)$  for all  $x \in X$  and  $\lambda$  in the above mentioned class.

From a computational view, nonlinear Lagrangians cause difficulty, and the unconstrained maximization in GLM becomes more complicated.

A similar extension of Gould's results holds true for the surrogate program. Consider the problem given by

$$S^1 : \max f(x) : \sum_{i=1}^{i=m} \lambda_i [g_i(x)] \leq \sum_{i=1}^{i=m} \lambda_i (b_i^0), \quad x \in X.$$

When  $\lambda$  is a monotone increasing function, then Theorem 1 applies. Moreover, Gould's theorem and Lemma 1 (extended slightly) show that there always exists a surrogate program to solve  $P$  so that no gaps arise. However, the same computational disadvantages occur when  $\lambda[g(x)]$  is a nonlinear functional form.

### SUMMARY AND CONCLUSIONS

IN THIS PAPER, we have organized a unified theory of surrogate mathematical programming. Theorems pertaining to the properties of  $\underline{\lambda}^*$  and  $F(\underline{\lambda})$  were presented, and  $S$  was related to  $P$  and GLM. For the case of

one multiplier, bisection is an efficient search scheme, and when  $P$  is separable, dynamic programming may be used to solve  $S$ .

It is our belief that useful algorithms may evolve from this study, and we are currently studying this aspect. One of the goals is to develop an algorithm that minimizes a quasiconvex function over the well structured set  $A$ , even when the function has discontinuities. One obvious use of surrogate programs is for gap reduction in GLM.

#### ACKNOWLEDGMENT

WE WISH to thank F. J. GOULD for many valuable comments and criticisms.

#### REFERENCES

1. K. J. ARROW AND L. HURWICZ, *Studies in Linear and Nonlinear Programming*, Stanford University Press, Stanford, California, 1958.
2. E. BALAS, "Discrete Programming by the Filter Method with Extension to Mixed-Integer Programming and Application to Machine Sequencing," ICC Report N66/10, International Computation Center, Centre International de Calcul (Nov., 1966).
3. M. BELLMORE, H. J. GREENBERG, AND J. J. JARVIS, "Generalized Penalty Function Concepts in Mathematical Optimization," *Opns. Res.* **18**, 229-252 (1970).
4. C. BERGE, *Topological Spaces*, Macmillan, New York, 1963.
5. R. BROOKS AND A. GEOFFRION, "Finding Everett's Lagrange Multipliers by Linear Programming," *Opns. Res.* **14**, 1149-1153 (1967).
6. H. EVERETT, III, "Generalized Lagrange Multiplier Method for Solving Problems of Optimum Allocation of Resources," *Opns. Res.* **11**, 399-471 (1963).
7. A. GEOFFRION, "Implicit Enumeration Using an Imbedded Linear Program," No. 120, Western Mgt. Sci. Inst. (May, 1967).
8. F. GLOVER, "A Multiphase-Dual Algorithm for the Zero-One Integer Programming Problem," *Opns. Res.* **13**, 879-919 (1965).
9. ———, "Surrogate Constraints," *Opns. Res.* **16**, 741-749 (1968).
10. F. J. GOULD, "Extensions of Lagrange Multipliers in Nonlinear Programming," *SIAM J. on Appl. Math.* **17**, 1280-1297 (1969).
11. H. J. GREENBERG, "On the Relation of Everett's Theorem and Conjugate Functions," Tech. Memo. CP-69002, SMU, Dallas, Texas (Jan., 1969).
12. H. W. KUHN AND A. W. TUCKER, "Nonlinear Programming," in J. NEYMAN (ed.), *Proceedings of the Second Berkeley Symposium in Mathematical Statistics and Probability*, 481-492, University of California Press, Berkeley, California, 1951.
13. D. G. LUENBERGER, "Quasi-Convex Programming," *SIAM J.* **16**, 1090-1095 (1968).
14. B. MARTOS, "Quasi-Convexity and Quasi-Monotonicity in Nonlinear Programming," *Studia Scientiarum Mathematicarum Hungarica* **2**, 265-273 (1967).