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# Stability Theorems for Infinitely Constrained Mathematical Programs

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**Abstract.** The primary concern of this paper is to investigate stability conditions for the mathematical program: find $x \in E^n$ that maximizes $f(x)$: $g^j(x) \leqslant 0$ for some $j \in J$, where $f$ is a real scalar-valued function and each $g$ is a real vector-valued function of possibly infinite dimension. It should be noted that we allow, possibly infinitely many, disjunctive forms. In an earlier work, Evans and Gould established stability theorems when $g$ is a continuous finite-dimensional real-vector function and $J = \{1\}$. It is pointed out that the results of this paper reduce to the Evans–Gould results under their assumptions. Furthermore, since we use a slightly more general definition of lower and upper semicontinuous point-to-set mappings, we can dispense with the continuity of $g$ (except in a few instances where it is implied by convexity assumptions).

**Key Words.** Stability of infinite programs, continuity of mathematical programs, nonlinear programming, infinitely constrained problems, stability analysis.

## 1. Introduction

As is well-known, mathematical programs usually represent an abstraction of some economic or physical phenomena. Hence, most of the literature in mathematical programming is devoted to the formulation and solution of these problems. An area which has received less emphasis,

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but which is quite important for the application and implementation o the solutions of mathematical programs, is the investigation of the continuity properties of the optimal solution set and the optimal response function. If a mathematical program lacks continuity, then small changes in parameters or functions (often due to inexact estimates of the parameters or functions) may result in large changes in the optima solutions or in the optimal objective function values or both. Such behavior could be very disconcerting to the user or the user's superiors Another possibly even more important need for continuity in mathematical programs is the fact that digital computers operate with finite arithmetic and often produce significant roundoff error over time Continuity of the mathematical program being solved gives credence to the belief that the algorithmic process being used may lead to an optima or *near-optimal* solution of the problem. Lack of continuity, on the other hand, could mean that the algorithm is yielding something far from optimal.

This paper is concerned with the investigation of continuity or *stability* properties of mathematical programs represented by

P: find $x \in S$ that maximizes $f(x) : \bar{g}^j(x) \leqslant 0$ for some $j \in J$,

where $S$ is a subset of Euclidean $n$-space, $f$ is a real scalar-valued function on $S$, and each $\bar{g}^j$ is a real vector-valued function on $S$ of possibly infinite dimension (we use the bar over $g$, i.e., $\bar{g}^j$, to denote a *specific* function). The set $J$ is an index set which allows for disjunctive forms i.e., constraints of the either-or variety. For example, let $J = \{1, 2\}$ Then, a disjunctive mathematical program would be $\max f(x)$, subject to either $\bar{g}^1(x) \leqslant 0$ or $\bar{g}^2(x) \leqslant 0$, where $\bar{g}^1(\cdot)$ and $\bar{g}^2(\cdot)$ are possibly infinite dimensional. Such constraints appear in many areas, such as multiple-choice programs, the theory of nonlinear programming, mathematica economics, and some problems in game theory. An example in the las area is given in Rosenthal (Ref. 1), where one wishes to solve the problem of selection of a social state or set of social states by society through voting. This problem reduces to finding $x \in E^n$ to

$$\min_{B \in A} \min_{\substack{\text{s.t.} Bx \geqslant e \\ x \geqslant 0}} (e, x),$$

that is, one wishes to find the minimum of the inner product $(e, x)$ over the disjunctive sets

$$\begin{matrix} B_1 x \geqslant e, & B_2 x \geqslant e, \ldots . \\ x \geqslant 0 & x \geqslant 0 \end{matrix}$$

The use of disjunctive constraints has not been widespread, due to the difficulty in devising efficient algorithms for such problems; however, many real problems have constraints which say either *this and this* or *that and that* must be satisfied, but not necessarily both sets of constraints.

Another widely applied class of mathematical programs which falls into the above formulation is the *semi-infinite* programs, which have a finite solution vector and a (countably) infinite number of constraints. Much work has gone into understanding and solving semi-infinite mathematical programs (see Refs. 2–5 and many other works), since these problems have wide application in such areas as numerical methods (Ref. 6), statistics (Refs. 7–9), and air pollution and meteorology (Refs. 10–13), to mention a few. Thus, a focus on the continuity and stability of the general mathematical programs presented in this paper will prove useful to investigators in many diverse application areas.

The analysis of the continuity and stability of the optimal solution set and optimal response function of mathematical programs focuses on the continuity and stability of the constraint region itself. If the constraint region behaves badly under small perturbations, it is not surprising that the optimal solution set, and quite possibly the optimal response function, will most often also behave badly. Furthermore, when the constraint region is stable, then under mild assumptions the optimal response function is continuous and often (under more restrictive assumptions) the optimal solution set is stable. This relationship of the stability of the constraint region to the stability of the optimal solution set and the optimal response function has been recognized by many authors (see Refs. 14–20). Indeed, in the study of stability, the concepts of closed and of upper and lower semicontinuous point-to-set mappings applied to the constraint region are fundamental to understanding the stability of the mathematical program. It should be noted that the various authors differ slightly as to their definitions of upper and lower semicontinuous (abbreviated to usc and lsc) point-to-set mappings; hence, their results are also slightly different. In this paper, we use the definitions given by Berge (Ref. 15). These definitions are stated explicitly in the following section. Furthermore, it should be noted that the concept of a closed map is closely related to that of an upper semicontinuous map; i.e., if a map is upper semicontinuous and if its image sets are closed, then the map is closed; conversely, if a map is closed and its range on a neighborhood is bounded, then it is upper semicontinuous. Meyer (Ref. 21) and Zangwill (Ref. 22) have used the concepts of closed and usc mappings in relation to the convergence of algorithms, thus demonstrating their importance in this companion area of stability analysis.

## 2. Background and Notation

Let $E^n$ denote Euclidean $n$-space with the usual Euclidean norm denoted by $\| x \|$.

Let $(F, \rho)$ be the metric space of all real-valued functions defined on a subset $S \subset E^n$ with metric

$$\rho(h_1, h_2) = \sup_{x \in S} | h_1(x) - h_2(x)|$$

for $h_1, h_2 \in F$.

In analyzing the continuity and stability of the feasibility region we are interested in families of real-valued functions which are *close* to our given functions and which have nonempty feasibility regions also. That is, we wish to perturb our original functions in a uniform manner and see what happens to the resulting feasibility regions. Hence, we define, for any $\epsilon > 0$ and $\bar{h} \in F$, the set $\theta(\bar{h}) \subset F$ given by

$$\theta(\bar{h}) = \{h \in F \mid \{x \mid h(x) \leqslant 0\} \neq \phi \text{ and } \sup_{x \in S} | h(x) - \bar{h}(x)| < \epsilon\}.$$

The set $\theta(\bar{h})$ is the set of all real-valued functions which differ from $\bar{h}$ at any $x \in S$ by at most $\epsilon$ and which have a nonempty zero level set.

Throughout this paper, we use extensively the level set mapping $\sigma$: $\theta(\bar{h}) \to 2^S$ (the power set of $S$) given by

$$\sigma(h) = \{x \in S \mid h(x) \leqslant 0\}$$

and the strict interior of the level set

$$\sigma^0(h) = \{x \in S \mid h(x) < 0\}.$$

It should be noted that, when $h$ is an upper semicontinuous, real-valued function, the strict interior $\sigma^0(h)$ is contained in, but is not necessarily equal to, the interior of $\sigma(h)$. If, for example, for $x \in E^1$,

$$h(x) = \begin{cases} x & \text{for } x \leqslant 0, \\ 0 & \text{for } 0 \leqslant x \leqslant 1, \\ x - 1 & \text{for } x \geqslant 1, \end{cases}$$

then $\sigma^0(h) = (-\infty, 0)$, but interior of $\sigma(h) = (-\infty, 1)$.

As mentioned previously, we will be concerned with certain continuity properties associated with $\sigma$ and other mappings closely related to $\sigma$, since the union $\bigcup_{j \in J}\{\sigma(g^j)\}$ defines the feasibility region.

Now, let us define the neighborhoods

$$N_\epsilon(x) = \{y \mid \| x - y \| < \epsilon\}, \qquad \text{where} \quad x \in S,$$

$$\eta_\epsilon(S) = \bigcup_{x \in S} N_\epsilon(x), \qquad \text{where} \quad S \subset E^n.$$

Observe that $\eta_\epsilon(S) = \eta_\epsilon(\text{cl } S)$, where cl $S$ denotes the closure of $S$.

Given any point-to-set map, $\sigma$, with domain $\theta(\bar{h})$, we say that $\sigma$ is usc [lsc] at $\bar{h}$ if, for any $\epsilon > 0$ and any sequence $\{h^k\} \subset \theta(\bar{h})$, where $\rho(h^k, \bar{h}) \to 0$, there exists a $k^*$ such that $k \geqslant k^*$ implies

$$\eta_\epsilon(\sigma(\bar{h})) \supset \sigma(h^k) \qquad [\eta_\epsilon(\sigma(h^k)) \supset \sigma(\bar{h})].$$

It should be noted that convergence in the sup metric means uniform convergence. If $\sigma$ is both lsc and usc at $\bar{h}$, we shall say that $\sigma$ is continuous at $\bar{h}$.

Now, let $\Gamma$ be an index set ($\Gamma$ may be finite or infinite). Let $\bar{g}^j(\cdot)$ for $j \in J$ be given by the set $\bar{g}^j(\cdot) = \{\bar{g}_\gamma{}^j(\cdot) \mid \gamma \in \Gamma\}$, where $\bar{g}_\gamma{}^j\colon S \to E^1$. Thus, if $\Gamma$ is finite,

$$\bar{g}^j(\cdot) = (\bar{g}_1{}^j(\cdot), \ldots, \bar{g}_m{}^j(\cdot)) \subset E^m.$$

For the following discussion, let $J = \{1\}$ and define $\bar{g}^1(\cdot) \equiv \bar{g}(\cdot) = \{\bar{g}_\gamma(\cdot) \mid \gamma \in \Gamma\}$.

There are several point-to-set maps associated with this collection which we shall consider. The two fundamental ones are: (a) the intersection map and (b) the union map.

We first define the *intersection map* $\sigma_i$ (where the subscript $i$ denotes *intersection*), given by the rule

$$\sigma_i(g) = \{x \mid g_\gamma(x) \leqslant 0, \gamma \in \Gamma\} = \bigcap_{\gamma \in \Gamma} \sigma(g_\gamma)$$

for all $g$ such that

$$g \in \theta_i(\bar{g}) = \left\{g \,\middle|\, \bigcap_{\gamma \in \Gamma} \{x \mid g_\gamma(x) \leqslant 0\} \neq \phi, g_\gamma \in F, \text{ and } \sup_{\gamma \in \Gamma} \rho(g_\gamma, \bar{g}_\gamma) < \epsilon\right\}.$$

Thus, $\sigma_i(g)$ is the feasibility region of $g$ given by the intersection of the nonempty feasibility regions of the collection of real-valued functions $g_\gamma$ which are *close* to $\bar{g}_\gamma$, $\gamma \in \Gamma$.

The *union map* $\sigma_u$ (where the subscript $u$ denotes *union*) is given by the rule

$$\sigma_u(g) = \{x \mid g_\gamma(x) \leqslant 0 \text{ for some } \gamma \in \Gamma\} = \bigcup_{\gamma \in \Gamma} \sigma(g_\gamma)$$

for all $g$ such that

$$g \in \theta_u(\bar{g}) = \Big\{ g \Bigm| \bigcup_{\gamma \in \Gamma} \{x \mid g_\gamma(x) \leqslant 0\} \neq \phi, g_\gamma \in F, \text{ and } \sup_{\gamma \in \Gamma} \rho(g_\gamma, \bar{g}_\gamma) < \epsilon \Big\}.$$

Thus, $\sigma_u(g)$ is the feasibility region of $g$ given by the union of the nonempty feasibility regions of the collection of real-valued functions $g_\gamma$ which are *close* to $\bar{g}_\gamma$, $\gamma \in \Gamma$.

The intersection map $\sigma_i(\bar{g})$ defines the most common form of feasibility region for mathematical programming problems found in the literature, especially when $\Gamma$ is finite. The union map $\sigma_u(\bar{g})$ is the simplest disjunctive form, i.e., where every constraint is of the *either-or* variety. Later, we will consider a family of point-to-set maps associated with the collection $\{\bar{g}_\gamma \mid \gamma \in \Gamma\}$ defined on $\theta(\bar{g})$ in terms of intersections and unions of subsets of the collection $\{\sigma(\bar{g}_\gamma) \mid \gamma \in \Gamma\}$.

We now define the strict interiors of $\sigma_i(\bar{g})$ and $\sigma_u(\bar{g})$ by

$$\sigma_i^0 = \{x \mid \bar{g}_\gamma(x) < 0, \gamma \in \Gamma\} = \bigcap_{\gamma \in \Gamma} \sigma^0(\bar{g}_\gamma)$$

and

$$\sigma_u^0 = \{x \mid \bar{g}_\gamma(x) < 0 \text{ for some } \gamma \in \Gamma\} = \bigcup_{\gamma \in \Gamma} \sigma^0(\bar{g}_\gamma),$$

respectively. Note that we have suppressed the dependence of $\sigma_i^0$ and $\sigma_u^0$ on $\bar{g}$, since we *always* use the strict interiors with respect to the specified function $\bar{g}$, whereas the definitions of the $\sigma_i$ and $\sigma_u$ mappings are in regard to $g \in \theta_i(\bar{g})$ and $g \in \theta_u(\bar{g})$, respectively.

Although the optimal response function $F_{\text{sup}}$ is not directly used in the results of this paper, it is defined here in order to point out the usefulness of our results with regard to this function and to the optimal solution set $\Omega(g)$.

The *optimal response function* $F_{\text{sup}}$: $\theta(\bar{g}) \to E^1 \cup \{+\infty\}$ is given by

$$F_{\text{sup}}(g) = \sup\{f(x) \mid x \in \sigma(g)\},$$

and the *optimal solution set* $\Omega(g) \subset E^n$ is given by

$$\Omega(g) = \{x \in \sigma(g) \mid f(x) = F_{\text{sup}}(g)\}.$$

In an earlier paper (Ref. 18), Evans and Gould derived stability theorems for the mathematical program P with the feasibility region $\sigma_i(\bar{g})$ and $\Gamma$ finite. That is, they considered the mathematical program

$$\text{maximize } f(x) : \bar{g}(x) - \bar{b} \leqslant 0, \qquad x \in E^n,$$

where $f(\cdot)$ is a continuous real-valued function, $\bar{g}$ is a continuous, real-valued, vector function $\bar{g}: E^n \to E^m$, and $\bar{b}$ is a given vector in $E^m$, and it is assumed that $P$ has a solution. They considered the usc and lsc of $\sigma_i(\bar{g})$ in regard to perturbations of the vector $\bar{b}$. They presented the following two theorems, under the assumption that $\sigma_i(\bar{g})$ is a compact set:

(1) $\sigma_i$ is usc at $\bar{b}$ iff there exists a $\tilde{b} > \bar{b}$ such that $\{x \mid \bar{g}(x) \leqslant \tilde{b}\}$ is compact.

(2) If $\sigma_i^0 \neq \varnothing$, then $\sigma_i$ is lsc at $\bar{b}$ iff closure $\sigma_i^0 = \sigma_i(\bar{g})$.

Thus, we say that $\sigma$ is *stable* at $\bar{b}$ if (a) $\sigma_i^0 \neq \varnothing$, (b) $\{x \mid \bar{g}(x) \leqslant \tilde{b}\}$ is compact for some $\tilde{b} > \bar{b}$, and (c) closure $\sigma_i^0 = \sigma_i(\bar{g})$.

Using these results (see also Berge, Ref. 15), Evans and Gould then established the following:

(3) If $f$ is an upper semicontinuous function and $\sigma_i(\bar{g})$ is an usc map at $\bar{b}$, then $F_{\text{sup}}(\cdot)$ is an upper semicontinuous function at $\bar{b}$.

(4) If $\sigma_i^0 \neq \varnothing$, $f$ is a lower semicontinuous function, and $\sigma_i(\bar{g})$ is a lsc map, then $F_{\text{sup}}$ is a lower semicontinuous function at $\bar{b}$.

Thus, the usc and lsc of the feasibility region play a very important role in the continuity of the optimal response function.

Later, Greenberg and Pierskalla (Ref. 19) extended the Evans–Gould results (1) and (2) to allow uniformly convergent perturbations of the vector function $\bar{g}$, namely:

(5) $\sigma_i$ is usc at $\bar{g}$ iff $\sigma_i$ is usc at $\bar{b}$.

(6) If $\sigma_i^0 \neq \varnothing$, $\sigma_i$ is lsc at $\bar{g}$ iff $\sigma_i$ is lsc at $\bar{b}$.

This means the Evans–Gould theorems are sufficiently general to allow uniform perturbations of the constraint functions. In the same paper (Ref. 19), Greenberg–Pierskalla proved the following:

(7) If $\sigma_i$ is usc at $\bar{b}$ and $F_{\text{sup}}$ is continuous at $\bar{b}$, then $\Omega$ is usc at $\bar{b}$.

(8) If $\Omega$ is usc at $\bar{b}$ and if there exists a $\delta > 0$ such that, for all $b \in N_\delta(\bar{b})$, $\Omega(g)$ consists of a single point where $g(x) = \bar{g}(x) - b$, then $\Omega$ is lsc at $\bar{b}$. (See also Berge, Ref. 15.)

Although all of these results were demonstrated for $\Gamma$ finite and the intersection map, results (3), (4), (7), and (8) will hold if $\Gamma$ is infinite and the feasibility region is given by other types of maps (such as the union map). Thus, to establish the continuity of $F_{\text{sup}}$ and $\Omega$, we must first examine the usc and lsc of $\sigma_i$ and $\sigma_u$ in general.

The next section presents usc and lsc of $\sigma_i$. The following section does the same for $\sigma_u$. The final section considers families of intersections and unions.

Before proceeding, it should be noted that all of the results of this paper are for general functional perturbations of $\bar{g}$ and include as a special case right-hand-side perturbations of $\bar{b}$.

## 3. Intersection Map

In this section, we are concerned with the intersection map $\sigma_i$. The extended Evans and Gould theorems do not apply when $\Gamma$ is infinite. For one thing, the set given by

$$\hat{S} = \{x \in \sigma_i(\bar{g}) \mid \bar{g}_\gamma(x) = 0 \text{ for some } \gamma \in \Gamma\}$$

need not be closed even if each $\bar{g}_\gamma$ is continuous and $S = E^n$. This property was used by Evans and Gould in developing and applying their *shrinkage lemma.*

To develop stability theorems for continuity properties in the infinite case, we shall define the function $\bar{g}_{\text{sup}}$: $S \to E^1 \cup \{+\infty\}$ as

$$\bar{g}_{\text{sup}}(x) = \sup\{\bar{g}_\gamma(x) \mid \gamma \in \Gamma\}, \qquad x \in S.$$

Let the strict interior of $\sigma(\bar{g}_{\text{sup}})$ be denoted by

$$\sigma^0_{\text{sup}} = \{x \in S \mid \bar{g}_{\text{sup}}(x) < 0\}.$$

We point out that, even if each $\bar{g}_\gamma$ is continuous on cl $S$, $\bar{g}_{\text{sup}}$ need not be.

We shall say that $\bar{g}_{\text{sup}}$ is *regular* if there exists $\tau$: $S \to \Gamma$, whereby

$$\bar{g}_{\text{sup}}(x) = \bar{g}_{\tau(x)}(x) \qquad \text{for each} \quad x \in S.$$

Observe that $\bar{g}_{\text{sup}}$ is regular if $\Gamma$ is finite.

**Lemma 3.1.** (i) $\sigma(\bar{g}_{\text{sup}}) = \sigma_i(\bar{g})$ and $\sigma^0_{\text{sup}} \subset \sigma_i{}^0$; and (ii) if $\bar{g}_{\text{sup}}$ is regular, then $\sigma^0_{\text{sup}} = \sigma_i{}^0$.

Now, we shall reduce our possibly infinite system to a single constraint function with the following theorem.

**Theorem 3.1.** If $\sigma^0_{\text{sup}} \neq \varnothing$, then $\sigma_i$ is lsc at $\bar{g}$ iff $\sigma$ is lsc at $\bar{g}_{\text{sup}}$.

The main consequence of Theorem 3.1 is that we can now deal with the case of a single level set $\sigma(\bar{g}_{\sup})$ whenever $\sigma^0_{\sup} \neq \varnothing$ (which we shall generally assume holds). Furthermore, in considering Lemma 3.1, if $\bar{g}_{\sup}$ is regular, then $\sigma_i{}^0 \neq \varnothing$ is equivalent to $\sigma^0_{\sup} \neq \varnothing$. This case could be useful if it is easier to establish the result $\sigma_i{}^0 \neq \varnothing$ than the result $\sigma^0_{\sup} \neq \varnothing$. Another consequence of the regularity of $\bar{g}_{\sup}$ was given by Greenberg and Pierskalla (Ref. 23). If $\bar{g}_{\sup}$ is regular and if each $\bar{g}_\gamma$ $(\gamma \in \Gamma)$ is explicitly quasiconvex, then $\bar{g}_{\sup}$ is explicitly quasiconvex. The importance of this rests with the fact established by Evans and Gould that a continuous, explicitly quasiconvex function yields a continuous point-to-set mapping (as defined by its level set) at 0 provided that $\sigma(\bar{g}_{\sup})$ is compact.

We cannot, however, directly apply the extended Evans–Gould results because $\bar{g}_{\sup}$ need not be continuous. However, using a different mode of proof, we shall develop a theorem similar to theirs involving a key *closure property* [*viz.*, $\operatorname{cl} \sigma^0_{\sup} = \operatorname{cl} \sigma(\bar{g}_{\sup})$]. Our results reduce to theirs when $\Gamma$ is finite and each $g_\gamma$ is continuous on $S = E^n$.

**Lemma 3.2.** If $\sigma_i{}^0 \neq \varnothing$, then $\operatorname{cl} \sigma_i{}^0 = \operatorname{cl} \sigma_i(\bar{g})$ iff, for all $x \in \operatorname{cl} \sigma_i(\bar{g})$ and any $\epsilon > 0$, there exists a $y \in \sigma_i{}^0$ such that

$$\| x - y \| < \epsilon, \qquad \text{i.e.,} \quad N_\epsilon(x) \cap \sigma_i{}^0 \neq \varnothing.$$

Note that this lemma also holds if $\sigma_i{}^0$ is replaced by $\sigma^0_{\sup}$ and $\sigma_i(\bar{g})$ by $\sigma(\bar{g}_{\sup})$, everywhere.

The lemma states a reasonably obvious fact that the closure of the nonempty strict interior of the feasibility region $\sigma_i{}^0$ is equal to the closure of the feasibility region $\sigma_i(\bar{g})$ iff there are points in $\sigma_i{}^0$ arbitrarily close to any points in $\operatorname{cl} \sigma_i(\bar{g})$. The lemma is used to establish the following theorem.

**Theorem 3.2.** Suppose that $\sigma^0_{\sup} \neq \varnothing$ and $\sigma(\bar{g}_{\sup})$ is bounded. Then, $\sigma$ is lsc at $\bar{g}_{\sup}$ iff $\operatorname{cl} \sigma^0_{\sup} = \operatorname{cl} \sigma(\bar{g}_{\sup})$.

It should be noted that the boundedness of $\sigma(\bar{g}_{\sup})$ is essential for the sufficiency of Theorem 3.2; it is not needed in the necessity. A simple example illustrating its need in the sufficiency is the following. Let $S = E^1$ and $\bar{g}_{\sup} = -\exp(x)$, so that $\sigma^0_{\sup} = E^1 = \operatorname{cl} \sigma(\bar{g}_{\sup})$. However, $\sigma$ is not lsc at $\bar{g}_{\sup}$, for let $h^k = \bar{g}_{\sup} + (1/k)$, so that $\sigma(h^k) = [\log(1/k), +\infty)$.

Now, let us identify situations where the closure property holds, assuming that $\sigma^0_{\sup} \neq \varnothing$.

**Theorem 3.3.** If $S$ is convex, $\sigma^0_{\sup} \neq \varnothing$, and if each $\bar{g}_\gamma$ is explicitly quasiconvex and $\bar{g}_{\text{sup}}$ is regular and continuous, then cl $\sigma^0_{\text{sup}} =$ cl $\sigma(\bar{g}_{\text{sup}})$.

A similar result holds for strictly isotonic (i.e., order-preserving) functions.

**Theorem 3.4.** If $\sigma^0_{\text{sup}} \neq \varnothing$, $S$ is convex, bounded, and a meet semilattice, and if each $g_\gamma$ is strictly isotonic on $S$, then cl $\sigma^0_{\text{sup}} =$ cl $\sigma(\bar{g}_{\text{sup}})$.

For these two classes of functions, explicitly quasiconvex and isotonic, with reasonable restrictions on the feasibility region, we see that $\sigma_i$ is a lsc map at $\bar{g}$.

Now, let us consider the usc of $\sigma_i$ at $\bar{g}$. It should be noted (Refs. 21–22) that successful convergence of primal algorithms depends upon upper semicontinuity of $\sigma_i$ rather than the lsc of $\sigma_i$.

**Theorem 3.5.** $\sigma_i$ is usc at $\bar{g}$ iff $\sigma$ is usc at $\bar{g}_{\text{sup}}$.

In seeking sufficient conditions for the usc of $\sigma$ at $\bar{g}_{\text{sup}}$, it is important to remember that $\bar{g}_{\text{sup}}$ need not be continuous, so the Evans–Gould theorems cannot be directly applied. Indeed, it is easy to see that $\bar{g}_{\text{sup}}$ must be well behaved (in some sense) in order for $\sigma$ to be usc at $\bar{g}_{\text{sup}}$. A simple example illustrating what can happen when $\bar{g}_{\text{sup}}$ is not well behaved is given by

$$\bar{g}_{\text{sup}}(x) = \begin{cases} (x-1)^2 - 1 & \text{for } x \leqslant 2, \\ 1 & \text{for } 2 < x \leqslant 3, \\ x - 3 & \text{for } x > 3. \end{cases}$$

Let $\bar{x} = 3$, so $\bar{g}_{\text{sup}}(\bar{x}) = 1$. However, if we consider the sequence $\{g^k\}$, where

$$g^k(x) = \bar{g}_{\text{sup}}(x) - 1/k,$$

then the points

$$x^k = \bar{x} + 1/k = 3 + 1/k$$

satisfy $x^k \in \sigma(g^k)$; but $x^k \notin \eta_\epsilon(\sigma(\bar{g}_{\text{sup}}))$ for all $k$, since $\bar{x} \notin \sigma(\bar{g}_{\text{sup}}) = (-\infty, 2]$. However, when each $\bar{g}_\gamma$ is lower semicontinuous on $S$, then $\bar{g}_{\text{sup}}$ is lower semicontinuous, and the Evans–Gould theorem establishing necessary and sufficient conditions for usc can then be extended to the following theorem.

**Theorem 3.6.** Suppose that $\bar{g}_\gamma$ is lower semicontinuous on $S$ for for each $\gamma \in \Gamma$ and that $\sigma(\bar{g}_{\text{sup}})$ is compact. Then, $\sigma$ is usc at $\bar{g}_{\text{sup}}$ iff there exists $\delta > 0$ such that $\sigma(\bar{g} - \delta)$ is compact.

## 4. Union Map

In this section, we are concerned with the union map $\sigma_u$: $\theta_u(\bar{g}) \to 2^S$, where

$$\sigma_u(g) = \bigcup_{\gamma \in \Gamma} \sigma(g_\gamma) = \{x \in S \mid g_\gamma(x) \leqslant 0 \text{ for some } \gamma \in \Gamma\}.$$

Recall that $\sigma_u{}^0 = \bigcup_{\gamma \in \Gamma} \sigma^0(\bar{g}_\gamma)$ and define

$$\bar{g}_{\text{inf}}(x) = \inf\{\bar{g}_\gamma(x) \mid \gamma \in \Gamma\}, \qquad x \in S.$$

We shall say that $\bar{g}_{\text{inf}}$ is *regular* if there exists $\tau$: $S \to \Gamma$, such that

$$\bar{g}_{\text{inf}}(x) = \bar{g}_{\tau(x)}(x), \qquad x \in S.$$

Let $\sigma_{\text{inf}}^0 = \sigma^0(\bar{g}_{\text{inf}})$. The following lemma is easily verified.

**Lemma 4.1.** (i) $\sigma(\bar{g}_{\text{inf}}) \supset \sigma_u(\bar{g})$ and $\sigma_{\text{inf}}^0 = \sigma_u{}^0$; and (ii) if $\bar{g}_{\text{inf}}$ is regular, then $\sigma(\bar{g}_{\text{inf}}) = \sigma_u(\bar{g})$.

**Theorem 4.1.** Suppose that $\sigma_u{}^0(\bar{g}) \neq \varnothing$ and $\sigma(\bar{g}_{\text{inf}}) = \sigma_u(\bar{g})$. Then, $\sigma_u$ is lsc at $\bar{g}$ iff $\sigma$ is lsc at $\bar{g}_{\text{inf}}$.

Again, we have only to deal with a single constraint. While convexity properties are not maintained by the infimum, monotonicity properties are; namely, if $S$ is a convex, meet semilattice, and if each $\bar{g}_\gamma$ is strictly isotonic on $S$, then $\bar{g}_{\text{inf}}$ is strictly isotonic on $S$ if it is regular.

Further, the infimum does preserve concavity. We therefore obtain the following lemma.

**Lemma 4.2.** Let $h$ be a scalar, strictly concave function on $S \equiv E^n$ such that $\sigma^0(h) \neq \varnothing$. Then, cl $\sigma^0(h) = \sigma(h)$.

It should be noted that strict concavity is needed. It can also be shown that the conclusion may fail if $h$ were to be a vector. Moreover, it is not sufficient for $h$ to be defined on a proper subset of $E^n$, say $S$. For example, let $h(x) = 1 - x^2$ and $S = [-10, 1]$. Observe that $\sigma^0(h) = [-10, -1)$, so that cl $\sigma^0(h) = [-10, -1]$. On the other hand, $\sigma(h)$ includes the point $x = 1$.

**Theorem 4.2.** Suppose that $\sigma_u(\bar{g}) = \sigma(\bar{g}_{\text{inf}})$. Then, $\sigma_u$ is usc at $\bar{g}$ iff $\sigma$ is usc at $\bar{g}_{\text{inf}}$.

## 5. Compositions of Unions and Intersections

Define

$$\theta_{ui}(\bar{g}) = \left\{ g \,\Big|\, \bigcup_{\lambda \in \Lambda} \left\{ \bigcap_{\gamma \in \Gamma(\lambda)} \sigma(g_\gamma) \right\} \neq \varnothing, g_\gamma \in F, \text{ and } \sup_{\substack{\gamma \in \Gamma(\lambda) \\ \lambda \in \Lambda}} \rho(g_\gamma, \bar{g}_\gamma) < \epsilon \right\}$$

and

$$\sigma_{ui} : \theta_{ui}(\bar{g}) \to 2^S,$$

where

$$\sigma_{ui}(g) = \bigcup_{\lambda \in \Lambda} \left\{ \bigcap_{\gamma \in \Gamma(\lambda)} \sigma(g_\gamma) \right\}.$$

We also define

$$\sigma_{ui}^0 = \bigcup_{\lambda \in \Lambda} \left\{ \bigcap_{\gamma \in \Gamma(\lambda)} \sigma^0(\bar{g}_\gamma) \right\},$$

and

$$\bar{g}_{\inf \sup}(x) = \inf\{\sup\{\bar{g}_\gamma(x) \mid \gamma \in \Gamma(\lambda)\} \mid \lambda \in \Lambda\}.$$

Define

$$\sigma_{\inf \sup}^0 = \sigma^0(\bar{g}_{\inf \sup}).$$

Then, we have the following theorem.

**Theorem 5.1.** Suppose that

$$\sigma_{\inf \sup}^0 = \sigma_{ui}^0 \neq \varnothing \qquad \text{and} \qquad \sigma(\bar{g}_{\inf \sup}) = \sigma_{ui}(\bar{g}).$$

Then, $\sigma_{ui}$ is usc (lsc) at $\bar{g}$ iff $\sigma$ is usc (lsc) at $\bar{g}_{\inf \sup}$.

The proof is straightforward. Moreover, we can establish regularity conditions with regard to $\bar{g}_{\inf \sup}$ that imply the equalities in the hypothesis. More importantly, we have established the approach to deal with any combinations of unions and intersections. That is, let $\sigma$ be a point-to-set map with image of the form

$$\sigma(g) = \bigcup \{\bigcap \{\cdots\{\bigcap \{\bigcup \{\bigcap \{Lg_\gamma\}\}\}\}\cdots\}\}.$$

Allowing the first or last index set to be vacuous, the above form is general (i.e., we could start with $\bigcup$ instead of $\bigcap$). Then, upon *replacing* inf for $\bigcup$ and sup for $\bigcap$, we can reduce to the level set of a single function. Our regularity that guarantees the continuity equivalence between

$\sigma_{ui\ldots ui}$ at $\bar{g}$ and $\sigma$ at $\bar{g}_{\inf\,\sup\ldots\inf\,\sup}$ holds for finite index sets (and a finite number of such operations). Moreover, if

$$g_\gamma(x) = h(x, \gamma),$$

where $h(x, \cdot)$ is continuous on $\Gamma$ and $\Gamma$ is compact for each $x \in S$, then regularity holds.

## 6. Appendix

The proof of Lemma 3.1 follows immediately from the definitions.

**Proof of Theorem 3.1.** **(i)** *Sufficiency*. Assume that $\sigma$ is lsc at $\bar{g}_{\sup}$. Consider any sequence $\{g^k\} \subset \theta(\bar{g})$ such that $\rho(g^k, \bar{g}) \to 0$, as $k \to +\infty$. Define

$$h^k(x) = \bar{g}_{\sup}(x) + \rho(g^k, \bar{g})$$

and note that $\rho(h^k, \bar{g}_{\sup}) \to 0$ as $k \to +\infty$. Since $\sigma^0_{\sup} \neq \varnothing$, there exists a $\bar{k}$ such that, for all $k \geqslant \bar{k}$, $h^k \in \theta(\bar{g}_{\sup})$. Furthermore, by the lsc of $\sigma$ at $\bar{g}_{\sup}$, for any $\epsilon > 0$ there exists a $k^0$ such that

$$\sigma(\bar{g}_{\sup}) \subset \eta_\epsilon(\sigma(h^k)) \qquad \text{for all} \quad k \geqslant k^0.$$

Thus, by Lemma 3.1, we can let $k^* = \max(\bar{k}, k^0)$ and deduce the following:

$$\sigma_i(\bar{g}) = \sigma(\bar{g}_{\sup}) \subset \eta_\epsilon(\sigma(h^k)) \qquad \text{for} \quad k \geqslant k^*.$$

Once we show that $\sigma(h^k) \subset \sigma_i(g^k)$ for $k > k^*$, then the lsc of $\sigma_i$ at $\bar{g}$ will have been established. For any $k \geqslant k^*$, choose $x \in \sigma(h^k)$. Note that $\sigma(h^k) \neq \varnothing$, since $h^k \in \theta(\bar{g}_{\sup})$. Thus,

$$\begin{aligned} 0 \geqslant h^k(x) &= \bar{g}_{\sup}(x) + \rho(g^k, \bar{g}) \\ &\geqslant \bar{g}_{\sup}(x) + g_\gamma{}^k(x) - \bar{g}_\gamma(x) \qquad \text{for all} \quad \gamma \in \Gamma \\ &\geqslant g_\gamma{}^k(x) \qquad \text{for all} \quad \gamma \in \Gamma, \end{aligned}$$

and we see $x \in \sigma_i(g^k)$ as required. Hence,

$$\sigma_i(\bar{g}) \subset \eta_\epsilon(\sigma(h^k)) \subset \eta_\epsilon(\sigma_i(g^k))$$

and $\sigma_i$ is lsc at $\bar{g}$.

**(ii)** *Necessity*. Assume that $\sigma_i$ is lsc at $\bar{g}$. Let $\{h^k\} \subset \theta(\bar{g}_{\sup})$, $\rho(h^k, \bar{g}_{\sup}) \to 0$. Define $g_\gamma{}^k = \bar{g}_\gamma + \rho(h^k, \bar{g}_{\sup})$ for $\gamma \in \Gamma$, and observe that $\rho(g^k, \bar{g}) \to 0$. Since $\sigma_i^0 \supset \sigma_{\text{sup}}^0 \neq \varnothing$, $\{g^k\} \subset \theta_i(\bar{g})$ for $k$ sufficiently large, so there exists $k^*$ such that $k \geqslant k^*$ implies that

$$\sigma_i(\bar{g}) \subset \eta_\epsilon(\sigma_i(g^k)).$$

Moreover, $\sigma_i(\bar{g}) = \sigma(\bar{g}_{\text{sup}})$, so $\sigma(\bar{g}_{\text{sup}}) \subset \eta_\epsilon(\sigma_i(g^k))$ for all $k \geqslant k^*$. We now will show that $\sigma_i(g^k) \subset \sigma(h^k)$. For $x \in \sigma_i(g^k)$, we have, by definition of $\rho(h^k, \bar{g}_{\text{sup}})$,

$$h^k(x) \leqslant \bar{g}_{\text{sup}}(x) + \rho(h^k, \bar{g}_{\text{sup}}).$$

For each $\beta > 0$, there exists $\gamma_\beta \in \Gamma$, where

$$\bar{g}_{\gamma\beta}(x) > \bar{g}_{\text{sup}}(x) - \beta.$$

Therefore, for any $\beta > 0$, we have

$$h^k(x) < \bar{g}_{\gamma_\beta}(x) + \rho(h^k, \bar{g}_{\text{sup}}) + \beta = g^k_{\gamma\beta}(x) + \beta \leqslant \beta.$$

This implies that

$$h^k(x) \leqslant 0,$$

so that $\sigma(h^k) \supset \sigma_i(g^k)$. Hence,

$$\sigma(\bar{g}_{\text{sup}}) \subset \eta_\epsilon(\sigma(h^k)) \qquad \text{for} \quad k > k^*$$

whence $\sigma$ is lsc at $\bar{g}_{\text{sup}}$.

**Proof of Lemma 3.2.** (i) *Necessity*. Assume that cl $\sigma_i^0 =$ cl $\sigma_i(\bar{g})$. Let $x \in$ cl $\sigma_i(\bar{g})$, and consider any $\epsilon > 0$. Since cl $\sigma_i(\bar{g}) =$ cl $\sigma_i^0$, $x$ is a limit point of $\sigma_i^0$. Hence, let $\{x^k\} \subset \sigma_i^0$ and $\{x^k\} \to x$. For $k$ sufficiently large, $\| x^k - x \| < \epsilon$.

**(ii)** *Sufficiency*. Assume that for all $x \in$ cl $\sigma_i(\bar{g})$, there exists a $y \in \sigma_i^0$, such that $\| x - y \| < \epsilon$. In general, cl $\sigma_i^0 \subset$ cl $\sigma_i(\bar{g})$, since $\sigma_i^0 \subset \sigma_i(\bar{g})$, so let us prove that cl $\sigma_i^0 \supset$ cl $\sigma_i(\bar{g})$. Consider $x \in$ cl $\sigma_i(\bar{g})$, so there exists $\{x^k\} \subset \sigma_i(\bar{g})$ such that $\{x^k\} \to x$. For each $x^k$, there must exist $y^k \in \sigma_i^0$ such that $\| x^k - y^k \| < 1/k$. Therefore, $\{y^k\} \subset \sigma_i^0$ and $\{y^k\} \to x$. This implies that $x \in$ cl $\sigma_i^0$.

**Proof of Theorem 3.2.** (i) *Sufficiency*. Assume that cl $\sigma_{\text{sup}}^0 =$ cl $\sigma(\bar{g}_{\text{sup}})$. Suppose, to the contrary, that there exist $\epsilon > 0$, $\{x^k\} \subset \sigma(h^k)$, $\{h^k\} \subset \theta(\bar{g}_{\text{sup}})$, $\rho(h^k, \bar{g}_{\text{sup}}) \to 0$ such that $\| x^k - y \| > \epsilon$ for all $y \in \sigma$. Since $\sigma(\bar{g}_{\text{sup}})$ is bounded, we may assume that $x^k \to x \in$ cl $\sigma(\bar{g}_{\text{sup}})$. Note that, for each $\delta > 0$, there exists $k_\delta^*$ such that $k \geqslant k_\delta^*$ implies that

$\sigma(h^k) \supset \sigma(\bar{g}_{\text{sup}} + \delta)$. To see this, recall $h^k$ converges uniformly to $\bar{g}_{\text{sup}}$. Hence, we have that, for all $\delta > 0$, there exists $k_\delta{}^*$ such that

$$k \geqslant k_\delta{}^* \Rightarrow \| x^k - y \| > \epsilon \qquad \text{for all} \quad y \in \sigma(\bar{g}_{\text{sup}} + \delta).$$

Since $x \in \text{cl}\, \sigma^0_{\text{sup}}$ by hypothesis, there exists $y \in \sigma^0_{\text{sup}}$ such that $\| x - y \| \leqslant \epsilon/2$. Choose $\delta = -\bar{g}_{\text{sup}}(y) > 0$, and note that $y \in \sigma(\bar{g}_{\text{sup}} + \delta)$. Then, choose $K$ such that $\| x^k - x \| \leqslant \epsilon/2$ for $k \geqslant K$. Let $k^* = \max\{K, k_\delta{}^*\}$, so $\| x^k - y \| \leqslant \epsilon$ for all $y \in \text{cl}\, \sigma(\bar{g}_{\text{sup}} + \delta)$, which is a contradiction.

**(ii)** *Necessity.* Assume that $\sigma$ is lsc at $\bar{g}_{\text{sup}}$. Choose $x \in \text{cl}\, \sigma(\bar{g}_{\text{sup}})$, so there exists $\{x^k\} \subset \sigma(\bar{g}_{\text{sup}})$ such that $\{x^k\} \to x$. Define

$$h^k(x) = \bar{g}_{\text{sup}}(x) + 1/k.$$

Clearly, $\{h^k\} \subset \theta(\bar{g}_{\text{sup}})$ and $\rho(h^k, \bar{g}_{\text{sup}}) \to 0$. Therefore, for each $\epsilon > 0$, there must exist $k^*$, such that, for all $k \geqslant k^*$, then

$$x^k \in \eta_\epsilon(\sigma_{\text{sup}}(h^k)).$$

Therefore, for each $k \geqslant k^*$, there exists $y^k \in \sigma(h^k)$ such that

$$\| x^k - y^k \| < \epsilon \qquad \text{and} \qquad \bar{g}_{\text{sup}}(y^k) + 1/k \leqslant 0.$$

This implies that $\{y^k\} \subset \sigma^0_{\text{sup}}$ and $y^k \to y \in \text{cl}\, \sigma^0_{\text{sup}}$. Therefore, we have shown that, for any $\epsilon > 0$, there exists $y \in \text{cl}\, \sigma^0_{\text{sup}}$ such that $\| x - y \| < \epsilon$. This implies that $x \in \text{cl}\, \sigma^0_{\text{sup}}$, so $\text{cl}\, \sigma^0_{\text{sup}} \supset \text{cl}\, \sigma(\bar{g}_{\text{sup}})$. Since $\text{cl}\, \sigma^0_{\text{sup}} \subset \text{cl}\, \sigma(\bar{g}_{\text{sup}})$ in general, equality must hold.

**Proof of Theorem 3.3.** Greenberg and Pierskalla (Ref. 23) have shown that $\bar{g}_{\text{sup}}$ is explicitly quasiconvex on $S$ under the given hypotheses. Then, Lemma 5 of Evans and Gould (Ref. 18) may be applied.

**Proof of Theorem 3.4.** We have, for any $x, y \in S$, and $x \neq y$, that their meet is given by $x^*y$ and

$$\begin{aligned} \bar{g}_{\text{sup}}(x^*y) &= g_{\tau(x^*y)}(x^*y) < g_{\tau(x^*y)}(x)^*\, g_{\tau(x^*y)}(y) \\ &\leqslant g_{\tau(x)}(x)^*\, g_{\tau(y)}(y) = \bar{g}_{\text{sup}}(x)^*\, \bar{g}_{\text{sup}}(y), \end{aligned}$$

so $\bar{g}_{\text{sup}}$ is strictly isotonic on $S$. The result in Ref. 23 then applies.

**Proof of Theorem 3.5.** **(i)** *Necessity.* Assume that $\sigma_i$ is usc at

$\bar{g}$. Then, for any $\epsilon > 0$ and $\{h^k\} \subset \theta(\bar{g}_{\sup})$, where $\rho(h^k, \bar{g}_{\sup}) \to 0$, define

$$g_\gamma^k = \bar{g}_\gamma - \rho(h^k, \bar{g}_{\sup}), \qquad \gamma \in \Gamma.$$

Observe that $\{g^k\} \subset \theta(\bar{g})$ for all $k$ and $\rho(g^k, \bar{g}) \to 0$, so for $k$ sufficiently large by usc $\sigma_i(g^k) \subset \eta_\epsilon(\sigma_i(\bar{g}))$. From Lemma 3.1, we have that $\sigma_i(\bar{g}) = \sigma(\bar{g}_{\sup})$. Further, if $x \in \sigma(h^k)$, then $h^k(x) \leqslant 0$, so

$$g_\gamma^k(x) \leqslant \bar{g}_\gamma(x) + h^k(x) - \bar{g}_{\sup}(x) \leqslant h^k(x) \leqslant 0.$$

This implies that

$$\sigma(h^k) \subset \sigma_i(g^k) \subset \eta_\epsilon(\sigma_i(\bar{g})) = \eta_\epsilon(\sigma(\bar{g}_{\sup})).$$

Hence, $\sigma$ is usc at $\bar{g}_{\sup}$.

**(ii)** *Sufficiency.* Assume that $\sigma$ is usc at $\bar{g}_{\sup}$. Consider any $\epsilon > 0$ and any sequence $\{g^k\} \subset \theta(\bar{g})$ such that $\rho(g^k, \bar{g}) \to 0$ as $k \to +\infty$. Define

$$h^k(x) = \bar{g}_{\sup}(x) - \rho(g^k, \bar{g}).$$

Note that $\rho(h^k, \bar{g}_{\sup}) \to 0$ as $k \to +\infty$ and that $\{h^k\} \subset \theta(\bar{g}_{\sup})$. Let $x \in \sigma_i(g^k)$ be given; then,

$$\begin{aligned} 0 &\geqslant g_\gamma^k(x) \qquad \text{for all} \quad \gamma \in \Gamma \\ &= \bar{g}_{\sup}(x) - \bar{g}_{\sup}(x) + g_\gamma^k(x) \qquad \text{for all} \quad \gamma \in \Gamma \\ &\geqslant \bar{g}_{\sup}(x) - \rho(g^k, \bar{g}) = h^k(x). \end{aligned}$$

Hence, $x \in \sigma(h^k)$ and, by usc of $\sigma$ at $\bar{g}_{\sup}$ and by Lemma 3.1,

$$\sigma_i(g^k) \subset \sigma(h^k) \subset \eta_\epsilon(\sigma(\bar{g}_{\sup})) = \eta_\epsilon(\sigma_i(\bar{g})),$$

and $\sigma_i$ is usc at $\bar{g}$.

**Proof of Theorem 4.1.** **(i)** *Necessity.* Assume that $\sigma_u$ is lsc at $\bar{g}$. Consider $\{h^k\} \subset \theta(\bar{g}_{\inf})$, $\rho(h^k, \bar{g}_{\inf}) \to 0$, and let $\epsilon > 0$. Define $g_\gamma^k = \bar{g}_\gamma + \rho(h^k, \bar{g}_{\inf})$, $\gamma \in \Gamma$. Since $\sigma_u^0 \neq \varnothing$, $\{g^k\} \subset \theta_u(\bar{g})$ for $k$ sufficiently large, so there exists $k^*$ such that $k \geqslant k^*$ implies that

$$\sigma_u(\bar{g}) \subset \eta_\epsilon(\sigma_u(g^k)).$$

We are assuming that $\sigma(\bar{g}_{\inf}) = \sigma_u(\bar{g})$, so

$$\sigma(\bar{g}_{\inf}) \subset \eta_\epsilon(\sigma_u(g^k)) \qquad \text{for} \quad k \geqslant k^*.$$

Now, consider $x \in \sigma_u(g^k)$. Then, there exists $\gamma \in \Gamma$, say $\gamma^*$, such that $g^k_{\gamma^*}(x) \leqslant 0$. This implies that

$$\bar{g}_{\gamma^*}(x) \leqslant -\rho(h^k, \bar{g}_{\inf}) \leqslant -h^k(x) + \bar{g}_{\inf}(x),$$

so

$$h^k(x) \leqslant \bar{g}_{\inf}(x) - \bar{g}_{\gamma^*}(x) \leqslant 0.$$

Therefore, $x \in \sigma(h^k)$. Hence, $\sigma_u(g^k) \subset \sigma(h^k)$, which implies that

$$\sigma(\bar{g}_{\inf}) \subset \eta_\epsilon(\sigma(h^k)) \qquad \text{for} \quad k \geqslant k^*.$$

Hence, $\sigma$ is lsc at $\bar{g}_{\inf}$.

**(ii)** *Sufficiency.* Assume that $\sigma$ is lsc at $\bar{g}_{\inf}$. Consider any $\epsilon > 0$ and any sequence $\{g^k\} \subset \theta_u(\bar{g})$ such that $\rho(g^k, \bar{g}) \to 0$ as $k \to +\infty$. Define

$$h^k(x) = \bar{g}_{\inf}(x) + \rho(g^k, \bar{g}) + 1/k.$$

Clearly, $\rho(h^k, \bar{g}_{\inf}) \to 0$ as $k \to +\infty$; and, since $\sigma^0_{\inf} \neq \varnothing$, there is a $\bar{k}$ such that, for all $k \geqslant \bar{k}$, $h^k \in \theta(\bar{g}_{\inf})$. By Lemma 4.1 and lsc, there is a $k^0$ such that

$$\sigma_u(\bar{g}) \subset \sigma(\bar{g}_{\inf}) \subset \eta_\epsilon(\sigma(h^k)).$$

Let $k^* = \max(\bar{k}, k^0)$. We shall show that $\sigma(h^k) \subset \sigma_u(g^k)$ for all $k \geqslant k^*$, and then the lsc of $\sigma_u$ at $\bar{g}$ will have been established. Let $x \in \sigma(h^k)$ [$\sigma(h^k) \neq \varnothing$, since $h^k \in \theta(\bar{g}_{\inf})$ for $k \geqslant k^*$].

Hence, the following relations hold:

$$\begin{aligned} 0 \geqslant h^k(x) &= \bar{g}_{\inf}(x) + \rho(h^k, \bar{g}_{\inf}) + 1/k \\ &\geqslant \bar{g}_{\inf}(x) + g_\gamma{}^k(x) - \bar{g}_\gamma(x) + 1/k, \qquad \text{for all} \quad \gamma \in \Gamma. \end{aligned}$$

Now, for any $k$ and any $x \in S$, there is a $\gamma_k$ such that

$$\bar{g}_{\inf}(x) > \bar{g}_{\gamma_k}(x) - 1/k,$$

or

$$\bar{g}_{\inf}(x) - \bar{g}_{\gamma_k}(x) + 1/k > 0.$$

Thus,

$$0 \geqslant g^k_{\gamma_k}(x) \qquad \text{and} \qquad x \in \sigma_u(g^k),$$

as required. We have

$$\sigma_u(\bar{g}) \subset \sigma(\bar{g}_{\inf}) \subset \eta_\epsilon(\sigma(h^k)) \subset \eta_\epsilon(\sigma_u(g^k)).$$

**Proof of Lemma 4.2.** Suppose that $x \in h$, and observe that

$$x = \tfrac{1}{2}(x + d) + \tfrac{1}{2}(x - d)$$

for any $d \in E^n$. Therefore,

$$0 \geqslant h(x) > \tfrac{1}{2}h(x + d) + \tfrac{1}{2}h(x - d).$$

This implies that $h(x + d) < 0$ or $h(x - d) < 0$. Therefore, choose $d = \epsilon \underline{1}$, where $\underline{1}$ is an $n$-vector of 1's, and observe that $N_\epsilon(x) \cap \sigma^0(h) \neq \varnothing$. From Lemma 3.2, our proof is complete.

**Proof of Theorem 4.2.** **(i)** *Necessity.* Assume that $\sigma_u$ is usc at $\bar{g}$. Let $\{h^k\} \subset \theta(\bar{g}_{\text{inf}})$, $\rho(h^k, \bar{g}_{\text{inf}}) \to 0$, and consider $\epsilon > 0$. Define $g_\gamma{}^k = \bar{g}_\gamma - \rho(h^k, \bar{g}_{\text{inf}}) - 1/k$ for $\gamma \in \Gamma$. Then, $\{g^k\} \subset \theta_u(\bar{g})$ for all $k$ and $\rho(g^k, \bar{g}) \to 0$. Therefore, there exists $k^*$ such that $k \geqslant k^*$ implies that

$$\sigma_u(g^k) \subset \eta_\epsilon(\sigma_u(\bar{g})) \subset \eta_\epsilon(\sigma(\bar{g}_{\text{inf}})).$$

The second containment follows from Lemma 4.1. Consider $x \in \sigma(h^k)$, so that, for each $\gamma \in \Gamma$, we have the following:

$$\begin{aligned} g_\gamma{}^k(x) &= \bar{g}_\gamma(x) - \rho(h^k, \bar{g}_{\text{inf}}) - 1/k \leqslant \bar{g}(x) + h^k(x) - \bar{g}_{\text{inf}}(x) - 1/k \\ &\leqslant \bar{g}_\gamma(x) - \bar{g}_{\text{inf}}(x) - 1/k. \end{aligned}$$

There exists $\gamma_k \in \Gamma$, such that

$$\bar{g}_{\gamma_k}(x) \leqslant \bar{g}_{\text{inf}}(x) + 1/k.$$

Therefore,

$$g^k_{\gamma_k}(x) \leqslant 0,$$

whence $x \in \sigma_u(g^k)$. This implies that $\sigma_u(g^k) \supset \sigma(h^k)$, so

$$\sigma_{\text{inf}}(h^k) \subset \eta_\epsilon(\sigma(\bar{g}_{\text{inf}})) \qquad \text{for} \quad k \geqslant k^*,$$

whence $\sigma_{\text{inf}}$ is usc at $\bar{g}_{\text{inf}}$.

**(ii)** *Sufficiency.* Assume that $\sigma$ is usc at $\bar{g}_{\text{inf}}$. Consider any $\epsilon > 0$ and any sequence $\{g^k\} \subset \theta_u(\bar{g})$ such that $\rho(g^k, \bar{g}) \to 0$ as $k \to +\infty$. Define

$$h^k(x) = \bar{g}_{\text{inf}}(x) - \rho(g^k, \bar{g}).$$

Hence, $\{h^k\} \subset \theta(\bar{g}_{\text{inf}})$ and $\rho(h^k, \bar{g}_{\text{inf}}) \to 0$ as $k \to +\infty$. By the usc of $\sigma$ at $\bar{g}_{\text{inf}}$, there is a $k^*$ such that, for all $k \geqslant k^*$,

$$\sigma(h^k) \subset \eta_\epsilon(\sigma(\bar{g}_{\text{inf}})) = \eta_\epsilon(\sigma_u(\bar{g})).$$

The last equality follows by assumption. Let $x \in \sigma_u(g^k)$. Thus, there is a $\gamma$ such that

$$\begin{aligned} 0 &\geqslant g_\gamma{}^k(x) = \bar{g}_{\inf}(x) - \bar{g}_{\inf}(x) + g_\gamma{}^k(x) \\ &\geqslant \bar{g}_{\inf}(x) - \bar{g}_\gamma(x) + g_\gamma{}^k(x) \geqslant \bar{g}_{\inf}(x) - \rho(g^k, \bar{g}) = h^k(x). \end{aligned}$$

Hence, $x \in \sigma(h^k)$, and we then have

$$\sigma_u(g^k) \subset \sigma(h^k) \subset \eta_\epsilon(\sigma(\bar{g}_{\inf})) \subset \eta_\epsilon(\sigma_u(\bar{g})),$$

whence $\sigma_u$ is usc at $\bar{g}$.

## References

1. ROSENTHAL, R. W., *On Voting Majority Sizes*, Center for Operations Research and Econometrics, Belgium, Discussion Paper No. 7337, 1973.
2. CHARNES, A., and COOPER, W. W., *Management Models and Industrial Applications of Linear Programming*, Vols. I and II, John Wiley and Sons, New York, New York, 1961.
3. CHARNES, A., COOPER, W. W., and KORTANEK, K. O., *Duality, Haar Programs and Fnite Sequence Spaces*, Proceedings of the National Academy of Science of USA, Vol. 48, pp. 783–786, 1962.
4. CHARNES, A., COOPER, W. W., and KORTANEK, K. O., *On the Theory of Semi-Infinite Programming and a Generalization of the Kuhn–Tucker Saddle-Point Theorem for Arbitrary Convex Functions*, Naval Research Logistics Quarterly, Vol. 16, pp. 41–51, 1969.
5. KORTANEK, K. O., and EVANS, J. P., *Pseudo-Concave Programming and Lagrange Regularity*, Operations Research, Vol. 15, pp. 882–891, 1967.
6. GUSTAFSON, S. Å., and KORTANEK, K. O., *Numerical Treatment of a Class of Semi-Infinite Programming Problems*, School of Urban and Public Affairs, Carnegie–Mellon University, IPP Report No. 21, 1971.
7. GUSTAFSON, S. Å., *On the Computational Solution of a Class of Generalized Moment Problems*, Society of Industrial and Applied Mathematics Journal on Numerical Analysis, Vol. 7, pp. 343–357, 1970.
8. GUSTAFSON, S. Å., KORTANEK, K. O., and ROM, W., *Non-Cebysevian Moment Problems*, Society of Industrial and Applied Mathematics Journal on Numerical Analysis, Vol. 7, pp. 335–342, 1970.
9. KARLIN, S., and STUDDEN, W. J., *Tchebycheff Systems with Applications in Analysis and Statistics*, John Wiley and Sons (Interscience Publishers), New York, New York, 1966.

10. Fortak, H. G., *Numerical Simulation of the Temporal and Spatial Distributions of Urban Air Pollution Concentration*, Proceedings of the Symposium on Multiple-Source Urban Diffusion Models, Research Triangle Park, North Carolina, 1970.
11. Gustafson, S. Å., and Kortanek, K. O., *Analytical Properties of Some Multiple-Source Urban Diffusion Models*, Environment and Planning, Vol. 4 pp. 31–41, 1972.
12. Rote, D. M., Gudenas, J. W., and Cooley, L. A., *Studies of the Argonne Integrated-Puff Model*, Argonne National Laboratory, Illinois, 1971.
13. Slade, D. H., Editor, *Meteorology and Atomic Energy 1968*, US Atomic Energy Commission, TID-24190, 1968.
14. Hausdorff, F., *Set Theory*, Chelsea, Publishing Company, New York New York, 1962.
15. Berge, C., *Topological Spaces*, The MacMillan Company, New York New York, 1963.
16. Debreu, G., *Theory of Value*, Cowles Foundation Monograph No. 17 Yale University Press, New Haven, Connecticut, 1959.
17. Dantzig, G., Folkman, J., and Shapiro, N., *On the Continuity of the Minimum Set of a Continuous Function*, Journal on Mathematical Analysis and Applications, Vol. 17, pp. 519–548, 1967.
18. Evans, J. P., and Gould, F. J., *Stability in Nonlinear Programming*, Operations Research, Vol. 18, pp. 107–118, 1970.
19. Greenberg, H. J., and Pierskalla, W. P., *Extensions of the Evans–Gould Stability Theorems for Mathematical Programs*, Operations Research, Vol. 20, pp. 143–153, 1972.
20. Böhm, V., *On the Continuity of the Optimal Policy Set for Linear Programs*, Institut für Gesellschafts- und Wirtschaftswissenschaften, Universität Bonn, Wirtschaftstheoretische Abteilung, Working Paper, Bonn, West Germany, 1973.
21. Meyer, R., *The Validity of a Family of Optimization Methods*, Society of Industrial and Applied Mathematics Journal on Control, Vol. 8, pp. 41–54, 1970.
22. Zangwill, W., *Nonlinear Programming—A Unified Approach*, Prentice-Hall, Englewood Cliffs, New Jersey, 1969.
23. Greenberg, H. J., and Pierskalla, W. P., *A Review of Quasi-Convex Functions*, Operations Research, Vol. 19, pp. 1553–1570, 1971.
24. Lootsma, F. A., *Boundary Properties of Penalty Functions for Constrained Minimization*, Phillips Research Reports Supplements, No. 3, Eindhoven, Netherlands, 1970.


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