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A REVIEW OF QUASI-CONVEX FUNCTIONS

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Many theorems involving convex functions have appeared in the literature since the pioneering work of JENSEN. Recently some results have been obtained for a larger class of functions: quasi-convex. This review summarizes in condensed form results known to date, providing some refinements to gain further generality. An additional objective of this review is to clarify the structure underlying quasi-convex functions by presenting analogues to properties of convex functions, and by illustrating where analogues do not exist.

THE CONVEX function, born from the theory of inequalities,[17] has been a subject of much investigation. The usual definition used for real scalar-valued functions over a convex subset of E^n, say X, is:

DEFINITION 1. f is *convex* if

$$f[\lambda x_1+(1-\lambda)x_2]\leqq \lambda f(x_1)+(1-\lambda)f(x_2)$$

for all $\lambda\epsilon[0, 1]$ and all $x_1, x_2\epsilon X$.

An equivalent definition, in terms of its epigraph, is given in Table II (property 1a). Both are communicating the geometric concept that, for any two points on the curve (surface), the line segment joining the two points lies entirely above the curve (surface) between the two points.

One should note that a function can be convex over each of its coordinates (taken one at a time), but not be convex over X. For example, $f(x, y)=xy$ is convex over x (for y fixed) and over y (for x fixed), but fails the defining inequality for $(x_1, y_1)=(1, 0)$, $(x_2, y_2)=(0, 1)$, and $\lambda=½$.

Many theorems have evolved[12] that describe interesting properties of convex functions. In many cases the defining inequality could have been weakened to require only $f[\lambda x_1+(1-\lambda)x_2]\leqq\max[f(x_1), f(x_2)]$, or, equivalently, that only the level sets need be convex. This led to the development of a wider class of functions that are given by:

DEFINITION 2. f is *quasi-convex* if

$$f[\lambda x_1+(1-\lambda)x_2]\leqq\max[f(x_1),f(x_2)]$$

for all $\lambda\epsilon[0, 1]$ and all x_1, $x_2\epsilon X$.

The negative of a (quasi-) convex function is a (quasi-) concave function.

The prefix 'quasi' means 'as if.' We thus expect quasi-convex functions to have some special properties similar to those for convex functions. Moreover, since every convex function is quasi-convex, we expect the convex function to be more highly structured in that certain properties do not carry over to the more general class of quasi-convex functions.

DEFINETTI[13] was one of the first persons to recognize some of the characteristics of functions having convex level sets. He did not name this class, but he did note that it includes all convex functions and some nonconvex functions. FENCHEL[12] was one of the early pioneers in formalizing, naming, and developing the class of quasi-convex functions.

One of the early generalizations that recognized the inherent structure of quasi-convex functions was NIKAIDO's[34] generalization of VON NEUMANN's minimax theorem using BROUWER's fixed-point theorem. Later theorems by NASH AND SION are described by BERGE[5] utilizing KAKUTANI's theorem.

SLATER[38] generalized the KUHN-TUCKER saddle-point equivalence theorem, and ARROW AND ENTHOVEN[1] presented an important paper dealing with quasi-concave optimization problems with an application to consumer demand.

More recently, MARTOS[29–33] has obtained many important results, particularly for quadratic functions. Of particular importance to mathematical programming is his algorithm for nonconvex quadratic programs that are quasi-convex.[32] COTTLE AND FERLAND[7,8] subsequently added to the study of quasi-convex quadratics. MANGASARIAN's recent book[26] has presented several of the key concepts that concern optimization problems.

Our analysis is intended to organize the key features of quasi-convexity and, by comparison with convex functions, add some insight into their structure. Some of the properties we state are slight generalizations of those found in the references. Later we shall consider 'explicitly quasi-convex' functions and present a new theorem of importance in semi-infinite nonlinear programming. First, let us complete our list of definitions in sections 1 and 2.

We see in property 1a (Table II) that an equivalent definition of a convex function is that its epigraph be a convex set. Thus, this cannot hold for arbitrary quasi-convex functions. However, an analogous equivalence holds for the level sets of f. Before discussing each entry in Tables II and III we first provide a table of notation (Table I), and we shall demonstrate classes of functions that are, in general, not convex but quasi-convex. Also, we wish to define the following:

TABLE I

NOTATION

Term	Meaning
$[X, f]$	The epigraph: $\{(z, x): x\epsilon X, f(x)\leqq z\}$ (*see* Fig. 1)
$[S^c, f]$	A level set: $\{x: x\epsilon X, f(x)\leqq c\}$ (*see* Fig. 2)
X^0	Relative interior of X
$\bar{X}$	Closure of X
$[X, f]'$ or $[X', f']$	The conjugate of $[X, f]$: $X'=\{y: \sup_{x\epsilon X}[xy-f(x)]<+\infty\}$ $f'(y)=\sup_{x\epsilon X}[xy-f(x)]$ for $y\epsilon X'$ (*see* Fig. 3)
E	The extreme points of X
ϕ	The empty set
∇f	The gradient of f: $\nabla f=\left(\frac{\partial f}{\partial x_1}, \cdots, \frac{\partial f}{\partial x_n}\right)$
H	The hessian of f assuming f has continuous second derivatives
D	The bordered hessian of f assuming f has continuous second derivatives: $D=\begin{bmatrix} 0 & \nabla f \\ \nabla f^T & H \end{bmatrix}$
$\lvert D_j\rvert$	The jth principal minor of $D(j=0, \cdots, n)$ (note that $\lvert D_0\rvert=0$)
δ_{ij}	$\delta_{ij}=\begin{cases} 1 & \text{if } i=j, \\ 0 & \text{if } i\neq j \end{cases}$

DEFINITION 3. f is *linear* (affine) if f is both concave and convex.

DEFINITION 4. f is *quasi-monotonic* if f is both quasi-concave and quasi-convex.

One might suppose that the term 'quasi-monotonic' should be 'quasi-linear' in order to keep the analogue intact. However, the term 'quasi-linearization' has been used (*see* BECKENBACH AND BELLMAN[3]). We follow the definition of Martos.[29] An example of a nonlinear quasi-monotonic function is given by:[33] $f(x)=(c^Tx+c_0)/(d^Tx+d_0)$ where X is a convex set that does not contain points such that the denominator vanishes.

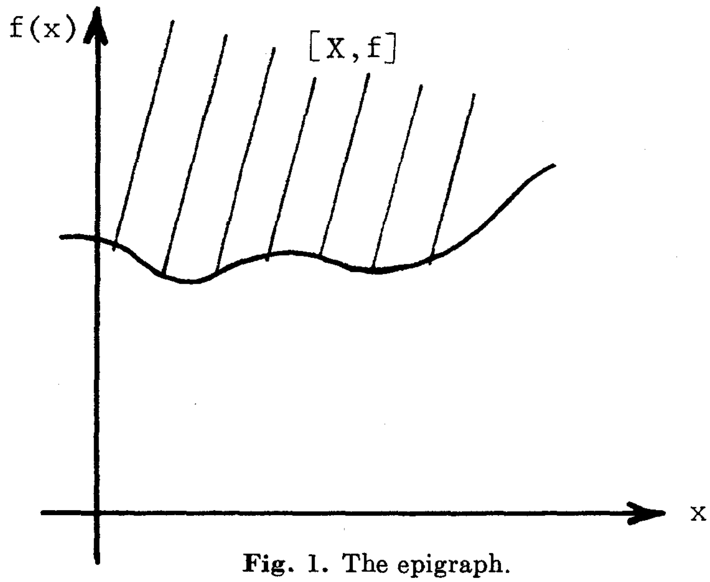

Fig. 1. The epigraph.

1. CLASSES OF FUNCTIONS

LET $E_r(x)$ be the rth elementary symmetric function given by:

$$E_1(x)=\sum_{i=1}^{i=n} x_i,$$
$$E_2(x)=\sum_{i=1}^{n-1}\sum_{j=i+1}^{j=n} x_i x_j,$$
$$\vdots$$
$$E_n(x)=\prod_{i=1}^{i=n} x_i.$$

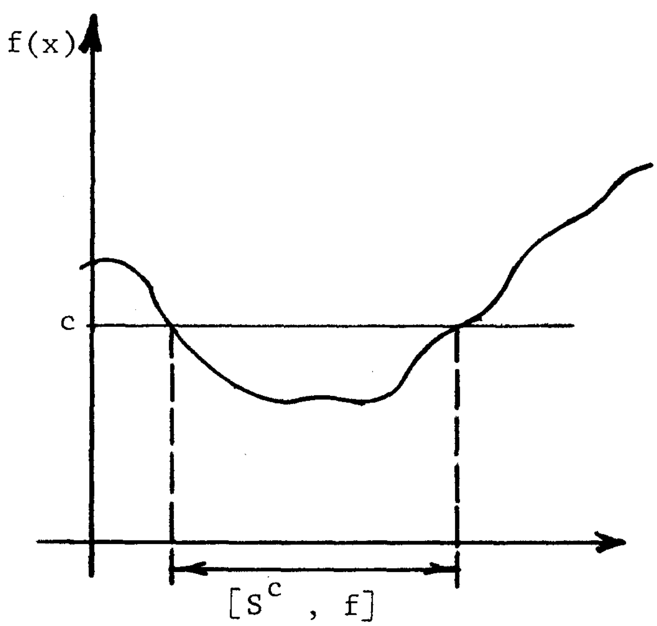

Fig. 2. The level set.

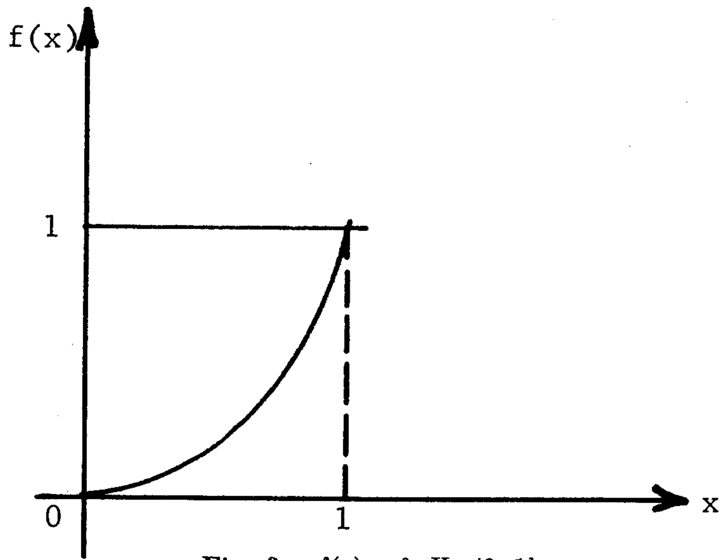

Fig. 3a. $f(x) = x^2$, $X = [0, 1]$.

Thus, $E_r(x)$ is the sum of distinct r-fold products for $1 \leqq r \leqq n$. Our domain, X, is taken to be $E_+{}^n = \{x : x \geqq 0\}$.

Marcus and Lopes[27] proved that $E_r(x)^{1/r}$ is concave over X. By property 19b of Table II we may conclude that $a_r E_r(x)^{k_r}$ is quasi-concave for $a_r k_r \geqq 0$ (i.e., $a_r \leqq 0$ and $k_r \leqq 0$, or $a_r \geqq 0$ and $k_r \geqq 0$). Moreover, since

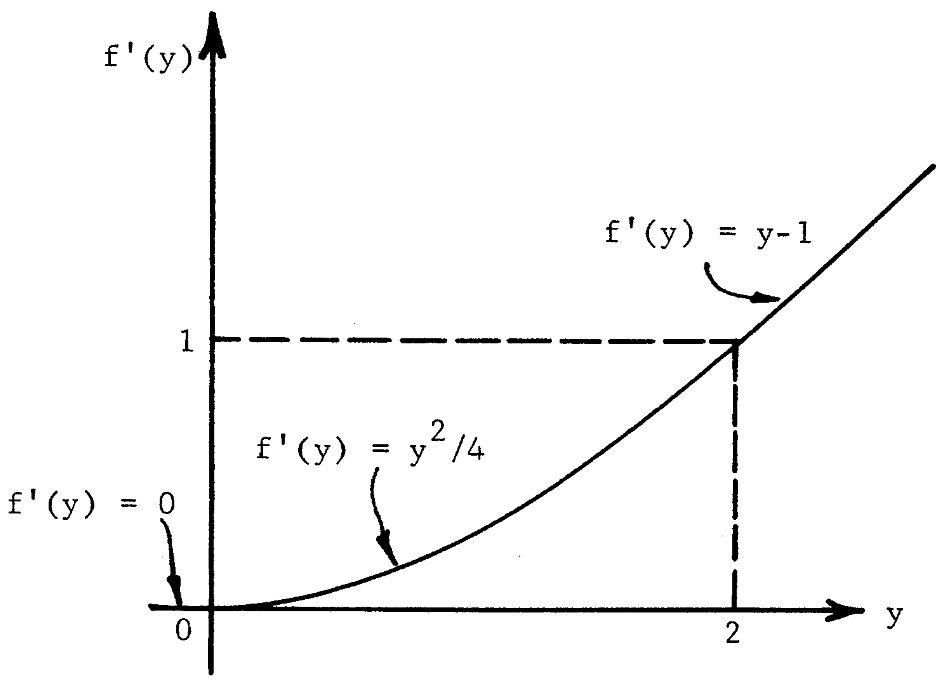

Fig. 3b. The conjugate of $f(x) = x^2$ over $[0, 1]$, $X' = (-\infty, \infty)$.

the reciprocal of a positive quasi-concave function is quasi-convex, it follows that $a_r E_r(x)^{k_r}$ is quasi-convex for $a_r k_r \leqq 0$ and $x \epsilon E_+{}^n \sim \{0\}$.

Another place where quasi-convexity arises is in surrogate mathematical programming. LUENBERGER[25] developed this approach for quasi-concave programs. GREENBERG AND PIERSKALLA[15] developed a theoretical foundation for arbitrary mathematical programs. A key result of this approach is the construction of a set X and a rule $f(x)$ as follows. Let $u(y)$ be a real-valued function mapping $E^n \to E^1$, and let $v(y)$ be real-valued function mapping $E^n \to E^m$. We define X and f as follows:

$$X = \{x \geqq 0\colon \sup_y \{u(y) : x^T v(y) \leqq 0\} < \infty\},$$

$$f(x) = \sup_y [u(y) : x^T v(y) \leqq 0] \quad \text{for} \quad x \epsilon X.$$

It can be easily proved[15] that X is a convex set and f is quasi-convex over X.

A third class of quasi-convex functions, which need not be convex, was recently shown by Martos;[31,32] namely, consider the quadratic function given by $f(x) = \frac{1}{2} xQx + cx$, where Q is a symmetric matrix. If $X = E^n$, then $f(x)$ is convex if and only if $f(x)$ is quasi-convex (whence Q is positive semi-definite). However, if $X = E_+{}^n$, then $f(x)$ may be quasi-convex, yet not be convex. For example, $f(x_1, x_2) = -x_1 x_2$ is quasi-convex over $E_+{}^2$, but is not convex.

MARTOS[32] proved that $f(x)$ is quasi-convex over $E_+{}^n$ if and only if for all $v \epsilon E^n$ we have $v^T Q v < 0$ implies $[Qv, c^T v] \gtreqless 0$, where the notation $u \gtreqless 0$ means that the vector u is either nonpositive or nonnegative (no two coordinates have opposite signs).

In reference 31 Martos established conditions by which a homogeneous quadratic (i.e., $c = 0$) is quasi-convex and not convex. This was extended in reference 32; that is, $f(x)$ is quasi-convex over $E_+{}^n$ and not convex, if and only if the following three conditions hold:

1. $Q \leqq 0$ and $c \leqq 0$.
2. Q has exactly one negative eigenvalue.
3. There is some $q \epsilon E^n$ such that $Qq = c$ and $cq \leqq 0$.

One can combine convex, concave and linear functions to form quasi-convex functions. For example,[4,18,26,30] let g and f be defined on a convex set X such that $f(x) \neq 0$ for all $x \epsilon X$. Then, g/f is quasi-convex on X if any of the following 6 hypotheses holds:

I. g is convex and $f(x) > 0$ for all $x \epsilon X$, *or* g is concave and $f(x) < 0$ for all $x \epsilon X$, and:

II. f is linear on X, *or* f is convex on X and $g(x) \leqq 0$ for all $x \epsilon X$, *or* f is concave on X and $g(x) \geqq 0$ for all $x \epsilon X$.

2. ANALOGOUS PROPERTIES

IN TABLE II we have grouped key properties in the categories: set relations, continuity, differentiability, boundedness, extreme values, inequalities, and transformations. We shall reference proofs and discuss some of them briefly to see where the defining inequalities apply. It is assumed that the domain of f, denoted X, is a convex subset of E^n.

We begin our discussion of Table II with properties 1–3. Fenchel[12] proves 1(a,b) and provides further discussion to add geometric insight. Martos[29] introduced quasi-monotonic functions, and property 2b was half given in the form: if f is quasi-monotonic and if X is polyhedral, then $[S^c, f]$ is polyhedral for all c. This is implied by the convexity of $[S^c, f]$ and $[S^c, -f]$. Because of the way Martos stated his theorem, a converse is not possible, since the fact that $[S^c, f]$ is polyhedral is not sufficient to guarantee that both $[S^c, f]$ and $[S^c, -f]$ are convex sets for all c. An example is given by $X=[-1, 1]$ and $f(x)=|x|$. The form given in Table II is indeed an equivalence theorem in the spirit of Martos' result and follows directly from definition 6. Property 3(a,b) follows directly from 2(a,b). However, continuity is needed for the sufficiency in 3b. To illustrate why this is so, consider $f(x)=0$ if $0 \leqq x \leqq 1$, 2 if $1<x\leqq 2$, 1 if $2<x\leqq 3$. The set Y is empty for $c\neq 0$, 1, 2 and for $c=0$, $Y=[0, 1]$; for $c=1$, $Y=(2, 3]$; for $c=2$, $Y=(1, 2]$. Thus, Y is convex for all c, but f is not quasi-convex and hence is not quasi-monotonic. Figure 4 illustrates this.

The proof of sufficiency in 3b proceeds as follows. It is necessary to show f is both quasi-convex and quasi-concave for all c. Since the proof of quasi-concavity parallels the proof of quasi-convexity, *mutatis mutandis*, we only demonstrate the latter. Consider any $x \epsilon X$, $y \epsilon X$, and $\alpha \epsilon (0, 1)$. Since X is convex, $w=\alpha x+(1-\alpha)y$ belongs to X. Let $c=\max[f(x), f(y)]$, we must show $f(w)\leqq c$ for all w on the line segment joining x and y. Assume, to the contrary, that for some $\alpha\epsilon(0, 1)$ we have $f(w)>c$. Without loss of generality let $\max[f(x), f(y)]=f(y)$. Then $f(w)>f(y)\geqq f(x)$. Since f is continuous on the compact line segment joining x and y, there exists u such that $\beta x+(1-\beta)w=u$ for some $\beta\epsilon(0, 1]$ and $f(u)=c$. Now $w = \alpha x+(1-\alpha)y=\{\alpha/[\beta+\alpha(1-\beta)]\}u+\{\beta(1-\alpha)/[\beta+\alpha(1-\beta)]\}y$, and by letting $\gamma=\alpha/[\beta+\alpha(1-\beta)]$ we see that $w=\gamma u+(1-\gamma)y$ and $\gamma\epsilon(0, 1)$. Since Y is convex and $u\epsilon Y$, $y \epsilon Y$ we must have $w \epsilon Y$, which contradicts $f(w)>c$.

Property 4a is well known and 4b was recently shown by EVANS and GOULD.[10]

The continuity of a convex function over the interior of its domain was an early result, and a proof appears in VALENTINE.[39] DEAK[9] proved 5b.

Property 7a has been known for some time, and we have been unable to find the first source. Arrow and Enthoven[1] prove 7b in slightly different form. They prove that $(-1)^j|D_j|\geqq 0$ for $j=1, \cdots, n$ in order that f be

TABLE II

ANALOGOUS (OR GENERALIZED) PROPERTIES

Convex	Quasi-convex
1a. f is convex if and only if $[X, f]$ is a convex set.	1b. f is quasi-convex if and only if $[S^c, f]$ is a convex set for all c.
2a. f is linear if and only if $[X, f]$ and $[X, -f]$ are convex sets.	2b. f is quasi-monotonic if and only if $[S^c, f]$ and $[S^c, -f]$ are convex sets for all c.
3a. f is linear if and only if $\{(z, x)\colon x \epsilon X, f(x) = z\}$ is a convex set.	3b. Let $Y = \{x \epsilon X\colon f(x) = c\}$. If f is quasi-monotonic, then Y is a convex set for all c. If Y is a convex set for all c and if f is continuous, then f is quasi-monotonic.
4a. $[S^c, f]$ is bounded for all c if and only if there exists a c^* such that $[S^{c^*}, f]$ is nonempty and bounded.	4b. If $[S^c, f]$ is nonempty and bounded, then there exists a $c^* > c$ such that $[S^{c^*}, f]$ is bounded.
5a. f is continuous over X^0.	5b. f is continuous almost everywhere over X^0.
6a. One-sided partial derivatives exist everywhere in X^0.	6b. One-sided partial derivatives exist almost everywhere in X^0.
7a. Suppose f is twice continuously differentiable on E^n. Then, f is convex on E^n if and only if its hessian $H(\cdot)$ is positive semidefinite throughout E^n.	7b. Suppose f is twice continuously differentiable on E^n. If f is quasi-convex on E_+^n, then $\lvert D_j \rvert \leqq 0$ for $j = 1, \cdots, n$. If $\lvert D_j \rvert < 0$ for $j = 1, \cdots, n$, then f is quasi-convex on E_+^n.
8a. Suppose f is once continuously differentiable on E^n. Then, f is convex on E^n if and only if $f(y) - f(x) \geqq \nabla f(x)^T (y - x)$.	8b. Suppose f is once continuously differentiable on E^n. Then f is quasi-convex on E^n if and only if $f(y) \leqq f(x)$ implies $\nabla f(x)^T (y - x) \leqq 0$.
9a. $\sup_{x \epsilon X^0} f(x) < +\infty$ if X is compact.	9b. $\sup_{x \epsilon X^0} f(x) < +\infty$ if X is compact.
10a. $\sup_{x \epsilon X} f(x) = \sup_{x \epsilon E} f(x)$ if X is compact.	10b. $\sup_{x \epsilon X} f(x) = \sup_{x \epsilon E} f(x)$ if X is compact.
11a. Every local minimum is a global minimum.	11b. Every local minimum is a global minimum or f is constant in a neighborhood of the local minimum.
12a. The set of global minima is convex.	12b. The set of global minima is convex.
13a. The Kuhn-Tucker constraint qualification holds for $\{x\colon f_i(x) \leqq 0$ for $i = 1, 2, \cdots, m\}$ if there exists an $x\colon f_i(x) < 0$ for all $i = 1, 2, \cdots, m$.	13b. The Kuhn-Tucker constraint qualification holds for $\{x\colon f_i(x) \leqq 0$ for $i = 1, 2, \cdots, m\}$ if $\nabla f_i(x) \neq 0$ for i such that $f_i(x) = 0, i = 1, 2, \cdots, m$.
14a. Let X, Y be compact subsets of E^n and E^m, respectively, and let $f\colon X \times Y \rightarrow E^1$, such that for each $y \epsilon Y$, $f(x, \cdot)$ is concave and for each $x \epsilon X$, $f(\cdot, y)$ is convex. Further, let $f(x, y)$ be continuous. Then, $f(x, y)$ has a saddle point, say $(x^*, y^*) \epsilon X \times Y$.	14b. Let X, Y be compact subsets of E^n and E^m, respectively, and let $f\colon X \times Y \rightarrow E^1$, such that for each $y \epsilon Y$, $f(x, \cdot)$ is quasi-concave and upper semi-continuous, and for each $x \epsilon X$, $f(\cdot, y)$ is quasi-convex and lower semi-continuous. Then, there exists a saddle point of $f(x.y)$, say $(x^*, y^*) \epsilon X \times Y$.

TABLE II—*Continued*

Convex	Quasi-convex
15a. $f(\lambda x) \leqq \lambda f(x)$ for $\lambda \epsilon [0, 1]$ if $f(0) \leqq 0$.	15b. $f(\lambda x) \leqq f(x)$ for $\lambda \epsilon [0, 1]$ if $f(x) \geqq f(0)$.
16a. $g(\lambda) = f(\lambda x)/\lambda$ is monotone increasing for $\lambda > 0$ if $f(0) = 0$, $\lambda \epsilon E^1$.	16b. $g(\lambda) = f(\lambda x)$ is monotone increasing for $\lambda \geqq 0$ if $f(x) \geqq f(0)$, $\lambda \epsilon E^1$.
17a. $g(\alpha) = f[\alpha x + (1-\alpha) y]$ is convex, over $\alpha \epsilon [0, 1]$ for any $x, y \epsilon X$ if and only if f is convex.	17b. $g(\alpha) = f[\alpha x + (1-\alpha) y]$ is quasi-convex over $\alpha \epsilon [0, 1]$ for any $x, y \epsilon X$ if and only if f is quasi-convex.
18a. $g(x) = \sup_{\gamma \epsilon \Gamma} f_\gamma(x)$ is convex, where Γ is any index set.	18b. $g(x) = \sup_{\gamma \epsilon \Gamma} f_\gamma(x)$ is quasi-convex, where Γ is any index set.
19a. $g(x) = F[f(x)]$ is convex if F is convex and nondecreasing.	19b. $g(x) = F[f(x)]$ is quasi-convex if F is nondecreasing.

quasi-concave. The corresponding statement for the quasi-convex function is given in 7b.

Property 8a is well known[36] (and implies property 7a). For a univariate function it is equivalent to saying that the first derivative is monotone nondecreasing. Property 8b was discovered more recently[1] and its converse is used to define the class of 'pseudo-convex' functions.[26]

Let us look at Fenchel's[12] proof of 9a. For any $x \epsilon X$, we may write $x = \sum_{j=1}^{n+1} \alpha_j x^j$, where $x^j \epsilon E$ and $\sum_{j=1}^{n+1} \alpha_j = 1$, $\alpha \geqq 0$. Further $f(x) \leqq \sum_{j=1}^{n+1} \alpha_j f(x^j)$. Therefore, we have our result. Now we observe that we only need the weaker inequality given by $f(x) \leqq \max_j f(x^j)$. Therefore, 9b follows directly.

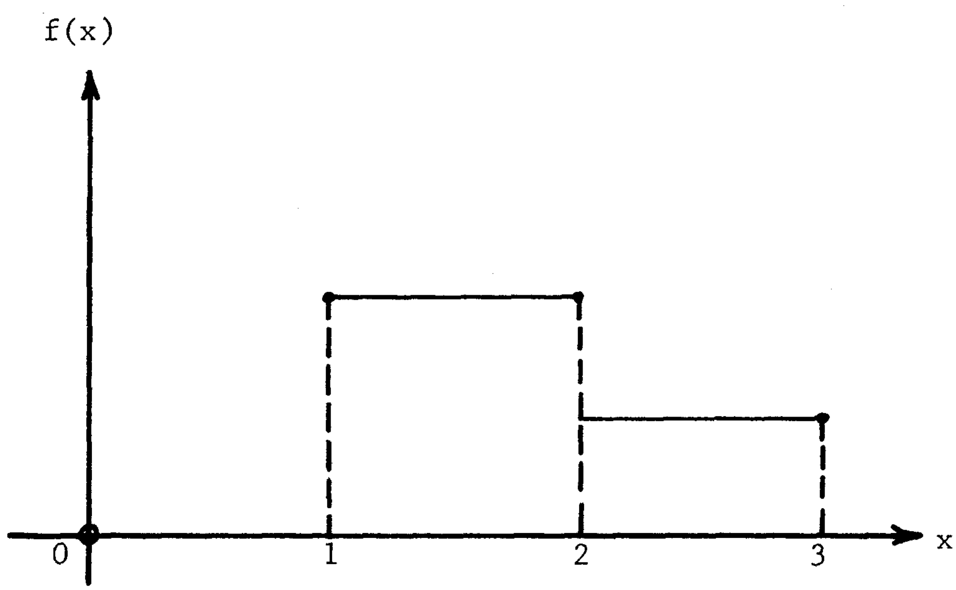

Fig. 4. Illustration of 3b, where $f(x)$ is not continuous.

The same argument proves 10(a,b). JERISON[19] first proved 10a in slightly different form, and Martos[29] extended the convexity result to quasi-convexity for polyhedral sets. The further extension to convex sets is immediate. Clearly, when E is finite (as in polyhedral sets), we may use 'max' instead of 'sup.' Moreover, we may use the Krein-Milman[5] theorem to prove the corresponding result for any compact subset (need not be convex) of X.

Property 11a says that f cannot have any 'flat regions' that are not global minima. This fails to be exactly true for quasi-convex functions due to the allowance of nonconvex epigraphs. PONSTEIN[35] observed the corresponding statement in 11b. We shall return to this point in section 4. Properties 12a and 12b follow from the convexity of level sets as follows. The optimality region is given by:

$$\{x^* \epsilon X:\ f(x^*) \leqq f(x) \quad \text{for all} \quad x \epsilon X\} = (\wedge_{x\epsilon X} \{x^*:\ f(x^*) \leqq f(x)\}) \wedge X.$$

Since the intersection of convex sets is a convex set, the desired properties follow.

Slater[38] proved 13a, and 13b follows from a theorem by ARROW, HURWICZ, AND UZAWA.[2] It should be observed that in 13a the existence of a strict interior solution implies $\nabla f_i(x) \neq 0$ for any x such that $f_i(x)=0$ (assuming differentiability). This follows from the fact that $\nabla f_i(x)=0$ implies x minimizes (globally) f_i. Therefore, if $f_i(x)=0$, there would not exist a strict interior. The stronger assumptions in 13b are needed, and Arrow et al.[2] proved that, when the feasibility region is convex, then their constraint qualification fails only if $\nabla f_i=0$ for some i such that $f_i(x)=0$. Their proof shows that the same statement applies to the Kuhn-Tucker constraint qualification because the cone of differentials they discuss is then convex. Further discussion with examples is given in Arrow and Enthoven.[1]

Property 14a is the classical saddle-point existence theorem by von Neumann. The generalization in 14b is due to Sion and is proven and further discussed by Berge.[5]

Property 15a is sometimes called a contraction property. It follows directly as $f(\lambda x)=f[\lambda x+(1-\lambda)0] \leqq \lambda f(x)+(1-\lambda)f(0) \leqq \lambda f(x)$. The analogue emphasizes that the inequality for quasi-convex functions is a weaker upper bound, since $f(x) > \lambda f(x)$ for $\lambda < 1$ and $f(x) > 0$. Thus, all we can say is that $f(\lambda x) \leqq f(x)$, but contraction does not hold in that $f(\lambda x)$ may be equal to $f(x)$ for $\lambda \epsilon (0, 1)$. The proof of 15b is equally direct.

A similar relation holds in 16(a,b). A proof of 16a appears in Beckenbach and Bellman,[3] and 16b may be proven as an extension.

Property 17a has been known for some time, but we are unable to determine priority; property 17b is proven in Arrow and Enthoven.[1]

Properties 18a and 18b appear in Fenchel.[12] They follow from the defining inequality of quasi-convex functions and the 'triangle inequality' given by:

$$\sup_x \{f(x)+g(x)\} \leqq \sup_x \{f(x)\} + \sup_x \{g(x)\}.$$

Properties 19a and 19b also appear in Fenchel,[12] and it is important to note that 19b does not require F to be convex.

In summary, the properties of Table II are founded upon the inequality definitions or the equivalent set definitions. By analyzing a proof that assumes convexity, it is possible in the cases cited to obtain a direct extension or an analogue for quasi-convexity.

MARCUS[28] considered certain 'matrix functions' and established consequences of convexity. We shall briefly examine two of his results and provide analogous theorems for quasi-convex matrix functions.

THEOREM (Marcus[28], p. 323). *Let H_n be a family of n-square complex Hermitian matrices that is convex in the n^2 unitary space. Let $f(t_1, \cdots, t_k)$ be concave on the k-cube generated by the k-fold Cartesian product of I with itself, where I is an open interval containing the field of every $A \epsilon H_n$. Define ψ_f on H_n by*

$$\psi_f(A) = \min\{f(x_1^T A x_1, \cdots, x_k^T A x_k) | x_i^T x_j = \delta_{ij} \quad \text{for} \quad i,j=1, \cdots, n\}.$$

Then, ψ_f is concave on H_n.

Upon examination of Marcus's proof, we can easily establish the analogous result: if f is quasi-concave on the k-cube, then ψ_f is quasi-concave on H_n.

A second result due to Marcus is the following:

THEOREM (Marcus[28], pp. 321–2). *Consider $\{A_j\}_1^k \subset H_n$ and define the eigenvalues of A_j as the n-vector $\lambda^{(j)}$. Let I_j be the interval given by $I_j = [\min_t \lambda_t^{(j)}, \max_t \lambda_t^{(j)}]$ and define $R = I_1 X \cdots X I_k$. Suppose f is convex on R, and let σ be a 1-1 function on the integers 1, $\cdots$, k to the integers 1, $\cdots$, n. Then,*

$$\max\{f(x_1^T A_1 x_1, \cdots, x_k^T A_k x_k) | x_i^T x_j = \delta_{ij} \quad \text{for} \quad i,j=1, \cdots, n\} \\ = \max_\sigma f(\lambda^{(1)}_{\sigma(1)}, \cdots, \lambda^{(k)}_{\sigma(k)}).$$

This theorem can be generalized to allow f to be quasi-convex on R.

3. PROPERTIES WITH NO ANALOGUES

BY CONTRAST to section 2, Table III shows some properties that do not extend, and for which no analogue is apparent.

The additive property (1) is an important property that does not extend to general quasi-convex functions. To the authors' knowledge, no nontrivial subclass of quasi-convex functions containing nonconvex functions has been identified as having the additive property.

The conjugate statement in (2) is well known and appears in Fenchel[12] and BRØNDSTED.[6] ROCKAFELLAR[36,37] provides further discussion. The second conjugate of $[X,f]$ is the closed convex hull of $[X, f]$ that equals $[X, f]$ if and only if $[X, f]$ is a closed convex set (hence, f is a closed convex function). One may note that the minimum of f over X, if it exists, is preserved in that it equals the minimum of f'' over X''.

The boundedness property in (3) is proven by Fenchel.[12] To illustrate where this does not hold, consider $f(x)=1/(x-1)$ for $x\epsilon[0, 1)$ and $f(1)=0$. Note that $f(x)$ is quasi-convex, but $\inf_{x\epsilon X^0} f(x)=-\infty$.

The transposition theorem (4) was proven two ways by FAN, GLICKSBERG, AND HOFFMAN.[11] Neither of their proofs generalizes, and no analogue is apparent.

TABLE III

PROPERTIES OF CONVEX FUNCTIONS WTIH NO ANALOGUE FOR QUASI-CONVEX FUNCTIONS

1. $\alpha f(x)+\beta g(x)$	is convex if $\alpha, \beta\geqq 0$
2. $[X, f]''=[X, f]$	if $[X, f]$ is closed
3. $\operatorname{Inf}_{x\epsilon X^0} f(x)>-\infty$	if X is bounded
4. Let $F(x)=[f_1(x), \cdots, f_m(x)]$, where $f_j(x)$ has domain X for each $j=1, \cdots, m$. Then, there exists $x\epsilon X$ such that $F(x)\leqq 0$ if and only if there exists no $\lambda\epsilon E_+^m$ and $\lambda\neq 0$, such that $\lambda F(x)>0$ for all $x\epsilon X$.	

In the spirit of Table III there are results that hold for quasi-convex functions, and the prefix 'quasi' may not be removed. One of these (related to property 3 of Table III) was observed by Martos:[33] *If, for each compact convex subset of $\bar{X}$, the global maximum of f is achieved and occurs at one of its extreme points, then f is quasi-convex.*

4. EXPLICITLY AND STRICTLY QUASI-CONVEX FUNCTIONS

THE LITERATURE ON quasi-convex functions contains two definitions: (1) explicitly quasi-convex and (2) strictly quasi-convex functions. The following paragraphs state these definitions and specify when they coincide and when they do not coincide. The discussion is then followed by some results illustrating the strong properties of explicitly quasi-convex functions.

DEFINITION 5. A function f with convex domain X is *explicitly quasi-convex*[33] if it is quasi-convex and if $f(x_1)\neq f(x_2)$, implies $f[\lambda x_1+(1-\lambda)x_2]<\max[f(x_1), f(x_2)]$ for all $\lambda\epsilon(0, 1)$.

DEFINITION 6. A function f with convex domain X is *strictly quasi-convex*[20] if $f(x_1)\neq f(x_2)$ implies $f[\lambda x_1+(1-\lambda)x_2]<\max[f(x_1), f(x_2)]$ for all $\lambda\epsilon(0, 1)$.

Note that a strictly quasi-convex function need not be quasi-convex. KARAMARDIAN[20] provided the following example: $f(x)=1$ if $x=0$, 0 if $x\neq 0$. The level set $\{x:f(x)\leqq 0\}$ is not convex, but f is strictly quasi-convex. Karamardian then proceeded to prove that a strictly quasi-convex function is quasi-convex if it is lower semicontinuous over S. Thus for functions that are lower semicontinuous on X the two definitions, explicitly quasi-convex and strictly quasi-convex, coincide.

We observe that every convex function is explicitly quasi-convex. To see this, observe that for any x_1, $x_2 \epsilon X$ such that $f(x_1)<f(x_2)$ and for any $\lambda\epsilon(0, 1)$ we have $f[\lambda x_1+(1-\lambda)x_2]\leqq \lambda f(x_1)+(1-\lambda)f(x_2)<f(x_2)$. Now let us consider what properties may be strengthened.

Property 11b in Table II may then be extended to the following statement: *If $f(x)$ is explicitly quasi-convex, then every local minimum is a global minimum.* We caution the reader not to draw the inference that every stationary point is a minimum. For example, $f(x)=x^3$ is explicitly quasi-convex but has an inflection point at the origin.

Second, the 'quasi-contraction' property in 15b (Table II) may be strengthened to: *$f(\lambda x)<f(x)$ for all $\lambda\epsilon(0, 1)$ if $f(x)>f(0)$.* This is still not as strong as the contraction property in 15a but it is stronger than 15b.

Let us consider property 18b. If $f_\gamma(x)$ is explicitly quasi-convex over X for each $\gamma\epsilon\Gamma$, then $g(x)=\sup_{\gamma\epsilon\Gamma} f_\gamma(x)$ need not be explicitly quasi-convex. For example, consider the following:

$$f_\gamma(x)=\begin{cases} x^3+\gamma(x-10) & \text{if } -1\leqq x\leqq 0,\\ \gamma(x-10) & \text{if } 0\leqq x\leqq 1,\\ (x-1)^3+\gamma(x-10) & \text{if } 1\leqq x\leqq 2,\end{cases}$$

$$X=\{x:-1\leqq x\leqq 2\},\quad \Gamma=\{\gamma:0<\gamma\}.$$

Observe that

$$g(x)=\sup_{\gamma>0} f(x)=\begin{cases} x^3 & \text{if } -1\leqq x\leqq 0,\\ 0 & \text{if } 0\leqq x\leqq 1,\\ (x-1)^3 & \text{if } 1\leqq x\leqq 2,\end{cases}$$

which is not explicitly quasi-convex. To see this, let $x_1=-1$, $x_2=1$, and $\lambda=\frac{1}{2}$. Then $g(-1)=-1<g(1)=0$, but $g(0)=0\nless\max[g(-1), g(1)]=0$.

It is obvious, however, that, if there exists $\gamma^*\epsilon\Gamma$ such that $g(x)=f_{\gamma^*}(x)$ for all $x\epsilon X$, then $g(x)$ is explicitly quasi-convex because $f_{\gamma^*}(x)$ is assumed to be so. (Note that γ^* is independent of x.)

More generally we can establish the following:

THEOREM. *Let $\{f_\gamma(x), \gamma\epsilon\Gamma\}$ be a collection of explicitly quasi-convex functions with common domain X, where Γ is an index set, and define $g(x)=\sup_\gamma \{f_\gamma(x):\gamma\epsilon\Gamma\}$ for each $x\epsilon X$. Then if, for each $x\epsilon X$, there exists at least one $\gamma\epsilon\Gamma$, say $\gamma^*(x)$, such that $g(x)=f_{\gamma^*(x)}(x)$, then $g(x)$ is explicitly quasi-convex on X.*

Proof. By 18b, $g(x)$ is quasi-convex, so it remains to establish the strict inequality given by $g[\alpha x_1+(1-\alpha)x_2]<\max[g(x_1), g(x_2)]$ for all $\alpha\epsilon(0, 1)$ if $g(x_1)\neq g(x_2)$.

Suppose, to the contrary, that $g(x_1)>g(x_2)$ and for some $x=\alpha x_1+(1-\alpha)x_2$, we have $g(x)=g(x_1)$.

By our hypothesis, there exists $\gamma^*(x)$, $\gamma^*(x_1)$ and $\gamma^*(x_2)\epsilon\Gamma$ such that $g(x)=f_{\gamma^*(x)}(x)$, $g(x_1)=f_{\gamma^*(x_1)}(x_1)$ and $g(x_2)=f_{\gamma^*(x_2)}(x_2)$. For notational convenience, let $\gamma^*(x)=\gamma^*$, $\gamma^*(x_1)=\gamma_1{}^*$ and $\gamma^*(x_2)=\gamma_2{}^*$. Then, we have, by our supposition, $f_{\gamma^*}(x)=g(x)=g(x_1)=f_{\gamma_1^*}(x_1)$. Since $f_{\gamma^*}(x)$ is explicitly quasi-convex, we have $f_{\gamma^*}(x)\leqq\max[f_{\gamma^*}(x_1), f_{\gamma^*}(x_2)]$. If $f_{\gamma^*}(x_1)\neq f_{\gamma^*}(x_2)$, then this inequality is strict. Further, by definition, we have $f_{\gamma^*}(x_1)\leqq f_{\gamma_1^*}(x_1)$ and $f_{\gamma^*}(x_2)\leqq f_{\gamma_2^*}(x_2)$. Therefore, if $f_{\gamma^*}(x_1)\neq f_{\gamma^*}(x_2)$, we have

$$g(x)=f_{\gamma^*}(x)<\max[f_{\gamma^*}(x_1), f_{\gamma^*}(x_2)] \leqq\max[f_{\gamma_1^*}(x_1), f_{\gamma_2^*}(x_2)]=\max[g(x_1), g(x_2)],$$

so that $g(x)<g(x_1)$, which is a contradiction. On the other hand, if $f_{\gamma^*}(x_1)=f_{\gamma^*}(x_2)$, then $g(x)\leqq f_{\gamma_2^*}(x_2)=g(x_2)<g(x_1)$, and we again have a contradiction. Hence, the proof is complete. Martos[33] established another interesting property of importance in nonlinear programming. A fundamental question is, 'What class of functions has the properties that it achieves its minimum at a vertex and that every local minimum is a global minimum?' The importance of this rests with the application of adjacent-vertex methods. In the previous section we quoted a theorem of Martos with regard to the vertex property. Martos[33] also established the following proposition:

Let X be a convex set with at least two points, and let f be lower semi-continuous on S. Suppose P is the class of compact, convex subsets of X containing at least 2 points. If, for each $S\epsilon P$, every local minimum of f on S is a global minimum of f on S, then f is explicitly quasi-convex.

A property of explicitly quasi-convex functions that need not hold for other quasi-convex functions occurs in E^1; namely, if there exists a global minimum x^*, then let I be an interval where $f(x)=f(x^*)$ for $x\epsilon I$. An explicitly quasi-convex function has the property that, for x to the left of I, $f(x)$ *strictly* decreases and, for x to the right of I, $f(x)$ *strictly* increases. This property allows certain sequential search schemes, such as Fibonacci search, to reduce successively a given interval $[a,b]$ to a subinterval containing an optimum point in I.

Finally, let us consider a property of explicitly quasi-convex functions that need not hold for quasi-convex functions in general. Namely, Evans and Gould[10] showed that, if $F(x)=[f_1(x), \cdots, f_m(x)]$ is a vector of m explicitly quasi-convex continuous functions with domain E^n, and if the strict interior of the level set given by $I=\{x:F(x)<0\}$ is not empty, then its closure has the property $\bar{I}=\{x:F(x)\leqq 0\}$. This need not hold for a quasi-convex function in general as illustrated by the function[10] $f(x)=x^3$ if $x\leqq 0$, 0 if $0\leqq x\leqq 1$, $(x-1)^3$ if $x\geqq 1$.

The importance of this property was developed by Evans and Gould,[10] where they established sufficient conditions for the continuity of the function given by $f^*(b)=\sup_x \{f_0(x)|F(x)\leqq b,\ x\epsilon E^n\}$ for $b\epsilon B=\{b:F(x)\leqq b$ for some $x\epsilon E^n\}$.

This has to do with stability in 'value space,' and GREENBERG AND PIERSKALLA[16] extended some of their results to perturbations of the functions $f_i(x)$ (rather than only allowing perturbations of b). A conclusion is that, if $F(x)$ is explicitly quasi-concave and lower semicontinuous and if $f_0(x)$ is lower semicontinuous, then $f^*(b)$ is lower semicontinuous at each $b\epsilon B$. (The upper semicontinuity follows from the compactness of the feasibility region.[10, 16])

In summary, the stronger inequality defining explicitly quasi-convex functions provides the needed stability properties not generally held by other quasi-convex functions. It still admits a larger class of functions than convex, and allows discontinuities.

5. CONCLUSIONS

IT WOULD BE an unnecessarily monumental task to gather each and every result for convex functions to see if it could be extended to quasi-convex functions. What we have done in this review is present some of the principal developments in the categories given by:

1. Relation to convex sets.
2. Continuity, boundedness, and differentiability.
3. Extreme values.
4. Inequalities.
5. Transforms.

We have drawn analogues in Table II to indicate how the weaker inequality of quasi-convexity may be sufficient to obtain a property possessed in related form by convex functions. Table III is included to illustrate where the stronger defining inequality is needed. Martos observed one case where the weaker inequality is needed to draw the given conclusion.

Berge[5] describes an interesting key to the relation of convex and quasi-convex functions. Let Φ be a class of functions satisfying

$$\left.\begin{matrix}\phi_1(x_1)<\phi_2(x_1)\\ \phi_1(x_2)<\phi_2(x_2)\end{matrix}\right\}\Rightarrow \phi_1(x)<\phi_2(x) \text{ for all } x=\alpha x_1+(1-\alpha)x_2,\ \alpha\epsilon[0,1]. \quad (1)$$

$$\left.\begin{matrix}\phi_1(x_1)=\phi_2(x_1)\\ \phi_1(x_2)=\phi_2(x_2)\end{matrix}\right\}\Rightarrow \phi_1(x)=\phi_2(x) \text{ for all } x=\alpha x_1+(1-\alpha)x_2,\ \alpha\epsilon[0,1]. \quad (2)$$

Then, f is a sub$-\Phi$ function (with respect to $\phi\epsilon\Phi$) if

$$\left.\begin{matrix} f(x_1)\leqq\phi(x_1) \\ f(x_2)\leqq\phi(x_2)\end{matrix}\right\}\Rightarrow f(x)\leqq\phi(x) \quad \text{for} \quad x=\alpha x_1+(1-\alpha)x_2,\ \alpha\epsilon[0,\ 1].$$

We see that linear affine functions are in Φ, i.e., $\phi(x)=ax+b$. Then, if $f(x_1)=\phi(x_1)$ and $f(x_2)=\phi(x_2)$, we require (for f to be sub$-\Phi$)

$$f[\alpha x_1+(1-\alpha)x_2]\leqq\phi[\alpha x_1+(1-\alpha)x_2]= \alpha\phi(x_1)+(1-\alpha)\phi(x_2)=\alpha f(x_1)+(1-\alpha)f(x_2).$$

Thus, the linear affine functions generate convex functions.

Further, constant functions are in Φ, i.e., $\phi(x)=b$. Then, if $\phi(x_1)=\phi(x_2)=b$ and $b=\max\{f(x_1),\ f(x_2)\}$ we require $f[\alpha x_1+(1-\alpha)x_2]\leqq\phi[\alpha x_1+(1-\alpha)x_2]=b=\max[f(x_1), f(x_2)]$ in order that f be sub$-\Phi$. Thus, the constant functions generate quasi-convex functions.

One may carry this further to generate further classes of functions with certain properties. For example, consider property 15 in Table II. Suppose ϕ is positively homogeneous of degree $p(\geqq 0)$, i.e., $\phi(\lambda x)=\lambda^p\phi(x)$ for all $\lambda\geqq 0$. Then, we generate functions of the form $f(\lambda x)\leqq\lambda^p f(x)$. For $p=0$, we obtain a result related to property 15b, and for $p=1$, we obtain a result related to 15a.

Also, if $X=E_+{}^n$, then Berge[5] reports a result of Bourbaki that states: If $f(x)>0$ for all $x\neq 0$ (and $x\epsilon E_+{}^n$), and if $f(\lambda x)=\lambda f(x)$ for all $\lambda\geqq 0$, then f is convex if and only if f is quasi-convex. Thus, positively homogeneous (of degree 1) positive functions (for $x\neq 0$), which are quasi-convex, are necessarily convex and hence possess all the properties of convex functions.

In reviewing this paper, Bela Martos kindly pointed out another possibility that yields a continuous-parameter transition from convex to quasi-convex functions. Furthermore, he proved the following theorems.

Let f be a numerical function defined on a convex set S in E^n; then f is a *w-convex* function $(w>0)$ on X if, for any $x_1,\ x_2\epsilon X$ and any $\alpha\epsilon[0,\ 1]$, it follows that

$$f[\alpha x_1+(1-\alpha)x_2]\leqq(1/w)\ln[\alpha e^{wf(x_1)}+(1-\alpha)e^{wf(x_2)}].$$

Let C^w be the class of w-convex functions on X and let C^0 be the class of functions given by: $f\epsilon C^0$ if and only if

$$f[\alpha x_1+(1-\alpha)x_2]\leqq\lim_{w\to 0}\,(1/w)\ln[\alpha e^{wf(x_1)}+(1-\alpha)e^{wf(x_2)}].$$

Similarly, let C^∞ be the class of functions satisfying

$$f[\alpha x_1+(1-\alpha)x_2]\leqq\lim_{w\to\infty}\,(1/w)\ln[\alpha e^{wf(x_1)}+(1-\alpha)e^{wf(x_2)}].$$

THEOREM. *C^0 is the class of convex functions on X and C^∞ is the class of quasi-convex functions on X.*

Further, let f and g be defined on X and consider

$$h(x) = (1/w)\ln[\beta e^{wf(x)} + \gamma e^{wg(x)}],$$

where $\gamma, \beta \geqq 0$. Call h a nonnegative w-combination of f and g. Property 1 of Table III ($w=0$) is analogous to property 18b in Table II ($w=\infty$) by noting the following:

THEOREM. *Nonnegative w-combinations of w-convex functions are w-convex.*

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