QUASI-CONJUGATE FUNCTIONS AND SURROGATE DUALITY

Harvey J. GREENBERG[1] and William P. PIERSKALLA[2]

1. *Introduction*

The results in conjugate function theory [1, 3, 10], particularly the work of Rockafellar, have had a profound effect on our understanding the structure of many optimization models as well as other aspects in functional analysis. The basic results stem from the convexity structure involved in the extremum problems. (We assume familiarity with [10], and we use his notation.)

This paper develops an analogous, though less ambitious, theory of quasi-conjugate functions based upon quasi-convexity structure. Whereas conjugates relate to epigraph supports, quasi-conjugates relate to level set supports; where conjugates provide a basis for Lagrangian duality, quasi-conjugates provide a basis for surrogate duality.

In the next section we begin with some fundamental definitions and elementary properties of quasi-conjugates. Section three introduces quasi-subdifferential theory, and section 4 applies our results to describe surrogate duality as developed by the authors [7].

2. *Definitions and fundamental properties*

Throughout our discussion, we consider a function $f : R^N \to R_\infty \equiv [-\infty,\infty]$. The epigraph and c-level set of f are given respectively by

$$\text{epi } f = \{(z, x) \in R^{n+1} \mid z \geq f(x)\}$$
$$L_c f = \{x \in R^n \mid c \geq f(x)\} .$$

[1] Computer Science/Operations Research Center, Southern Methodist University.
[2] IE/MS Department, Northwestern University.

Definition : The *z-quasi-conjugate* of f is a function $f_z^* : R^n \to R_\infty$, where

$$f_z^*(y) \equiv z - \text{Inf}\,\{f(x) \mid xy \geq z\} \ .$$

Note : The infimum of a function on the empty set $\emptyset$ is defined to be $+\infty$.

Although geometric properties of the z-quasi-conjugate function are not a readily apparent as those of the conjugate function, these two functions have strong ties. This is not surprising since (as we shall demonstrate) the z-quasi-conjugate function relates to quasi-convexity in much the same manner as the conjugate function relates to convexity. Of course, the theorems do not assume any special structure unless so stated.

Letting f^v denote the (convex) conjugate of f, the following theorem lists some elementary properties.

Theorem 1 : The following statements are true :

(i) $f^v(y) \geq f_z^*(y)$ for all $(z, y) \in R^{n+1}$.

(ii) $f^v(y) = \text{Sup}\,\{f_z^*(y) \mid z \in R^1\}$.

(iii) $f \leq g$ implies $f_z^* \geq g_z^*$ for all $z \in R^1$.

(iv) $(\lambda f)_z^*(y) = \lambda\, f_{z/\lambda}^*\,(y/\lambda)$ for all $\lambda \in R^1 : \lambda > 0$.

(v) f_z^* is quasi-convex for each $z \in R^1$; that is, $y \in [y_1, y_2]$ implies $f_z^*(y) \leq \text{Sup}\,\{f_z^*(y_1), f_z^*(y_2)\}$ (equivalently, $L_c f_z^*$ is convex for all c).

(vi) $\{y \mid f_z^*(y) = +\infty\}$ is convex, but the effective domain of f_z^* need not be convex.

Thus, we see that the z-quasi-convex function provides a lower bound for the conjugate function, and indeed the conjugate is the supremum over z of the z-quasi-conjugate (ref., properties (i) and (ii)). All of the above elementary properties will prove useful in our study. Although we omit a complete proof of Theorem 1, we shall substantiate property (v) since it describes a strong tie to quasi-convexity structure to which we alluded earlier.

If $wy \geq z$ and $y \in [y_1, y_2]$, then $wy_1 \geq z$ or $wy_2 \geq z$. Therefore, $wy \geq z \Rightarrow f(w) \geq \text{Inf}\ \{z - f_z^*(y_1),\ z - f_z^*(y_2)\}$ for all w. This implies

$$z - \text{Inf}\ \{f(w) \mid wy \geq z\} \quad \text{Sup}\ \{f_z^*(y_1),\ f_z^*(y_2)\}$$

But the left hand side is $f_z^*(y)$, so property (v) follows.

Now let us consider the second z-quasi-conjugate of f and a normalization having an important impact on our analysis. We have (by definition)

$$(f_z^*)_z^*(x) = z - \text{Inf}\ \{f_z^*(y) \mid xy \geq z\}\ ,$$

and it is easy to show that this can be written as

$$(f_z^*)_z^*(x) = \underset{y}{\text{Sup}}\ \{\underset{w}{\text{Inf}}\ \{f(x) \mid wy \geq z\} \mid\ xy \geq z\}\ .$$

Define the *normalized* second quasi-conjugate of f as

$$f^{**}(x) = \underset{z}{\text{Sup}}\ (f_z^*)_z^*(x).$$

Theorem 2 : The following statements are true :

(i) $f^{**}(x) = \underset{y}{\text{Sup}}\ \text{Inf}\ \{f(w) \mid wy \geq xy\}$.

(ii) f^{**} is quasi-convex.

(iii) $f(x) \geq f^{**}(x) \geq f^{\vee\vee}(x)$.

Note that property (iii) means that f^{**} provides a better quasi-convex approximation of f from below than does $f^{\vee\vee}$, the second (convex) conjugate. Of course, if $f(x) = f^{\vee\vee}(x)$ (e.g., f closed and convex), then $f(x) = f^{**}(x)$ as well. Later, we shall establish weaker conditions for which this latter equality holds.

Example : Let $f(x) = -e^{-x^2}$. Then, $f^{**}(x) = f(x)$ and $f^{\vee\vee}(x) = -1$ for all $x \in R^1$.

A convenient property of conjugates is that odd order conjugates equal the first conjugates, and even order conjugates equal the second.

We now prove a similar property holds for quasi-conjugates.

Theorem 3 : $(f^{**})^*_z(y) = f^*_z(y)$ and $(f^{**})^{**}(x) = f^{**}(x)$ for all x, y, z.

Proof : By property (iii) in Theorem 1 and (iii) in Theorem 2, it suffices to prove $(f^{**})^*_z(y) \leq f^*_z(y)$ in order to establish the first assertion. This can be verified by direct substitution as follows :

$$\begin{aligned}(f^{**})^*_z(y) &= z - \operatorname*{Inf}_x \{\operatorname*{Sup}_v \operatorname*{Inf}_w \{f(w)|wv \geq xv\}|xy \geq z\} \\ &\leq z - \operatorname*{Inf}_x \{\operatorname*{Inf}_w \{f(w)|wv \geq xy\}|xy = z\} \\ &= z - \operatorname*{Inf}_w \{f(w)|wy \geq z\} \\ &= f^*_z(y) .\end{aligned}$$

The second part follows from the first and by the definitions. Q.E.D

Unlike the second conjugate, f^{vv}, the normalized second quasi-conjugate, f^{**}, need not close f. For example, let f be the univariate function given by

$$f(x) = \begin{cases} 0 \text{ if } x < 0 \\ \alpha \text{ if } x = 0 \\ 1 \text{ if } x > 0 \end{cases}$$

By choice of α we can make f upper or lower semi-continuous or neithe For $\alpha \in [0,1]$ we have $f^{**}(x) = f(x)$ and $f^{vv}(x) = 0$. This also shows that $L_c f^{**}$ need not be closed. The next theorem below demonstrates conditions under which $L_c f^{**}$ is closed.

Theorem 4 : $L_c f^{**}$ is closed if f is upper semi-continuous and $L_{c+\delta} f$ is bounded for some $\delta > 0$.

Proof : Let $\{x^k\} \subset L_c f^{**}$ and $x^k \to x$. For any $y \in R^n$ we have (cf., property (i) of Theorem 2) :

$$\operatorname{Inf} \{f(w)|wy \geq x^k y\} \leq c.$$

For any $\theta > 0$ there exists w_ε^k such that $f(w_\varepsilon^k) \leq c + \theta$ and $w_\varepsilon^k y \geq x^k y$.

Choose $\theta < \delta$, so there exists a limit point of $\{w_\varepsilon^k\}$, say w_ε, such that $w_\varepsilon y \geq xy$ and because f is upper semi-continuous, $f(w_\varepsilon) \leq c + \theta$. Therefore,

$$\text{Inf } \{f(w) | wy \geq xy\} \leq c + \theta$$

for any $\theta \in (0,\delta]$ and hence $f^{**}(x) \leq c$, so $L_c f^{**}$ is closed. Q.E.D.

One of the important results in conjugate function theory is that epi f^{vv} = cl conv epi f. We have the following analogy.

Theorem 5 : Assume $L_c f$ is compact for all c. Then, $L_c f^{**}$ = conv $L_c f$ for all c.

Proof : In general $L_c f^{**} \supset L_c f$, and since f^{**} is quasi-convex, $L_c f^{**} \supset$ conv $L_c f$.

Now suppose $x \notin$ conv $L_c f$. Let y be given such that $w \in$ conv $L_c f$ implies $wy < xy$, and let $w^* \in$ argmin $\{f(w) | wy \geq xy\}$, existence guaranteed by the compactness assumption. Clearly, $w^* y \geq xy$ so $w^* \notin$ conv $L_c f$.

Further, $f^{**}(x) \geq f(w^*) > c$, so $x \notin L_c f^{**}$. Q.E.D.

Corollary 5.1. : If f is quasi-convex and $L_c f$ is compact for all c, then $f^{**} = f$. Later we shall alter the hypothesis in corollary 5.1 and still deduce $f = f^{**}$, which we apply to surrogate duality theory.

3. *Quasi-subdifferential theory.*

Analogous to the subgradient, which is a slope of an epigraph support, we introduce the quasi-subgradient, which has an analogous property of level set support.

Definition : A quasi-subgradient of f at x is a vector $y \in R^n$ such that

$$f(x) + f_{xy}^*(y) = xy.$$

The set of subgradients of f at x is the *quasi-subdifferential* of f at x and is denoted by $\partial^* f(x)$. We say f is *quasi-subdifferentiable* at x if $\partial^* f(x) \neq \emptyset$.

Theorem 6 : The following statements are true :

(i) $\partial^* f(x) = \{y | wy \geq xy \text{ implies } f(w) \geq f(x)\}$.

(ii) $\lambda\, \partial^* f(x) = \partial^*(\lambda f)(x) = \partial^* f(x)$ for all $\lambda \in R^1 : \lambda > 0$.

(iii) $\partial\, f(x) \subset \partial^* f(x)$, with proper inclusion possible.

(iv) $\partial^* f(x)$ is convex.

(v) $0 \in \partial^* f(x) \Longleftrightarrow x \in \text{argmin } f$.

The proof of Theorem 5 is straightforward, and we omit it. Propert (i) is a simple restatement of the optimality condition which defines $\partial^* f(x)$. Property (ii) tells us that the quasi-subdifferential only cont rays, sans the origin; if f is quasi-subdifferentiable, then $\partial^* f(x)$ must unbounded; this simply reflects the fact that

$$\{w | wy \geq xy\} = \{w | \lambda wy \geq \lambda xy\}$$

for any $\lambda > 0$. Thus, we could scale quasi-subgradients to let only direc tions serve as representatives. One place this is used is to compactify sequence of quasi-subgradients.

Property (iii) is illustrated by Rockafellar's example ([10], p.21 namely,

$$f(x) = \begin{cases} -(1 - |x|)^{\frac{1}{2}} \text{ if } |x| \leq 1 \\ +\infty \qquad \text{else.} \end{cases}$$

In this case the subdifferential at $x = 1$, $\partial f(1) = \emptyset$, but $\partial^* f(1) = (0, \infty$

Property (iv) follows from the quasi-convexity of f_z^* for any z. Pr perty (v) can be interpreted as a minimum-preserving property. Notice th these properties apply to any function and do not assume any special stru ture. Of course, f may not be quasi-subdifferentiable anywhere; however, despite its triviality property (v) will prove useful later when establi the quasi-subdifferentiability of a function under certain conditions.

We now establish certain duality relations that parallel those in subdifferential theory. This will provide a structural basis for surroga duality skin to the role of subdifferentials in Lagrangian duality. Defir $C(x,y) \equiv \text{argmin } \{f(w) | wy \geq xy\}$. We can see that $x \in C(x,y)$ is equivalent $y \in \partial^* f(x)$. As greenberg's [4,5] analysis of Lagrangian subgradients prov insights to the Lagrangian approach, so shall we find a similar fruitful

Thus, we define $C^*(x) \equiv \{y | x \in \text{conv } C(x, y)\}$. The first part of Theorem 7 below shows that $C^*(x)$ contains the quasi-subgradients of f at x.

Additionally, define

$$\partial^{*^{-1}} f(x) \equiv \{y | x \in \partial^* f^*_{xy}(y)\} .$$

Recall that an important duality relation for conjugates of convex functions is

$$y \in \partial f(x) \iff x \in \partial f^V(y) .$$

A similar result follows from Theorem 7 below using $\partial^{*-1} f(x)$ and later theorems we shall prove that establish conditions for which $f = f^{**}$.

Theorem 7 : $\partial^* f(x) \subset C^*(x) \subset \text{argmax } \{xv - f^*_{xv}(v)\} \subset \partial^{*-1}$, with equality throughout if $f(x) = {}^V f^{**}(x)$.

Proof : If $y \in \partial^* f(x)$, then $x \in C(x, y)$ directly. Next we first recall the definition,

$$xv - f^*_{xv}(v) = \text{Inf } \{f(w) | wv \geq xv\}.$$

If $y \in C^*(x)$, then there exists $\{w^i\}_0^n \subset C(x,y)$ such that $x = \sum_{i=0}^{n} \alpha_i w^i$, where is in the standard n-simplex. By definition of w^i we have $w^i v \geq xv$ for some i (otherwise we would obtain the contradiction $xv < xv$). Therefore,

$$xv - f^*_{xv}(v) \leq f(w^i) = xy - f^*_{xy}(y) ,$$

so the second inclusion follows. Now suppose $y \in \underset{v}{\text{argmax}} \{xv - f^*_{xv}(v)\}$, so

$$\begin{aligned} xy - f^*_{xy}(y) &= \underset{v}{\text{Sup}} \{xv - f^*_{xv}(v)\} \\ &= \underset{v}{\text{Sup}} \underset{w}{\text{Inf}} \{f(w) | wv \geq xv\} \\ &= f^{**}(x) . \end{aligned}$$

By theorem 3 we have $f^{**}(x) = (f^*_{xy})^*_{xy}(x)$, so

$$f^*_{xy}(y) + (f^*_{xy})^*_{xy}(x) = xy .$$

Thus, $x \in \partial^* f^*_{xy}(y)$ as desired.
Now suppose $f(x) = f^{**}(x)$. Then we shall prove $\partial^* f(x) \supset \partial^{*-1} f(x)$.
By the last argument above we can write

$$f(x) + f^*_{xy}(y) = xy$$

If $y \in \partial^{*-1} f(x)$. But this implies $y \in \partial^* f(x)$ as desired. Q.E.D.

Define $L^o_c f = \{x | f(x) < c\}$, and let $H_x(y)$ be a hyperplane of slope passing through x. Given a set S we say, "$H_x(y)$ supports S at x" if $x \in c\ell\ S$ and $w \in S$ implies $wy \leq xy$. We now demonstrate a level set supp property of quasi-subgradients to which we alluded earlier. This is fol by an important theorem that establishes the equivalence between quasi-subdifferentiability and the equality : $f = f^{**}$. This is, in fact, our basis for surrogate duality; for we then concentrate upon quasi-subdiff rentiability with Theorem 10 and its three corollaries.

Theorem 8 : Assume $L^o_{f(x)} f \neq \emptyset$.

(i) If $c\ell\ L^o_{f(x)} f = L_{f(x)} f$ and $y \in \partial^* f(x)$, then $H_x(y)$ supports $L_{f(x)} f$ at x.

(ii) If f is lower semi-continuous at x and if $H_x(y)$ supports $L_{f(x)} f$ at x, then $y \in \partial^* f(x)$.

Proof :

(i) Since $y \in \partial^* f(x)$, $f(w) < f(x) \Rightarrow wy < xy$.

Therefore,

$L^o_{f(x)} f \subset \{w | wy < xy\}$,

so (i) follows by taking the closure.

(ii) If $f(x) < f(x)$, then by lower semi-continuity, $w \in \text{int}\ L_{f(x)} f$
Since $H_x(y)$ supports $L_{f(x)} f$ at x, it follows that $wy < xy$, so $uy \geq xy = f(u) \geq f(x)$. Hence, $y \in \partial^* f(x)$. Q.E.D.

Theorem 9 : $\partial^* f(x) \neq \emptyset \Leftrightarrow f(x) = f^{**}(x)$.

Proof :

If $y \in \partial^* f(x)$, then $f(x) = \text{Inf}\{f(w) | wy \geq xy\}$, so $f(x) = f^{**}(x)$. Conversely, let $f(x) = f^{**}(x)$. Then, $\exists \{y^k\} \ni \text{Inf}\{f(w) | wy^k \geq xy^k\} \to f(x)$. If $\|y^k\| = 0$ for all but a finite number of k, then $0 \in \partial^* f(x)$. Otherwise, let $\{y^{k_j}\} \subset \{y^k\}$, where $\|y^{k_j}\| \neq 0 \ \forall_j$. Define $v^j = y^{k_j}/\|y^{k_j}\|$ and note

$$\text{Inf}\{f(w) | wy^{k_j} \geq xy^{k_j}\} = \text{Inf}\{f(w) | wv^j \geq xv^j\}$$

for all j. Thus, $\text{Inf}\{f(w) | wv^j \geq xv^j\} \to f(x)$. Further, since $\{v^j\}$ lies in a compact set, there exists a limit v. Without loss in generality we shall suppose $v^j \to v$. Since $f(x) \geq \text{Inf}\{f(w) | wv^j \geq xv^j\}$ for all j, it follows for any $\epsilon > 0$ there exists $N(\epsilon)$ such that $j > N(\epsilon)$ implies

$$f(x) - \epsilon \leq \text{Inf}\{f(w) | wv^j \geq xv^j\} \leq f(x).$$

This implies $\text{Inf}\{f(w) | wv \geq xv\} = f(x)$, so $v \in \partial^* f(x)$. Q.E.D.

Using Theorem 9 we can now alter corollary 5.1 by establishing conditions under which $\partial^* f(x) \neq \emptyset$.

Theorem 10 : If f is quasi-convex and lower semi-continuous at x, and if $x \in \text{bd}\, L_{f(x)} f$, then $\partial^* f(x) \neq \emptyset$.

Proof :

If $L^o_{f(x)} f \neq \emptyset$, then $0 \in \partial^* f(x)$ by property (v) in Theorem 6. Suppose now $L^o_{f(x)} f \neq \emptyset$. There exists y such that $H_x(y)$ supports $L_{f(x)} f$ at x, so $f(w) < f(x)$ implies $wy < xy$. Equivalently, $wy \geq xy$ implies $f(w) \geq f(x)$, so $y \in \partial^* f(x)$. Q.E.D.

Corollary 10.1 : If f is lower semi-continuous and explicitly quasi-convex, then $\partial^* f(x) \neq \emptyset$ for all x.

Proof :

We need only consider $L^o_{f(x)} f \neq \emptyset$. In that case $c\ell\, L^o_{f(x)} f = L_{f(x)} f$ (see [2]), and we shall use this to show $x \in \text{bd}\, L_{f(x)} f$. Assume to the contrary that there exists $\epsilon > 0$ such that $N_\epsilon(x) \subset L_{f(x)} f$. Consider $r > 0$ such that $[x-r, x+r] \subset N_\epsilon(x)$. Then,

$$f(x) \leq \text{Sup}\{f(x-r), f(x+r)\} \leq f(x),$$

so equality holds throughout. Since f is explicitly quasi-convex, $f(x-r) = f(x+r)$ so $f(w) = f(x)$ for $w \in N_\delta(x)$ for some $\delta > 0$. This yields the contradiction, $x \notin c\ell L^{o}_{f(x)}f$. Q.E.D.

Now to establish some important cases for monotonic functions, we first consider the following :

Lemma : If f is isotonic, then

$$f^*(x) = \sup_{y \geq 0} \operatorname{Inf} \{f(w) | wy \geq xy\},$$

and $y \in \partial^* f(x)$ implies $y^+ \in \partial^* f(x)$, where $y_i^+ \equiv \operatorname{Max} \{0, y_i\}$.

Proof :

The second part follows from our proof of the first. Consider any $y \in R^n$, we shall prove

$$\operatorname{Inf} \{f(w) | wy \geq xy\} \leq \operatorname{Inf} \{f(w) | wy^+ \geq xy^+\}.$$

Consider w such that $wy^+ \geq xy^+$. We shall prove there exists v such that $vy \geq vx$ and $f(v) \leq f(w)$. Without loss in generality partion y so that $y = (y_1, y_2)$, where $y_1 \geq 0$ (so $y^+ = (y_1, 0)$). Then, similarly partition and w and note

$$w_1 y_1 \geq x_1 y_1 .$$

Define $v = (v_1, v_2)$, where $v_1 \equiv w_1$ and $(v_2)_i \equiv \operatorname{Min} \{(w_2)_i, (x_2)_i\}$. Then, $v \leq w$ so $f(v) \leq f(w)$. Further, $vy = w_1 y_1 + v_2 y_2$

$$\geq x_1 y_1 + x_2 y_2 = xy .$$

Q.E.D.

Definition : f is *strongly isotonic* at x if $h > 0$ implies $f(x+h) > f(x)$. Note that strong isotonicity is defined locally, so f need not be isoton Further, we allow $x \geq w$, $x \neq w$ and $f(x) = f(w)$, so f need not be strictl isotonic even though it be strongly isotonic at every x (e.g., let $f(x)$ $x_1 x_2 \geq 0$ and $+\infty$ otherwise; then, f is strongly isotonic on the non-negat orthant but not strictly isotonic).

Corollary 10.2 : If f is quasi-convex, lower semi-continuous and strongly isotonic at x, then $\partial^* f(x) \neq \emptyset$.

Proof :

We have only to show $x \in \text{bd}\, L_{f(x)}f$. If $x \in \text{int} L_{f(x)}f$, then $\exists\, h > 0$ such that $x + h \in L_{f(x)}f$, so $f(x+h) \leq f(x)$, a contradiction. Q.E.D.

Corollary 10.3 : Suppose f is quasi-convex, isotonic, lower semi-continuous at x and strongly isotonic at x + h for some $h \geq 0$. Then, $\partial^* f(x) \neq \emptyset$.

Proof :

From Corollary 10.2, $\partial^* f(x+h) \neq \emptyset$. Now let $y \in \partial^* f(x+h)$, where $y \geq 0$ (cf. Lemma above). We have

$$f(x+h) = \text{Inf}\, \{f(w) | wy \geq xy + hy\}.$$

Since $hy \geq 0$, we note $wy \geq xy + by \Longrightarrow wy \geq xy$, so

$$f(x+h) \leq \text{Inf}\, \{f(w) | wy \geq xy\} \leq f(x) .$$

However, since f is siotonic, $f(x+h) \geq f(x)$, so equality holds throughout, implying $y \in \partial^* f(x)$. Q.E.D.

4. *Surrogate duality.*

Define

$$f(b) = \text{Inf}\, \{f(x) | g(x) \geq b,\ x \in S\},$$

where $f: S \to R$, $g: S \to R^m$ and $b \in R^m$.

The surrogate dual is defined to be

$$D(b) = \sup_{\lambda \geq 0} \text{Inf}\, \{f(x) | \lambda g(x) \geq \lambda b,\ x \in S\},$$

where $\lambda \in R^m$. This is equivalent to (see [4]) :

$$D(b) = \sup_{\lambda \geq 0} \text{Inf}\, \{F(\beta) | \lambda \beta \geq \lambda b\}.$$

Since F is isotonic, we have

$$D(b) = F^{**}(b) .$$

Many of the results in [7] are now immediate from the foregoing theory. notable instance is the gap theory (Theorem 2 and corollary 2.1 in [7]) and our corollary 10.3. When g is bounded, $F(b) = +\infty$ for b sufficiently large, so the strong isotonicity condition holds. Therefore, we can conclude that $F(b) = D(b)$ when, in addition, F is quasi-convex and lower semi-continuous. The assumptions in [7] ensure these conditions are met. In that case, each maximizing multiplier λ, is a quasi-subgradient of F at b. More generally, if the range of (f,g) is compact, then Theorem 7 tells us that

$$b \in \text{conv argmin } \{F(\beta) | \lambda\beta \geq \lambda b\}$$

if and only if $\lambda \in \partial^* F^{**}(b)$. So even in the presence of a surrogate dual gap we can interpret the optimal multiplier and ensure gap detection.

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