Reprinted from OPERATIONS RESEARCH
Vol. 20, No. 1, January–February, 1972
Printed in U.S.A.

Extensions of the Evans-Gould Stability Theorems for Mathematical Programs

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(Received September 23, 1969)

This paper extends the results of Evans and Gould for stability in mathematical programming. In particular, it shows that their conditions apply to functional perturbation, to equality constraints, and to policy stability under certain conditions. Further, it shows that strictly monotonic programs and positively homogeneous programs possess the closure property needed for stability. Finally, some necessary and sufficient conditions are presented for lower and upper semicontinuity of certain point-to-set mappings.

WE ARE concerned with the mathematical program given by

$$P\text{: Maximize } f(x) : \bar{g}(x) \leqq \bar{b}, \quad x \epsilon E^n,$$

where $f(\cdot)$ is a continuous real-valued function, $\bar{g}$ is a continuous, real-valued, vector function $\bar{g}: E^n \rightarrow E^m$, and $\bar{b}$ is a given vector in E^m. We assume that P has a solution.

EVANS AND GOULD[3] have recently developed stability theorems for P. Since many of the results of this paper are extensions of their work, we shall use notation consistent with their paper. Let a class of feasibility regions be given by $S_b = \{x : \bar{g}(x) \leqq b\}$. Thus, $S_{\bar{b}}$ is the feasibility region of P. Further, the 'strict interior' is given by $I_b = \{x : \bar{g}(x) < b\}$. Let int S_b denote the interior of S_b. Since $\bar{g}$ is continuous, we have $I_b \subset \text{int} S_b$. To illustrate that I_b is not equal to intS_b, consider the curve in Fig. 1; we have $I_b = \{(0, 1)\} \cup \{(1, 2)\}$ and int$S_b = \{(0, 2)\}$.

We define the set of 'feasible right-hand sides' as $B = \{b : S_b \neq \emptyset\}$, where $\emptyset$ is the empty set. We then define a point-to-set map S from B to the power set of E^n with the rule $S(b) = S_b$ for $b \epsilon B$. In our subsequent developments, we assume $S_{\bar{b}}$ is compact, but S_b may not be compact for $b \neq \bar{b}$.

The supremum function is defined to be $f_{\sup}(b) = \sup f(x) : x \epsilon S_b$. Thus, $f_{\sup}: B \rightarrow E^1 \cup \{+\infty\}$.

We use the notions of distance as:

$\rho(x, y) = [\sum_{i=1}^{i=n} (x_i - y_i)^2]^{1/2}$ for $x, y \epsilon E^n$.

$\rho(x, A) = \min_{y \epsilon A} \rho(x, y)$ for $x \epsilon E^n$, $A \subset E^n$, and A closed.

$\rho(A, D) = \min_{x \epsilon D} \rho(x, A)$ for $A, D \subset E^n$, A closed, and D compact.

Finally, an ϵ-neighborhood of a point x is the sphere given by $N_\epsilon(x) = \{y : \rho(x, y) < \epsilon\}$, and for a closed set A we have $\eta_\epsilon(A) = \{y : \rho(y, A) < \epsilon\}$. Further notation will be introduced as needed.

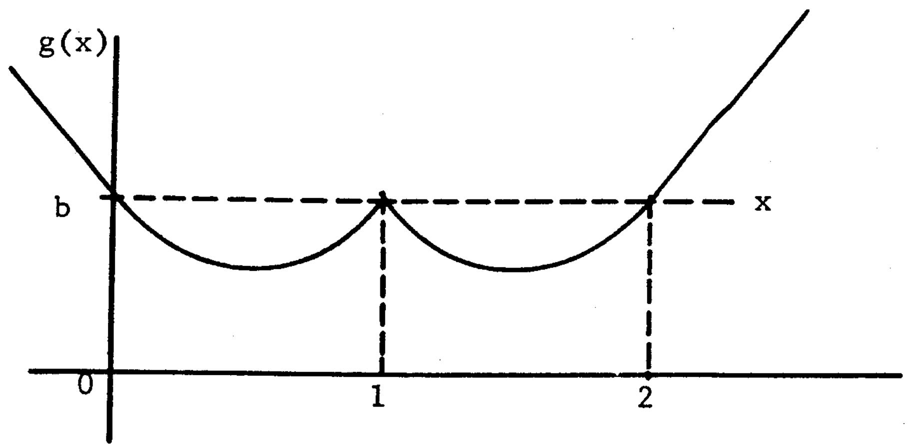

Figure 1

Evans and Gould approached the stability question of P with the concepts of upper and lower semicontinuity of S at $\bar{b}$ as found in BERGE[1]; namely, given $b^n \to \bar{b}$, where $\{b^n\} \subset B$ and $S_{\bar{b}}$ is compact, and given any $\epsilon > 0$, then (1) S is *usc* at $\bar{b}$ if there exists n^0 such that, for $n > n^0$, $S_{b^n} \subset \eta_\epsilon(S_{\bar{b}})$, and (2) S is *lsc* at $\bar{b}$ if there exists n^0 such that, for $n > n^0$, $S_{\bar{b}} \subset \eta_\epsilon(S_{b^n})$.

If S is both *usc* and *lsc* at $\bar{b}$, we shall say S is continuous at $\bar{b}$. Evans and Gould elaborate upon this and distinguish the above definitions from the semicontinuity of a function at a point. We shall say S is *stable*[3] at $\bar{b}$ if (1) $I_{\bar{b}} \neq \emptyset$, (2) $S_{\bar{\bar{b}}}$ is compact for some $\bar{\bar{b}} > \bar{b}$, and (3) closure $I_{\bar{b}} = S_{\bar{b}}$.

Two of the Evans-Gould theorems are: *Suppose $S_{\bar{b}}$ is compact, then*:

(1) *S is usc at $\bar{b}$ if and only if there exist $\bar{\bar{b}} > \bar{b}$ such that $S_{\bar{\bar{b}}}$ is compact.*

(2) *If $I_{\bar{b}} \neq \emptyset$, then S is lsc at $\bar{b}$ if and only if closure $I_{\bar{b}} = S_{\bar{b}}$.*

They provide examples and later apply their theorems to establish upper and lower semicontinuity of $f_{\sup}(b)$ at $\bar{b}$.

Our forthcoming comments extend their results to (1) allow perturbations of the functions $\bar{g}_i(\cdot)$, (2) allow equality constraints, (3) consider policy stability, (4) establish other classes of stable programs, and (5) characterize certain *usc* and *lsc* point-to-set mappings in an analytically useful manner.

FUNCTIONAL STABILITY

LET US DEFINE C as the class of real valued continuous m-vector functions defined on E^n and $G = \{g \epsilon C : \{x : g(x) \leqq \bar{b}\} \neq \emptyset$ and $\max_i \sup_{x \epsilon E^n} |g_i(x) - \bar{g}_i(x)| < \infty\}$. Let the metric on G be given by $d(u, v) = \max_i \sup_{x \epsilon E^n} |u_i(x) - v_i(x)|$. Further, let us define $\sigma_g = \{x : g(x) \leqq \bar{b}\}$ for $g \epsilon G$. Let σ be a point-to-set map from G into the power set of E^n with rule given by $\sigma(g) = \sigma_g$ for $g \epsilon G$.

Let $\{g^n\} \subset G$ be a sequence of functions uniformly converging to $\bar{g} \epsilon G$ (where $\bar{g}$ is the original constraint vector in P). We define, for any $\epsilon > 0$, (1) σ is *usc* at $\bar{g}$ if there exists n^0 such that for $n > n^0$ $\sigma_{g^n} \subset \eta_\epsilon(\sigma_{\bar{g}})$, and (2) σ is *lsc* at $\bar{g}$ if there exists n^0 such that for $n > n^0$ $\sigma_{\bar{g}} \subset \eta_\epsilon(\sigma_{g^n})$. If σ is both *usc* and *lsc* at $\bar{g}$, we will say σ is continuous at $\bar{g}$ (following the terminology in references 1 and 3).

It is then easily shown that the following theorem is true.

THEOREM 1. (a) *σ is usc at $\bar{g}$ if and only if S is usc at $\bar{b}$.* (b) *Let $I_{\bar{b}} \neq \emptyset$. Then σ is lsc at $\bar{g}$ if and only if S is lsc at $\bar{b}$.*

This means that the Evans-Gould results are sufficiently general to allow perturbations of the constraint functions $g_i(\cdot)$. The proof of Theorem 1, as well as the proof of all subsequent results, may be found in the appendix. Note that part (b) assumes $I_{\bar{b}} \neq \emptyset$. The following example illustrates why this is needed.

Example.

$$\bar{g}(x) = \begin{cases} x-1, & \text{if } 1 \leqq x, \\ 0, & \text{if } -1 \leqq x \leqq 1, \\ -x-1, & \text{if } x \leqq -1. \end{cases}$$

Let $\bar{b}=0$; then $S_0 = [-1, 1]$ and $I_0 = \emptyset$. Now S is *lsc* at 0 because, for any sequence $\{b^k\} \to 0$ with $b^k \epsilon B$ (hence, $\{b^k\}$ converges to 0 from above), $S_{b^k} = [-b^k-1, b^k+1]$. So for any $\epsilon > 0$ and (in this case) all k, $S_0 \subset \eta_\epsilon(S_{b^k})$. However, consider the sequence of functions

$$g^k(x) = \begin{cases} x-1, & \text{if } 1 \leqq x, \\ (1/k)(x-1), & \text{if } 0 \leqq x \leqq 1, \\ (1/k)(x+1), & \text{if } -1 \leqq x \leqq 0, \\ -x-1, & \text{if } x \leqq -1. \end{cases}$$

Now $g^k \epsilon C$ for all k and $\sup_x |g^k(x) - \bar{g}(x)| = 1/k$; hence g^k converges uniformly to $\bar{g}$. Furthermore, $\sigma_{g^k} = \{-1\} \cup \{1\}$ for all g^k. Now it is not difficult to see that, for every $1 > \epsilon > 0$, there is *no* k such that

$$[-1, 1] = \sigma_{\bar{g}} \subset \eta_\epsilon(\sigma_{g^k}) = \eta_\epsilon(\{-1\} \cup \{1\}).$$

COROLLARY 1.1. (a) *If there exists $\bar{\bar{b}}$ such that $\bar{\bar{b}} > \bar{b}$ and $S_{\bar{\bar{b}}}$ is compact, then σ is usc at $\bar{g}$.*

(b) *If $I_{\bar{b}} \neq \emptyset$ and cl $I_{\bar{b}} = S_{\bar{b}}$, then σ is lsc at $\bar{g}$.*

The proof of Corollary 1.1 is immediate, and we omit it.

EQUALITY CONSTRAINTS

SUPPOSE P has equality constraints of the form

$$T_{\bar{e}} = \{x : h(x) = \bar{e}\},$$

where $h: E^n \to E^k$ and h is continuous on E^n. Define $T_e^1 = \{x : h(x) \leqq e\}$ and $T_e^2 = \{x : h(x) \geqq e\}$. We observe that $T_e = T_e^1 \cap T_e^2$.

THEOREM 2. *If T^1 and T^2 are usc at $\bar{e}$, then T is usc at $\bar{e}$.*

This can easily be extended to consider upper semicontinuity of $S_b \cap T$ at $(\bar{b}, \bar{e})$, as well as functional perturbation (as in Theorem 1). Berge[1] (*see* Chapter 6) proved Theorem 2.

The lower semicontinuity of T at $\bar{e}$ is more difficult, and extensions to functional perturbation need not follow. One of the problems is that the right-hand-side vector $\bar{e}$ may not be perturbed arbitrarily.

For example, suppose we have

$$2x_1 + x_2 = 3 = \bar{e}_1,$$
$$x_1 + x_2 = 2 = \bar{e}_2,$$
$$x_1 + 2x_2 = 3 = \bar{e}_3.$$

The only sequences $\{e^n\}$ for which $T_{e^n} \neq \emptyset$ are of the form

$$e^n = (e_1{}^n,\ e_2{}^n,\ 3e_2{}^n - e_1{}^n).$$

Any other sequence yields inconsistent equations, so that $T_{e^n} = \emptyset$ for all n.

Observe further than we cannot apply the Evans-Gould result directly by using $T_{\bar{e}}^1 \cap T_{\bar{e}}^2$, because it is not possible to have a nonempty strict interior.

To deal with the possibility of $T_e n = \emptyset$ for all n, we define (1) for any $\epsilon > 0$, T is *lsc* at $\bar{e}$ relative to the sequence $\{e^n\} \to \bar{e}$ if there exists $\bar{n}$ such that $n \geqq \bar{n}$ implies $T_{\bar{e}} \subset \eta_\epsilon(T_{e^n})$, and (2) $\{e^n\} \to \bar{e}$ is a feasible sequence if there exists $\bar{n}$ such that $n \geqq \bar{n}$ implies $T_{e^n} \neq \emptyset$.

A real scalar-valued function θ is *quasiconvex* on a convex set $X \subset E^n$ if, for all $\alpha \epsilon [0, 1]$ and any x, $y \epsilon X$, $\theta[\alpha x + (1-\alpha)y] \leqq \text{maximum}\ [\theta(x),\ \theta(y)]$. We say θ is *explicitly quasiconvex* on a convex set $X \subset E^n$ if θ is quasiconvex on X and if, for $\theta(x) \neq \theta(y)$, $\theta[\alpha x + (1-\alpha)y] < \text{maximum}[\theta(x),\ \theta(y)]$ for all $\alpha \epsilon (0, 1)$.

The function θ is *quasiconcave* or *explicitly quasiconcave* if $-\theta$ is quasiconvex or explicitly quasiconvex respectively. (When θ *is* lower semicontinuous the definitions of explicit quasiconvexity, as given above, and strict quasiconvexity, as given in reference 5 and used by Evans and Gould,[3] are equivalent.) We say that h is *explicitly quasimonotonic* if it is both *explicitly* quasiconvex and *explicitly* quasiconcave. For a more complete discussion of explicit quasiconvexity and quasimonotonicity, *see* reference 6 or reference 4. Let us begin with $k=1$ (one constraint). Define $I_e{}^1 = \{x : h(x) < e\}$ and $I_e{}^2 = \{x : h(x) > e\}$.

THEOREM 3. *Let $g(\cdot)$ be an m-vector of continuous quasiconvex functions and let $h(\cdot)$ be any continuous function. If $S \cap T^1$ and $S \cap T^2$ are each lsc at $(\bar{b}, \bar{e})$, then $S \cap T$ is lsc at $(\bar{b}, \bar{e})$ relative to any feasible sequence. Moreover, if $I_{\bar{b}} \neq \emptyset$, then every sequence [converging to $(\bar{b}, \bar{e})$] is feasible if and only if $I_{\bar{b}} \cap I_{\bar{e}}{}^1 \neq \emptyset$ and $I_{\bar{b}} \cap I_{\bar{e}}{}^2 \neq \emptyset$.*

In the absence of any inequality constraints, this theorem holds for the single equality constraint $h(\cdot)$, where all references to $\bar{b}$ and $I_{\bar{b}}$ are omitted, and $S_{\bar{b}} \equiv E^n$.

The extension to more than one constraint is not direct. We have found it necessary to restrict the class of functions further.

Define $\tau_e{}^j = \{x : h_i(x) = e_i \text{ for } i = 1, \cdots, j\}$, $I_k{}^1 = \{x : h_k(x) < \bar{e}_k\}$, and $I_k{}^2 = \{x : h_k(x) > \bar{e}_k\}$.

THEOREM 4. *If (1) $S_{\bar{b}}$ is compact, (2) $I_{\bar{b}} \cap \tau_{\bar{e}}^{k-1} \cap I_k{}^1 \neq \emptyset$ and $I_{\bar{b}} \cap \tau_{\bar{e}}^{k-1} \cap I_k{}^2 \neq \emptyset$, (3) g is explicitly quasiconvex and continuous for $i = 1, \cdots, m$, and (4) h_i is explicitly quasimonotonic and continuous for $i = 1, 2, \cdots, k$, then $S \cap \tau^k$ is lsc at $(\bar{b}, \bar{e})$ relative to any feasible sequence.*

To see why hypothesis (2) is needed, consider

$$S_{\bar{b}} = \{(x, y) : 1/x + y/x \leqq 1,\ 2 \geqq x \geqq 1,\ 0 \leqq y \leqq 2\}, \qquad T_{\bar{e}} = \{(x, y) : y - x = -1\}.$$

The function $g(x, y) = 1/x + y/x$ is explicitly quasiconvex on $\{2 \geqq x \geqq 1, 0 \leqq y \leqq 2\}$. Note that

$$I_{\bar{b}} \cap I^2{}_{\bar{e}} = \emptyset.$$

Consider the sequence where

$$S_{b^n} = \{(x, y) : 1/x + y/x \leqq 1 + 1/2n,\quad 2 \geqq x \geqq 1,\quad 0 \leqq y \leqq 2\}$$
$$T_{e^n} = \{(x, y) : y - x = -1 + 1/n\}.$$

Then $S_{b^n} \cap T_{e^n}$ consists of the single point set $\{(2, 1 + 1/n)\}$ for all $n > 0$. However,

$$S_{\bar{b}} \cap T_{\bar{e}} = T_{\bar{e}} \cap \{2 \geqq x \geqq 1 \quad \text{and} \quad 0 \leqq y \leqq 1\}.$$

Therefore

$$(x, y) = (1, 0) \epsilon S_{\bar{b}} \cap T_{\bar{e}},$$

but

$$(1, 0) \notin \eta_\epsilon (S_{b^n} \cap T_{e^n})$$

for $\epsilon < \frac{1}{2}$.

It should be noted that the hypotheses of Theorem 4, in conjunction with the work of Evans and Gould, provide sufficient conditions for the two mappings $S \cap \tau^{k-1} \cap T_k^1$ and $S \cap \tau^{k-1} \cap T_k^2$ to be *lsc* [where $T_k^{1(2)} = \{x : h_k(x) \leqq (\geqq) e_k\}$]. If one is willing to assume that these two mappings are *lsc* to start with, then one can weaken Theorem 4 to allow precisely one of the equality constraints, say $h_k(x) = \bar{e}_k$, to be continuous but not necessarily quasimonotonic. The proof of this last statement parallels the proofs of Theorems 3 and 4.

POLICY STABILITY

DEFINE THE optimality region as $\Omega_b = \{x \epsilon S_b : f(x) = f_{\sup}(b)\}$. Since $S_{\bar{b}}$ is compact, and since f is continuous, $\Omega_{\bar{b}}$ is nonempty and compact.

THEOREM 5. *If S is usc at $\bar{b}$ and if $f_{\sup}$ is continuous at $\bar{b}$, then Ω is usc at $\bar{b}$.*

In the proof of Theorem 5 we use a lemma which is interesting and perhaps useful in other respects; for these reasons we state it here.

LEMMA. *If* (i) *$A \subset E^n$ and $B \subset E^n$ are closed sets,* (ii) *A or B is bounded, and* (iii) *$A \cap B \neq \emptyset$, then, for all $\epsilon > 0$, there exists a $\delta > 0$ such that $\eta_\delta(A) \cap \eta_\delta(B) \subset \eta_\epsilon(A \cap B)$.*

To illustrate why the continuity of $f_{\sup}$ at $\bar{b}$ is needed, consider the following example: $\max x$: $g_1(x) \leqq 0$, $x \leqq 10$ and $-x \leqq 10$, where

$$g_1(x) = \begin{cases} x^3, & \text{if } x \leqq 0, \\ 0, & \text{if } 0 \leqq x \leqq 1, \\ (x-1)^3, & \text{if } x \geqq 1. \end{cases}$$

Note $\Omega(0, 10, 10) = \{1\}$. Consider $b^k = (-1/k^3, 10, 10)$. Note that $f_{\sup}$ is discontinuous at $\bar{b} = (0, 10, 10)$ and $\Omega(b^k) = \{-1/k\}$ for all $k \geqq 1$. Clearly, Ω is *not* usc at $\bar{b}$.

Now let us consider lower semicontinuity. Note that the results of the previous section do not apply, because the strict interiority assumption cannot be met. Moreover, it is possible for Ω to be *lsc* at $\bar{b}$ and not be able to extend to functional perturbation (as in Theorem 1).

For example, consider

$$\max\{x + y : x + y \leqq b,\ x,\ y \geqq 0\}.$$

Note that

$$\Omega_b = \{(x, y) \geqq 0 : x + y = b\}$$

for all b. Therefore, if $b^n \to b$ and $\epsilon > 0$, then

$$\Omega_{b^n} \subset \eta_\epsilon \Omega_b$$

for n sufficiently large. However, consider

$$\max\{x + y : x + \alpha y \leqq 1,\ x,\ y \geqq 0\}.$$

Observe that

$$\Omega(\alpha) = \{(x, y) \geqq 0 : x + y = 1\} \quad \text{if} \quad \alpha = 1,$$

$$\Omega(\alpha) = \{(1, 0)\} \quad \text{if} \quad \alpha > 1,$$

and

$$\Omega(\alpha)=\{(0, 1/\alpha)\} \quad \text{if} \quad \alpha<1.$$

Therefore, if $\alpha^n \to 1$, we do not satisfy $\Omega(\alpha) \subset \eta_\epsilon \Omega(\alpha^n)$ for any n if $\epsilon < 1/2$.

COROLLARY 5.1. *If Ω is usc at $\bar{b}$ and if there exists $\delta>0$ such that, for all $b \epsilon N_\delta(\bar{b})$, Ω_b consists of a single point, then Ω is lsc at $\bar{b}$.*

Note that, if f is strictly concave and if g is explicitly quasiconvex, then we have policy stability. Theorem 5 is essentially the same as Theorem I2.2 in DANTZIG, FOLKMAN, AND SHAPIRO.[2] They proved closedness of the mapping Ω at $\bar{b}$ and we show *usc* of the mapping Ω at $\bar{b}$. In general, if a map is *usc* and if the image sets are closed, then the map is closed. Conversely, if a map is closed and its range on a neighborhood is bounded, then it is *usc*. Furthermore, Berge[1] essentially proves Theorem 5 (he assumes S is continuous for all $b \epsilon B$) and Corollary 5.1; however, we have included the proof of Theorem 5 here for completeness of the stability presentation.

MEYER[7] has considered this question in relation to the convergence of algorithms, thus demonstrating its importance. However, only the usc of Ω is considered in the same manner as ZANGWILL[9] uses the closedness of Ω.

STABLE CLASSES OF PROGRAMS

IN THIS SECTION we establish the closure property for several classes of functions. First, we define a function $u(x)$ to be *increasing* if $x \geqq y$ and $x \neq y$ implies $u(x) > u(y)$. If $-u(x)$ is increasing, then we say $u(x)$ is decreasing.

THEOREM 6. *If $g(x)$ is increasing (decreasing) and if $I_{\bar{b}} \neq \emptyset$, then cl $I_{\bar{b}} = S_{\bar{b}}$.*

Next we define a function $u(x)$ to be *positively homogeneous of degree p* if for any scalar $\lambda \geqq 0$ we have $u(\lambda x) = \lambda^p u(x)$.

THEOREM 7. *If $g_i(x)$ is positively homogeneous of degree p_i, $\bar{b}>0$, and either $p_i>0$ for all i or $p_i<0$ for all i, then $I_{\bar{b}} \neq \emptyset$ and cl $I_{\bar{b}} = S_{\bar{b}}$.*

Now we shall consider 'periodic functions.' Consider the 'fractional function' $\theta: E^1 \to [0, 1)$, where $\theta(z) = z - [z]$ ($[z]$ denotes the greatest integer less than or equal to z). Let $\{a^j\} \subset E^n$. A function $u(x)$ is *periodic relative to a basis* $A = [a^1, \cdots, a^n]$ if, for all $\lambda \epsilon E^n$, we have $u(\sum_{j=1}^{j=n} \lambda_j a^j) = u[\sum_{j=1}^{j=n} \theta(\lambda_j) a^j]$.

For example, consider $u(x, y) = \sin x + \sin y$. Let $A = \begin{bmatrix} 2\pi & 0 \\ 0 & 2\pi \end{bmatrix}$. Then, let $(x, y) = (\lambda_1, \lambda_2) \cdot A = 2\pi(\lambda_1, \lambda_2)$. We have $u(x, y) = \sin\theta(\lambda_1)2\pi + \sin\theta(\lambda_2)2\pi = u[\theta(\lambda_1)a^1 + \theta(\lambda_2)a^2]$. The length of the period is defined to be the volume of the parallelepiped given by $P(A) = \{x : x = \lambda A, 0 \leqq \lambda \leqq 1\}$. It is well known that this volume is the value of the determinant of A.

Define $I_b(A) = \{x \epsilon P(A) : g(x) < b\}$ and $S_b(A) = \{x \epsilon P(A) : g(x) \leqq b\}$.

THEOREM 8. *Let g be continuous on E^n and periodic relative to a basis A for E^n. If cl $I_{\bar{b}}(A) = S_{\bar{b}}(A)$, then cl $I_{\bar{b}} = S_{\bar{b}}$.*

The preceding paragraphs discussed the closure property needed to establish the *lsc* of the map whose images are level sets for certain classes of functions. It is possible to obtain converse statements that emphasize the problem with flat regions of the constraint functions, as the following result shows.

THEOREM 9. *If g is continuous and quasiconvex and if S is lsc at each $b \epsilon B$, then g is explicitly quasiconvex.*

As is well known, stability of mathematical programs involves both the *lsc* and the *usc* of S. Now the *usc* of S at $\bar{b}$ can be deduced if we redefine $S_b = \{x \epsilon X : g(x) \leqq b\}$, where $X = \{x : a \leqq x \leqq c\}$; a and c are (finite) n-vectors. Clearly, without introducing X (or some such set), it is not possible for $S_{\bar{b}}$ to be compact when g is, say, increasing (decreasing).

USC AND LSC POINT-TO-SET MAPPINGS

Evans and Gould point out the fact that a monotonic increasing function is an upper semicontinuous (lower semicontinuous) function at $\bar{b}$ if and only if it is right-continuous (left-continuous) at $\bar{b}$. A similar result holds for *usc* and *lsc* point-to-set mappings.

Let Γ be a mapping of $X \subset E^m$ into the power set of E^n. We define:

(1) Γ is a *monotonic increasing mapping* if, for every $x \geqq y$, $x, y \epsilon X$, $\Gamma(x) \supset \Gamma(y)$.

(2) Γ is a *right-continuous mapping at* y if, for any $\epsilon > 0$, there exists a $\delta > 0$ such that, whenever $\rho(x, y) < \delta$ and $x \geqq y$, then $\Gamma(x) \subset \eta_\epsilon[\Gamma(y)]$.

(3) Γ is a *left-continuous mapping at* y if, for any $\epsilon > 0$, there exists a $\delta > 0$ such that, whenever $\rho(x, y) < \delta$ and $x \leqq y$, $\Gamma(y) \subset \eta_\epsilon[\Gamma(x)]$.

It should be noted that the mapping S defined earlier is a monotonic increasing mapping since, for any $b \geqq \bar{b}$, $S_b \supset S_{\bar{b}}$.

The following theorem characterizes *usc* and *lsc* monotonic increasing mappings. It should be pointed out that, in the definition of *usc* and *lsc* mappings, the restriction to compact image spaces is not needed for this theorem.

Theorem 10. *Let* Γ *be a monotonic increasing mapping.* Γ *is an usc* (*lsc*) *mapping at* y *if and only if* Γ *is a right-continuous* (*left-continuous*) *mapping at* y.

One can define a monotonic decreasing mapping similarly, and this theorem will again hold, *mutatis mutandis*.

APPENDIX

Proof of Theorem 1. Note that $\sigma_{\bar{g}} = S_{\bar{b}}$ is a compact set by assumption and that $\eta_\epsilon(\sigma_{\bar{g}}) = \eta_\epsilon(S_{\bar{b}})$. Let $h \epsilon G$ be given such that $\max_i d(\bar{g}_i, h_i) < \delta$, and let Δ be the m-vector whose elements are δ. Thus $-\Delta < h(x) - \bar{g}(x) < \Delta$ for all x.

First it will be shown that $S_{\bar{b}-\Delta} \subset \sigma_h \subset S_{\bar{b}+\Delta}$. Choose any $x \epsilon S_{\bar{b}-\Delta} : \bar{g}(x) \leqq \bar{b} - \Delta$. Then since $h(x) < \bar{g}(x) + \Delta \leqq \bar{b}$, we have $x \epsilon \sigma_h$. Now choose $x \epsilon \sigma_h : h(x) \leqq \bar{b}$. Then $\bar{g}(x) - \Delta < h(x) \leqq \bar{b}$ implies $\bar{g}(x) \leqq \bar{b} + \Delta$. Thus $x \epsilon S_{\bar{b}+\Delta}$.

The necessity is obvious; only the sufficiency will be demonstrated. Given S is *usc* (*lsc*). That is, given any $\epsilon > 0$, there is a $\delta > 0$ such that $|\bar{b} - b| < \delta$ implies $S_b \subset \eta_\epsilon(S_{\bar{b}})[S_{\bar{b}} \subset \eta_\epsilon(S_b)]$. Choose $b_i^0 = \bar{b}_i + \delta/2\sqrt{m}$. Then $|\bar{b} - b^0| = \delta/2$ and $S_{b^0} \subset \eta_\epsilon(S_{\bar{b}})[S_{\bar{b}} \subset \eta_\epsilon(S_{b^0})]$. Also choose any $h \epsilon G$ such that $\max_i d(\bar{g}_i, h_i) < \delta/2\sqrt{m}$. By the preceding result, σ is *usc* at $\bar{g}$, since we have

$$\sigma_h \subset S_{\bar{b}+\Delta/2\sqrt{m}} = S_{b^0} \subset \eta_\epsilon(S_{\bar{b}}) = \eta_\epsilon(\sigma_{\bar{g}}).$$

[For *lsc* choose $b_i^0 = \bar{b}_i - \delta/2\sqrt{m}$. For δ sufficiently small, $b^0 \epsilon B$, since $I_{\bar{b}} \neq \emptyset$ and $\bar{g}$ is continuous. Thus $\sigma_{\bar{g}} = S_{\bar{b}} \subset \eta_\epsilon(S_{b^0}) = \eta_\epsilon(S_{\bar{b}} - \Delta/2\sqrt{m}) \subset \eta_\epsilon(\sigma_h)$, since $S_{\bar{b}-\Delta/2\sqrt{m}}) \subset \sigma_h$.]

Proof of Theorem 2. See Berge.[1]

Proof of Theorem 3. Let us suppose $(b^n, e^n) \to (\bar{b}, \bar{e})$ and $S_{b^n} \cap T_{e^n} \neq \emptyset$ for $n \geq \bar{n}$. Consider an $x \in S_{\bar{b}} \cap T_{\bar{e}}$ and define for $n \geq \bar{n}$:

$$\rho(u^n, x) = \inf_u \{\rho(u, x) : g(u) \leq b^n \quad \text{and} \quad h(u) \leq e^n\},$$

$$\rho(v^n, x) = \inf_v \{\rho(v, x) : g(v) \leq b^n \quad \text{and} \quad h(v) \geq e^n\}.$$

Since $S \cap T^1$ is *lsc* at $(\bar{b}, \bar{e})$, $\rho(u^n, x) \to 0$ as $(b^n, e^n) \to (\bar{b}, \bar{e})$. Similarly $\rho(v^n, x) \to 0$, so there exists an $n^* \geq \bar{n}$ such that, for $n \geq n^*$, we have $\rho(u^n, x) < \epsilon$ and $\rho(v^n, x) < \epsilon$, where $\epsilon > 0$. Let $w^n = \alpha u^n + (1-\alpha) v^n$ for $\alpha \in [0, 1]$. Then, $\rho(w^n, x) < \epsilon$ for all α and all $n \geq n^*$. We have $g(u^n) \leq b^n$, $h(u^n) \leq e^n$, $g(v^n) \leq b^n$ and $h(v^n) \geq e^n$. Since h is continuous, for each n there exists an $\alpha_n \in [0, 1]$ such that $h(w^n) = e^n$; furthermore, since $g(\cdot)$ is quasiconvex, $g(w^n) \leq b^n$. Therefore, $x \in \eta_\epsilon(S_{b^n} \cap T_{e^n})$. This implies $S_{\bar{b}} \cap T_{\bar{e}} \subset \eta_\epsilon(S_{b^n} \cap T_{e^n})$ for $n \geq n^*$, so $S \cap T$ is *lsc* at $(\bar{b}, \bar{e})$.

To prove the second part of Theorem 3, consider the feasibility of an arbitrary sequence $\{(b^n, e^n)\}$. Let $1/\boldsymbol{n}$ be an m-vector with each component equal to $1/n$.

Necessity. Since $S_{b^n} \cap T_{e^n} \neq \emptyset$ for all $n \geq \bar{n}$ and all sequences $\{(b^n, e^n)\}$ converging to $(\bar{b}, \bar{e})$, let $b^n = \bar{b} - 1/\boldsymbol{n}$ and $e^n = \bar{e} - 1/\boldsymbol{n}$. For any $x \in S_{b^n} \cap T_{e^n}$, $g(x) \leq \bar{b} - 1/\boldsymbol{n} < \bar{b}$ and $h(x) = \bar{e} - 1/\boldsymbol{n} < \bar{e}$. Therefore, $x \in I_{\bar{b}} \cap I_{\bar{e}}^1$. Now let $b^n = \bar{b} - 1/\boldsymbol{n}$ and $e^n = \bar{e} + 1/\boldsymbol{n}$; then $(b^n, e^n) \to (\bar{b}, \bar{e})$ and for any $x \in S_{b^n} \cap T_{e^n}$, $g(x) \leq \bar{b} - 1/\boldsymbol{n} < \bar{b}$ and $h(x) = \bar{e} + 1/\boldsymbol{n} > \bar{e}$. Hence, $x \in I_{\bar{b}} \cap I_{\bar{e}}^2$.

Sufficiency. Let $\{(b^n, e^n)\}$ be any sequence converging to $(\bar{b}, \bar{e})$. Since $S_{b^n} \cap T_{e^n} = (S_{b^n} \cap T_{e^n}^1) \cap (S_{b^n} \cap T_{e^n}^2)$, we will establish the theorem by first showing $S_{b^n} \cap T_{e^n}^i \neq \emptyset$ for all $n \geq \bar{n}$, $i = 1, 2$, and then show that the intersection is not empty. Since $I_{\bar{b}} \cap I_{\bar{e}}^1 \neq \emptyset$ and $I_{\bar{b}} \cap I_{\bar{e}}^2 \neq \emptyset$, there are u and v such that $g(u) < \bar{b}$, $h(u) > \bar{e}$, $g(v) < \bar{b}$, $h(v) < \bar{e}$. Define β and γ as $\bar{b}_i - g_i(v) = \beta_i > 0$, $i = 1, \cdots, m$; $\bar{e} - h(v) = \beta_{m+1} > 0$; $\bar{b}_i - g_i(u) = \gamma_i > 0$, $i = 1, \cdots, m$; $h(u) - \bar{e} = \gamma_{m+1} > 0$. Let $\beta = \min_{1 \leq i \leq m+1} \{\beta_i, \gamma_i\} > 0$ and choose any δ such that $\beta > \delta > 0$. Then, since $(b^n, e^n) \to (\bar{b}, \bar{e})$, there exists numbers n_i, $i = 1, \cdots, m+1$, such that: $|\bar{b}_i - b_i^n| < \delta$ for all $n \geq n_i$, $i = 1, \cdots, m$; $|\bar{e} - e^n| < \delta$ for all $n \geq n_{m+1}$.

Let $\bar{n} = \max_{1 \leq i \leq m+1} \{n_i\}$. Then $g_i(v) - b_i^n = g_i(v) - \bar{b}_i + (\bar{b}_i - b_i^n) = -\beta_i + (\bar{b}_i - b_i^n) < -\beta^i + \delta < 0$ for all $n \geq \bar{n}$, $g_i(u) - b_i^n < -\gamma_i + \delta < 0$ for all $n \geq \bar{n}$, $h(u) - e^n > \gamma_{m+1} - \delta > 0$ for all $n \geq \bar{n}$, and $h(v) - e^n < -\beta_{m+1} + \delta < 0$ for all $n \geq \bar{n}$. Thus, $v \in S_{b^n} \cap T_{e^n}^1$ and $u \in S_{b^n} \cap T_{e^n}^2$ for all $n \geq \bar{n}$.

We now show there exists an x^n such that $h(x^n) = e^n$ and $g(x^n) \leq b^n$ for each $n \geq \bar{n}$, and the sequence $\{(b^n, e^n)\}$ is then a feasible sequence. Consider any $n \geq \bar{n}$ and choose $u^n \in S_{b^n} \cap T_{e^n}^2$ and $v^n \in S_{b^n} \cap T_{e^n}^1$. Define $[u^n, v^n] = \{x : x = \alpha u^n + (1-\alpha) v^n$ for some $\alpha \in [0, 1]\}$. Clearly, $[u^n, v^n] \subset S_{b^n}$ because $g(\cdot)$ is quasiconvex. Since $h(u^n) \geq e^n$ and $h(v^n) \leq e^n$, there exists $x^n \in [u^n, v^n]$ such that $h(x^n) = e^n$. Hence, $x^n \in S_{b^n} \cap T_{e^n}$, and our proof is complete.

Proof of Theorem 4. We shall proceed by induction. First consider $k = 1$. We must prove that, for n sufficiently large, we have $S_{\bar{b}} \cap \tau_{\bar{e}}^1 \subset \eta_\epsilon(S_{\bar{b}} n \cap \tau_{\bar{e}}^1 n)$, where $\epsilon > 0$ is specified.

Since $I_{\bar{b}} \cap I_1^1 \neq \emptyset$ and $I_{\bar{b}} \cap I_1^2 \neq \emptyset$ and $g(\cdot)$ and $h(\cdot)$ are explicitly quasiconvex, then by a theorem and a lemma of Evans and Gould, $S \cap T^1$ and $S \cap T^2$ are each *lsc* at $(\bar{b}, \bar{e})$. Thus by Theorem 3 the mapping $S \cap T \equiv S \cap \tau^1$ is *lsc* at $(\bar{b}, \bar{e})$ relative to any feasible sequence.

Assume the theorem is true for $k-1$ equations (and any finite collection of inequalities satisfying the hypotheses). Let $x \in S_{\bar{b}} \cap \tau_{\bar{e}}^k$ and define

$$\rho(u^n, x) = \inf_u \{\rho(u, x) : g(u) \leq b^n,\ h_k(u) \geq e_k^n,\ h_i(u) = e_i^n \quad \text{for} \quad i = 1, \cdots, k-1\}$$

and

$$\rho(v^n, x) = \inf_v \{\rho(v, x) : g(v) \leq b^n,\ h_k(v) \leq e_k^n,\ h_i(v) = e_i^n \quad i = 1, \cdots, k-1\}.$$

By the induction hypothesis, $\rho(u^n, x) \to 0$ and $\rho(v^n, x) \to 0$ as $n \to \infty$, since there are only $k-1$ equations in each case. Thus, for any $\epsilon > 0$ there exists an $\bar{n}$ such that, for all $n \geq \bar{n}$, $\rho(u^n, x) < \epsilon$ and $\rho(v^n, x) < \epsilon$. Define $x^n = \alpha u^n + (1-\alpha) v^n$. Observe that, for any $\alpha \in [0, 1]$, $g(x^n) \leq b^n$

and $h_i(x^n)=e_i^n$ for $i=1, \cdots, k-1$, since $g(\cdot)$ and $h_i(\cdot)$ are quasiconvex and quasimonotonic, respectively. But the continuity of h_k and the facts that $h_k(u^n)\geqq e_k^n$ and $h_k(v^n)\leqq e_k^n$ imply that there is an $x^n=\alpha_n u^n+(1-\alpha_n)v^n$ for some $\alpha_n\epsilon[0, 1]$ such that $h_k(x^n)=e^n$. Thus, $x\epsilon \eta_\epsilon(S_{b^n}\cap\tau^k_{e^n})$, so $S_{\bar b}\cap\tau_{\bar e}^k\subset\eta_\epsilon(S_{b^n}\cap\tau^k_{e^n})$. Hence, $S\cap\tau^k$ is *lsc* at $(\bar b, \bar e)$ relative to any feasible sequence.

Proof of Theorem 5. Since S is *usc* at $\bar b$, there exists $\bar{\bar b}>\bar b$ such that $S_{\bar{\bar b}}$ is compact. Therefore, if $b^n\to\bar b$, S_{b^n} is compact for $n\geqq\bar n$ and $\Omega_{b^n}\subset S_{b^n}\subset S_{\bar{\bar b}}$. Define $\Psi_b=\{x\epsilon S_{\bar{\bar b}}:f(x)\geqq f_{\sup}(b)\}$. Note that $\Omega_{b^n}=S_{b^n}\cap\psi_{b^n}$ for $n\geqq\bar n$. Further, ψ is *usc* at $\bar b$, since $\{x\epsilon S_{\bar{\bar b}}:f(x)\geqq f_{\sup}(\bar b)-\beta\}$ is compact for any $\beta>0$. Since $f_{\sup}$ is continuous at $\bar b$, $f_{\sup}(b^n)\to f_{\sup}(\bar b)$. Therefore, for any $\gamma>0$ there exists n^* such that for $n\geqq n^*$, $S_{b^n}\subset\eta_\gamma(S_{\bar b})$ and $\psi_{b^n}\subset\eta_\gamma(\psi_{\bar b})$. Hence $\Omega_{b^n}\subset(\eta_\gamma S_{\bar b})\cap(\eta_\gamma\psi_{\bar b})$. By the Lemma (proved below), for any $\epsilon>0$ there exists $\gamma>0$ such that $(\eta_\gamma S_{\bar b})\cap(\eta_\gamma\psi_{\bar b})\subset\eta_\epsilon(S_{\bar b}\cap\psi_{\bar b})$. This then implies $\Omega_{b^n}\subset\eta_\epsilon(\Omega_{\bar b})$, as desired.

Proof of the Lemma. Assume that, for all $\delta>0$, $\eta_\delta(A)\cap\eta_\delta(B)\not\subset\eta_\epsilon(A\cap B)$. Let $x_\delta\epsilon\eta_\delta(A)\cap\eta_\delta(B)$ and $x_\delta\notin\eta_\epsilon(A\cap B)$. Let $\{\delta_i\}$ be any positive sequence converging monotonically to zero. Then there is a subsequence of $\{x_{\delta_i}\}$, say $\{x_{\delta_{i_j}}\}$, such that $\{x_{\delta_{i_j}}\}\to x$, since any infinite sequence on a bounded set has a limit point. Now, since A is closed, B is closed, and $\eta_{\delta_i}(A)\supset\eta_{\delta_{i+1}}(A)$ and $\eta_{\delta_i}(B)\supset\eta_{\delta_{i+1}}(B)$, then $\bigcap_i\eta_{\delta_i}(A)=A$, $\bigcap_i\eta_{\delta_i}(B)=B$, $\bigcap_i(\eta_{\delta_i}(A)\cap\eta_{\delta_i}(B))=A\cap B$, which is compact. Since $A\cap B$ is nonempty and compact, we have $x\epsilon A\cap B$.

Thus, there exists j_0 such that, for all $j>j_0$, $x_{\delta_{i_j}}\epsilon N_\epsilon(x)\cap\eta_\epsilon(A\cap B)\subset\eta_\epsilon(A\cap B)$, which yields the desired contradiction.

Proof of Theorem 6. Clearly $cl\ I_{\bar b}\subset S_{\bar b}$, so consider $x\epsilon S_{\bar b}$. Our task is to construct a sequence $\{x^k\}\subset I_{\bar b}$ such that $\{x^k\}\to x$. Let $x^0\epsilon I_{\bar b}$ and assume each $g_i(x)$ is increasing. (If each g_i is descreasing, a similar proof follows.) Define $x_j^*=\min\{x_j^0, x_j\}$ for $j=1, \cdots, n$. Then, $g(x^*)\leqq g(x^0)<\bar b$, so $x^*\epsilon I_{\bar b}$. If $x^*=x$, then $x\epsilon cl I_{\bar b}$. Otherwise, define $x^k=\theta^k x^*+(1-\theta^k)x$, $\theta^k\epsilon(0, 1]$. Note $x^k\leqq x$ and, since $x_j>x_j^*$ for at least one j, $x_j^k<x_j$ for at least one j. Therefore, $g(x_k)<g(x)\leqq\bar b$. Hence, $\{x^k\}\subset I_{\bar b}$ as long as $\theta^k>0$. Let $\theta^k\to0$, so $x^k\to x$; and our proof is complete.

Proof of Theorem 7. Clearly $I_{\bar b}\neq\emptyset$ since $g(0)=0<\bar b$. As before, our task is to construct $\{x^k\}\subset I_{\bar b}$ such that $\{x^k\}\to x$, where x is an arbitrary point specified from $S_{\bar b}$. Consider $x^k=\alpha^k x$. If $g_i(x)<\bar b_i$, then there exists $\epsilon>0$ such that $g_i(x^k)<\bar b_i$ for $|\alpha^k-1|<\epsilon$. If $g_i(x)=\bar b_i$, then $g_i(x^k)=(\alpha^k)^{p_i}g_i(x)=(\alpha^k)^{p_i}\bar b_i$. If $p_i>0$ for all i, choose $\alpha^k=1-1/k$, and, if $p_i<0$ for all i, choose $\alpha^k=1+1/k$. In either case, since $\bar b>0$, we have $g(x^k)<\bar b$ for k sufficiently large. Therefore, $\{x^k\}\subset I_{\bar b}$ and $\{x^k\}\to x$ as $k\to\infty$.

Proof of Theorem 8. Since $cl\ I_{\bar b}\subset S_{\bar b}$, it suffices to prove that, for any $x\epsilon S_{\bar b}$, there exists a sequence $\{x^k]\to x$ such that $\{x^k\}\subset I_{\bar b}$. Let $x\epsilon S_{\bar b}$ be given by $x=\lambda A$. Define $y=\sum_{j=1}^{j=n}\theta(\lambda_j)a^j$, so $y\epsilon P(A)$. Moreover, since $g(y)=g(x)\leqq\bar b$, we have $y\epsilon S_{\bar b}(A)$. Therefore, there exists a sequence $\{y^k\}\subset I_{\bar b}(A)$ such that $\{y^k\}\to y$. Let $y^k=\sum_{j=1}^{j=n}\alpha_j^k u^j$, $0\leqq\alpha_j^k\leqq1$. Observe $\{\alpha_j^k\}\to\theta(\lambda_j)$ for each $j=1, \cdots, n$, since the representation of a vector in terms of a given basis is unique. Now define $\lambda_j^k=\lambda_j-\theta(\lambda_j)+\alpha_j^k$ and $x^k=\sum_{j=1}^{j=n}\lambda_j^k a^j$. Since $\{\alpha_j^k\}\to\theta(\lambda_j)$ for all $j=1, \cdots, n$, $\{\lambda^k\}\to\lambda$, and hence $\{x^k\}\to x$. Moreover,

$$\theta(\lambda_j^k)=\theta(\lambda_j-\theta(\lambda_j)+\alpha_j^k)=\theta(\lambda_j)-\theta(\lambda_j)+\alpha_j^k=\alpha_j^k.$$

Therefore, $g(x^k)=g(y^k)<\bar b$. Hence, $\{x^k\}\subset I_{\bar b}$, as required.

Proof of Theorem 9. Let $g(\cdot)$ be quasiconvex and consider any $x^1\neq x^2$, such that, for some i $(i=1, \cdots, m)$, $g_i(x^1)<g_i(x^2)$. We will show $g_i(\lambda x^1+(1-\lambda)x^2)<g_i(x^2)$ for all $\lambda\epsilon(0, 1)$.

Choose $b_j=\max\{g_j(x^1), g_j(x^2)\}$ for $j=1, \cdots, m$. Observe $b\epsilon B$ and consider $b^n=b-(1/n)e_i$, where e_i is the ith column of the identity matrix. For n sufficiently large, $b^n\epsilon B$, since $g_i(x^2)>g_i(x^1)$. Since S is *lsc* at b, $S_b\subset\eta_\epsilon(S_{b^n})$ for n sufficiently large. In particular, there must exist x^ϵ such that $\rho(x^2, x^\epsilon)<\epsilon$ and $g(x^\epsilon)\leqq b-(1/n)e_i$.

Consider $y^\epsilon=(1-\lambda)x^\epsilon+\lambda x^1$. Observe that $g_i(y^\epsilon)\leqq b_i-(1/n)<b_i$, since g is quasiconvex. Moreover, as $\epsilon\to 0_+$ we obtain a sequence $\{x^\epsilon\}$ such that $\{x^\epsilon\}\to x^2$, since $\rho(x^2, x^\epsilon)\to 0$. Therefore,

$$y^\epsilon=\lambda x^1+(1-\lambda)x^2+(1-\lambda)(x^\epsilon-x^2)\to\lambda x^1+(1-\lambda)x^2.$$

We have, for all $\epsilon>0$,

$$g_i(y^\epsilon)<b_i=g_i(x^2).$$

Therefore,

$$g(\lambda x^1+(1-\lambda)x^2+(1-\lambda)(x^\epsilon-x^2))<g_i(x^2).$$

Choose any $\gamma^\epsilon>0$ such that $\gamma^\epsilon<g_i(y^\epsilon)$. Since g_i is continuous and since $\rho(x^\epsilon, x^2)<\epsilon$, we have, for some $\hat{\epsilon}$,

$$g_i(\lambda x^1+(1-\lambda)x^2)\leqq g_i(\lambda x^1+(1-\lambda)x^2+(1-\lambda)(x^{\hat{\epsilon}}-x^2))+\gamma^\epsilon<g_i(x^2).$$

Therefore,

$$g_i(\lambda x^1+(1-\lambda)x^2)<g_i(x^2),$$

and, since $\lambda\epsilon(0, 1)$ was arbitrary, the proof is complete.

Proof of Theorem 10. The necessity is obvious from the definition of *usc* of Γ at y.

(*Sufficiency.*) For any $\epsilon>0$, let $\delta>0$ define $N_\delta(y)$ such that Γ is right-continuous at y for all $x\geqq y$, $x\epsilon N_\delta(y)$. Consider any $x\epsilon N_\delta(y)$ and define

$$\bar{x}_j=\begin{cases} x_j, & \text{if } x_j\geqq y_j,\\ y_j+(y_j-x_j), & \text{if } x_j<y_j.\end{cases}$$

Then $\bar{x}\geqq x$ and $\bar{x}\geqq y$ and $\bar{x}\epsilon N_\delta(y)$. Since $\bar{x}\geqq y$, then by right-continuity of Γ at y,

$$\Gamma(\bar{x})\subset\eta_\epsilon(\Gamma(y)),$$

and, since Γ is monotonically increasing,

$$\Gamma(x)\subset\Gamma(\bar{x}).$$

Thus the sufficiency is proved for *usc*. For *lsc*, define

$$\bar{x}_j=\begin{cases} x_j, & \text{if } x_j\leqq y_j,\\ y_j+(y_j-x_j), & \text{if } x_j>y_j.\end{cases}$$

Then $\bar{x}\leqq x$, $\bar{x}\leqq y$, and $\bar{x}\epsilon N_\delta(y)$, and

$$\Gamma(y)\subset\eta_\epsilon[\Gamma(\bar{x})]\subset\eta_\epsilon[\Gamma(x)],$$

since $\Gamma(\bar{x})\subset\Gamma(x)$.

ACKNOWLEDGMENT

THE AUTHORS gratefully acknowledge the numerous constructive comments made by the two referees.

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