

OPTIMAL ISSUING POLICIES IN INVENTORY MANAGEMENT—I*

WILLIAM PIERSKALLA

Case Institute of Technology

The general inventory depletion problem can be described as the problem of finding an issue policy which maximizes or minimizes a prescribed function when the inventory itself is changing in quality over time. Earlier authors writing on this subject have placed many restrictive assumptions on the model. The assumption of one demand source withdrawing items from the stockpile is removed and the case of several demand sources is considered. Next, it is assumed that there is a constant penalty cost, p , each time an item is issued. It can be described as an installation or work stoppage cost. Finally, the assumption that the field life, $L(S)$, is a concave or convex function is removed. A more general type of function is considered. $L(S)$ is a concave nonincreasing function for $S \in [0, t]$ and $L(S) = L(t) = c > 0$ for $S \geq t$. When $L(S)$ has this form, it provides a good approximation to the general decreasing S -shaped curve. In all of the foregoing cases, optimal policies or bounds on the optimal policies are presented.

The general inventory depletion problem can be described as the problem of finding an issue policy which maximizes or minimizes a prescribed function when the inventory itself is changing in quality over time. The change in quality may be either an appreciation or a deterioration of the useful life, the field life, of each item in the inventory as long as the item remains in the stockpile. An issue policy is a selected order of issue of the items in the stockpile when demands for the items are made from the field.

In order to be more specific, the particular depletion model discussed is characterized by: (1) At the beginning of the process, a stockpile has n indivisible identical items of varying ages $S_1 < S_2 < \dots < S_n$ where $S_1 > 0$. The ages S_i are called the initial ages of the items. (2) Each item has a field life $L(S)$ which is a known non-negative function of the age S of the item upon being issued. (3) Items are issued successively until either the entire stockpile is depleted or the remaining items in the stockpile have no further useful life, i.e., $L(S) = 0$ for the remaining items. (4) No penalty or installation costs are associated with the issuance of an item from the stockpile. (5) New items are never added to the stockpile after the process starts. (6) An item is issued from the stockpile only when the entire life of the preceding item issued is ended. (7) At the beginning of the process each item has positive field life, i.e., $L(S_i) > 0$ for all $i = 1, 2, \dots, n$.

The objective is to find the issue policy which maximizes the total field life of the stockpile. An issue policy which achieves this maximum is called an optimal policy.

* Received January 1966.

Since $L(S)$ must be nonnegative, then when $L(S)$ is a decreasing function of S and $L(0) > 0$, we say $S_0 \leq +\infty$ is a truncation point for $L(S)$ if and only if $S_0 = \inf \{S \in [0, \infty) \mid L(S) \leq 0\}$ and then $L(S)$ is redefined [13] to be

$$\begin{aligned} L(S) &= L(S) > 0 && \text{for all } S \in [0, S_0) \\ &= 0 && \text{for all } S \geq S_0 \end{aligned}$$

From a practical point of view, it makes little sense to permit $L(S)$ to be arbitrarily large for some S . Hence it is assumed that there is some number $k < \infty$ such that $L(S) < k$ for all S of interest. If $L(S) = 1/S$ we will assume this $L(S)$ applies only to those $S > 0$ such that $L(S) = 1/S < k$. Then if a finite number, n , of items are issued by any policy A , the total field life, Q_A , is bounded by $0 < Q_A < nk = K$ for all policies A and any n items $0 \leq S_1 < S_2 < \dots < S_n$.

Multiple Demands on the Stockpile and Bounds on the Optimal Policy

The model contains the implicit hypothesis that there is only one demand source withdrawing items from the stockpile. Except for Zehna [13] and Eilon [4], the previous work done on the deterministic inventory depletion model necessarily requires this single demand source assumption. They, however, proved that when $L(S) = aS + b$ ($b > 0 > a > -1$), FIFO is optimal for one or more demand sources. In addition, Zehna showed that if $L(S)$ is either a convex or a concave differentiable function with $L'(S) < -1$, LIFO is optimal for one or more demand sources.

In this section the assumption of a single demand source is removed. The number of demand sources requesting items from the stockpile is denoted by the letter " ν " (ν is an integer $1 \leq \nu \leq n$). We do not consider $\nu > n$ since the policy of issuing the n items to n demand sources cannot be improved. The demand sources will be denoted by M_1, M_2, \dots, M_ν .

Assumption (6) is modified as follows: (6)' An item is issued from the stockpile whenever any demand source has consumed the entire useful field life of the item previously issued to it. If two or more demand sources request a new item at the same time, the new items will be issued to them in the same order as they received their last previously issued items.

A policy is said to be *feasible* if a demand on the stockpile is always satisfied, provided the stockpile is not empty. It is easily shown that any policy which is not feasible must yield a lower total field life than some feasible policy.

It will be useful to define the notation which is used to describe a policy: (1) List the items assigned to a particular demand source in their order of use from the first item used until the last item used, and (2) separate the items for different demand sources by a semicolon. For example, a policy A can be described as follows:

$$A = [S_{11}, S_{12}, \dots, S_{1i_1}; S_{21}, \dots, S_{2i_2}; \dots; S_{\nu 1}, \dots, S_{\nu i_\nu}].$$

Note that $\sum_{j=1}^{\nu} i_j = n$ if all items are assigned. It is obvious that the choice of

M_1, M_2, \dots, M_ν , for the particular assignment of items above was arbitrary. Hence the $\nu!$ policies obtained by permuting the M_i 's are equivalent policies in the sense that the total field life obtained from the n items is unchanged regardless of how the demand sources are labelled.

It is assumed that the process begins by issuing ν items, one to each M_1, M_2, \dots, M_ν . Furthermore, we will say that $L(S)$ has *property Ω* if $L(S)$ is a continuous nonincreasing function for $0 \leq S \leq S_0$ and $L^-(S) \geq -1$ for $0 < S \leq S_0$.

The following lemmas will be used in proving the theorems of this section. The lemma proofs are in the Appendix.

Lemma 1.1: Let $L(S)$ have property Ω . Let $\nu = 1$. If the items in the stockpile are issued according to FIFO, the field life of any item at the time of issue is strictly positive.

Lemma 1.2 gives the ordering of items by FIFO when $\nu > 1$ and $L(S)$ has property Ω .

Lemma 1.2: Let $L(S)$ have property Ω . Let $\nu \geq 1$. Then starting from the oldest item S_n , FIFO assigns every ν^{th} item to the same demand source, i.e., without loss of generality we can arbitrarily let M_1 receive S_n , M_2 receive S_{n-1} , etc., to start, then demand source M_j receives items indexed by $n - k\nu - j + 1$ for all $j = 1, \dots, \nu$ and for $k = 0, 1, 2, \dots$ until all items have been assigned. Conversely, if the assignment of items is as given above, then the assignment is FIFO.

There is an interesting corollary to this lemma which states exactly how many items each demand source receives under FIFO.

Corollary 1.1: Let $L(S)$ have property Ω . If FIFO is used to assign the n items to $\nu \geq 1$ demand sources, then demand source M_j receives exactly $[(n - j)/\nu] + 1$ items ($j = 1, \dots, \nu$) where $[x]$ denotes the greatest integer $\leq x$.

Lemma 1.3: Let $L(S)$ have property Ω . Let $\nu \geq 1$. Consider two sets of n items with the following characteristics:

$$I = \{S_1, \dots, S_n \mid S_i < S_{i+1} < S_0 \quad \text{for all } i = 1, \dots, n - 1\}$$

$$II = \{S'_1, \dots, S'_n \mid S'_i < S'_{i+1} < S_0 \quad \text{for all } i = 1, \dots, n - 1\}$$

and $S_i \leq S'_i$ for all $i = 1, \dots, n$. Denote by $Q_F(n, \nu)$ and $Q'_F(n, \nu)$ the total field life by FIFO issuance of the n items to the ν demand sources with the items from Sets I and II respectively. Then

$$Q_F(n, \nu) \geq Q'_F(n, \nu).$$

Lemma 1.4: Let $L(S)$ be concave and have property Ω . Let $\nu = 2$ and $n = 3$ or $n = 4$. Then FIFO is the optimal issue policy.

As Zehna points out, the extension of the results for $\nu = 1$ to the case $\nu \geq 1$ when $L(S)$ is concave nonincreasing is not a simple matter. He gives a counter example to show that such an extension is not possible in general.

Presented below are a set of theorems which provide upper and lower bounds on the optimal policy when $\nu > 1$ and $L(S)$ is concave nonincreasing for $S \leq S_0$. These bounds for the optimal policy coincide with the bounds for the FIFO policy for the same n items and $\nu > 1$. Since we have not in general been able to

find the optimal policy analytically, FIFO may be used as a good approximation to the optimal policy.

Define $Q_{n,\nu}^*$ and $Q_F(n, \nu)$ as the total field life obtained from n items and ν demand sources when an optimal policy and a FIFO policy are followed, respectively.

Theorem 1.1: Let $L(S)$ have property Ω . Let $\nu \geq 1$. Then

$$(1) \quad Q_{n,\nu}^* \leq Q_{n,\nu+1}^* \quad \text{for any } \nu = 1, \dots, n-1.$$

$$(2) \quad Q_F(n, \nu) \leq Q_F(n, \nu + 1) \quad \text{for any } \nu = 1, \dots, n-1.$$

If one item is added to the initial stockpile prior to the issuance of any item, then

$$(3) \quad Q_{n,\nu}^* \leq Q_{n+1,\nu}^*$$

$$(4) \quad Q_F(n, \nu) \leq Q_F(n+1, \nu).$$

Proof: Only parts (1) and (2) will be proved. The proofs for parts (3) and (4) are similar, although not quite identical with (1) and (2).

Part (1): Let A be the optimal policy which achieves $Q_{n,\nu}^*$. Now $n > \nu$ implies that at least one demand source receives more than one item (or $n - \nu$ items deteriorate to zero in inventory in which case it is obvious that $Q_{n,\nu}^* \leq Q_{n,\nu+1}^*$). Consider a feasible policy $B_{\nu+1}$ of the form: use policy A for demand sources M_1, \dots, M_ν except assign the last item of policy A to demand source $M_{\nu+1}$. Since the last item is assigned at an earlier time than under policy A , the total field life of $B_{\nu+1}$ is greater. If $M_{\nu+1}$ requires a second item, assign the next to last item of policy A to $M_{\nu+1}$. Continue this method until all items are feasibly assigned. But the items assigned to $M_{\nu+1}$ have a greater total field life than under policy A , and since the field lives of the other items are unchanged,

$$Q_{n,\nu}^* \leq Q_B(n, \nu + 1) \leq Q_{n,\nu+1}^*.$$

Part (2): By Lemma 1.1 (applied to each demand source separately), the FIFO issuance of the n items in the stockpile results in *each* item having positive field life on issuance under either $F_{n,\nu}$ or $F_{n,\nu+1}$.

Let $L(S_i + x_i)$ and $L(S_i + y_i)$ for $i = 1, \dots, n$ be the field life on issuance of item S_i under policies $F_{n,\nu}$ and $F_{n,\nu+1}$, respectively. We will show $L(S_i + y_i) \geq L(S_i + x_i)$, but since $L(\cdot)$ is nonincreasing, it is only necessary to show $x_i \geq y_i$ for all i .

Case 1: $i \in \{n - \nu, n - \nu + 1, \dots, n\}$. Then $y_i = 0$, and since $x_i \geq 0$, we have $x_i \geq y_i$.

Case 2: $1 \leq i \leq n - \nu - 1$. Using Lemma 1.2, for all i , x_i and y_i have the following policies

$$F(x_i) = [S_{i+t\nu}, S_{i+(t-1)\nu}, \dots, S_{i+\nu}]$$

$$F(y_i) = [S_{i+s(\nu+1)}, S_{i+(s-1)(\nu+1)}, \dots, S_{i+\nu+1}].$$

Now $s \leq t$ since by Lemma 1.2 every ν^{th} item is assigned to the j^{th} demand source (say M_j received S_i) under $F_{n,\nu}$ and every $(\nu + 1)^{\text{st}}$ item is assigned

under $F_{n,\nu+1}$. Hence, when the $F_{n,\nu}$ policy is followed, the demand source which receives S_i will have already received more (or equal) items than the demand source which receives S_i under $F_{n,\nu+1}$.

Furthermore, $i + q\nu < i + q(\nu + 1) \Rightarrow S_{i+q\nu} < S_{i+q(\nu+1)}$ for all $q = 1, 2, \dots, s$. Now consider the FIFO policy of issuing only the s items $S_{i+\nu}, \dots, S_{i+s\nu}$ and denote this policy by A , i.e.,

$$A = [S_{i+s\nu}, S_{i+(s-1)\nu}, \dots, S_{i+\nu}].$$

Hence, by Lemma 1.3, $Q_A \geq y_i$. Furthermore, since $s \leq t$, then by part (4) of this theorem and Corollary 1.1, $Q_A \leq x_i$. Thus $x_i \geq y_i$ for $i = 1, \dots, n - \nu - 1$. Q.E.D.

Corollary 1.2: Let $L(S)$ have property Ω . Let $\nu \geq 1$.

$$(1) \quad Q_{n,\nu}^* \leq Q_{n,\nu+M}^* \quad \text{for any } \nu + M \leq n$$

$$(2) \quad Q_F(n, \nu) \leq Q_F(n, \nu + M) \quad \text{for any } \nu + M \leq n.$$

If $M \geq 1$ items are added to the stockpile prior to the issuance of any of the items, then

$$(3) \quad Q_{n,\nu}^* \leq Q_{n+M,\nu}^*$$

$$(4) \quad Q_F(n, \nu) \leq Q_F(n + M, \nu).$$

Theorem 1.2: Let $L(S)$ be concave and have property Ω . Let $\nu \geq 1$. If $[\frac{1}{2}(n + 1)] \leq \nu \leq n$, then any feasible policy which assigns more than two items to any demand source has a lower total field life than some policy which assigns at most two items to each demand source.

Proof: The proof will be outlined briefly.

Since ν is an integer $\geq \frac{1}{2}$ times the number of items in the stockpile, then when i demand sources have $k_i > 2$ items assigned to them, there are at least $\sum_i (k_i - 2)$ demand sources which have only one item assigned to them (since all demand sources must have at least one item by the initial assignment).

Consider one demand source with $k_i > 2$ items and $k_i - 2$ demand sources with only one item each. The same procedure will apply to all other demand sources with $k_j > 2$ items assigned to them. Let $i > 2$ items be assigned to M_k . In particular, let these items be denoted by $S_{k_1} < S_{k_2} < \dots < S_{k_i}$. Let M_j be a demand source with only one item assigned to it. It is easily shown that the total field life to these two demand sources is improved if the S_{k_i} 's are issued in FIFO order.

Therefore, if we let policy A be $A = [S_{k_i}, \dots, S_{k_2}, S_{k_1}; S_{j_1}]$, then $Q_A \geq Q_\psi$ where Q_ψ is any arbitrary feasible policy.

The theorem then follows since it can be shown that if $S_{j_1} < S_{k_1}$, then policy $D = [S_{k_i}, S_{k_{i-1}}, \dots, S_{k_3}, S_{k_1}; S_{k_2}, S_{j_1}]$ results in a greater total field life than policy A . And if $S_{j_1} > S_{k_1}$, then policy

$$C = [S_{k_i}, S_{k_{i-1}}, \dots, S_{k_3}, S_{j_1}; S_{k_2}, S_{k_1}]$$

results in a greater total field life than policy A .

This reduction process continues in the same manner as above until all demand sources have at most two items assigned to them. Q.E.D.

Theorem 1.3: Let $L(S)$ be concave and have property Ω . If $\lfloor \frac{1}{2}(n+1) \rfloor \leq \nu \leq n$, then FIFO is the optimal issuing policy.

Proof: Note that FIFO is feasible and that FIFO issues all n of the items, i.e., none of the items deteriorate to zero in the stockpile.

We will now show that an optimal policy for the conditions given in this theorem also must issue all of the items. This last statement is proved by contradiction. Assume that the optimal policy allows at least one item, say S_j , to expire in the stockpile. Then since $\lfloor \frac{1}{2}(n+1) \rfloor \leq \nu \leq n$ there is at least one demand source which receives only one item, say S_i . In addition $S_i < S_j$ or else by Lemma 1.1, S_j would have positive field life upon the consumption of S_i , i.e., $S_j + L(S_i) < S_0$, and S_j would then be issued. Thus, assume $S_i < S_j$. Now by Lieberman [9] Theorem 3, we have $L(S_j) + L(S_i + L(S_j)) \geq L(S_i)$ where equality holds only if $L'(S) = -1$ over the range of S_i and S_j , and strict inequality holds at all other times. Therefore, letting S_j deteriorate to zero in the stockpile can not be optimal. We obtain a contradiction to the assumption of optimality. However, S_j was a general item which deteriorated in the stockpile; thus the contradiction obtained applies to all S_j , and the optimal policy must issue all n items.

Thus, the optimal policy as well as the FIFO policy issues all items in the stockpile. Thus we restrict our attention to looking at those policies which issue all n items. Let A be one of these policies, and consider any two demand sources M_i and M_j under policy A .

Case 1: M_i receives S_{i_1}, S_{i_2} ; M_j receives S_{j_1}, S_{j_2} .

Case 2: M_i receives S_{i_1} ; M_j receives S_{j_1}, S_{j_2} .

Case 3: M_i receives S_{i_1} ; M_j receives S_{j_1} .

If the items are not assigned to M_i and M_j according to FIFO, then the total field life can be increased by a FIFO assignment. By Lemma 1.4, FIFO is optimal for $n = 3$ or 4 , $\nu = 2$.

Thus, the total field life from all demand sources can be improved until every demand source has a FIFO ordering of its items relative to every other demand source. We will call such an ordering a pairwise-FIFO ordering. Any other ordering results in a lower total field life, hence pairwise-FIFO is optimal.

We must now show that pairwise-FIFO is the same as FIFO for the total assignment of the n items to the ν demand sources. Assume the items are in pairwise-FIFO order. Now relabel the demand sources such that M_ν receives item S_n , $M_{\nu-1}$ receives item S_{n-1} , \dots , M_p receives item S_{n-i+1} , \dots , M_1 receives item $S_{n-\nu+1}$. This relabelling is possible since no two of the items $S_n, \dots, S_{n-\nu+1}$ can be assigned to the same demand source under pairwise-FIFO. Now consider the demand source M_p which has the two items S_{n-i+1} and S_{i_1} assigned to it, for any $p = 1, \dots, \nu$. We must show that $S_{i_1} = S_{n-\nu-i+1}$; then by Lemma 1.2, we have a FIFO ordering for the total assignment (since p was arbitrary). The proof of $S_{i_1} = S_{n-\nu-i+1}$ is by contradiction. Assume $S_{i_1} \neq S_{n-\nu-i+1}$.

Case 1: $S_{i_1} > S_{n-\nu-i+1}$. From above $S_{i_1} > S_{n-\nu+1}$ and since $S_{i_1} > S_{n-\nu-i+1}$,

there are at most $i - 2$ items, with initial life greater than S_{i_1} , which are available for assignment to demand sources M_ν, \dots, M_{p+1} . Now M_ν, \dots, M_{p+1} are the first $i - 1$ demand sources to consume their initial items, hence, some item $S_{j_1} < S_{i_1}$ must be assigned to one of these $i - 1$ demand sources, say demand source M_{p+j} ($j \geq 1$). Then the pairwise ordering for M_p and M_{p+j} is

$$[S_{n-i+1}, S_{i_1}; S_{n-i+1+j}, S_{j_1}];$$

but $S_{n-i+1} < S_{n-i+1+j}$ and $S_{i_1} > S_{j_1}$ is not a FIFO ordering, hence we obtain a contradiction to the assumption of pairwise-FIFO. Therefore $S_{i_1} \succ S_{n-p-i+1}$.

Case 2: $S_{i_1} < S_{n-p-i+1}$. A similar argument shows $S_{i_1} \prec S_{n-p-i+1}$. Thus $S_{i_1} = S_{n-p-i+1}$. Q.E.D.

Addition of Penalty Costs

The removal of assumption (4) which states that there are no penalty costs is important not only because it is often the case in practical situations where there is an installation or work-stoppage cost, but also because the optimal policy in the model without penalty costs may no longer be optimal when penalty costs are added. For example, in the no penalty cost case, the optimal policy may issue a large number of items, whereas some other policy may issue only a few items. Then if the penalty cost is sufficiently large, the policy which was optimal could easily become the worst policy after subtracting the penalty costs.

It is assumed that there is a constant penalty cost, p , associated with the issuance of each item from the stockpile. Furthermore, it is assumed that p is defined in the same units of measure as $L(S)$.

Denote by $R_A(i, \nu)$ the total return obtained from the issuance of i items to ν demand sources in accordance with policy A ; $R_A(i, \nu) = Q_A(i, \nu) - ip$. The objective will be to find a policy which will maximize R over all possible policies.

It is conceivable that in issuing an item which has positive field life, the net increase (if any) in the total field life may be more than offset by the penalty cost incurred. Because of this event, we will also remove the assumption that an item must be issued if it has positive field life. In its place, we will merely assume that any item with zero field life will not be issued. Furthermore, we will assume that there is no cost associated with the disposal of items which are not issued.

It should be noted that to start the process, it may no longer be optimal to issue ν items to M_1, \dots, M_ν . If the optimal policy calls for the issuance of only $i < \nu$ items, then the i items would be issued immediately and the process would terminate.

Define: for $j \leq n$, (i) $A_{j,\nu}$ is any policy of issuing j items to ν demand sources, (ii) $F_{j,\nu}$ is the policy of issuing the *same* j items as are issued in (i) to ν demand sources by FIFO, (iii) $F_{(j,\nu)^*}$ is the policy of issuing the *youngest* j items to ν demand sources by FIFO.

Theorem 2.1: Let $L(S)$ be concave and have property Ω . Let $\nu \geq 1$. If FIFO is the issuing policy which maximizes the total field life for any j items in inventory, then the optimal issuing policy must be one of the n policies $F_{(1,\nu)^*}, F_{(2,\nu)^*}, \dots, F_{(n,\nu)^*}$.

Proof: The proof of this theorem merely consists of showing

$$(2.1.1) \quad R_F(j, \nu)^* \geq R_F(j, \nu) \geq R_A(j, \nu) \quad \text{for all } j = 1, \dots, n \text{ and } \nu \geq 1.$$

By hypothesis $Q_F(j, \nu) \geq Q_A(j, \nu)$ and by Lemma 1.3 $Q_F(j, \nu)^* \geq Q_F(j, \nu)$, then (2.1.1) follows from the definition of R . Q.E.D.

Corollary 2.1: If $L(S) = aS + b$ where $b > 0 > a > -1$ and $\nu \geq 1$, then the optimal issue policy must be one of the n policies $F_{(1,\nu)^*}, F_{(2,\nu)^*}, \dots, F_{(n,\nu)^*}$.

Corollary 2.2: Let $L(S)$ be concave and have property Ω . Let $\nu = 1$. Then the optimal policy is one of the n policies $F_{(1,1)^*}, F_{(2,1)^*}, \dots, F_{(n,1)^*}$.

Thus, Theorem 2.1 and Corollaries 2.1 and 2.2 state that it is only necessary to search n policies until the optimal $F_{(j,\nu)^*}$ is found; then issue the j newest items by FIFO and discard the remaining $n - j$ items without issuing them even if they have positive field life. Their positive field life is offset by the penalty cost of installation.

In certain cases it is possible to select analytically the optimal policy from the n policies $F_{(1,\nu)^*}, \dots, F_{(n,\nu)^*}$.

Lemma 2.1: Let $L(S)$ be a concave nonincreasing function and a be any real number with $0 > a > -1$. Let S_{j_1}, \dots, S_{j_i} be any i items with $S_{j_k} < S_{j_{k+1}}$ for all $k = 1, \dots, i - 1$ and $S_{j_i} < S_0$. $F_i = [S_{j_i}, S_{j_{i-1}}, \dots, S_{j_1}]$ and $F_{i-1} = [S_{j_{i-1}}, \dots, S_{j_1}]$ are two FIFO policies for issuing the i and $i - 1$ youngest items, respectively:

(1) If $L^-(S) \geq a$ for $0 < S \leq S_0$ then

$$Q_F(i) - Q_F(i - 1) - (1 + a)^{i-1}L(S_{j_i}) > 0,$$

(2) if $L^-(S) \geq -1$ and $L^+(S) \leq a$ for $0 \leq S \leq S_0$ then

$$Q_F(i - 1) - Q_F(i) + (1 + a)^{i-1}L(S_{j_i}) \geq 0,$$

(3) if $L(S) = aS + b$ for $0 \leq S \leq S_0$ and $b > 0$ then

$$Q_F(i) - Q_F(i - 1) - (1 + a)^{i-1}L(S_{j_i}) = 0.$$

The proof of each part (1), (2) and (3) follows easily by induction and will not be given here.

Theorem 2.2: Let $L(S)$ be a concave nonincreasing function, and let $0 > a > -1$ be any real number. Let $\nu \geq 1$. Item S_{j+1} is the $t + 1^{\text{st}}$ item remaining to be issued to some M_k (hence $t = [j/\nu]$). If FIFO maximizes the total field life, then

(1) if $L^-(S) \geq a$ for $0 < S \leq S_0$ and if for some j , $0 < p \leq (1 + a)^t L(S_{j+1})$ then

$$R_F(j + 1, \nu)^* \geq R_F(j - i, \nu)^* \quad \text{for all } i = 0, \dots, j - 1;$$

(2) if $L^-(S) \geq -1$ and $L^+(S) \leq a$ for $0 \leq S \leq S_0$ and if $p \geq (1 + a)^t L(S_{j+1})$ for some j then

$$R_F(j, \nu)^* \geq R_F(j + i, \nu)^* \quad \text{for all } i = 1, \dots, n - j$$

Proof: Part (1) is proved by showing $R_F(k, \nu)^* \geq R_F(k - 1, \nu)^*$ for all

$k = 2, \dots, j + 1$; and this latter follows by writing out $R_F(k, \nu)^* - R_F(k - 1, \nu)^*$ for $k \leq \nu$ and for $j \geq k > \nu$.

Part (2). Let $\nu = 1$. By Lemma 2.1

$$R_F(k, 1)^* - R_F(k + 1, 1)^* = Q_F(k, 1)^* - Q_F(k + 1, 1)^* \\ + p \geq Q_F(k, 1)^* - Q_F(k + 1, 1)^* + (1 + a)^k L(S_{k+1}) \geq 0.$$

Let $\nu > 1$, and reindex the items as follows:

Demand Source	Items Assigned to M_k Using Lemma 1.2	Items Assigned to M_k Using Corollary 1.1
M_k	$[S_{n-k+1}, S_{n-k-\nu+1}, \dots]$	$[S_{[(n-k)/\nu]+1}^{(k)}, \dots, S_1^{(k)}]$

Then in this notation $S_{j+1} \equiv S_{i+1}^{(k)}$, and $p \geq (1 + a)^t L(S_{i+1}^{(k)}) \geq (1 + a)^t L(S_{i+1}^{(r)}) > (1 + a)^{t+1} L(S_{i+1}^{(r)}) \geq (1 + a)^{t+1} L(S_{i+2}^{(s)})$ for $r = 1, \dots, k$ and $s = k + 1, \dots, \nu$.

Using these inequalities then the case for $\nu = 1$ above can be applied to each demand source separately. Hence, all items older than S_j may be discarded immediately and $R_F(j, \nu)^* \geq R_F(j + 1, \nu)^*$ for all $i = 1, \dots, n - j$. Q.E.D.

An algorithm for obtaining the optimal policy when $L(S)$ is linear and $\nu \geq 1$ is presented below. It is useful to define an augmented set of items and a search procedure. Assumption (1) of the model states that the process starts initially with n items of initial ages $0 < S_1 < S_2 < \dots < S_n$. The augmented set of items is the set of $n + \nu$ items $0 < S_1 < S_2 < \dots < S_{n+\nu-1} < S_{n+\nu}$ where $L(S_i) > 0$ for all $i = 1, \dots, n + \nu$ and where the penalty cost p has $p \geq (1 + a)^{[(n-\nu)/\nu]+1} L(S_{n+1}) = (1 + a)^{[n/\nu]} L(S_{n+1})$.

Note that it is always possible to find items $S_{n+1}, \dots, S_{n+\nu}$ which satisfy the augmented system since $S_n < S_0$ and p is a fixed positive constant. It will be shown in Theorem 2.3 that the optimal policy for the augmented set is the same as for the original set. We now define the search procedure.

Search Procedure: Using the reindexed method of labelling the items issued to each demand source, consider all adjacent pairs of items for each demand source starting with the oldest adjacent pair for M_1 , viz. $S_{[(n-1)/\nu]+2}^{(1)}$ and $S_{[(n-1)/\nu]+1}^{(1)}$, then the oldest pair for M_2 viz. $S_{[(n-2)/\nu]+2}^{(2)}$ and $S_{[(n-2)/\nu]+1}^{(2)}$, etc., for M_3 through M_ν . Then consider the second oldest adjacent pair for M_1 viz. $S_{[(n-1)/\nu]+1}^{(1)}$ and $S_{[(n-1)/\nu]}^{(1)}$, etc., for M_2 through M_ν . Continue in this manner searching all adjacent pairs in order of their age from oldest pair to the youngest pair.

Theorem 2.3: Let $L(S) = aS + b$ for $0 \leq S \leq S_0$ and $b > 0 > a > -1$. Let $\nu \geq 1$.

Two cases are possible:

- (i) If $p \geq L(S_1)$, then the penalty cost is greater than the value received from any item, hence it does not pay to issue any item; but if ν items must be issued, then $F_{(\nu, \nu)^*}$ is optimal.
- (ii) If $p \not\geq L(S_1)$, then apply the *Search Procedure* to the *augmented* set of items. If for some demand source, M_i , it is the *first* demand source such that for some $j = 1, \dots, [(n - j)/\nu]$

$$(2.3.1) \quad (1 + a)^{j-1}L(S_j^{(i)}) > p \geq (1 + a)^jL(S_{j+1}^{(i)})$$

then the FIFO policy which issues all items of initial age less than or equal to $S_j^{(i)}$ and discards all items strictly older than $S_j^{(i)}$ is the optimal issuing policy. That is, if $S_j^{(i)} \equiv S_t$, then $F_{(t,\nu)^*}$ is optimal.

Proof: Part (i) is obvious.

Part (ii): Assume there exists an $S_j^{(i)}$ and $S_{j+1}^{(i)}$ with the property that upon application of the Search Procedure these two items are the first adjacent pair found to satisfy (2.3.1). If not, we have part (i) above. Now by Theorem 2.2, $p < (1 + a)^{j-1}L(S_j^{(i)})$ implies $R_F(t, \nu)^* \geq R_F(t - i, \nu)^*$ for all $i = 1, \dots, t - 1$; also by Theorem 2.2, $p \geq (1 + a)^jL(S_{j+1}^{(i)})$ implies

$$R_F(t, \nu)^* \geq R_F(t + i, \nu)^*$$

for all $i = 1, \dots, n + \nu - t$. Therefore $F_{(t,\nu)^*}$ is an optimal policy. Q.E.D.

We now consider cases when LIFO-type policies are optimal. Define $L_{(i,\nu)^*}$ as the LIFO issuance of the i youngest items in the stockpile to the ν demand sources. Then we obtain Theorem 2.4 for LIFO which is similar to Theorem 2.1 for FIFO.

Theorem 2.4: Let $L(S)$ be a concave nonincreasing function. Let $\nu \geq 1$. If LIFO is the issuing policy which maximizes the total field life for any i items in inventory, then the optimal issuing policy which maximizes the total return must be one of the n policies $L_{(1,\nu)^*}, L_{(2,\nu)^*}, \dots, L_{(n,\nu)^*}$.

The proof of this theorem merely consists of showing that if $L_{i,\nu}$ is the policy of issuing any i items to ν demand sources by LIFO, then the total field life from $L_{(i,\nu)^*}$ is greater than the total field life from $L_{i,\nu}$.

Corollary 2.2: Let $L(S)$ be linear on $[0, S_0]$ with $L'(S) = -1$ on $[0, S_0]$. Let $\nu \geq 1$. The optimal policy which maximizes the total return must be one of the n policies $L_{(1,\nu)^*}, \dots, L_{(n,\nu)^*}$.

Theorem 2.5: Let $L(S)$ be a convex or a concave differentiable function on $[0, S_0]$ with $L'(S) < -1$ on $[0, S_0]$. Let $\nu \geq 1$. The optimal policy which maximizes the total return must be one of the ν policies $L_{(1,\nu)^*}, \dots, L_{(\nu,\nu)^*}$.

Proof: By Zehna [13] Theorems 2.4, 2.6, 4.2 and 4.3, LIFO maximizes the total field life for any i items and ν demand sources. Now LIFO also issues the minimum number of items since under LIFO after the first ν items are issued to start the process, all other items $S_k > S_\nu$ have field life of zero when they are to be issued. Hence LIFO maximizes the total field life and minimizes the total penalty costs. Q.E.D.

It is also interesting to point out that if only i items where $i \leq \nu$ have the property that $L(S_j) > p$ for $j = 1, \dots, i$, then $L_{(i,\nu)^*}$ is the optimal policy for the general case of Theorems 2.4 and 2.5 and for Corollary 2.2.

The proof of this statement is the same as given in Theorem 2.5.

S-shaped Field Life Functions

It may be the case that for a certain type of inventory item, the actual field life function may not be convex or concave but is an S-shaped function. For

example, $L(S) = (-S + 3)^{(1/3)} + 2$ for $0 \leq S \leq 11$, $L(S) = 0$ otherwise.

In this section, a particular type of S -shaped function is examined. It has the property that when there are n items in inventory, the optimal policy must be one of n policies. It has the added property that it can be used as an approximation to more general S -shaped functions. The S -shaped function considered is: $L(S)$ is concave nonincreasing for all $S \in [0, t]$ where $t > 0$ and $L(S) = c > 0$ for all $S \in [t, \infty)$.

It will be useful to define two models for the field life function, $L(S)$.

Model I: $L_1(S)$ is concave nonincreasing for all $S \in [0, S_0]$, $L(S) = 0$ for all $S \in [S_0, \infty)$ and $L_1^-(S) \geq -1$ for all $S \in (0, S_0]$.

Model II: $L_1(S)$ coincides with $L_1(S)$ in Model I for all $S \in [0, t]$ where $t < S_0$ and $L_2(S) = c > 0$ for all $S \in [t, \infty)$.

We will now state a series of lemmas which will be used in the proofs for the theorems in this section. The lemma proofs can be found in the Appendix. Furthermore, since the words "field life" are used so frequently throughout this section, they will be abbreviated f.l.

Lemma 3.1: Let $\nu \geq 1$. If a FIFO issuing policy is used in both Model I and Model II, then $Q_F(II) \geq Q_F(I)$.

Define F_iA as the policy which issues the i youngest items by FIFO first and then the remaining $n - i$ items by any arbitrary policy A . F_iL implies $A = LIFO$.

Lemma 3.2: Let $\nu \geq 1$. If B is any arbitrary policy which results in exactly i items having f.l. $> c$ on issuance and the remaining $n - i$ items having f.l. $= c$ on issuance and if FIFO is optimal for Model I, then in Model II, $Q_{F_iA} \geq Q_B$.

Lemma 3.3: Let $L(S) = aS + b$ for all $S \in [0, S_0]$ where $b > 0 > a > -1$ and $S_0 = -(b/a)$. Let $\nu = 1$. If a FIFO issuing policy is used then the total f.l. is $Q_F(n) = a \sum_{i=1}^n (1+a)^{i-1} S_i + (b/a)[(1+a)^n - 1]$.

Lemma 3.4: Let c, b, a be given real numbers such that $b > c > 0 > a > -1$. Then the function $H_i = [c - b(1+a)^{i-1}]/[a(1+a)^{i-1}]$ for $i = 1, 2, 3, \dots$ is a strictly decreasing function of i .

Lemma 3.5: Let $L(S) = aS + b$ for all $S \in [0, t]$ and $L(S) = c > 0$ for all $S \in [t, \infty)$ where $b > c > 0 > a > -1$. Let $\nu = 1$.

- (i) If $S_i \leq H_i$ for some $i = 2, \dots, n$ and if a FIFO issuing policy is used for S_i, S_{i-1}, \dots, S_1 , then the f.l. on issuance of each of the items S_i, S_{i-1}, \dots, S_1 is strictly greater than c .
- (ii) If $S_1 < (c - b)/a = t$, then the f.l. of S_1 is strictly greater than c , and if $S_1 > t$, then the f.l. of all items on issuance is equal to c .

Lemma 3.6: Let $L(S) = aS + b$ for all $S \in [0, t]$ and $L(S) = c$ for all $S \in [t, \infty)$ where $b > c > 0 > a > -1$. Let $\nu = 1$. If $S_i \leq H_i$ and $S_{i+1} \geq H_{i+1}$ for some $i = 1, \dots, n - 1$, then for any F_jL policy with $j \geq i$, the age of item S_{i+1} when it is issued is $\geq t$. Hence, item S_{i+1} has f.l. $= c$ on issuance. Consequently, all S_{i+1+k} for $k = 0, 1, \dots, n - (i + 1)$ have f.l. equal to c on issuance.

We will now use the preceding lemmas to obtain (i) the set of n policies which contains the optimal policy for Model II and (ii) the specific optimal policy when $L_1(S)$ is linear.

Theorem 3.1: Let $L(S)$ be a concave nonincreasing function for all $S \in [0, t]$ and $L(S) = c > 0$ for all $S \in [t, \infty)$. Let $L'(S) \geq -1$ for all $S \in (0, t]$. Let $\nu \geq 1$. Assume FIFO is optimal for Model I. Then the optimal policy is one of the n policies F_iL where $i = 1, \dots, n$. Furthermore, if $S_i < t$, the optimal policy F_iL has the property that the i FIFO issued items have f.l. $> c$ on issuance and the remaining $n - i$ LIFO issued items have f.l. $= c$ on issuance.

Proof: Lemma 3.2. reduces the search for the optimal policy to F_iA for $i = 1, \dots, n$. By repeated application of Lemma 3.2 and this theorem, the optimal policy must have the property that the $n - i$ items issued by A all have f.l. $= c$ on issuance. Hence, A no longer needs to be an arbitrary policy but can be reduced to any fixed policy. Thus, we arbitrarily let $A = \text{LIFO}$, and we only need to search the policies F_1L, F_2L, \dots, F_nL such that the LIFO issued items have f.l. $= c$ on issuance. The only part of the theorem remaining to be proved is that the first i items issued by the optimal policy have f.l. $> c$ on issuance. This last point follows since any F_jL policy where the first j items do not have f.l. $> c$ yields less total f.l. than some F_iL policy which has first i items issued with f.l. $> c$. Q.E.D.

At this point, it is worth noting that if $\nu = 1$ or if $L_1(S)$ is linear, then the assumption that FIFO is optimal for Model I can be removed in Theorem 3.1.

Theorem 3.1 reduces the search for the optimal policy to those F_iL 's with the property that the first i items have f.l. $> c$ and the last $n - i$ items have f.l. $= c$ on issuance. That this property is not unique to the optimal policy is shown by the following example:

$$\begin{aligned} L(S) &= 1.5 && \text{for } 0 \leq S \leq 1.5 \\ &= -\frac{1}{3}S + 2 && \text{for } 1.5 \leq S \leq 4.5 \\ &= 0.5 && \text{for } 4.5 \leq S \end{aligned}$$

For $S_1 = 2.0$, $S_2 = 4.0$, $S_3 = 5.0$ and $S_4 = 6.0$ and $\nu = 1$, then $Q_{F_1L} = 2.8333$, $Q_{F_2L} = 2.777$, $Q_{F_3L} = 2.500$, and $Q_{F_4L} = 2.333$. $F_1L = [S_1, S_2, S_3, S_4]$ is optimal. But both F_1L and F_2L have the property that the FIFO issued items have f.l. $> c$ and the LIFO issued items have f.l. $= c$ on issuance. Hence it is not sufficient to locate any F_iL with the requisite properties; it is necessary to check all such F_iL policies. However, in Model II when we let $L_1(S) = aS + b$ with $b > c > 0 > a > -1$, we are able to isolate the unique optimal F_iL policy. Furthermore, if $L_1(S)$ is concave or convex and $L'_1(S) \leq -1$, then F_1L is optimal.

Theorem 3.2: Let $L(S) = aS + b$ for all $S \in [0, t]$ and $L(S) = c$ for all $S \in [t, \infty)$ where $b > c > 0 > a > -1$. Let $\nu \geq 1$. Using the item indexing notation of Theorem 2.3:

- (a) If $S_j \equiv S_{[(n-j)/\nu]+1}^{(j)} \leq H_{[(n-j)/\nu]+1}$ and $S_{[(n-j+1)/\nu]+1}^{(j-1)} > H_{[(n-j+1)/\nu]+1}$ for some $j = 1, \dots, \nu$, then F_jL is the optimal policy.

- (b) If $S_j \equiv S_1^{(j)} \geq (c - b)/a$ for some $j = 1, \dots, \nu$, then F_jL is the optimal policy. (In this case $F_jL = \text{LIFO}$.)
- (c) If neither (a) nor (b) is satisfied, then use the *Search Procedure* (defined previously) and consider all adjacent pairs of items for each demand source starting with the oldest adjacent pair and ending with the newest. Then if M_j is the *first* demand source such that for two adjacent items $S_i^{(j)} \equiv S_I$ and $S_{i+1}^{(j)} \equiv S_{I+\nu}$ assigned to M_j

$$(3.2.1) \quad S_i^{(j)} \leq H_i$$

$$(3.2.2) \quad S_{i+1}^{(j)} > H_{i+1}$$

for some $I \in \{\nu + 1, \dots, n - \nu - 1\}$, then $F_I L$ is the optimal policy.

Proof: We defer the proof of (a) until after we have proved (b) and (c).

Part (b): $S_j \equiv S_1^{(j)} \geq t$ implies all $S_i > S_1^{(j)} \geq t$ and $L(S_i) = c$ for all $i \geq J$. But then less than ν items have initial f.l. $> c$ and all $n - \nu$ or more items have initial f.l. $= c$. It is optimal to issue immediately the $J - 1$ or less items with f.l. $> c$ and then issue the remaining items by any policy. But policy F_jL does precisely this. Hence F_jL is optimal.

Part (c): Since $S_i^{(j)}$ is the first (in the sense of oldest) item for which (3.2.1) and (3.2.2) hold, then for all $0 < j - k < j$ where $(k = 1, \dots, j - 1)$

$$(3.2.3) \quad S_i^{(j-k)} > H_i$$

for all $j + k > j$ where $(k = 1, \dots, \nu - j)$

$$(3.2.4) \quad S_{i+1}^{(j+k)} > H_{i+1}.$$

In addition, for all $S_p \leq S_i^{(j)}$, we have

$$(3.2.5) \quad S_p \leq H_i.$$

In (3.2.5) we consider in particular

$$(3.2.6) \quad S_i^{(j+k)} > H_i \quad \text{for all } k = 1, \dots, \nu - j$$

$$(3.2.7) \quad S_{i-1}^{(j-k)} < H_i < H_{i-1}$$

for $k = 1, \dots, j - 1$, by Lemma 3.4. Hence, combining (3.2.3) with (3.2.7) and (3.2.4) with (3.2.6), we have the case that all $\nu - 1$ pairs of items following the *first* pair, $S_i^{(j)}$ and $S_{i+1}^{(j)}$, also satisfy conditions (3.2.1) and (3.2.2) of the theorem. We will now show

$$(3.2.8) \quad Q_{F_I L} \geq Q_{F_{I-k} L} \quad \text{for all } k = 1, \dots, I - 1$$

$$(3.2.9) \quad Q_{F_I L} \geq Q_{F_{I+k} L} \quad \text{for all } k = 1, \dots, n - I.$$

We first prove (3.2.8). By Lemma 3.5 the first I items issued under $F_I L$ have f.l. $> c$. Now $F_I L$ and any $F_{I-k} L$ policy have S_{I+1}, \dots, S_n with f.l. $= c$ on issuance. Furthermore, Theorem 3.1 allows us to restrict our search for the optimal policy to those $F_{I-k} L$'s which have the properties (i) the first $I - k$ items have f.l. $> c$ on issuance and (ii) the remaining $n - I + k$ items have f.l. $= c$ on issuance. Form $Q_{F_I L} - Q_{F_{I-k} L}$ for $k \geq 0$.

It will be convenient to change our notation in regard to the items assigned to M_q to say M_q receives items indexed by $q + h\nu$ for $h = 0, 1, 2, \dots$ where $q + h\nu < n$.

Under $F_I L$, M_q receives i items in the FIFO part of the policy and under $F_{I-k} L$, M_q receives, say $i - k_q$ items in the FIFO part of the policy where $\sum_{r=1}^{\nu} k_r = k$. Now denote by $Q_M(q, i)$ and $Q_M(q, i - k_q)$ the total f.l. of the first i items and the first $i - k_q$ items issued to M_q by FIFO under $F_I L$ and $F_{I-k} L$, respectively.

Applying Lemma 2.1 we obtain

$$\begin{aligned} Q_M(q, i) - Q_M(q, i - k_q) &= \sum_{j=0}^{k_q-1} [Q_M(q, i - j) - Q_M(q, i - j - 1)] \\ &= \sum_{j=0}^{k_q-1} (1 + a)^{i-j-1} L(S_{i-j}^{(q)}) \geq \sum_{j=0}^{k_q-1} (1 + a)^{i-j-1} L(S_I) \\ &> k_q(1 + a)^{i-1} L(S_I) \end{aligned}$$

since $1 > 1 + a > 0$ and $L(S_{i-j}^{(q)}) \geq L(S_I)$.

But then we have $Q_{F_I L} - Q_{F_{I-k} L} > (1 + a)^{i-1} L(S_I) \sum_{r=1}^{\nu} k_r - kc$ where $-kc$ appears since $Q_{F_{I-k} L}$ has k more items with f.l. = c than does $Q_{F_I L}$. Thus,

$$\begin{aligned} Q_{F_I L} - Q_{F_{I-k} L} &> (1 + a)^{i-1} L(S_I)k - kc = k[(1 + a)^{i-1}(aS_I + b) - c] \\ &\geq k[(1 + a)^{i-1}\{aH_i + b\} - c] \\ &= k[(1 + a)^{i-1}b + c - b(1 + a)^{i-1} - c] = 0. \end{aligned}$$

Therefore (3.2.8) holds.

We now prove (3.2.9). By Theorem 3.1 we only need to consider policies where the first $I + k$ items have f.l. $> c$ on issuance and remaining $n - I - k$ items have f.l. = c . Now under the FIFO part of $F_{I+k} L$, each demand source will have k_q more items assigned than under $F_I L$, where $k_q \geq 0$ and $\sum_{i=1}^{\nu} k_i = k$. Then by applying Lemma 2.1 again we obtain

$$\begin{aligned} Q_M(q, i) - Q_M(q, i + k_q) &= \sum_{j=0}^{k_q-1} [Q_M(q, i + j) - Q_M(q, i + j + 1)] \\ &= -\sum_{j=0}^{k_q-1} (1 + a)^{i+j} L(S_{i+j+1}^{(q)}) \\ &\geq -\sum (1 + a)^{i+j} L(S_{I+1}) > -k_q(1 + a)^i L(S_{I+1}) \end{aligned}$$

since $L(S_{I+1}) \geq L(S_{i+j+1}^{(q)})$. Now

$$\begin{aligned} Q_{F_I L} - Q_{F_{I+k} L} &> -\sum_{r=1}^{\nu} k_r(1 + a)^i L(S_{I+1}) + (I + k - I)c \\ &= -k(1 + a)^i(aS_{I+1} + b) + kc > -k(1 + a)^i(aH_{i+1} + b) + kc = 0. \end{aligned}$$

Therefore (3.2.9) holds. Thus for part (c), $F_I L$ is optimal since (3.2.8) and (3.2.9) hold. We now prove part (a). But (a) is just a special case of part (c). Q.E.D.

Theorem 3.3: Let $L(S)$ be a concave or convex decreasing function with $L^-(S) \leq -1$ and $L^+(0) \leq -1$ for all $S \in [0, t]$. Let $L(S) = c$ for all $S \in [t, \infty)$. Let $\nu \geq 1$. Then LIFO is the optimal policy.

The proof of this theorem follows by proving it for $\nu = 1$ and for $\nu = 2$ and

then by Zehna [13] Theorem 4.3, LIFO is optimal for all $1 \leq \nu \leq n$, ν integer. Since this proof is quite easy, it will not be presented here.

Appendix

Proof of Lemma 1.1: If $S_0 = +\infty$, the lemma is trivially true. For $S_0 < +\infty$ the lemma follows easily by induction.

Proof of Lemma 1.2: The proof of this lemma can be accomplished by the use of induction in several parts. Let $k = 1$. Then it can be shown that the first round of items issued after the initial assignment is issued in the required order. Then assume the lemma is true for $k = t$, and it is easily proved true for $k = t + 1$.

Proof of Corollary 1.2.1: By Lemma 1.2, any demand source M_j receives items indexed by $n - k\nu - j + 1$ for $k = 0, 1, \dots, t$ where t is the largest integer such that $n - t\nu - j + 1 \geq 1$. Thus $n - t\nu - j \geq 0$ and $t \leq (n - j)/\nu$. But t is the largest integer satisfying this condition, hence $t = [(n - j)/\nu]$. Now since k takes $t + 1$ values, M_j receives exactly $1 + [(n - j)/\nu]$ items. Q.E.D.

Proof of Lemma 1.3: For the case $\nu = 1$, the lemma is easily proved by induction on n .

Thus let $\nu > 1$. Using Lemma 1.2, the following assignment of items holds:

Demand Source	Set I	Set II	Total Field Life Contributed by M_i	
			Set I	Set II
M_j	$[S_{n-j+1}, \dots, S_{n-k\nu-j+1}]$	$[S'_{n-j+1}, \dots, S'_{n-k\nu-j+1}]$	x_j	x'_j

where the subscripts on the S 's are such that $n - k\nu - i \geq 1$ for all $k = 0, 1, \dots$ and $i = 0, 1, \dots, \nu - 1$, i.e., the inventory is exhausted. Note that Lemma 1.2 says that the subscripts on the items for a particular demand source are the same for both Sets I and II. Hence, the items assigned to M_j from Sets I and II obey the conditions (i) $S_{n-k\nu-j+1} \leq S'_{n-k\nu-j+1}$ for all $k = 0, 1, 2, \dots$, and (ii) there is the same number of items assigned to M_j from Set I as there is from Set II. But these conditions hold for all M_j , $j = 1, \dots, \nu$. Hence by the case for $\nu = 1$, $x_j \geq x'_j$ for all $j = 1, \dots, \nu$. Q.E.D.

Proof of Lemma 1.4: The proof of Lemma 1.4 is obtained by the enumeration and elimination of all non-FIFO policies.

Proof of Lemma 3.1: For $\nu = 1$, the proof follows by induction on n . Let $\nu > 1$. By Lemma 1.2 each demand source receives the same indexed items (and in the same order) under both models. Hence we may consider each demand source separately. But for $\nu = 1$, $Q_{II}(M_i) \geq Q_I(M_i)$ for all $i = 1, \dots, \nu$; therefore, $Q_F(II) = \sum_{i=1}^{\nu} Q_{II}(M_i) \geq \sum_{i=1}^{\nu} Q_I(M_i) = Q_F(I)$. Q.E.D.

Proof of Lemma 3.2: Let the i items in policy B which have f.l. $> c$ be distributed by the rule: demand source M_j receives items $[S_{j1}, \dots, S_{jk_j}]$ for $j = 1, \dots, \nu$, where $i = \sum_{j=1}^{\nu} k_j$.

Now for any M_j , we can locate k_j ages $S'_{j1}, S'_{j2}, \dots, S'_{jk_j}$ in Model I such that the field life from each of the k_j items in Model I is the same as the f.l. of each of the k_j items in Model II under policy B and in the same order.

Denote the total f.l. of the i items in Model *II* under policy *B* by $x_B(i)$. Denote the total f.l. of the i relocated items in Model *I* by Q_i . Thus $x_B(i) = Q_i$.

Furthermore, denote by $Q_F(i)$ and $Q_F^*(i)$ the total field lives of the i relocated items in Model *I* and the i youngest items (S_i, \dots, S_1) , respectively, where in both cases FIFO is used. [We know that the i youngest items must have $S_i < S_0$ or else in Model *II* under policy *B* there could not be i items with f.l. $> c$.]

Since FIFO is optimal in Model *I*, then $Q_F(i) \geq Q_i$ and by Lemma 1.3 $Q_F^*(i) \geq Q_F(i)$. But $Q_F^*(i) \equiv Q_F(I)$ of Lemma 3.1; thus $Q_F(II) \geq Q_F(I) \equiv Q_F^*(i) \geq Q_F(i) \geq Q_i = x_B(i)$, where $Q_F(II)$ is the total f.l. from the FIFO issuance of the i youngest items in Model *II*. If we denote the total f.l. from the remaining $n - i$ items in policy F_{iA} by $Q_A(n - i)$, then

$$Q_{F_{iA}} = Q_F(II) + Q_A(n - i) \geq x_B(i) + (n - i)c = Q_B.$$

Since *B* was any arbitrary policy with exactly i items with f.l. $> c$, then F_{iA} dominates any policy with this characteristic. Q.E.D.

Proof of Lemma 3.3: Let $n = 1$, then $Q_F(1) = aS_1 + (b/a)(1 + a - 1) = aS_1 + b$ as required. Assume the lemma is true for $n = k$. Then

$$\begin{aligned} Q_F(k + 1) &= aS_{k+1} + b + a \sum_{i=1}^k (1 + a)^{i-1} (S_i + aS_{k+1} + b) \\ &+ (b/a)[(1 + a)^k - 1] = a \sum_{i=1}^{k+1} (1 + a)^{i-1} S_i + (b/a)[(1 + a)^{k+1} - 1]. \end{aligned}$$

Q.E.D.

Proof of Lemma 3.4:

$$\begin{aligned} H_i - H_{i+1} &= [c - b(1 + a)^{i-1}]/[a(1 + a)^{i-1}] \\ &- [c - b(1 + a)^i]/[a(1 + a)^i] = c/(1 + a)^i > 0. \end{aligned}$$

Q.E.D.

Proof of Lemma 3.5: Part (ii) is obvious.

Part (i): By Lemma 3.4, $S_i \leq H_i < t$ for $i = 2, \dots, n$; hence $L(S_i) > c$. Let x_{i-j} be the total f.l. from the FIFO issuance of $S_i, S_{i-1}, \dots, S_{i-j}$. Assume item S_{i-j} has f.l. $> c$ on issuance. It will be proved that item S_{i-j-1} has f.l. $> c$ on issuance. Now

$$\begin{aligned} x_{i-j} &= a \sum_{k=1}^{j+1} (1 + a)^{k-1} S_{i+k-j-1} + (b/a)[(1 + a)^{j+1} - 1] \\ &< aS_{i-j-1} \sum_{k=1}^{j+1} (1 + a)^{k-1} + (b/a)[(1 + a)^{j+1} - 1] \\ &= S_{i-j-1}[(1 + a)^{j+1} - 1] + (b/a)[(1 + a)^{j+1} - 1]. \end{aligned}$$

Then

$$S_{i-j-1} + x_{i-j} < S_i(1 + a)^{j+1} + (b/a)[(1 + a)^{j+1} - 1].$$

Now

$$S_i \leq H_i < H_{i-1} < \dots < H_{i+2} = [c - b(1 + a)^{j+1}]/[a(1 + a)^{j+1}]$$

where $j = 0, 1, \dots, i - 2$;

hence,

$$S_{i-j-1} + x_{i-j} < ([c - b(1+a)^{j+1}]/[a(1+a)^{j+1}])(1+a)^{j+1} \\ + (b/a)[(1+a)^{j+1} - 1] = (c-b)/a = t.$$

Q.E.D.

Proof of Lemma 3.6: Let $j = i$, and consider two cases. Case 1: $S_i \leq H_{i+1}$. By Lemma 3.3 and Lemma 3.5, the total f.l. for the first i items issued by FIFO is $Q_F(i) = a \sum_{k=1}^i (1+a)^{k-1} S_k + (b/a)[(1+a)^i - 1]$. Since $S_k \leq H_{i+1}$, $k \leq i$,

$$Q_F(i) \geq a \sum_{k=1}^i (1+a)^{k-1} \{ [c - b(1+a)^i]/[a(1+a)^i] \} \\ + (b/a)[(1+a)^i - 1] = (c/a) - c/[a(1+a)^i].$$

Now

$$S_{i+1} + Q_F(i) \geq L_{i+1} + Q_F(i) \geq c/[a(1+a)^i] - (b/a) \\ + (c/a) - c/[a(1+a)^i] = (c-b)/a = t.$$

Case 2: $H_{i+1} = \leq S_i \leq H_i$ (where $H_{i+1} < H_i$). Let $0 \leq \beta \leq 1$ and let $S_i = P_0 = \beta H_i + (1-\beta)H_{i+1}$ for some β . Then $P_0 = \beta c/(1+a)^i + c/[a(1+a)^i] - b/a$. Again by Lemma 3.3 and 3.5 and $S_k \leq P_0$, $k \leq i$,

$$Q_F(i) \geq a \sum_{k=1}^i (1+a)^{k-1} \{ \beta c/(1+a)^i + c/[a(1+a)^i] - b/a \} \\ + (b/a)[(1+a)^i - 1] = \beta c + c/a - c/(1+a)^i - \beta c/[a(1+a)^i].$$

Now $S_{i+1} > S_i \Rightarrow S_{i+1} > P_0$, hence $S_{i+1} + Q_F(i) > P_0 + Q_F(i) \geq (c-b)/a + \beta c \geq t$. Thus for all $S_i \leq H_i$, we have $S_{i+1} + Q_F(i) \geq t$. Let $j < i$, and again consider two cases. Case 1: Some item S_k , where $j < k \leq i$, under policy $F_j L$ has f.l. = c on issuance. But then all items S_{k+p} for $p = 1, \dots, n-k$ must have f.l. = c on issuance. Case 2: All items S_k for $j < k \leq i$ under policy $F_j L$ have f.l. $> c$ on issuance. By Theorem 2.1 and Lemma 3.5 each of the first j items of $F_j L$ must have f.l. $> c$ on issuance for all $j \leq i$. By Lemma 3.3, the first j items issued have total f.l.

$$(L.3.6.1) \quad B_j = a \sum_{p=1}^j (1+a)^{p-1} S_p + (b/a)[(1+a)^j - 1].$$

Since $F_j L$ says to issue in the order $S_j, S_{j-1}, \dots, S_2, S_1, S_{j+1}, S_{j+2}, \dots, S_i, \dots, S_n$, then by induction it is easily shown that the total f.l. for items S_{j+1}, \dots, S_i is given by

$$(L.3.6.2) \quad C_i = a \sum_{p=1}^{i-j} (1+a)^{p-1} S_{i-p+1} + B_j [(1+a)^{i-j} - 1] \\ + (b/a)[(1+a)^{i-j} - 1].$$

Combining (L.3.6.1) and (L.3.6.2), we obtain the total f.l. of the first i items issued by $F_j L$.

$$\begin{aligned}
B_j + C_i &= a \sum_{p=1}^{i-j} (1+a)^{p-1} S_{i-p+1} + a(1+a)^{i-j} \sum_{p=1}^j (1+a)^{p-1} S_p \\
&\quad + (b/a)[(1+a)^i - 1] \\
&> a \sum_{p=1}^{i-j} (1+a)^{p-1} S_{i+1} + a(1+a)^{i-j} \sum_{p=1}^j (1+a)^{p-1} S_{i+1} \\
&\quad + (b/a)[(1+a)^i - 1] \\
&= (S_{i+1} + (b/a))[(1+a)^i - 1].
\end{aligned}$$

Now since $S_{i+1} \geq H_{i+1}$, we have

$$\begin{aligned}
S_{i+1} + B_j + C_i &> S_{i+1} + S_{i+1}[(1+a)^i - 1] + (b/a)[(1+a)^i - 1] \\
&\geq \{[c - b(1+a)^i]/[a(1+a)^i]\}(1+a)^i + (b/a)[(1+a)^i - 1] \\
&= (c - b)/a = t. \quad \text{Q.E.D.}
\end{aligned}$$

References

1. BOMBERGER, E. E., "Optimal Inventory Depletion Policies," *Management Science*, Vol. 7, No. 3 (April 1961), pp. 294-303.
2. DERMAN, C. AND KLEIN, M., "Inventory Depletion Management," *Management Science*, Vol. 4, No. 4 (July 1958), pp. 450-456.
3. — AND —, "Discussion: A Note on the Optimal Depletion of Inventory," *Management Science*, Vol. 5, No. 2 (January 1959), pp. 210-213.
4. EILON, S., "Stock Depletion for Multi-Channel Outlets in Maintenance Work," *Proceedings of the 3rd International Conference on Operational Research*, Oslo, 1963, Dunod (Paris) and English Universities Press, Ltd. (London), 1964, pp. 238-246.
5. —, "Obsolescence of Commodities Which are Subject to Deterioration in Store," *Management Science*, Vol. 9, No. 4 (July 1963), pp. 623-642.
6. —, "Problems of Stock Depletion," *The Engineer*, (United Kingdom), Vol. 214, No. 5570 (October 1962), pp. 706-708.
7. —, "FIFO and LIFO Policies in Inventory Management," *Management Science*, Vol. 7, No. 3 (April 1961), pp. 304-315.
8. GREENWOOD, J. A., "Issue Priority," *Naval Research Logistics Quarterly*, Vol. 2, No. 4, pp. 251-268.
9. LIEBERMAN, G. J., "LIFO vs. FIFO in Inventory Depletion Management," *Management Science*, Vol. 5, No. 1 (October 1958), pp. 102-105.
10. PIERSKALLA, W. P., "Optimal Issuing Policies in Inventory Management—I," Tech. Memo No. 44, Operations Research Group, Case Institute of Technology, Cleveland, Ohio, January 1966.
11. —, "Optimal Issuing Policies in Inventory Management," Technical Report No. 7, Program in Operations Research, Stanford University, Stanford, California, August 28, 1965.
12. VEINOTT, A. F., JR., *Optimal Ordering, Issuing, and Disposal of Inventory with Known Demand*, Unpublished Ph.D. Dissertation, Columbia University, 1960.
13. ZEHNA, P. W., "Inventory Depletion Policies," Chapter 15 in Arrow, K. J., S. Karlin, and H. Scarf (eds.), *Studies in Applied Probability and Management Science*, Stanford, California, Stanford University Press, 1962.