Optimal Issuing Policies In

Inventory Management--II

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ABSTRACT

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We are concerned with further generalizations of the inventory
depletion model where the inventory deteriorates in time as it remains
on the shelf. Once put in use, an item from the inventory of  n
items has a nonnegative field life. $L(S)$ , which is a function of
the age,  S ,  of the item upon issuance to the field. The objective
is to determine the order of issue that maximizes the total field life
of the stockpile. Optimal policies of the form LIFO, last in first
out, or FIFO, first in first out, are sought.

Earlier writers have considered a static model where there are
n  items of differing ages in the stockpile. In this paper the model
is broadened to become a dynamic model where new items are added and
batches of items of the same age are allowed. It is shown that in
many cases LIFO and FIFO remain optimal.

In addition the assumption that  $L(S)$  is a known function is
dropped and the field life of an item now becomes a nonnegative random
variable,  $X(S)$ . $X(S)$  may take on any one of a countable number of
values  $L_i(S)$  with probability  $p_i$ . For  $L_i(S)$  linear it is shown
that FIFO is optimal for the dynamic inventory depletion model where
batches are allowed.

# OPTIMAL ISSUING POLICIES IN INVENTORY MANAGEMENT--II

The general inventory depletion problem can be described as the
problem of finding an issue policy which maximizes or minimizes a
prescribed function when the inventory itself is changing in quality
over time. The change in quality may be either an appreciation or a
deterioration of the useful life, the field life, of each item in the
inventory as long as the item remains in the stockpile. An issue
policy is a selected order of issue of the items in the stockpile
when demands for the items are made from the field.

In 1958 Derman and Klein [2] and Lieberman [9] presented some
analytic results concerning a more specific formulation of the general
model. They obtained optimal policies of the form LIFO, last in first
out, or FIFO, first in first out. However their model contains several
restrictive assumptions which limit the application of the results to
most real situations.

It is the primary purpose of this paper to eliminate and/or modify
some of the assumptions of the model and still, whenever possible,
obtain LIFO or FIFO as optimal policies.

In order to be more specific as to which assumptions will be changed
or removed, it will be advantageous to characterize the original model
explicitly. It is assumed that: (1) At the beginning of the process, a

stockpile has $n$ indivisible identical items of varying ages $S_1 < S_2 < \ldots < S_n$ where $S_1 > 0$ . The ages $S_i$ are called the initial ages of the items. (2) Each item has a field life $L(S)$ which is a known nonnegative function of the age $S$ of the item upon being issued. (3) Items are issued successively until either the entire stockpile is depleted or the remaining items in the stockpile have no further useful life, i.e., $L(S) = 0$ for the remaining items. (4) No penalty or installation costs are associated with the issuance of an item from the stockpile. (5) New items are never added to the stockpile after the process starts. (6) An item is issued from the stockpile only when the entire life of the preceding item issued is ended. (7) At the beginning of the process each item has positive field life, i.e., $L(S_i) > 0$ for all $i = 1, 2, \ldots, n$ .

The objective is to find the issue policy which maximizes the total field life of the stockpile. An issue policy which achieves this maximum is called an optimal policy.

Assumption (2) of the model requires that $L(S)$ be a nonnegative function of $S$ . In the case where we assume that $L(S)$ is a concave decreasing function then there is a point, say $S_o$ , such that $L(S) > 0$ for all $S \epsilon [0, S_o)$ and $L(S) \leq 0$ for all $S \geq S_o$ . Thus for all $S \geq S_o$ , $L(S)$ must be redefined to be identically zero and $S_o$ is a finite truncation point. In another case, e.g., $L(S) = 1/S$ for all $S > 0$ then as $S \longrightarrow + \infty$ , $L(S) \longrightarrow 0$ and $S_o = + \infty$ is called the truncation point.

In general, if $L(S)$ is a decreasing function of $S$ and $L(0) > 0$, then $S_o \leq +\infty$ is a truncation point for $L(S)$ if and only if $S_o = \inf \{S \in [0, \infty) : L(S) \leq 0\}$ and then $L(S)$ is redefined to be

$$L(S) = \begin{cases} L(S) > 0 & \text{for all } S \in [0, S_o) \\ 0 & \text{for all } S \geq S_o \end{cases}$$

(Ref. Zehna [13].)

From a practical point of view it makes little sense to permit $L(S)$ to be arbitrarily large for some $S$. Hence we will assume that there is some number $k < \infty$ such that $L(S) < k$ for all $S$ of interest. If $L(S) = 1/S$ as shown in the example above, we will assume this $L(S)$ applies only to those $S > 0$ such that $L(S) = 1/S < k$. Then if a finite number, $n$, of items are issued by any policy $A$, the total field life, $Q_A$, is bounded by $0 < Q_A < nk = K$ for all policies $A$ and any $n$ items $0 \leq S_1 < S_2 < \ldots < S_n$.

In all of the following sections the number of sources demanding items from the stockpile will be denoted by $\nu$ ($\nu$ is an integer $\geq 1$).

In a previous paper, Pierskalla [10], we investigated the modification of assumption (6) and the removal of assumption (4). Probably the most restrictive assumption of the model is assumption (5). Assumption (5) states that new items are never added to the stockpile after the process starts. In the next section this assumption is removed and the dynamic inventory depletion model is presented. We assume $L(S)$ concave nonincreasing and the left-hand derivative of

$L(\cdot)$ , $L^-(S) \geq -1$ , then if FIFO is optimal in the static model, FIFO is optimal when $N$ items are added to the inventory at different times in the future. When $L(S)$ is concave or convex and $L^-(S) < -1$ , then a policy called modified-LIFO (ML) is optimal. ML is the policy where LIFO is used until a new item arrives then the new item is immediately issued to the demand source which has the least life remaining on its item currently in consumption.

In Pierskalla [10] and in the second section of this paper we were concerned with the deterministic model, i.e., $L(S)$ is a known function. In the third section we consider the case where the field life of an item is a nonnegative random variable, $X(S)$ , dependent on the age, $S$ , of the item upon being issued. In this case the objective function now becomes: maximize the total expected return (utility) of the stockpile. Throughout the third section it is assumed that $X(S)$ can take on any one of a countable number of values, $L_i(S)$ with probability $p_i$ where $i = 1, \ldots, M$ ; $\sum_{i=1}^{M} p_i = 1$ , and $p_i \geq 0$ (M may be replaced by $+\infty$). If $L_i(S) = a_i S + b_i$ where $b_i > 0 > a_i > -1$ , $L_i(S) < L_{i+1}(S)$ for all $i$ and any $S < S_o$ and $L_i(S_o) = L_{i+1}(S_o) = 0$ for all $i$ , then FIFO is optimal for $n$ items in the stockpile and one demand source. This result says that if we know that each item deteriorates according to some linear field life function then even though the specific function is unknown for any item, FIFO is a policy which maximizes the total expected return. If we change $L_i(S) = a_i S + b_i$ above to $L_i(S)$ is concave and differentiable with $0 \geq L_i'(S) > L_{i+1}'(S) \geq -1$ for all $i$ and $S < S_o$ ,

FIFO is optimal for  n = 2 .

In the final section we consider the case when there are batches of items of the same age in the stockpile. Eilon [5] considered this problem in regard to the obsolescence of commodities which are subject to deterioration in the stockpile. However he did not consider the batch assumption's effect on the optimality of LIFO or FIFO. In the final section we obtain the general result is proved: if  L(S)  is continuous and if FIFO (LIFO) is optimal in the case of no batches, then FIFO (LIFO) is optimal when batches are permitted.


## The Dynamic Inventory-Depletion Model

For the results of this section, we remove the assumption that new items are never added and substitute the new assumption:

> "If a new item is added to inventory, it has age  S = 0
> and initial field life  L(0)  immediately upon entry to
> the inventory."

In addition we will assume that only a finite number,  N ,  of new items are ever added to the inventory. The ages of the new items are all assumed to be different and are denoted by  $F_1$, $F_2$, ... , $F_N$ where  $F_i > F_{i+1}$  means that item  $F_i$  arrives at the inventory before item  $F_{i+1}$ .

The assumption that  N  is finite is not necessarily restrictive since  N  can be chosen so large as to encompass the "going life" of any business concern.

We now construct two different models called (i) the "dynamic" model and (ii) the "extended" static model. The dynamic model is the dynamic inventory problem of finding the optimal issuing policy for the n items $S_1 < S_2 < \ldots < S_n < S_0$ which are originally in the stockpile and the N items $F_1 > F_2 > \ldots > F_N$ which are added at arbitrary times in the future. The time of arrival of item $F_N$ will be denoted by T (We are presently at time zero). The extended static model is the static inventory problem of finding the optimal issuing policy for the N + n items $F_N < F_{N-1} < \ldots < F_1 < S_1 < \ldots S_n$ where all n + N items are originally in the stockpile and new items are never added. If we consider $S < 0$ as future time in the dynamic model, then the extended static model can be thought of as the dynamic model under the transformation $L(S^*) = L(S+T)$ .

Theorem 1.1: Let $L(S)$ be a continuous concave nonincreasing function with $L^-(S) \geq -1$ for all $S \leq S_0$ . Let $\nu \geq 1$ . If

    (i)   FIFO is optimal in the extended static model and

    (ii)  the arrival of items $F_1, \ldots, F_N$ in the dynamic model are timed so that no stockouts occur

then FIFO is the optimal issuing policy in the dynamic model.

Proof: Since no stockouts occur, then the FIFO issuance of $S_n, \ldots, S_1$ and $F_1, \ldots, F_N$ in the dynamic model results in the same total field life as FIFO in the extended static model. Assume that there exists a policy A in the dynamic model which gives a greater total field life than FIFO. Then policy A must give a greater total field life than FIFO in the extended static model. This last statement follows

from the fact that the set of all possible policies in the extended

static model includes all policies of the dynamic model. But FIFO is

optimal in the extended static model hence we have a contradiction.    q.e.d.

Corollary 1.1: Let $L(S) = aS + b$ for all $S \leq S_o$ and with

$b > 0 > a > -1$ . Let $\nu \geq 1$ . If no stockouts occur in the dynamic

model, then FIFO is optimal.

Proof: By Zehna [13] Theorems 4.1 and 4.3, FIFO is optimal for the

extended static model.                                                        q.e.d.

Corollary 1.2: Let $L(S)$ be a concave nonincreasing function with

$L'(S) \geq -1$ for all $S \leq S_o$ . Let $\nu = 1$ . If no stockouts occur in

the dynamic model, then FIFO is optimal.

Proof: By Lieberman [9] Theorem 3, FIFO is optimal for the extended

static model.                                                                 q.e.d.

Corollary 1.3: Let $L(S)$ be concave nonincreasing with $L^-(S) \geq -1$ for

all $S \leq S_o$ . Denote by $[x]$ the largest integer $\leq x$ where $x$ is a real

number. Then if the number of demand sources $\nu$ has $[1/2(N+n+1)] \leq \nu \leq n$

and if no stockouts occur in the dynamic model then FIFO is optimal.

Proof: By Pierskalla [10] Theorem 1.3, FIFO is optimal for the extended

static model.                                                                 q.e.d.

The preceding theorem and corollaries were concerned with $L(S)$

concave with slope $\geq -1$ and in the linear case with $L'(S) > -1$ . We

now consider the linear case for $L'(S) = -1$ , and show that FIFO is

optimal for this case also. It is only necessary to prove that FIFO is

optimal for the extended problem and then apply Theorem 1.1. For the time being assume that new items are not added to the stockpile.

**Lemma 1.1:** Let $L(S)$ be linear, with $L'(S) = -1$ for all $S \leq S_o$. Let $\nu = 1$. Then any issue policy $A$ given by $A = [S_{i_1}, \ldots, S_{i_j}]$ where $S_{i_k} > S_{i_{k+1}}$ ($k = 1, \ldots, j-1$) has a total field life of

$$Q_A = L(S_{i_j}) = S_o - S_{i_j} .$$

**Proof:** Let $x$ denote the total field life up to but not including the issue of item $S_{i_j}$. Then $Q_A = L(S_{i_j}+x) + x$ and by lemma 1.1 of Pierskalla [10] $S_{i_j} + x < S_o$. Hence

$$[L(S_{i_j}+x) - L(S_o)]/(S_{i_j}+x-S_o) = -1 \implies Q_A = L(S_{i_j}+x) + x = S_o - S_{i_j} ,$$

but $[L(S_{i_j}) - L(S_o)]/(S_{i_j}-S_o) = -1 \implies L(S_{i_j}) = S_o - S_{i_j} = Q_A$.     q.e.d.

**Lemma 1.2:** Let $L(S)$ be linear and $L'(S) = -1$ for all $S \leq S_o$. Let $\nu \geq 1$. Then any issuing policy which issues items $S_1, S_2, \ldots, S_\nu$ (i.e., the $\nu$ youngest items) each to a different demand source is optimal and the total field life from an optimal policy, $Q^*$, is given by

(1.1) $$Q^* = \Sigma_{i=1}^{\nu} L(S_i) = \nu S_o - \Sigma_{i=1}^{\nu} S_i .$$

Furthermore $Q_{FIFO,\nu} = Q_{LIFO,\nu} = Q^*$.

**Proof:** We will first show (1.1).

Consider any policy which issues $S_1, \ldots, S_\nu$ each to different demand sources say $M_1, \ldots, M_\nu$ respectively. Let demand source $M_j$ receive the $c$ items $[S_{j_1}, \ldots, S_{j_c}]$. Now for any two items in

inventory with current age $S_i < S_j$ , we have two properties:

1. If $S_i$ is issued first, then at the expiration of the field life of $S_i$ , $S_j$ will have no field life remaining, and

2. If $S_j$ is issued first, then at the expiration of the field life of $S_j$ , $S_i$ will still have positive field life remaining.

These two cases follow from $[L(S_o) - L(S_k)]/(S_o - S_k) = -1$ , for any $k$ which implies $S_i + L(S_j) < S_o < S_j + L(S_i)$ .

Hence in any issue policy $A = [S_{j_1} , \ldots , S_{j_c}]$ we can omit any items $S_{j_i}$ for which there is some $S_{j_{i-k}}$ ($k = 1, \ldots , i-1$) such that $S_{j_{i-k}} < S_{j_i}$ , since for these $S_{j_i}$ , there will be no field life remaining when they are ready to be issued. Thus of all possible policies, we only have to consider policies where each succeeding item is younger than the previously issued item.

Now by the second property since $S_j$ is the youngest item of the $S_{j_i}$'s we must have that upon issue at any time, $S_j$ will have positive field life. But as shown above any item issued after $S_j$ has field life of zero and can be discarded without issuance, hence $S_j$ is the last item to be issued under all policies which we need to consider.

Thus instead of considering policy $A$ for $M_j$ it is only necessary to consider the policy $B_j = [S_{j_{i1}} , \ldots , S_j]$ where $S_{j_{ik}} > S_{j_{i,k+1}}$ . By Lemma 1.1, the total field life obtained from policy $B_j$ is

$Q_{B_j} = L(S_j) = S_o - S_j$ . But $M_j$ was picked arbitrarily, thus for all

$j = 1, \ldots, \nu$ the total field life for all $M_j$'s is

$$Q = \Sigma_{j=1}^{\nu} Q_{B_j} = \Sigma_{j=1}^{\nu} L(S_j) = \Sigma_{j=1}^{\nu} (S_o - S_j) = \nu S_o - \Sigma_{j=1}^{\nu} S_j \ ,$$

which is (1.1) as required.

Now let $C$ be any policy which does not issue $S_1, \ldots, S_{\nu}$ each to different $M_1, \ldots, M_{\nu}$. Hence $C$ must issue at least two of the items $S_1, \ldots, S_{\nu}$ to the same demand source, say $S_i$ and $S_j$ are issued to $M_k$ where $S_1 \leq S_i < S_j \leq S_{\nu}$. Now by properties (1) and (2) and by Lemma 1.1 we have

$$Q_C(M_k) = L(S_i) = S_o - S_i \ .$$

And since $S_i$ and $S_j$ are issued to $M_k$, there is at least one $M_t$ such that the youngest item issued to $M_t$ has initial age $S_t > S_{\nu}$, and

$$Q_C(M_t) = L(S_t) = S_o - S_t \ .$$

Thus the total field life for policy $C$ is at most

$$Q_C \leq \Sigma_{\substack{i=1 \\ i \neq j}}^{\nu} L(S_i) + L(S_t) = \nu S_o - S_t - \Sigma_{\substack{i=1 \\ i \neq j}}^{\nu} S_i < \nu S_o - \Sigma_{i=1}^{\nu} S_i = Q^* \ .$$

Therefore the policy of issuing items $S_1, \ldots, S_{\nu}$ each to a different demand source is optimal. Now $Q_{LIFO,\nu}$ issues only $S_1, \ldots, S_{\nu}$ and each to different demand sources since for all $k > \nu$,

$$S_k + L(S_{\nu}) > S_{\nu} + L(S_{\nu}) = S_o \implies L(S_k + L(S_{\nu})) = 0 \ .$$

Hence $Q_{LIFO,\nu} = \Sigma_{i=1}^{\nu} L(S_i) = Q^*$. Furthermore by lemma 1.2 of Pierskalla [10], we note that FIFO belongs to the class of policies such that items $S_1, \ldots, S_{\nu}$ are each issued to different $M_1, \ldots, M_{\nu}$;

hence $Q_{FIFO,\nu} = \Sigma_{i=1}^{\nu} L(S_i) = Q^*$ .                    q.e.d.

We are now able to state:

<u>Corollary 1.4</u>: Let $L(S)$ be linear with $L'(S) = -1$ for all $S \leq S_0$ . Consider the dynamic model and the extended static model given previously. Let $\nu \geq 1$ . If no stockouts occur in the dynamic model, then FIFO is optimal.

Note that by lemma 1.2 if the $F_i$ are known for all $i = N - \nu + 1, \ldots, N$ then the total field life for the model of Corollary 1.4 is

$$Q^* = Q_{FIFO,\nu} = \Sigma_{i=0}^{\nu-1} L(F_{N-i}) .$$

We now seek the optimal issuing policy for the dynamic inventory model when $L(S)$ is concave or convex and has slope $< -1$ for all $0 \leq S \leq S_0$ . It is interesting to note that it is no longer necessary to assume that no stockouts occur; the reason for this will be discussed later.

<u>Lemma 1.3</u>: Let $L(S)$ be a convex or concave differentiable function on $[0, S_0]$ with $L'(S) < -1$ on $[0, S_0]$ . Let there be $n$ items $0 < S_1 < S_2 < \ldots < S_n < S_0$ in inventory and no new items are ever added to the inventory. Let $\nu \geq 1$ . If $x_i$ is the total field life contributed by demand source $M_i$ under any arbitrary policy A and if the $x_i$ are ordered $x_1 \geq x_2 \geq \ldots \geq x_\nu$ , then $x_i \leq L(S_i)$ for all $i = 1, \ldots, \nu$ .

By Zehna [13] Theorems 4.2 and 4.3, LIFO maximizes the total field life. This lemma states that in addition LIFO maximizes the field life for each demand source.

<u>Proof</u>: Assume to the contrary that $x_i > L(S_i)$ for some $i = 1, \ldots, \nu$.
Then $x_i$ must contain one or more items $S_j < S_i$ for if all items
$S_k$ assigned to $M_i$ under policy A are such that $S_k \geq S_i$, then
since LIFO is optimal for $\nu = 1$ (cf. Zehna [13] Theorems 2.4 and
2.6) we would have $L(S_i) \geq x_i$ contrary to the assumption $x_i > L(S_i)$.

But if $S_j < S_i$ is assigned to $M_i$ then there are at most $i - 2$
S's which have $S_k < S_i$, $k \neq j$, available for assignment to the
$i - 1$ $M_t$'s <u>viz</u>. $M_1, \ldots, M_{i-1}$. Hence some $M_t$, $t = 1, \ldots, i - 1$,
does not receive any $S_k < S_i$. Therefore as stated in the preceding
paragraph we must have $x_t \leq L(S_i)$ and $x_i > L(S_i) \geq x_t$ where $t < i$.
But $x_i > x_t$ for $t < i$ contradicts the hypothesis of the lemma.
Therefore $x_i \leq L(S_i)$ for all $i = 1, \ldots, \nu$.            q.e.d.

Let A be any arbitrary policy for issuing the n items originally
in the inventory and the N items added to the inventory in the future.
We define a modified-A policy, MA, for issuing items to the $\nu \geq 1$
demand sources in the following way: Use policy A until a new item
arrives, then discard the item currently in use in the field which has
the least field life remaining and immediately replace it with the new
item. When A ≅ LIFO we denote MA by ML.

<u>Theorem 1.2</u>: Let L(S) be a convex or concave differentiable function
on $[0, S_o]$ with $L'(S) < -1$ on $[0, S_o]$. Let $\nu \geq 1$. Then ML
is the optimal issuing policy for the dynamic model.

<u>Proof</u>: The proof will be by induction on N. Let N = 1. And let
the time of arrival of the new item be denoted by t.

We first show that under any policy A it is always better to discard some item currently in use and use the new item immediately.

Let $T$ be the field life remaining to demand source $M_i$ when the new item arrives. There are three cases:

Case (i): $0 < T < S_0$ then $[L(0) - L(T)]/-T < -1$ implies $L(0) > L(T) + T$ .

Case (ii): $S_0 \leq T$ which implies $L(T) = 0$ . Then $L(0) \geq L(S_1) \geq T = T + L(T)$ .

Case (iii): $T = 0$ then $L(0) = T + L(T)$ . In all cases it is better to use the new item immediately. For $j \neq i$ the field lives of the other $M_j$'s are not affected by this change.

In the above we have implicitly assumed for $j \neq i$ that all $M_j$'s have items currently in use. If some $M_j$ did not have any items left and if $T > 0$ for $M_i$ the new item would be assigned to $M_j$ .

Let $M_i$ be the demand source with the least field life remaining at time $t$ . Denote this remaining field life to $M_i$ by $T_{min}$ . $T_{min} \geq 0$ . For any $j \neq i$ , let $T_j$ be the field life remaining to $M_j$ . Then $T_j \geq T_{min}$ . Let $Q$ be the total field life obtained by all the $M_k$'s , $k = 1, \ldots, \nu$ , if the new item is not issued until the current items issued to $M_i$ and $M_j$ expire. Then $Q + L(0) - T_{min} \geq Q + L(0) - T_j$ , for any $j \neq i$ . Hence under any policy A we obtain

Statement (1): The new item should be issued immediately

to the demand source which must discard the
least field life.

Thus the optimal policy for the case $N = 1$ must belong to the class
of modified policies.

We now show ML is optimal for $N = 1$. Consider any policy MA
with MA $\neq$ ML. Let $x_1 \geq x_2 \geq \ldots \geq x_\nu$ be the field life contributed
by $M_1, \ldots, M_\nu$ under policy A when the new item is not considered.
By MA the new item will be assigned to $M_\nu$ since $x_\nu$ is the smallest
field life. We consider three mutually exclusive and exhaustive cases.
Recall by lemma 1.3 that $x_i \leq L(S_i)$ for all i.

Case 1:  $\qquad x_\nu \leq L(S_\nu)$ and $t \leq L(S_\nu)$

Then $\qquad Q_{ML} = \Sigma_{i=1}^{\nu-1} L(S_i) + t + L(0) \geq \Sigma_{i=1}^{\nu-1} x_i + t + L(0) = Q_{MA}$.

Case 2:  $\qquad x_\nu \leq L(S_\nu)$ and $t > L(S_\nu)$

Then $\qquad Q_{ML} = \Sigma_{i=1}^{\nu} L(S_i) + L(0) \geq \Sigma_{i=1}^{\nu} x_i + L(0) = Q_{MA}$.

Case 3:  $\qquad x_\nu < L(S_\nu)$ and $x < t \leq L(S_\nu)$

Then $\qquad Q_{ML} = \Sigma_{i=1}^{\nu-1} L(S_i) + t + L(0) > \Sigma_{i=1}^{\nu} x_i + L(0) = Q_{MA}$.

In all cases $Q_{ML} \geq Q_{MA}$ and since A was any arbitrary policy ML is
optimal for $N = 1$.

Assume ML is optimal for adding $N = k$ items to inventory and
consider $N = k + 1$.

We will first establish that the optimal policy for $k + 1$ must
belong to the class of modified policies. Let T be the field life

remaining to $M_i$ when the $k + 1^{st}$ item arrives. Let $t_k$ and $t_{k+1}$ denote the time of arrival of the $k^{th}$ and $k + 1^{st}$ items respectively. Since arrivals are distinct events $t_k < t_{k+1}$ and all items in use or in the stockpile, except the $k + 1^{st}$ item, have age greater than zero at time $t_{k+1}$, then $L(0) > T \geq 0$, and we have the same three cases (i), (ii) and (iii) as before.

Hence the new item should always be installed immediately on arrival. Furthermore by the same argument given in $N = 1$, the new item should be assigned to the demand source which loses the least field life. Thus Statement (1) applies also to the case of $N = k + 1$ and the optimal policy belongs to the class of modified policies.

In order to proceed further it is necessary to develop some additional notation. Let $Q_{M_i,N}$ and $x_{i,N}$ denote the total field life for $M_i$ under ML and MA respectively. In addition relabel the $M_i$'s in ML and in MA such that $Q_{M_i,N} \geq Q_{M_{i+1},N}$ and $x_{i,N} \geq x_{i+1,N}$ for all $i = 1, \ldots, \nu - 1$. It is possible that $M_1$ under ML is not the same $M_1$ as under MA but this fact is of no importance in the following.

In the case $N = 1$ we showed $Q_{ML} \geq Q_{MA}$ but also in conjunction with lemma 1.3 we showed that the total field life for each $M_i$ under ML is greater than under MA, i.e., using the notation above $Q_{M_{i,1}} \geq x_{i,1}$ for all $i = 1, \ldots, \nu$. It will now be proved that

(1.2)     $Q_{M_{i,k+1}} \geq x_{i,k+1}$ for all $i = 1, \ldots, \nu$

where we inductively assume

(1.3) $\qquad Q_{M_{i,k}} \geq x_{i,k}$ for all $i = 1, \ldots, \nu$.

Now (1.3) and Statement (1) inform us that the $k+1^{st}$ arrival is immediately assigned to $M_\nu$. We only need to show $Q_{M_{\nu,k+1}} \geq x_{\nu,k+1}$.

Let $T_{ML}$ and $T_{MA}$ be the total field life remaining to $M$ at time $t_{k+1}$ when ML and MA are being followed respectively. By (1.3) $T_{ML} \geq T_{MA} \geq 0$. We consider the three mutually exclusive and exhaustive cases:

Case 1: $\qquad x_{\nu,k} \leq Q_{M_{\nu,k}}$ and $t_{k+1} > Q_{M_{\nu,k}}$

Then $\qquad Q_{M_{\nu,k+1}} = Q_{M_{\nu,k}} + L(0) \geq x_{\nu,k} + L(0) = x_{\nu,k+1}$

Case 2: $\qquad x_{\nu,k} \leq Q_{M_{\nu,k}}$ and $t_{k+1} \leq x_{\nu,k}$

Then $\qquad Q_{M_{\nu,k}} - x_{\nu,k} = T_{ML} - T_{MA}$

and $\qquad Q_{M_{\nu,k+1}} = Q_{M_{\nu,k}} - T_{ML} + L(0) = x_{\nu,k} - T_{MA} + L(0) = x_{\nu,k+1}$.

Case 3: $\qquad x_{\nu,k} < Q_{M_{\nu,k}}$ and $x_{\nu,k} < t_{k+1} \leq Q_{M_{\nu,k}}$

Then $\qquad T_{MA} = 0$ and $Q_{M_{\nu,k}} - x_{\nu,k} > T_{ML} - T_{MA} = T_{ML}$

$$Q_{M_{\nu,k+1}} = Q_{M_{\nu,k}} - T_{ML} + L(0) > x_{\nu,k} + L(0) = x_{\nu,k+1}.$$

Hence in all cases

(1.4) $\qquad Q_{M_{\nu,k+1}} \geq x_{\nu,k+1}$

Now since the field life for the other $M_i$ $i = 1, \ldots, \nu - 1$ are

unchanged then by (1.3)

(1.5) $\quad Q_{M_{i,k}} = Q_{M_{i,k+1}} \geq x_{i,k+1} = x_{i,k} \quad$ for all $i = 1, \ldots, \nu - 1$ .

Combining (1.4) and (1.5) we see that (1.2) holds for all $i$ , and

since MA was any arbitrary policy, ML is optimal. $\qquad$ q.e.d.

In the earlier FIFO results, it was assumed that the ordering

schedule for new items was arranged so that stockouts did not occur.

This assumption was essential for FIFO optimality as the following

example shows: $L(S) = -(1/3) S + 3$ for $S \epsilon [0,9]$ and $L(S) = 0$ otherwise;

$F_1 = -2.3956,$ $F_2 = -2.6667,$ $S_1 = 1,$ $S_2 = 5,$ $S_3 = 6,$ $S_4 = 7,$ $S_5 = 8$ .

Then for $\nu = 2$ FIFO $= [S_5, S_3, S_1, F_2; S_4, S_2, F_1]$ which yields

$Q_F = 10.9883$ as compared to $A = [S_5, S_4, S_3, S_2, F_1; S_1, F_2]$ which

yields $Q_A = 11.0623$ is definitely not optimal. In the case of FIFO ,

however, a stockout occurred because the total field life for $[S_4, S_2]$

is 1.7781 whereas item $F_1$ does not arrive until $t_1 = 2.3956$.

It is interesting to note, however, that when $L'(S) < -1$ we can

drop the no stockout assumption. The reason for this is essentially

contained in lemma 1.3 which states that each demand source receives

more field life under LIFO than from any other policy. Thus if we

followed a non-LIFO policy, say MA , we could expect stockouts to

be more frequent and of a much longer duration. But, the new arriving

item under any modified policy is used to its fullest extent. There-

fore the policy which minimizes the total stockout duration will

maximize the total field life; and as shown this optimal policy is ML

for the dynamic model.

As the concluding statement in this section, it should be noted that results were not presented for the case $L(S)$ convex decreasing with slope $L'(S) \geq -1$. Even if we assumed that LIFO was optimal for the static depletion model, there are numerous counterexamples for $\nu = 1$, $n = 2$, and $N = 1$ in the dynamic model where neither LIFO nor ML nor any of the other possible policies is optimal in all cases. If we desire to find the conditions in this simple case where LIFO or ML is optimal, it is necessary to make very restrictive assumptions on $S_1$, $S_2$ and $F_1$. We have not done this because the transition to general $n$ and $N$ even keeping $\nu = 1$ does not appear to be interesting from a practical point of view.

## Batches of Items of the Same Age in the Stockpile

It has always been assumed in the literature and in this paper that the $n$ items in inventory all have different initial ages. In general, this assumption is not necessary. With minor modifications, the theorems, lemmas and corollaries of the preceding chapters as well as the results of the papers referenced in the Bibliography can be stated for batches of items of the same age.

More specifically we modify assumption (1) of the model as follows:

Assumption (1)': At the start of the process a stock-pile has $N$ sets of indivisible identical items where the items in the $i^{th}$ set all have the same initial age, $S_i$ , for $i = 1, \ldots , N$ . The initial age of the items in any set, say the $i^{th}$ set, is different than the initial age of the items in any other set. Assume $0 \leq S_1 < S_2 < \ldots < S_N < S_o$ and that the $i^{th}$ set contains $n_i$ items for $i = 1, \ldots , N$ . Then $\Sigma_{i=1}^{N} n_i = n$ .

For ease of adapting the previous results to the batch problem we make the following ordering of the $n$ items.

Let the first $n_1$ items be numbered from 1 to $n_1$ , the next $n_2$ items from $n_1 + 1$ to $n_1 + n_2$ , the next $n_3$ items from $n_1 + n_2 + 1$ to $n_1 + n_2 + n_3$ , etc., until all the items possess a number from 1 to $n$ and such that $S_i \leq S_{i+1}$ for all $i = 1, \ldots , n - 1$ . We note that there are $\pi_{i=1}^{N}(n_i)!$ ways to complete the above ordering; hence choosing any one of these ways is somewhat arbitrary. However the total field life realized from the $n$ items by any policy under any one of the $\pi(n_i)!$ ways is the same. Thus we choose an ordering and define FIFO (LIFO) as the policy which issues the highest (lowest) indexed item in the stockpile each time an item is issued.

We now prove a general theorem which applies to most of the previous results in inventory depletion theory.

**Theorem 2.1:** Let $L(S)$ be a continuous function. If FIFO (LIFO) is optimal when assumption (1) of the model holds, i.e., when there are no batches in the inventory, then FIFO (LIFO) is optimal when assumption (1)' holds, i.e., when batches are allowed.

**Proof:** We will prove the theorem for FIFO; the theorem for LIFO follows _mutatis mutandis_.

Let $\epsilon_0 = \min_{1 \leq i \leq N-1} [S_{i+1} - S_i, S_0 - S_N] > 0$ . $\epsilon_0$ exists since $0 \leq S_1 < S_2 < \ldots < S_N < \infty$ . Consider any $\epsilon$ such that

$$(2.1) \qquad \epsilon_0 > \epsilon > 0$$

and consider the $n$ items defined by

$$(2.2) \qquad T_{ij} = S_i + (\epsilon/[n_i - j + 1])$$

for all $j = 1, \ldots, n_i$ and $i = 1, \ldots, N$ . Then from (2.2) we have

$$(2.3) \quad 0 < T_{11} < T_{12} < \ldots < T_{1n_1} < T_{21} < \ldots < T_{N1} < \ldots < T_{Nn_N} < S_0 .$$

Denote by $Q_F(\epsilon)$ and $Q_A(\epsilon)$ the total field life from the issuance of the $n$ items of (2.3) by FIFO and by an arbitrary policy $A$ , respectively. Denote by $Q_{F_B}$ and $Q_{A_B}$ the total field life from the issuance of the $n$ items in batches by FIFO and by an arbitrary policy $A$ , respectively.

Since $L(\cdot)$ is a continuous function, then $Q_A$ is also a continuous function for any policy $A$ .

Hence for any $\delta > 0$ and $\delta$ sufficiently small there is an $\epsilon > 0$ such that $\epsilon$ satisfies (2.1) and such that $|Q_{F_B} - Q_F(\epsilon)| < \delta$, $|Q_{A_B} - Q_A(\epsilon)| < \delta$.

Then

(2.4)  $$Q_{F_B} + \delta > Q_F(\epsilon) \quad \text{and}$$

(2.5)  $$Q_A(\epsilon) > Q_{A_B} - \delta \ .$$

By hypothesis however

(2.6)  $$Q_F(\epsilon) \geq Q_A(\epsilon)$$

for all $\epsilon$ satisfying (2.1). Thus

(2.7)  $$Q_{F_B} > Q_{A_B} - 2\delta \ .$$

Now (2.4), (2.5), and (2.6) hold for all $\delta > 0$ where $Q_F(\epsilon)$ and

$Q_A(\epsilon)$ are defined by the issuance of the items in (2.3). And since

$\delta > 0$ can be made arbitrarily small, we have $Q_{F_B} \geq Q_{A_B}$ for any

arbitrary policy A .                                           q.e.d.

The foregoing proof also holds when we consider the stochastic case

since $U_A$ is the sum of continuous functions, $Q_{A_i}$ ; hence $U_A$ is

also continuous. In the case of the stochastic field life functions of

the previous section, the continuity of $U_A$ in the countably infinite

case follows from:

$$\left| U_{A_B} - U_A(\epsilon) \right| \leq \Sigma_{i=1}^{\infty} \left| Q_{A_{B,i}} - Q_{A_i}(\epsilon) \right| P^{(i)}$$

but since $\Sigma_{i=1}^{\infty} \left| Q_{A_{B,i}} - Q_{A_i}(\epsilon) \right| P^{(i)} \leq 2K$ where $K$ is an upper bound

for all $Q_A$ and any A , then there exists an N such that $\Sigma_{i=N+1}^{\infty} P^{(i)} < \epsilon/4K$

and we have $|U_{A_B} - U_A(\epsilon)| \leq \Sigma_{i=1}^{\infty} |Q_{A_{B,i}} - Q_{A_i}(\epsilon)| \; P^{(i)}$

$$\leq \Sigma_{i=1}^{N} |Q_{A_{B,i}} - Q_{A_i}(\epsilon)| \; P^{(i)} + 2K(\epsilon/4K) \; .$$

But $Q_A$ is continuous hence we can choose $|Q_{A_{B,i}} - Q_{A_i}(\epsilon)| < (\epsilon/2N)$

then $|U_{A_B} - U_A(\epsilon)| \leq \Sigma_{i=1}^{N} |Q_{A_{B,i}} - Q_{A_i}(\epsilon)| \; P^{(i)} + (\epsilon/2)$

$$\leq \Sigma_{i=1}^{N} P^{(i)} (\epsilon/2N) + (\epsilon/2) < N (\epsilon/2N) + (\epsilon/2) = \epsilon \; .$$

This paper and the previous paper [10], have presented changes to assumptions (1), (2), (4), (5) and (6). We did not consider field life functions which become negative since they do not seem to have a nice interpretation in the inventory context of the model. However it may be possible that negative $L(S)$ would be an interesting representation of some other model based on costs or profits.

Also we have not sought to change assumption (3). Assumption (3) in conjunction with assumption (6) defines the demand function on the inventory and this assumption is what gives this particular model its interest and its problems. It makes any current decision as to the optimal issuing policy entirely dependent upon all past decisions. Other authors have removed assumption (3) and assumption (6) to make the demands on the stockpile independent of the ages of the items in the stockpile. They have obtained some very interesting results (Ref. [2], [3], [5], [12] and [13]).

It should be noted that the theorems presented in the preceding chapters do not exhaust the possible changes to the model nor do they exhaust the possible variations on the changes already suggested. For example, the stochastic models presented in Chapter 5 of Zehna [11] and in Chapter 6 of this work are stated only for the case $\nu = 1$ and for special probability density functions and $L_i(S)$ . Since the stochastic case is very important from a practical point of view, extensions to $\nu \geq 1$ and other field life functions would be desirable. It should be mentioned that these extensions appear to be difficult since the truncation point $S_o$ makes most proofs rather complicated.

Some other areas of future research should be

(i)   for the case $\nu > 1$ find the optimal policy for various $L(S)$'s and/or find more sufficient conditions for LIFO or FIFO optimality,

(ii)  investigate other types of S-shaped functions,

(iii) consider the dynamic depletion model in relation to removing other assumptions of the model,

(iv)  investigate appreciating field life functions, and

(v)   look at a fixed (or minimum) spacing, $\delta$ , between the initial ages of the items, i.e., $S_{i+1} \geq S_i + \delta$ .

This last area would correspond to the case when inventory items arrived at the stockpile in some fixed pattern as would be the case, e.g., if the production facility was turning out one item or a batch of items every time period.

As a final consideration some research should be devoted to different types of objective functions. An important case of this is pointed

out in Eilon [6] and Eilon [5].  He suggested that if we are a seller

(rather than consumer) of the items in the stockpile and if the holding

costs per item are high, then we may wish to minimize the time the

items spend in inventory.  This type of objective function is, in a

sense, the polar image of the objective function considered in the

preceding chapters since we now wish to minimize the total field life

of the stockpile (at least of the first  $n - \nu$  items issued from the

stockpile).  However it is not, in general, true that if FIFO (LIFO)

is optimal for the one objective function that LIFO (FIFO) is optimal

for the other.

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