

OPTIMAL ISSUING POLICIES IN INVENTORY MANAGEMENT

W. P. Pierskalla*

1. INTRODUCTION

In 1958 Derman and Klein [2] and Lieberman [9] presented some analytic results concerning an inventory depletion model. They obtained optimal policies of the form LIFO, last in first out, or FIFO, first in first out. However their model contains several restrictive assumptions which limit the application of the results to most real situations. This paper modifies some of the assumptions of their model.

The original depletion model is characterized by these assumptions:

(1) At the beginning of the process, a stockpile has n indivisible identical items of varying ages $S_1 < S_2 < \dots < S_n$ where $S_1 > 0$. The ages S_i are called the initial ages of the items. (2) Each item has a field life $L(S)$ which is a known nonnegative function of the age S of the item upon being issued. (3) Items are issued successively until either the entire stockpile is depleted or the remaining items in the stockpile have no further useful life, i.e., $L(S) = 0$ for the remaining items. (4) No penalty or installation costs are associated with the issuance of an item from the stockpile. (5) New items are never added to the stockpile after the process starts. (6) An item is issued from the stockpile only when the entire life of the preceding item issued is ended. (7) At the beginning of the process each item has positive field life, i.e., $L(S_i) > 0$ for all $i = 1, 2, \dots, n$.

The objective is to find the issue policy which maximizes the total field life of the stockpile. An issue policy which achieves this maximum is called an optimal policy.

Assumption (2) of the model requires that $L(S)$ be nonnegative. Thus if $L(S)$ is a decreasing function of S and $L(0) > 0$, we must define a truncation point, S_0 , for $L(S)$. $S_0 \leq +\infty$ is a truncation point for $L(S)$ if and only if $S_0 = \inf \{S \in [0, \infty) : L(S) \leq 0\}$ and then $L(S)$ is redefined to be

$$L(S) = \begin{cases} L(S) > 0 & \text{for all } S \in [0, S_0) \\ 0 & \text{for all } S \geq S_0 \end{cases} \quad (\text{Ref. Zehna [13]}).$$

In the following sections the number of sources demanding items from the stockpile will be denoted by v (v is an integer ≥ 1).

In a previous paper, Pierskalla [10], we investigated the modification of assumption (6) and the removal of assumption (4). Probably the most restrictive assumption of the model is assumption (5). In the next section assumption (5) is removed and the dynamic inventory depletion model is presented. We assume $L(S)$ is concave nonincreasing and the left-hand derivative of $L(S)$, $L^-(S)$, is ≥ -1 , then if FIFO is optimal in the static model, FIFO is optimal when N items are added to the inventory at different times in the future. When $L(S)$ is concave or convex and $L^-(S) < -1$, then a policy called modified-LIFO (ML) is optimal. ML is the policy where LIFO is

* Case Institute of Technology, Cleveland, Ohio

used until a new item arrives then the new item is immediately issued to the demand source which has the least life remaining on its item currently in consumption.

In the final section we consider the case when there are batches of items of the same age in the stockpile. We obtain the general result: if $L(S)$ is continuous and if FIFO (LIFO) is optimal in the case of no batches, then FIFO (LIFO) is optimal when batches are permitted.

2. THE DYNAMIC INVENTORY-DEPLETION MODEL

For the results of this section, we remove the assumption that new items are never added and substitute the new assumption:

"If a new item is added to inventory, it has age $S = 0$ and initial field life $L(0)$ immediately upon entry to the inventory."

In addition we will assume that only a finite number, N , of new items are ever added to the inventory. The ages of the new items are all assumed to be different and are denoted by F_1, F_2, \dots, F_N where $F_i > F_{i+1}$ means that item F_i arrives at the inventory before item F_{i+1} .

The assumption that N is finite is not necessarily restrictive since N can be chosen so large as to encompass the "going life" of any business concern.

We now construct two different models called (i) the "dynamic" model (d.m.) and (ii) the "extended" static model (e.s.m.). The d.m. is the dynamic inventory problem of finding the optimal issuing policy for the n items $S_1 < S_2 < \dots < S_n < S_0$ which are originally in the stockpile and the N items $F_1 > F_2 > \dots > F_N$ which are added at arbitrary times in the future. The time of arrival of item F_N will be denoted by T (We are presently at time zero). The e.s.m. is the static inventory problem of finding the optimal issuing policy for the $N+n$ items $F_N < F_{N-1} < \dots < F_1 < S_1 < \dots < S_n$ where all $n+N$ items are originally in the stockpile and new items are never added. If we consider $S < 0$ as future time in the d.m., then the e.s.m. can be thought of as the d.m. under the transformation $L(S^*) = L(S+T)$.

Theorem 1.1: Let $L(S)$ be a continuous concave nonincreasing function with $L'(S) \geq -1$ for all $S \leq S_0$. Let $v \geq 1$. If

- (i) FIFO is optimal in the e.s.m., and
- (ii) the arrival of items F_1, \dots, F_N in the d.m. are timed so that no stockouts occur

then FIFO is the optimal issuing policy in the d.m.

Proof: Since no stockouts occur, then the FIFO issuance of S_n, \dots, S_1 and F_1, \dots, F_N in the d.m. results in the same total field life as FIFO in the

e.s.m. Assume that there exists a policy A in the d.m. which gives a greater total field life than FIFO. Then policy A must give a greater total field life than FIFO in the e.s.m. This last statement follows from the fact that the set of all possible policies in the e.s.m. includes all policies of the d.m. But FIFO is optimal in the e.s.m. hence we have a contradiction.

Corollary 1.1: Let $L(S) = aS + b$ for all $S \leq S_0$ and with $b > 0 > a > -1$. Let $v \geq 1$. If no stockouts occur in the d.m., then FIFO is optimal.

Proof: By Zehna [13] FIFO is optimal for the e.s.m.

Corollary 1.2: Let $L(S)$ be a concave nonincreasing function with $L'(S) \geq -1$ for all $S \leq S_0$. Let $v = 1$. If no stockouts occur in the d.m., then FIFO is optimal.

Proof: By Lieberman [9] FIFO is optimal for the e.s.m.

Corollary 1.3: Let $L(S)$ be concave nonincreasing with $L'(S) \geq -1$ for all $S \leq S_0$. Denote by $[x]$ the largest integer $\leq x$ where x is a real number. Then if the number of demand sources v has $[1/2(N+n+1)] \leq v \leq n$ and if no stockouts occur in the d.m. then FIFO is optimal.

Proof: By Pierskalla [10] FIFO is optimal for the e.s.m.

The preceding theorem and corollaries were concerned with $L(S)$ concave with slope ≥ -1 and in the linear case with $L'(S) > -1$. We now consider the linear case for $L'(S) = -1$, and show that FIFO is optimal for this case also. It is only necessary to prove that FIFO is optimal for the e.s.m. and then apply Theorem 1.1. For the time being assume that new items are not added to the stockpile.

Lemma 1.1: Let $L(S)$ be linear, with $L'(S) = -1$ for all $S \leq S_0$. Let $v = 1$. Then any issue policy A given by $A = [S_{i_1}, \dots, S_{i_j}]$ where $S_{i_k} > S_{i_{k+1}}$ ($k=1, \dots, j-1$) has a total field life of $Q_A = L(S_{i_j}) = S_0 - S_{i_j}$.

Proof: Let x denote the total field life up to but not including the issue of item S_{i_j} . Then $Q_A = L(S_{i_j} + x) + x$ and by Lemma 1.1 of Pierskalla [10]

$S_{i_j} + x < S_0$. Hence

$[L(S_{i_j} + x) - L(S_0)] / (S_{i_j} + x - S_0) = -1 \implies Q_A = L(S_{i_j} + x) + x = S_0 - S_{i_j}$, but

$[L(S_{i_j}) - L(S_0)] / (S_{i_j} - S_0) = -1 \implies L(S_{i_j}) = S_0 - S_{i_j} = Q_A$.

Lemma 1.2: Let $L(S)$ be linear and $L'(S) = -1$ for all $S \leq S_0$. Let $v \geq 1$. Then any issuing policy which issues items S_1, S_2, \dots, S_v each to a different demand source is optimal and the total field life from an optimal

policy, Q^* , is given by

$$Q^* = \sum_{i=1}^v L(S_i) = vS_0 - \sum_{i=1}^v S_i. \quad (1.1)$$

Furthermore $Q_{\text{FIFO},v} = Q_{\text{LIFO},v} = Q^*$.

Proof: We will first show (1.1). Consider any policy which issues S_1, \dots, S_v each to different demand sources say M_1, \dots, M_v respectively. Let demand source M_j receive the c items $[S_{j_1}, \dots, S_{j_c}]$. Now for any two items in inventory with current age $S_i < S_j$, we have two properties:

1. If S_i is issued first, then at the expiration of the field life of S_i , S_j will have no field life remaining, and
2. If S_j is issued first, then at the expiration of the field life of S_j , S_i will still have positive field life remaining.

Hence in any issue policy $A = [S_{j_1}, \dots, S_{j_c}]$ we can omit any items S_{j_i} for which there is some $S_{j_{i-k}}$ ($k=1, \dots, i-1$) such that $S_{j_{i-k}} < S_{j_i}$, since for these S_{j_i} , there will be no field life remaining when they are ready to be issued. Now by the second property since S_j is the youngest item of the S_{j_i} 's we must have that upon issue at any time, S_j will have positive field life. And by the first property S_j is the last item to be issued under all policies which we need to consider.

Thus instead of considering policy A for M_j it is only necessary to consider the policy $B_j = [S_{j_{i1}}, \dots, S_j]$ where $S_{j_{i1}} > S_{j_{i,k+1}}$. By Lemma 1.1, the total field life obtained from policy B_j is $Q_{B_j} = L(S_j) = S_0 - S_j$. But M_j was picked arbitrarily, thus for all $j = 1, \dots, v$ the total field life for all M_j 's is

$$Q = \sum_{j=1}^v Q_{B_j} = \sum_{j=1}^v L(S_j) = \sum_{j=1}^v (S_0 - S_j) = vS_0 - \sum_{j=1}^v S_j,$$

which is (1.1) as required.

Let C be any policy which does not issue S_1, \dots, S_v each to different M_1, \dots, M_v . Hence C must issue at least two of the items S_1, \dots, S_v to the same demand source, say S_i and S_j are issued to M_k where

$S_1 \leq S_i < S_j \leq S_v$. Now by properties (1) and (2) and by Lemma 1.1 we have $Q_C(M_k) = L(S_i) = S_0 - S_i$. And since S_i and S_j are issued to M_k , there is at least one M_t such that the youngest item issued to M_t has initial age $S_t > S_v$, and $Q_C(M_t) = L(S_t) = S_0 - S_t$. Thus the total field life for policy C is at most

$$Q_C \leq \sum_{i \neq j}^v L(S_i) + L(S_t) = vS_0 - S_t - \sum_{i \neq j}^v S_i < vS_0 - \sum_{i=1}^v S_i = Q^*.$$

Therefore the policy of issuing items S_1, \dots, S_v each to a different demand source is optimal. Now $Q_{LIFO, v}$ issues only S_1, \dots, S_v and each to different demand sources. Hence $Q_{LIFO, v} = Q^*$. Furthermore by Lemma 1.2 of Pierskalla [10], FIFO belongs to the class of policies such that items S_1, \dots, S_v are each issued to different M_1, \dots, M_v ; hence $Q_{FIFO, v} = Q^*$.

We are now able to state:

Corollary 1.4: Let $L(S)$ be linear with $L'(S) = -1$ for all $S \leq S_0$. Let $v \geq 1$. If no stockouts occur in the d.m., then FIFO is optimal.

Note by Lemma 1.2 if the F_i are known for all $i = N - v + 1, \dots, N$ the total field life for the model of Corollary 1.4 is $Q^* = Q_{FIFO, v} = \sum_{i=0}^{v-1} L(F_{N-i})$.

We now seek the optimal issuing policy for the d.m. when $L(S)$ is concave or convex and has slope < -1 for all $0 \leq S \leq S_0$. It is interesting to note that it is no longer necessary to assume that no stockouts occur; the reason for this will be discussed later.

Lemma 1.3: Let $L(S)$ be a convex or concave differentiable function on $[0, S_0]$ with $L'(S) < -1$ on $[0, S_0]$. Let there be n items $0 < S_1 < S_2 < \dots < S_n < S_0$ in inventory and no new items are ever added to the inventory. Let $v \geq 1$. If x_i is the total field life contributed by demand source M_i under any arbitrary policy A and if the x_i are ordered $x_1 \geq x_2 \geq \dots \geq x_v$, then $x_i \leq L(S_i)$ for all $i = 1, \dots, v$.

By Zehna [13] Theorems 4.2 and 4.3, LIFO maximizes the total field life. This lemma states that in addition LIFO maximizes the field life for each demand source.

Proof: Assume to the contrary that $x_i > L(S_i)$ for some $i = 1, \dots, v$. Then x_i must contain one or more items $S_j < S_i$ for if all items S_k assigned to M_i under policy A are such that $S_k \geq S_i$, then since LIFO is optimal for $v = 1$ (Zehna [13] Theorems 2.4 and 2.6) we would have $L(S_i) \geq x_i$ contrary to the assumption $x_i > L(S_i)$.

But if $S_j < S_i$ is assigned to M_i then there are at most $i - 2$ S 's which have $S_k < S_i$, $k \neq j$, available for assignment to the $i - 1$ M_t 's viz. M_1, \dots, M_{i-1} . Hence some M_t , $t = 1, \dots, i - 1$, does not receive any $S_k < S_i$. Therefore as stated in the preceding paragraph we must have $x_t \leq L(S_i)$ and $x_i > L(S_i) \geq x_t$ where $t < i$. But $x_i > x_t$ for $t < i$ contradicts the hypothesis of the lemma. Therefore $x_i \leq L(S_i)$ for all $i = 1, \dots, v$. |

Let A be any arbitrary policy for issuing the n items originally in the inventory and the N items added to the inventory in the future. We define a modified- A policy, MA , for issuing items to the $v \geq 1$ demand sources in the following way: Use policy A until a new item arrives, then discard the item currently in use in the field which has the least field life remaining and immediately replace it with the new item. When $A \equiv$ LIFO we denote MA by ML .

Theorem 1.2: Let $L(S)$ be a convex or concave differentiable function on $[0, S_0]$ with $L'(S) < -1$ on $[0, S_0]$. Let $v \geq 1$. Then ML is the optimal issuing policy for the dynamic model.

Proof: The proof is by induction. Let $N = 1$. And let the time of arrival of the new item be denoted by t .

We first show that under any policy A it is always better to discard some item currently in use and use the new item immediately.

Let T be the field life remaining to demand source M_i when the new item arrives. There are three cases:

Case (i): $0 < T < S_0$ then $L(0) > L(T) + T$.

Case (ii): $S_0 \leq T$ which implies $L(T) = 0$. Then $L(0) \geq L(S_1) \geq T = T + L(T)$.

Case (iii): $T = 0$ then $L(0) = T + L(T)$. In all cases it is better to use the new item immediately. For $j \neq i$ the field lives of the other M_j 's are not affected by this change.

In the above we have implicitly assumed for $j \neq i$ that all M_j 's have items currently in use. If some M_j did not have any items left and if $T > 0$ for M_i the new item would be assigned to M_j .

Let M_i be the demand source with the least field life remaining at time t . Denote this remaining field life to M_i by T_{\min} . $T_{\min} \geq 0$. For any $j \neq i$, let T_j be the field life remaining to M_j . Then $T_j \geq T_{\min}$. Let Q be the total field life obtained by all the M_k 's, $k = 1, \dots, v$, if the new item is not issued until the current items issued to M_i and M_j expire.

Then $Q + L(0) - T_{\min} \geq Q + L(0) - T_j$, for any $j \neq i$. Hence under any policy A we obtain

Statement (1): The new item should be issued immediately to the demand source which must discard the least field life.

Thus the optimal policy for the case $N = 1$ must belong to the class of modified policies.

We now show ML is optimal for $N = 1$. Consider any policy MA with $MA \neq ML$. Let $x_1 \geq x_2 \geq \dots \geq x_v$ be the field life contributed by M_1, \dots, M_v under policy A when the new item is not considered. By MA the new item will be assigned to M_v since x_v is the smallest field life. Recall by Lemma 1.3 that $x_i \leq L(S_i)$ for all i .

Case 1: $x_v \leq L(S_v)$ and $t \leq L(S_v)$

Then $Q_{ML} = \sum_{i=1}^{v-1} L(S_i) + t + L(0) \geq \sum_{i=1}^{v-1} x_i + t + L(0) = Q_{MA}$.

Case 2: $x_v \leq L(S_v)$ and $t > L(S_v)$

Then $Q_{ML} = \sum_{i=1}^v L(S_i) + L(0) \geq \sum_{i=1}^v x_i + L(0) = Q_{MA}$.

Case 3: $x_v < L(S_v)$ and $x < t \leq L(S_v)$

Then $Q_{ML} = \sum_{i=1}^{v-1} L(S_i) + t + L(0) > \sum_{i=1}^v x_i + L(0) = Q_{MA}$.

In all cases $Q_{ML} \geq Q_{MA}$ and since A was any arbitrary policy ML is optimal for $N = 1$.

Assume ML is optimal for adding $N = k$ items to inventory and consider $N = k + 1$.

We will first establish that the optimal policy for $k + 1$ must belong to the class of modified policies. Let T be the field life remaining to M_i when the $k + 1^{\text{st}}$ item arrives. Let t_k and t_{k+1} denote the time of arrival of the k^{th} and $k + 1^{\text{st}}$ items respectively. Since arrivals are distinct events $t_k < t_{k+1}$ and all items in use or in the stockpile, except the $k + 1^{\text{st}}$ item, have age greater than zero at time t_{k+1} , then $L(0) > T \geq 0$, and we have the same three cases (i), (ii) and (iii) as before.

Hence the new item should always be installed immediately on arrival. Furthermore by the same argument given in $N = 1$, the new item should be assigned to the demand source which loses the least field life. Thus Statement (1) applies also to the case of $N = k + 1$ and the optimal policy belongs to the class of modified policies.

Let $Q_{M_i, N}$ and $x_{i, N}$ denote the total field life for M_i under ML and MA respectively. In addition relabel the M_i 's in ML and in MA such that $Q_{M_i, N} \geq Q_{M_{i+1}, N}$ and $x_{i, N} \geq x_{i+1, N}$ for all $i = 1, \dots, v-1$. It is possible that M_1 under ML is not the same M_1 as under MA but this fact is of no importance in the following.

In the case $N = 1$ we showed $Q_{ML} \geq Q_{MA}$ but also in conjunction with Lemma 1.3 we showed that the total field life for each M_i under ML is greater than under MA, i.e., using the notation above $Q_{M_i, 1} \geq x_{i, 1}$ for all $i = 1, \dots, v$. It will now be proved that

$$Q_{M_i, k+1} \geq x_{i, k+1} \text{ for all } i = 1, \dots, v \quad (1.2)$$

where we inductively assume

$$Q_{M_i, k} \geq x_{i, k} \text{ for all } i = 1, \dots, v. \quad (1.3)$$

Now (1.3) and Statement (1) inform us that the $k+1^{\text{st}}$ arrival is immediately assigned to M_v . We only need to show $Q_{M_v, k+1} \geq x_{v, k+1}$. Let T_{ML} and T_{MA} be the total field life remaining to M at time t_{k+1} when ML and MA are being followed respectively. By (1.3) $T_{ML} \geq T_{MA} \geq 0$.

Case 1: $x_{v, k} \leq Q_{M_v, k}$ and $t_{k+1} > Q_{M_v, k}$

Then $Q_{M_v, k+1} = Q_{M_v, k} + L(0) \geq x_{v, k} + L(0) = x_{v, k+1}$

Case 2: $x_{v, k} \leq Q_{M_v, k}$ and $t_{k+1} \leq x_{v, k}$

Then $Q_{M_v, k} - x_{v, k} = T_{ML} - T_{MA}$

and $Q_{M_v, k+1} = Q_{M_v, k} - T_{ML} + L(0) = x_{v, k} - T_{MA} + L(0) = x_{v, k+1}$.

Case 3: $x_{v, k} < Q_{M_v, k}$ and $x_{v, k} < t_{k+1} \leq Q_{M_v, k}$

Then $T_{MA} = 0$ and $Q_{M_v, k} - x_{v, k} > T_{ML} - T_{MA} = T_{ML}$

$Q_{M_v, k+1} = Q_{M_v, k} - T_{ML} + L(0) > x_{v, k} + L(0) = x_{v, k+1}$.

Hence in all cases

$$Q_{M_v, k+1} \geq x_{v, k+1} \quad (1.4)$$

Now since the field life for the other M_i $i = 1, \dots, v-1$ are unchanged

then by (1.3)

$$Q_{M_{i,k}} = Q_{M_{i,k+1}} \geq x_{i,k+1} = x_{i,k} \text{ for all } i = 1, \dots, v-1. \quad (1.5)$$

Combining (1.4) and (1.5) we see that (1.2) holds for all i , and since MA was any arbitrary policy, ML is optimal.

In the earlier FIFO results, it was assumed that the ordering schedule for new items was arranged so that stockouts did not occur. This assumption was essential for FIFO optimality as the following example shows:

$L(S) = -(1/3)S + 3$ for $S \in [0,9]$ and $L(S) = 0$ otherwise; $F_1 = -2.3956$, $F_2 = -2.6667$, $S_1 = 1$, $S_2 = 5$, $S_3 = 6$, $S_4 = 7$, $S_5 = 8$. Then for $v = 2$ FIFO = $[S_5, S_3, S_1, F_2; S_4, S_2, F_1]$ which yields $Q_F = 10.9883$ as compared to A = $[S_5, S_4, S_3, S_2, F_1; S_1, F_2]$ which yields $Q_A = 11.0623$ is definitely not optimal. In the case of FIFO, however, a stockout occurred because the total field life for $[S_4, S_2]$ is 1.7781 whereas item F_1 does not arrive until $t_1 = 2.3956$.

It is interesting to note, however, that when $L'(S) < -1$ we can drop the no stockout assumption. The reason for this is essentially contained in Lemma 1.3 which states that each demand source receives more field life under LIFO than from any other policy. Thus if we followed a non-LIFO policy, say MA, we could expect stockouts to be more frequent and of a much longer duration. But, the new arriving item under any modified policy is used to its fullest extent. Therefore the policy which minimizes the total stockout duration will maximize the total field life; and as shown this optimal policy is ML for the dynamic model.

3. BATCHES OF ITEMS OF THE SAME AGE IN THE STOCKPILE

It has always been assumed in the literature and in this paper that the n items in inventory all have different initial ages. In general, this assumption is not necessary. With minor modifications, the theorems, lemmas and corollaries of the preceding sections as well as the results of the papers referenced in the Bibliography can be stated for batches of items of the same age. More specifically we modify assumption (1) of the model as follows:

Assumption (1)': At the start of the process a stockpile has N sets of indivisible identical items where the items in the i^{th} set all have the same initial age, S_i , for $i = 1, \dots, N$.

The initial age of the items in any set, say the i^{th} set, is different than the initial age of the items in any other set. Assume $0 \leq S_1 < S_2 < \dots < S_N < S_0$ and that the i^{th} set contains n_i items for $i = 1, \dots, N$. Then $\sum_{i=1}^N n_i = n$.

For ease of adapting the previous results to the batch problem we make the following ordering of the n items.

Let the first n_1 items be numbered from 1 to n_1 , the next n_2 items from $n_1 + 1$ to $n_1 + n_2$, the next n_3 items from $n_1 + n_2 + 1$ to $n_1 + n_2 + n_3$, etc., until all the items possess a number from 1 to n and such that $S_i \leq S_{i+1}$ for all $i = 1, \dots, n-1$. We note that there are $\prod_{i=1}^N (n_i)!$ ways to complete the above ordering; hence choosing any one of these ways is somewhat arbitrary. However the total field life realized from the n items by any policy under any one of the $\prod_{i=1}^N (n_i)!$ ways is the same. Thus we choose an ordering and define FIFO (LIFO) as the policy which issues the highest (lowest) indexed item in the stockpile each time an item is issued.

We now prove a general theorem which applies to most of the previous results in inventory depletion theory.

Theorem 2.1: Let $L(S)$ be a continuous function. If FIFO (LIFO) is optimal when assumption (1) of the model holds, i.e., when there are no batches in the inventory, then FIFO (LIFO) is optimal when assumption (1)' holds, i.e., when batches are allowed.

Proof: We will prove the theorem for FIFO; the theorem for LIFO follows mutatis mutandis.

Let $\epsilon_0 = \min_{1 \leq i \leq N-1} [S_{i+1} - S_i, S_0 - S_N] > 0$. ϵ_0 exists since $0 \leq S_1 < S_2 < \dots < S_N < \infty$. Consider any ϵ such that

$$\epsilon_0 > \epsilon > 0 \quad (2.1)$$

and consider the n items defined by

$$T_{ij} = S_i + (\epsilon/[n_i - j + 1]) \quad (2.2)$$

for all $j = 1, \dots, n_i$ and $i = 1, \dots, N$. Then from (2.2) we have

$$0 < T_{11} < T_{12} < \dots < T_{1n_1} < T_{21} < \dots < T_{N1} < \dots < T_{Nn_N} < S_0. \quad (2.3)$$

Denote by $Q_F(\epsilon)$ and $Q_A(\epsilon)$ the total field life from the issuance of the n items of (2.3) by FIFO and by an arbitrary policy A , respectively. Denote by Q_{FB} and Q_{AB} the total field life from the issuance of the n items in batches by FIFO and by an arbitrary policy A , respectively.

Since $L(\cdot)$ is a continuous function, then Q_A is also a continuous function for any policy A .

Hence for any $\delta > 0$ and δ sufficiently small there is an $\epsilon > 0$ such that ϵ satisfies (2.1) and such that $Q_{FB} - Q_F(\epsilon) < \delta$, $Q_{AB} - Q_A(\epsilon) < \delta$.

Then

$$Q_{FB} + \delta > Q_F(\epsilon) \quad \text{and} \quad Q_A(\epsilon) > Q_{AB} - \delta. \quad (2.4)$$

By hypothesis, however

$$Q_F(\epsilon) \geq Q_A(\epsilon) \quad (2.5)$$

for all ϵ satisfying (2.1). Thus $Q_{F_B} > Q_{A_B} - 2\delta$. Now (2.4) and (2.5) hold for all $\delta > 0$ where $Q_F(\epsilon)$ and $Q_A(\epsilon)$ are defined by the issuance of the items in (2.3). Since $\delta > 0$ can be made arbitrarily small, we have $Q_{F_B} \geq Q_{A_B}$ for any arbitrary policy A.

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