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AN INVENTORY PROBLEM WITH OBSOLESCENCE*

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ABSTRACT

A stochastic single product convex cost inventory problem is considered in which there is a probability, π_j, that the product will become obsolete in the future period j. In an interesting paper, Barankin and Denny essentially formulate the model, but do not describe some of its interesting and relevant ramifications. This paper is written not only to bring out some of these ramifications, but also to describe some computational results using this model. The computational results show that if obsolescence is a distinct possibility in the near future, it is quite important that the probabilities of obsolescence be incorporated into the model before computing the optimal policies.

1. INTRODUCTION

In certain inventory situations it is the case that technological change or changes in the techniques of production may make a product obsolete almost overnight. In some of these cases it may be possible to state a probability mass function (pmf) which gives the probability π_j that obsolescence will occur in some period j in the future. π_j is the pmf of the random variable N which describes the length of the horizon. Thus the primary inventory problem to be considered here is the stochastic single-product convex-cost N period problem, where the number of periods N is a random variable. In an interesting paper, Barankin and Denny [3] essentially formulate the model, but do not describe some of its interesting and relevant ramifications.

This paper is written not only to bring out some of these ramifications, but also to describe some computational results using this model versus the well known n period model of Arrow, Harris, and Marschak [1] and Bellman, Glicksberg, and Gross [6] and the infinite period model so capably presented in Arrow, Karlin, and Scarf [2].

The author's original interest in this model was stimulated by a paper of Professor Amnon Rapoport [10]. Rapoport describes an experiment he performed with a group of students in an attempt to determine their abilities to think dynamically. The students who had not had an inventory course were confronted with a sequence of six stochastic, single-product, convex-cost, inventory problems and at each stage they had to decide how much inventory to order. He compared their performance with the infinite stage model and found that they did not perform particularly well. Unfortunately he did not use the correct model. These students knew that no problem would last an infinite number of periods; furthermore the actual number of periods, to them, was a random variable independent of the demands in each period. Hence, in this case also, the appropriate inventory model is the obsolescence situation soon to be described in detail.

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In the next section the model formulation is given. In the subsequent section some characteris of the model are presented and some particular obsolescence distributions are discussed. Then in final section some computational results are given, which in themselves are worthy of note due to manner in which the critical numbers behave for various types of obsolescence distributions.

2. MODEL FORMULATION

The following definitions, assumptions, and conventions will be used:

(a) In discussing the finite period problem the periods will be numbered backwards. numbers M and T are integers representing the maximum number of periods that the problem last and the first period in which the probability of obsolescence π_T first becomes positive, respecti $(T \leq M)$.

(b) $h(\cdot)$ are $p(\cdot)$ are continuously differentiable convex increasing holding and shortage functions, respectively, and will be charged at the end of the period after the period's demand occurred, but prior to delivery of stock for the next period. It is assumed $h(0)=p(0)=0$.

(c) There is no backlogging of excess demand.

(d) Delivery is immediate.

(e) c is the marginal cost of one unit of stock purchased, r is the marginal revenue of one of stock sold $(r > c)$, and α is the discount factor $0 \leq \alpha \leq 1$.

(f) π_j is the probability of obsolescence in period j after ordering and after the occurrenc demand $(j=1, \ldots, M)$. Let w denote the period in which obsolescence occurs and let p_r be the co tional probability that the problem terminates in period $r \leq t \leq m$ given that the problem has not minated in periods $M, M-1, \ldots, t+1$. Then

$$P_r = P\{w=r|w \leq t\}$$
$$= \frac{P\{w=r \text{ and } w < t\}}{P\{w \leq t\}} = \frac{\pi_r}{\sum_{k=1}^{t} \pi_k} \quad ; \qquad r=1, \ldots, t.$$

Of course $1-p_r$ is the conditional probability that the problem will not terminate in period r, give has survived to period t.

(g) $\phi(\xi)$ is the probability density function (pdf) of the nonnegative demand random vari D. The demands are assumed to be independent and identically distributed among the periods.

$$\text{(h) } L(y) = \int_0^y h(y-\xi)\phi(\xi)d\xi - r\int_0^y \xi\phi(\xi)d\xi - ry\int_y^\infty \phi(\xi)d\xi$$
$$+\int_y^\infty p(\xi-y)\phi(\xi)d\xi. \tag{1}$$

(i) $\bar{f}_j(x)$ is the minimum discounted conditional expected cost from period j onward g the initial stock on hand in period j is x and given that obsolescence has not occurred in prior peri

$$\text{(j) } \bar{G}_j(y) = cy + L(y) + \alpha(1-p_j)\left[\bar{f}_{j-1}(0)\int_y^\infty \phi(\xi)d\xi \right.$$
$$\left. + \int_0^y \bar{f}_{j-1}(y-\xi)\phi(\xi)d\xi\right]. \tag{2}$$

(k) $\bar{f}_j(x) = \min_{y \geq x} \{\bar{G}_j(y) - cx\}$.

(1) It is assumed that $h'(0)+p'(0)+r \geqslant \alpha c(1-p_n)$ for all $n=1,2, \ldots$. This condition holds if $h'(0)+p'(0)+r \geqslant \alpha c$ as required and justified in [2], [6], and others. Furthermore, it is assumed that $E[p'(D)] > c-r$, where $p'(x)=dp(x)/dx$. This latter condition ensures that the one period problem has a positive finite solution y_1^*.

The inventory problem is to find $y \geqslant x$ to minimize $\bar{G}_j(y)-cx$. This problem as well as the Arrow, Harris, and Marschak [1] problem has a single critical number independent of the initial stock x.

3. CHARACTERISTICS OF THE MODEL

Some of the characteristics of the model are independent of the type of distribution chosen for $\{\pi_j\}$. In these cases the same conclusions hold as in the Bellman, Glicksberg, and Gross [6] and Arrow, Karlin, and Scarf [2] models and for the same reasons. Other results, however, are dependent on the nature of the $\{\pi_j\}$. In the former cases the results will be given without proofs (proofs for the various parts are available in [8], [7], [6], [1], [2]) and in the latter cases proofs will be supplied in the appendix.

THEOREM 1: $\bar{f}_n(x)$ is a convex function and the optimal policy in each period for the n-period problem is a single critical number, $y^*(p_n, p_{n-1}, \ldots, p_1) \equiv y^*(P_n)$, where $y^*(P_n)$ is defined as the smallest number satisfying $\bar{G}_n'(y)=0$.

LEMMA 1: $\bar{f}_n'(x)=\dfrac{d\bar{f}_n(x)}{dx} \geqslant -c$ for all x, and all n.

Let $p=\lim\limits_{n\to\infty} p_n$, $\quad (0 \leqslant p \leqslant 1)$, $\quad$ and let $P_\infty=(p_1, p_2, p_3, \ldots)$.

Define $y^*(P_\infty)$ as the smallest number y satisfying

$$c+L'(y)+\alpha(1-p)\int_0^y \bar{f}'(y-\xi)\phi(\xi)d=0 \tag{3}$$

(i.e., $c+L'(y)-\alpha c(1-p)\phi(y)=0$), where $\bar{f}(x)$ is the minimum discounted expected costs for an infinite horizon given an initial stock of x units and given that obsolescence has not occurred. When discussing the infinite period case the periods will be numbered forward.

It will be assumed that $L'(y)=0$ has a solution and the smallest y satisfying $L'(y)=0$ will be denoted by $\bar{y}$. Define

$$\bar{G}(y)=c\cdot y+L(y)+\alpha(1-p)[\bar{f}(0)\int_y^\infty \phi(\xi)d\xi+\int_0^y \bar{f}(y-\xi)\phi(\xi)d\xi]. \tag{4}$$

THEOREM 2: If $p>0$ or if $\alpha<1$, or both, then

(a) $\lim\limits_{n\to\infty} \bar{f}_n=\bar{f}(x)$: $\lim\limits_{n\to\infty} G_n(x)=\bar{G}(x)$,

(5) (b) $\bar{f}(x)=\min\limits_{y\geqslant x}\{\bar{G}(y)-cx\}$

(c) $\bar{f}(x)$ is a convex function,

(d) $y^*(P_\infty)$ is an optimal solution to (5), and

(e) the sequence $\{y^*(P_n)\}$ contains convergence subsequences and every limit point of $\{y^*(P_n)\}$ satisfies (3). Furthermore if (3) has a unique solution, then $\{y^*(P_n)\}$ converges to $y^*(P_\infty)$. Equation (3) has a unique solution if $L(y)$ is strictly convex.

The following results hold under certain assumptions on the probabilities .

$$[\pi_j]_{j=1}^{M}$$

It will be assumed that the random variable N has an increasing failure rate (IFR) distribution. Th definition of IFR is as follows: A discrete random variable has an increasing failure rate distributio provided the function

$$p(j)=\frac{\pi_j}{\sum_{i=j+1}^{\infty}\pi_j}$$

is increasing in j. In an interesting paper, Barlow, Marshall, and Proschan [4] describe many propertie of IFR distributions. Among these properties is the following: "If N is a time variable and time is re versed, then the random variable N has an increasing failure rate if, and only if, $-N$ has the decreasin ratio

$$\pi_j/\sum_{i=1}^{j}\pi_i."$$

For this inventory model then N has an IFR distribution if, and only if, p_j is decreasing, since time ha been reversed in the n-period model.

THEOREM 3: If $1-p_j$ is a nondecreasing function of j then

(a) $y^*(P_1)\leq y^*(P_2)\leq \ldots \leq y^*(P_n)\leq y^*(P_\infty)\leq \bar{y}$.

(b) $\lim_{n\to\infty} y^*(P_n)=y^*(P_\infty)\leq \bar{y}$.

Some examples of IFR distributions which might prove useful in this obsolescence context ar

(a) the equally likely distribution

$$\pi_j=\begin{cases}\frac{1}{T} \text{ for } j=1,\ldots,T\\ 0 \text{ otherwise.}\end{cases}$$

(b) the truncated Poisson distribution

$$\pi_j=\begin{cases}\left(\frac{\lambda^{j-1}}{(j-1)!}\right)\cdot\left(\sum_{k=1}^{T}\frac{\lambda^{k-1}}{(k-1)!}\right)^{-1} & \text{for } j=1,\ldots,T\\ & \lambda>0\\ 0 & \text{otherwise.}\end{cases}$$

(c) the discrete triangular distribution

$$\pi_j=\begin{cases} j\left(\sum_{k=1}^{T} k\right)^{-1} & \text{for } j=1,\ldots,T\\ 0 & \text{otherwise.}\end{cases}$$

(d) the truncated geometric distribution

$$\pi_j=\begin{cases}\gamma^{j-1}\left(\sum_{k=1}^{T}\gamma^{k-1}\right)^{-1} & \text{for } j=1,\ldots,T\\ & 0<\gamma<1\\ 0 & \text{otherwise.}\end{cases}$$

Actually pmf (*d*) would perhaps be less reasonable since it weights the very last periods most heavily; however, there could be obsolescence situations in which the last periods should receive the greatest weights. It will be seen later on that because of this property pmf (*d*) behaves more like the n-period model (n known) than any of the distributions (*a*), (*b*), or (*c*).

There are many other IFR distributions besides the foregoing. The book by Barlow and Proschan [5] gives a listing and discussion of them.

Of these four distributions perhaps the most useful is the truncated Poisson since it allows one to choose the parameters λ and T in such a way as to locate the mean and range of the distribution as desired. In this case,

$$E[N]=\frac{\lambda\sum_{j=1}^{T-1}\frac{\lambda^{j-1}}{(j-1)!}}{\sum_{j=1}^{T}\frac{\lambda^{j-1}}{(j-1)!}}$$

and

$$E[N^2]=\frac{\lambda\sum_{j=1}^{T-1}\frac{\lambda^{j-1}}{(j-1)!}+\lambda^2\sum_{j=1}^{T-2}\frac{\lambda^{j-1}}{(j-1)!}}{\sum_{j=1}^{T}\frac{\lambda^{j-1}}{(j-1)!}}.$$

Distributions (*a*) and (*c*) because of their single parameter T do not have the same flexibility. Of course many other multi-parameter distributions for the nonnegative random variable N could be used (see [5]).

For these four distributions the conditional probabilities of obsolescence, p_j's, *given the process has j periods to go* are

$$\text{(a) } p_j=\frac{1}{j} \qquad j=1,\ldots,T$$

$$\text{(b) } p_j=\left(\frac{\lambda^{j-1}}{(j-1)!}\right)\left(\sum_{k=1}^{j}\frac{\lambda^{k-1}}{(k-1)!}\right)^{-1} \qquad j=1,\ldots,T,\ \lambda>0$$

$$\text{(c) } p_j=\frac{2}{j+1} \qquad j=1,\ldots,T$$

$$\text{(d) } p_j=\frac{\gamma^{j-1}}{\sum_{k=1}^{j}\gamma^{k-1}} \qquad j=1,\ldots,T,\ 0<\gamma<1.$$

It should be noted that as $T\to+\infty$ we are numbering foreward and the distributions (b) and (d) become the Poisson and geometric distributions respectively. Furthermore for

(*b*) $p=e^{-\lambda}>0$ for $0<\lambda<\infty$,
(d) $p=1-\gamma>0$ for $0<\gamma<1$,

and (a) and (c) $p=0$.

The foregoing results, Lemma 1 and Theorems 1, 2, and 3, hold under various modifications of the assumptions of the model. If assumption (c) is removed and complete backlogging of excess demand is allowed, then lags in delivery are permissible. It is also possible to obtain some bounds on the critical numbers $y^*(P_n)$ which indicate the rate of convergence of the $y^*(P_n)$ to $y^*(P_\infty)$ when IFR distributions are used for $\{\pi_j\}$. Before stating the bounds, it is useful to give the following Lemma.

LEMMA 2: If backlogging is allowed and if $1-p_j$ is nondecreasing in j then for any $a \geq 0$ and all x such that $x \leq x+a \leq y^*(P_\infty)$, $0 \leq f_n(x+a)-f_n(x)+ac \leq \mu_n(a)$, $n=0, 1, \ldots$ where

$$\mu_0(a)=ac$$

$$\mu_n(a)=\alpha[ac(1-p)+(1-p_n)(\mu_{n-1}(a)-ac)].$$

The lemma states that for any two points in the domain lying below the critical number for the infinite problem $y^*(P^\infty)$ call these points x_1 and x_2, where $x_2-x_1=a \geq 0$, the difference in the functional, values $f(x_2)-f(x_1)$ is bounded by

$$-c(x_2-x_1) \leq f_n(x_2)-f_n(x_1) \leq \mu_n(x_2-x_1)-c(x_2-x_1).$$

It is not difficult to show by induction that $\mu_n(a)$ can be given by the following formula. First define $1-p_j \equiv q_j$ and $1-p=q$ then

$$\mu_n(a)=ac\left[\alpha(q-q_n)+\sum_{j=2}^{n}\alpha^j(q-q_{n-j+1})\prod_{k=1}^{j-1}q_{n-k+1}+\alpha^n\prod_{k=1}^{n}q_{n-k+1}\right]$$

for all $n=1, 2, 3, \ldots, \mu_0(a)=ac$.

For Theorem 3, if $1-p_j$ is nondecreasing then $y^*(P_n) \leq y^*(P_\infty)$ and it is perfectly acceptable to let $y^*(P_\infty)-y^*(P_n)=a$ in Lemma 2 above.

LEMMA 3: If backlogging is allowed, $L(y)$ is a convex function ϵC^2, $L''(y)>0$ for $y \epsilon [0, \bar{y}]$, and $1-p_j$ is nondecreasing in j, then

$$0 \leq y^*(P_\infty)-y^*(P_n) \leq \frac{c}{L''(\hat{y})}\left[\alpha(q-q_n)+\sum_{j=2}^{n}\alpha^j(q-q_{n-j+1})\prod_{k=1}^{j-1}q_{n-k+1}+\alpha^n\prod_{k=1}^{n}q_{n-k+1}\right],$$

where $\hat{y}$ is the point in the open interval $(y^*(P_n), y^*(P_\infty))$ given by the second order Taylor expansion

$$L(y^*(P_\infty))=L(y^*(P_n))+aL'(y^*(P_n))+\frac{a^2}{2}L''(\hat{y}).$$

Other modifications to the assumptions are possible. For example, a fixed setup cost $K \cdot \delta(y-x)$ may be added to the production cost

$$\left(\delta(y-x)=\begin{cases}1 \text{ if } y>x \\ 0 \text{ if } y=x\end{cases}\right).$$

In this case the optimal policies will be (s, S) and the cost functions K-convex; however, the preceding results will not be valid.

4. COMPUTATIONAL RESULTS

The results shown in this section are quite revealing. In general they show that if obsolescence is a distinct possibility in the near future, it is important that the probabilities of obsolescence be incorporated into the model before computing the optimal M period policies.

The computations were obtained on the Case Western Reserve University Univac 1107. In the dynamic programming algorithm the search for the optimal policy $y^*(P_j)$ in each stage was confined to integers and lattice search was used [11]. For this reason the actual $y^*(P_j)$'s so obtained may vary slightly from the true $y^*(P_j)$'s.

Two examples were run. The following parameters were chosen.

	c	h	p	r	$M=T$	Demand Erlang	Expected demand	Standard deviation of demand
Example 1...	$17.80	$2.10	$46.00	$22.50	15	$r=11.0$ $\beta=0.044$	250	75
Example 2...	$17.80	$2.10	$46.00	$22.50	25	$r=4.0$ $\beta=1/3$	12	6

The costs above coincide with the costs for Rapoport's [10] Example Number 4. Other costs (not given here) were run which yield the same type of results. The Erlang distribution was chosen for the demand distribution since it is a two parameter distribution:

$$\phi(\xi)=\frac{\beta}{(r-1)!}(\beta\xi)^{r-1}e^{-\beta\xi} \qquad \text{for } \xi \geq 0$$

$$=0 \qquad \text{otherwise.}$$

The critical numbers for each example were computed using the M period problem without obsolescence and using the four obsolescence distributions (a)–(d) above. In distribution (b), the truncated Poisson distribution, λ was chosen $\lambda=10$; in distribution (d), the truncated geometric distribution, γ was chosen $\gamma=1/2$.

The critical numbers are portrayed on the following two graphs (Figures 1 and 2).

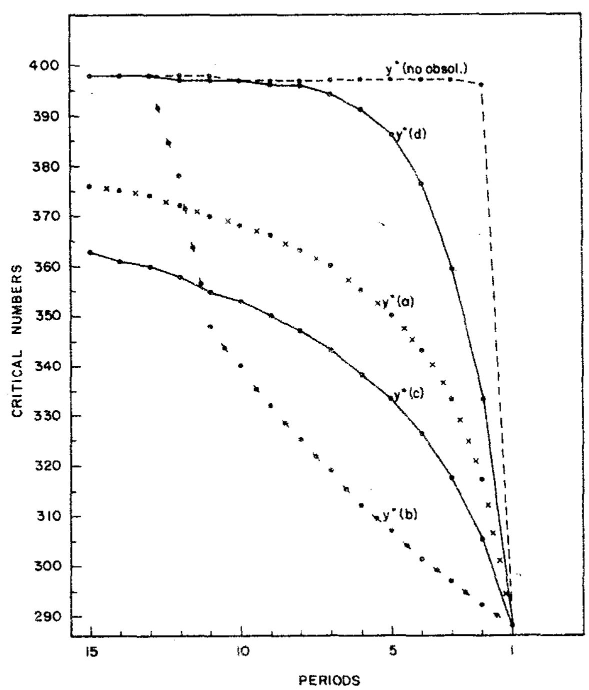

Figure 1. Example 1: Critical Numbers— Demand is Gamma (11.0, 0.044)

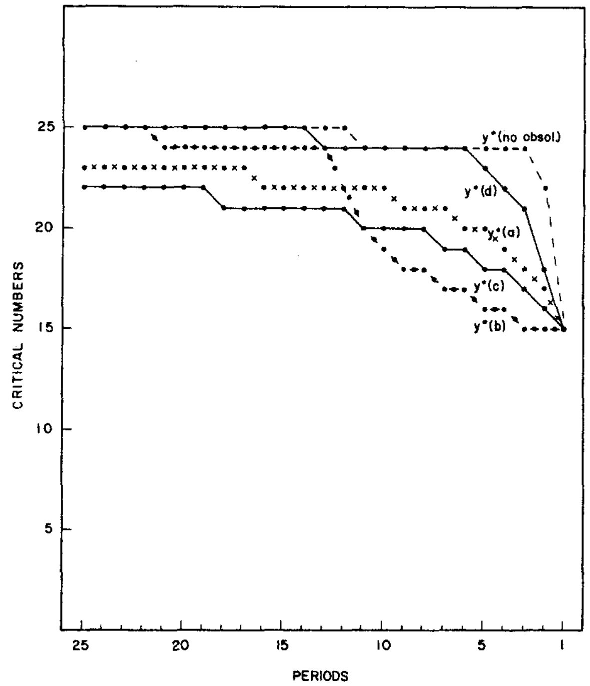

Figure 2. Example 2: Critical Numbers—Demand is Gamma (4, 1/3)

It is apparent in both examples that the critical numbers without obsolescence rapidly approac the steady state number y^*. Furthermore the critical numbers for the truncated geometric are clos to the critical numbers for the no obsolescence case except near the end of the process. As mentione earlier, this phenomenon is caused by the heavy weighting attached to the ending periods by th truncated geometric distribution.

On the other hand, the other three obsolescence distributions (a), (b), and (c) differ markedl from the no obsolescence results and indeed over the range of M studied here, distributions (a) an (c) have not achieved the steady state.

The conclusion reached by these examples is that it is important to consider obsolescence prob abilities in computing the critical numbers for an inventory problem. This conclusion is particularl applicable to spare parts stocking problems where the threat of obsolescence is high.

As a final point, an example is given below where an IFR distribution for π is not used and th critical numbers are not ordered as in Theorem 3.

	c	h	p	r	M=T	Demand Erlang	π_1	π_2	π_3
Example 3	$17.80	$2.10	$46.00	$22.50	3	$\gamma=4.0$ $\lambda=1/3$	0.019	0.001	0.980

By using these parameters we find the optimal three period policies to be $y^*(P_3)=15$, $y^*(P_2)=21$, and $y^*(P_1)=15$, and we do not have the ordering given in Theorem 3; however, the lack of ordering among the $y^*(P_j)$'s in no way detracts from the importance of considering the obsolescence probabilities when computing the critical numbers.

Appendix

PROOF OF THEOREM 3:

(a) The theorem will be proved by induction on n. We first show $y^*(P_1) \leqslant y^*(P_2)$. Recall that $y^*(P_1)$ and $y^*(P_2)$ are the smallest y's satisfying $\bar{G}_1'(y) = L'(y) + c = 0$ and $\bar{G}_2'(y) = c + L'(y) + \alpha(1-p_2) \int_0^y \bar{f}_1'(y-\xi)\phi(\xi)d\xi = 0$. Furthermore, $L'(0) = -r - E[p'(D)] < -c = L'(y^*(P_1))$ and since $L(\cdot)$ is convex $y^*(P_1) > 0$. Consider any y in the right-closed interval $(0, y^*(P_1)]$.

$$\begin{aligned}\bar{G}_2'(y) - \bar{G}_1'(y) &= \alpha(1-p_2)\int_0^y \bar{f}_1'(y-\xi)\phi(\xi)d\xi \\ &= \alpha(1-p_2)\int_0^y (-c)\phi(\xi)d\xi \\ &= -\alpha c(1-p_2)\Phi(y) \leqslant 0.\end{aligned}$$

Hence $\bar{G}_2'(y) \leqslant \bar{G}'(y)$ for all $y \epsilon (0, y^*(P_1)]$ and $\bar{G}_2'(y^*(P_1)) \leqslant \bar{G}_1'(y^*(P_1)) = 0$. Since $\bar{G}_2$ is convex, then $y^*(P_2) \geqslant y^*(P_1)$. In this case the proof is independent of the behavior of p_2.

We will now show $y^*(P_n) \geqslant y^*(P_{n-1})$. Again note $y^*(P_n)$ is obtained as the smallest y satisfying $\bar{G}_n'(y) = 0$. $y^*(P_n)$ exists since $\bar{G}_n'(0) = L'(0) < 0$ and $\bar{G}_n(+\infty) = +\infty$ and $\bar{G}_n$ is convex. Similarly $y^*(P_{n-1})$ is obtained as the smallest y satisfying $\bar{G}_{n-1}'(y) = 0$ and it also exists. Now

$$\bar{G}_n'(y) - \bar{G}_{n-1}'(y) = \alpha[(1-p_n)\int_0^y \bar{f}_{n-1}'(y-\xi)\phi(\xi)d\xi - (1-p_{n-1})\int_0^y \bar{f}_{n-2}'(y-\xi)\phi(\xi)d\xi].$$

Furthermore, for all y such that $y \leqslant y^*(P_{n-1})$

$$\bar{f}_{n-1}'(y) = -c$$

and $\bar{f}_{n-1}'(y-\xi) = -c$ for all $\xi \geqslant 0$

From lemma 1 $\bar{f}_{n-2}'(x) \geqslant -c$ for all x. Thus for all y such that $y \leqslant y^*(P_{n-1})$

$$\begin{aligned}&\bar{G}_n'(y) - \bar{G}_{n-1}'(y) \\ &= \alpha[(1-p_n)(-c)\Phi(y) - (1-p_{n-1})\int_0^y \bar{f}_{n-2}'(y-\xi)\phi(\xi)d\xi] \\ &\leqslant \alpha[(1-p_n)(-c)\Phi(y) + (1-p_{n-1})\int_0^y c\phi(\xi)d\xi] \\ &= -\alpha c\Phi(y)[(1-p_n) - (1-p_{n-1})] \\ &\leqslant 0 \text{ since } 1-p_j \text{ is nondecreasing in } j.\end{aligned}$$

Then $\bar{G}_n'(y) \leqslant \bar{G}_{n-1}'(y)$ for all $y \leqslant y^*(P_{n-1})$; in particular,

$$\bar{G}_n'(y^*(P_{n-1})) \leqslant \bar{G}_{n-1}'(y^*(P_{n-1})) = 0$$

and since $\bar{G}_j(y)$ is convex for all j, $y^*(P_n) \geqslant y^*(P_{n-1})$. We now show

$$y^*(P_n) \leqslant y^*(P_\infty) \qquad \text{for all } n.$$

The proof is the same as given above for the n period case with $\bar{G}'$ replacing $\bar{G}_n'$ and $\bar{G}_n'$ replacing $\bar{G}_{n-1}'$.

Finally, we show $y^*(P_\infty) \leqslant \bar{y}$.

$$L'(y^*(P_\infty)) = -c - \alpha(1-p) \int_0^{y^*(P_\infty)} \bar{f}'(y^*(P_\infty)-\xi)\phi(\xi)d\xi$$

$$= -c + \alpha c(1-p)\Phi(y^*(P_\infty))$$

$$\leqslant -c + \alpha c(1-p) < 0 = L'(\bar{y}).$$

Since $L(\cdot)$ is convex, $y^*(P_\infty) \leqslant \bar{y}$.

(b) Since the $\{y^*(P_n)\}$ is monotonic by part (a) above and bounded from above by $y^*(P_\infty)$, it has a limit point y^*. We must show $y^* = y^*(P_\infty)$. (Since we already know $y^* \leqslant y^*(P_\infty)$, we only must show $y^* \not< y^*(P_\infty)$.) Assume $y^* < y^*(P_\infty)$. Since $y^*(P_\infty)$ is the smallest y satisfying $\bar{G}'(y) = 0$, then $\bar{G}'(y^*) < \bar{G}'(y^*(P_\infty)) = 0$; however, we know $\bar{G}_n'(y^*(P_n)) = 0$ for all n and $\lim_{n\to\infty} \bar{G}_n'(x) = \bar{G}'(x)$. Thus by the uniform continuity of $\bar{G}_n'$ on $[0, \bar{y}]$, we have $\lim_{n\to\infty} \bar{G}_n'(y^*(P_n)) = \bar{G}'(y^*) = 0$ which contradicts $\bar{G}'(y^*) < 0$. Hence $y^* = y^*(P_\infty)$. Q.E.D.

PROOF OF LEMMA 2:

Since $\bar{f}_n(\cdot)$ is convex and $\bar{f}_n(\cdot) \geqslant -c$, then for $a \geqslant 0$, $\bar{f}_n(x+a) - \bar{f}_n(x) + ac \leqslant \bar{f}_n(x+a) - \bar{f}_n(x) - a\bar{f}'(x) \leqslant 0$ for all n. We must show that the second inequality holds. It is clearly true for $n = 0$. Hence assume it is true for $n-1$. We will consider three cases. ($\mu_n(a)$ will be abbreviated by μ_n.)

CASE 1: $y_n^* \leqslant x \leqslant x+a \leqslant y^*$

$$\bar{f}_n(x+a) - \bar{f}_n(x) + ac$$

$$= L(x+a) - L(x) + ac + \alpha q_n \int_0^\infty [\bar{f}_{n-1}(x+a-\xi) - \bar{f}_{n-1}(x-\xi)]\phi(\xi)d$$

$$\leqslant L(x+a) - L(x) + ac + \alpha q_n \int_0^\infty [\mu_{n-1} - ac]\phi(\xi)d\xi$$

$$= L(x+a) - L(x) + ac + \alpha q_n[\mu_{n-1} - ac]$$

$$\leqslant aL'(y^*) + ac + \alpha q_n[\mu_{n-1} - ac] \text{ since } L(\cdot) \text{ is convex}$$

$$= ac[\alpha q - 1] + ac + \alpha q_n[\mu_{n-1} - ac]$$

$$= \alpha[acq + q_n(\mu_{n-1} - ac)] = \mu_n \text{ as required.}$$

CASE 2: $x \leqslant y_n^* \leqslant x+a \leqslant y^*$

$$\bar{f}_n(x+a) - \bar{f}_n(x) + ac$$

$$= L(x+a) + \alpha q_n \int_0^\infty \bar{f}_{n-1}(x+a-\xi)\phi(\xi)d\xi$$

$$- c(y_n^* - x) - L(y_n^*) - \alpha q_n \int_0^\infty \bar{f}_{n-1}(y_n^* - \xi)\phi(\xi)d\xi + ac$$

$$= L(x+a) - L(y_n^*) + ac - c(y_n^* - x)$$

$$+ \alpha q_n \int_0^\infty [\bar{f}_{n-1}(x+a-y_n^*+y_n^*-\xi) - \bar{f}_{n-1}(y_n^*-\xi)]\phi(\xi)d\xi$$

$$\leqslant L(x+a) - L(y_n^*) + ac - c(y_n^* - x)$$

$$+ \alpha q_n \int_0^\infty [\mu_{n-1}(x+a-y_n^*) - (x+a-y_n^*)c]\phi(\xi)d\xi$$

$$= L(x+a) - L(y_n^*) + ac - c(y_n^* - x) + \alpha q_n[\mu_{n-1}(x+a-y_n^*) - (x+a-y_n^*)c]$$

$$\leq aL'(y^*) + c(x+a-y_n^*) + \alpha q_n[\mu_{n-1}(x+a-y_n^*) - (x+a-y_n^*)c]$$

$$= ac[\alpha q - 1] + c(x+a-y_n^*) + \alpha q_n[\mu_{n-1}(x+a-y_n^*) - (x+a-y_n^*)c]$$

$$= ac[\alpha q - 1] + c(x+a-y_n^*)$$

$$+ \alpha q_n[(x+a-y_n^*)c\alpha(q-q_{n-1})$$

$$+ (x+a-y_n^*)c \sum_{j=2}^{n-1} \alpha^j (q-q_{n-j}) \prod_{k=1}^{j-1} q_{n-k}$$

$$+ (x+a-y_n^*)c\alpha^{n-1} \prod_{j=1}^{n-1} q_{n-j} - (x+a-y_n^*)c]$$

Since $q \geq q_t \geq 0$ and $x - y_n^* \leq 0$, then $x + a - y_n^* \leq a$

$$\leq ac[\alpha q - 1] + ac + \alpha q_n \Big[ac\alpha(q-q_{n-1})$$

$$+ ac \sum_{j=2}^{n-1} \alpha^j (q-q_{n-j}) \prod_{k=1}^{j-1} q_{n-k}$$

$$+ ac\alpha^{n-1} \prod_{k=1}^{n-1} q_{n-k} - ac \Big]$$

$$+ c(x-y_n^*) - \alpha q_n c(x-y_n^*).$$

(Now, $c(x-y^*) - \alpha q_n c(x-y^*) \leq 0$.)

$$\leq \alpha[acq + q_n \Big\{ ac\alpha(q-q_{n-1}) + ac \sum_{j=2}^{n-1} \alpha^j (q-q_{n-j}) \prod_{k=1}^{j-1} q_{n-k}$$

$$+ ac\alpha^{n-1} \prod_{k=1}^{n-1} q_{n-k} - ac \Big\}$$

$$= \alpha[acq + q_n(\mu_{n-1} - ac)] = \mu_n \text{ as required.}$$

CASE 3: $x \leq x + a \leq y_n^* \leq y^*$. On this interval $\bar{f}(y) = \bar{f}_n'(x+a) = \bar{f}_n'(x) = -c$ for $y \in [x, x+a]$. Thus $\bar{f}_n(y)$ is linear on $[x, x+a]$, *and we have*

$$\frac{\bar{f}_n(x+a) - \bar{f}_n(x)}{x} = -c,$$

i.e.,

$$\bar{f}_n(x+a) - \bar{f}_n(x) + ac = 0 \leq \mu_n. \qquad \text{Q.E.D.}$$

PROOF OF LEMMA 3:

Define $y^* \equiv y^*(P_\infty)$, $y_n^* \equiv y^*(P_n)$,

$$\delta_n = \alpha(q-q_n) + \sum_{j=2}^{n} \alpha^j (q-q_{n-j+1}) \prod_{k=1}^{j-1} q_{n-k+1} + \alpha^n \prod_{j=1}^{n} q_{n-j+1},$$

$a = y^* - y_n^*$. Thus since $\bar{f}_n$ and L are convex:

$$\begin{aligned}
\mu_n(a) &\geq \bar{f}_n(y^*) - \bar{f}_n(y_n^*) + ac \\
&= L(y^*) + \alpha q_n \int_0^\infty \bar{f}_{n-1}(y^* - \xi)\phi(\xi)\,d\xi \\
&\qquad - L(y_n^*) - \alpha q_n \int_0^\infty \bar{f}_{n-1}(y_n^* - \xi)\phi(\xi)\,d\xi + ac \\
&\geq L(y^*) - L(y_n^*) - aL'(y_n^*) \\
&\qquad + \alpha q_n \int_0^\infty [\bar{f}_{n-1}(y_n^* - \xi) - \bar{f}_{n-1}(y_n^* - \xi) - a\bar{f}'_{n-1}(y_n^* - \xi)]\phi(\xi)\,d\xi \\
&\geq L(y^*) - L(y_n^*) - aL'(y_n^*) \\
&= \frac{a^2}{2} L''(\hat{y}) \text{ where } \hat{y} \epsilon [y_n^*, y^*].
\end{aligned}$$

Hence

$$\frac{a^2}{2} L''(\hat{y}) \leq \mu_n(a) = ac\delta_n$$

or

$$\frac{a}{2} L''(\hat{y}) \leq c\delta_n \qquad \text{implies}$$

$$0 \leq a = y^* - y_n^* \leq \frac{2c\delta_n}{L''(\hat{y})}. \qquad \text{Q.E.D}$$

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