MANAGEMENT SCIENCE
Vol. 13, No. 11, July, 1967
Printed in U.S.A.

INVENTORY DEPLETION MANAGEMENT WITH STOCHASTIC FIELD LIFE FUNCTIONS*

WILLIAM P. PIERSKALLA

Case Institute of Technology

Most inventory depletion analysis has been concerned with the case of a deterministic field life function. Since it is rarely the case that the exact field life function of a stock of items would be known, it is more reasonable to treat the case where the field life of an item is a random variable, $X(S)$, which depends only on the age, S, of the item upon issuance. In this stochastic case, it is shown that the policy of issuing the oldest item first (FIFO) is optimal provided $E[X(S)] = aS + b$ where $0 \geqq a > -1$, $b > 0$, i.e., where the items have a linearly decreasing mean value function, and where some additional, but behaviorally not restrictive, assumptions are made.

1. Introduction

The inventory depletion problem can be stated as follows: once put in use, an item from an inventory of n items has a nonnegative field life, $L(S)$, which is a function of the age, S, of the item upon issuance to the field. The objective is to determine the order of issue of the n items of differing ages such that the total field life of the stockpile is maximized. Optimal policies of the form LIFO, last in first out, or FIFO, first in first out are sought.

Most of the previous papers concerning this inventory depletion problem require that $L(S)$ be a known function. Thus, these earlier papers were concerned with the deterministic inventory depletion model. An exception to this last statement is [14]. Zehna considers the problem where the field life of an item is a nonnegative random variable, $X(S)$, dependent on the age, S, of the item upon being issued. In this paper we will also consider this case. Furthermore, it should be noted that the total field life, Q, of any issue policy is the sum of n dependent random variables. Let $U = E[Q]$; i.e., U is the expected value of Q. Then a policy which maximizes U will be said to be an optimal policy.

Before proceeding, it will be advantageous to characterize the model explicitly. It is assumed that: (1) At the beginning of the process, a stockpile has n indivisible identical items of varying ages $S_1 < S_2 < \cdots < S_n$ where $S_1 > 0$. The ages S_i are called the initial ages of the items. (2) Each item has a field life $X(S)$ which is a nonnegative random variable dependent on the age S of the item upon being issued. (3) Items are issued successively until either the entire stockpile is depleted or the remaining items in the stockpile have no further useful life, i.e., $X(S) \equiv 0$ for the remaining items. (4) No penalty or installation costs are associated with the issuance of an item from the stockpile. (5) New items are never added to the stockpile after the process starts. (6) An item is

* Received July 1966 and revised December 1966.

issued from the stockpile only when the entire life of the preceding item issued is ended. These assumptions define the model and will be implicitly and/or explicitly contained in every theorem unless an explicit statement to the contrary is made.

If one were to apply the results of inventory depletion theory, it would be quite important to consider the case of stochastic field life functions, since it would be an unusual situation where exact knowledge of the field life of an item would be known prior to issuance. It is more likely that the field life is a nonnegative random variable and should be treated as such (we only consider items which appreciate or depreciate in field life as they remain the stockpile). Fortunately, however, for the practioner, the cases investigated in this paper lead to the same optimal policies which occur for these types of cases in deterministic inventory depletion theory. It is hoped that all or most of the deterministic results will carry over to the case of stochastic field life functions; however, a means for proving this has not, in general, been found.

In the next section it is assumed that although the specific field life function for a given item may not be known, it is known that any given item must obey one of a family of field life functions. Under certain conditions it is shown that FIFO is optimal. In the final part of the section it is assumed that the number of items in the stockpile, n, is neither large enough nor is the deterioration fast enough such that any item deteriorates to zero field life prior to its issuance. In this case it is again shown that FIFO is optimal under certain conditions.

Since Theorem 3 of Lieberman [9] is used frequently, we state this theorem here: "If (i) $L'(S) \geqq -1$ and (ii) $L(S)$ is a nonincreasing or nondecreasing concave function, then FIFO is an optimal policy." In this theorem $L(S)$ is a known function.

2. Decreasing Field Life Functions

We now consider the case that, although the specific field life function for a given item may not be known, it may be known that any given item must obey one of a family of decreasing field life functions. More specifically, we will assume that $X(S)$ takes on a value $L_i(S) = a_i S + b_i$ with probability p_i ($i = 1, 2, \cdots; \sum_{i=1}^{\infty} p_i = 1, p_i \geqq 0$). Furthermore, it will be assumed that $0 < L_1(S) < L_2(S) < \cdots$ for $0 \leqq S < S_0$, and $L_i(S) = 0$, for $S_0 \leqq S < \infty$ with $S_0 = -b_i/a_i$ for all $i = 1, 2, \cdots$. S_0 is called the finite truncation point.

Note that, in general, if $L(S)$ is a decreasing function of S and $L(0) > 0$, then $S_0 \leqq +\infty$ is a truncation point for $L(S)$ if and only if

$$S_0 = \inf \{S \,\varepsilon\, [0, \infty) \mid L(S) \leqq 0\}$$

and then $L(S)$ is redefined to be

$$\begin{aligned} L(S) &= L(S) > 0 \quad \text{for all} \quad S \,\varepsilon\, [0, S_0) \\ &= 0 \qquad\qquad\quad \text{for all} \quad S \geqq S_0 \qquad\qquad \text{(Zehna [14]).} \end{aligned}$$

Diagrammatically for the case of two field life functions $L_1(S)$ and $L_2(S)$, see Figure 1.

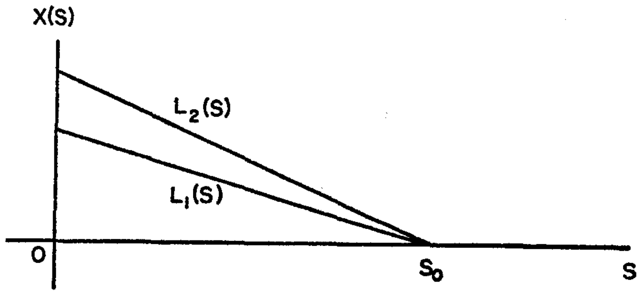

FIGURE 1

The interpretation of this stochastic model can be given as follows. We have n items in inventory each of which deteriorates in accordance with one of a set of continuous nonincreasing field life functions. It is not known which specific function an item will follow; however, the probability of it following a given function is fixed and independent of the age of the item. When an item is young, its range of possible values is large with respect to its actual field life, but, as the items age, those with lower quality initially, do not deteriorate as fast. The process of deterioration continues until all of the field life functions approach the same truncation point. Note that when $p_i = 1$ for some i, this model reduces to the linear case of Lieberman's [9] deterministic model.

Before proceeding to the case where the field life of an item is a random variable, we will state and prove a lemma in which we assume the field life function $L_i(S)$, $i = 1, 2, \cdots$ is known for each item $S_1, \cdots, S_n$.

Lemma 1. Let $L_i(S)$ be a nonnegative decreasing function with the following properties. $L_i(S) = a_iS + b_i$, $0 \leqq S \leqq S_0$ and $L_i(S) = 0$, $S_0 \leqq S < \infty$, $i = 1, 2, \cdots$ where $0 > a_i \geqq -1$. Let there be n items in the stockpile with initial ages $0 \leqq S_1 < S_2 \cdots < S_n < S_0$. Suppose the field life function of item j is $L_{i(j)}(S)$ and that for each j, $i(j)$ is known. Then FIFO is optimal.

Proof. Suppose $n = 2$. Let $i(1) \equiv s$ and $i(2) \equiv t$. We must show

$$(1.1) \quad Q_F - Q_L = L_t(S_2) + L_s(S_1 + L_t(S_2)) - L_s(S_1) - L_t(S_2 + L_s(S_1)) \geqq 0,$$

where Q_F and Q_L are the total field lives from FIFO and LIFO policies respectively.

Note that for any i and j

$$S_1 + L_j(S_2) = S_1 + a_jS_2 + b_j < S_2 + a_jS_2 + b_j < S_0 \quad \text{implies}$$

$$(1.2) \qquad L_i(S_1 + L_j(S_2)) > 0.$$

If $L_t(S_2 + L_s(S_1)) = 0$ then

$$Q_F - Q_L = a_tS_2 + b_t + a_s(S_1 + a_tS_2 + b_t) + b_s - a_sS_1 - b_s = (1 + a_s)(a_tS_2 + b_t) > 0, \quad \text{and (1.1) holds.}$$

If $L_t(S_2 + L_s(S_1)) > 0$, then

$$\begin{aligned} Q_F - Q_L &= a_tS_2 + b_t + a_s(S_1 + a_tS_2 + b_t) + b_s - a_sS_1 - b_s \\ &\quad - a_t(S_2 + a_sS_1 + b_s) - b_t \\ &= a_ta_s(S_2 - S_1) + a_sb_t - a_tb_s = a_ta_s(S_2 - S_1) > 0, \end{aligned}$$

since $a_sb_t = a_tb_s$ and (1.1) again holds. Therefore FIFO is optimal for $n = 2$.

Assume FIFO is optimal for n items, we must show FIFO is optimal for $n + 1$ items. Let $Q_F(n + 1)$ and $Q_A(n + 1)$ be the total field lives from FIFO and an arbitrary policy A respectively when there are $n + 1$ items in the stockpile.

Let S denote the initial age of the last item issued under policy A, and let S have field life function $L_r(S) = a_rS + b_r$. Moreover, let x denote the total field life from the first n items issued. Then $Q_A(n + 1) = x + a_r(S + x) + b_r$ and $[dQ_A(n + 1)/dx] = 1 + a_r > 0$. Hence $Q_A(n + 1)$ is an increasing function of x, and $Q_A(n + 1)$ is maximized by making x as large as possible. But by the inductive assumption x is maximized by FIFO; therefore $Q_A(n + 1)$ is maximized by using FIFO on the first n items issued.

Now, if $S \neq S_1$, then S_1 is issued next to last and since $Q_F(2) - Q_L(2) \geqq 0$, then the total field life of $Q_A(n + 1)$ can be improved by interchanging the order of issue of the last two items. Hence, S_1 is issued last, and the first n items are issued by FIFO by the application, again, of the inductive assumption.

This lemma in itself can be used as a generalization of the linear case of Lieberman's model in deterministic problems. However, in this paper we use it as a stepping stone to the following theorem.

Theorem 1. Let $L_i(S)$ and the n items in the stockpile be defined as in Lemma 1. Let $X(S)$ be a random variable which takes on any one of the values $L_i(S)$ with probability p_i ($\sum_{i=1}^{\infty} p_i = 1$, $p_i \geqq 0$, $i = 1, 2, \cdots$). Then FIFO maximizes the total expected return for all $n \geqq 2$.

Proof. Consider any realization of the random variables, i.e., the first item has realization $L_{i(1)}(S)$, the second $L_{i(2)}(S)$, etc. By Lemma 1, if this realization were known in advance FIFO would be the optimal policy. Thus, if we let all possible realizations K be indexed by $k = 1, 2, \cdots, K$ (K may be infinite) and the probability of the k^{th} realization be denoted by $P^{(k)}$, then

$$U_F - U_A = \sum_{k=1}^{K} [Q_F^{(k)} - Q_A^{(k)}]P^{(k)} \geqq 0.$$

It is possible to establish a form of Theorem 3.1 for $L_i(S)$ concave nonincreasing when there are but two items in the stockpile.

Theorem 2. Let $X(S)$ be a random variable which takes on any one value of the values $L_i(S)$ with probability p_i ($i = 1, 2, \cdots, \sum_{i=1}^{\infty} p_i = 1, p_i \geqq 0$). Let $n = 2$ items in the stockpile.

Let $L_i(S)$ have the following properties:

(i) $L_i(S)$ is concave for all $S \varepsilon [0, S_0]$

(ii) $L_i(S) = 0$ for all $S \varepsilon (S_0, \infty)$ and $L_i(S) > 0$ for all $S \varepsilon (0, S_0)$

(iii) $0 \geqq L'_i(S) \geqq -1$ for all $i = 1, 2, \cdots$ and all $S \varepsilon [0, S_0]$.

Then FIFO maximizes the total expected return.

The proof of Theorem 2 follows directly by writing out $U_F - U_L = E[Q_F - Q_L]$ and showing that this difference is nonnegative.

A further and more important generalization of Theorem 1 can be given where we consider the dynamic inventory depletion model of adding new items to the stockpile at times in the future.

Corollary 1. Let the hypotheses of Theorem 1 hold. Then FIFO is optimal for the dynamic model where N new items are added one each to the inventory at the N different future times $F_1, \cdots, F_N$. The new items have age $S = 0$ when they arrive.

Before indicating the proof of this corollary, it will be useful to prove a theorem for the case of a single known field life function $L(S)$, i.e., the deterministic case. First some definitions are needed.

A stockout occurs if the customer (demand source for the items in the stockpile) demands an item from the stockpile, and the stockpile is empty; therefore, the customer must wait until a new item arrives, at which time the new item is immediately assigned to the customer. A customer demands an item from the stockpile only when the life of the previously issued item is totally used up.

Theorem 3 following proves that for the case of a single customer and known field life function $L(S)$, FIFO is the optimal policy provided $L(S)$ is concave nonincreasing and $L'(S) \geqq -1$. In a previous paper [10] Corollary 1.2 we had assumed also that no stockouts may occur in order to obtain this conclusion. Thus, Theorem 3 removes the no stockout assumption for the case of a single customer.

Define the dynamic model as the dynamic inventory problem of finding the optimal issuing policy for the n items of initial ages $0 \leqq S_1 < S_2 < \cdots < S_n$ which are originally in the stockpile and the N items which are added one at a time to the stockpile at age 0 at times $0 \leqq t_1 < t_2 \cdots < t_N$ which are represented by numbers $F_i = -t_i$ and can be thought of as future times measured on the negative half of the S axis (see Figure 2). Thus the item with F_1 arrives first, F_2 second, etc. Denote by $L^-(S)$ the left hand derivative of $L(\cdot)$ evaluated at S.

Theorem 3. Let $L(S)$ be a continuous concave nonincreasing function with $L^-(S) \geqq -1$ for all $S \leqq S_0$ and $L(0) > 0$. Then FIFO is optimal in the dynamic model.

Proof. By Lieberman [9], Theorem 3 FIFO maximizes the total field life of

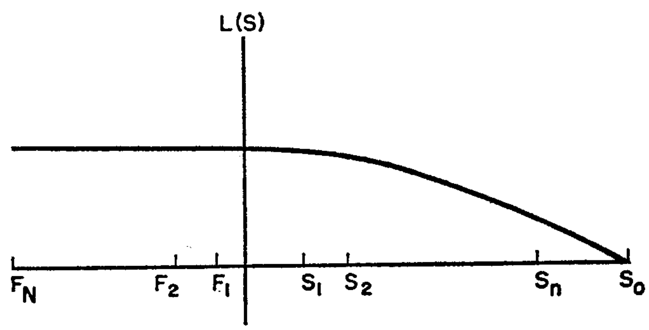

FIGURE 2

the first n items in the stockpile. If item F_1 arrives before the first n items are consumed by any policy A then by [10], Theorem 1.1, FIFO maximizes the total field life of the first $N + 1$ items. [Note that Theorem 1.1 of [10] states that under the conditions of Theorem 3 above and if, in addition, FIFO is optimal in the static model where new items are not considered and if no stockouts occur in the dynamic model, then FIFO is optimal for the dynamic model.]

If there is some policy A such that all n items are used up before F_1 arrives, then we must show FIFO is still optimal for the $n + 1$ items. Let x_F and x_A denote the total field life of the original n items in the stockpile when issued by FIFO and by policy A, respectively. Let Q_F and Q_A denote the total field life of the $n + 1$ items issued by FIFO and by policy A respectively.

Case 1

$$x_A < -F_1 \quad \text{and} \quad x_F < -F_1$$

Then since $x_F \geqq x_A$ from Lieberman [9] Theorem 3

$$Q_F = x_F + L(0) \geqq x_A + L(0) = Q_A .$$

Case 2

$$x_A < -F_1 \quad \text{and} \quad x_F \geqq -F_1$$

Let $V_1 = x_F + F_1 \geqq 0$ and $U_1 = -F_1 - x_A > 0$ then $V_1 + U_1 = x_F - x_A > 0$. Since $L(\cdot)$ is concave nonincreasing with $L^-(S) \geqq -1$ for all $S \leqq S$ and by Lemma 1.1 of [11] $L(x_F + F_1) > 0$ and since $S_0 > 0, L(-U_1) = L(x_A + F_1) > 0$ then

$$[L(x_F + F_1) - L(x_A + F_1)]/(x_F - x_A) \geqq -1 \quad \text{implies}$$

$$L(x_F + F_1) + x_F \geqq L(x_A + F_1) + x_A ,$$

but $L(\cdot)$ is nonincreasing and $0 > x_A + F_1$,

$$Q_F = L(x_F + F_1) + x_F \geqq L(0) + x_A = Q_A .$$

Hence if one item is added at time F_1 then FIFO is optimal.

We now assume FIFO is optimal for adding $N - 1$ items. Now let x_F and x_A denote the total field life of issuing the first $n + N - 1$ items by FIFO and by A, respectively. Let Q_F and Q_A denote the total field life of issuing the $n + N$ items by FIFO and by A, respectively.

Case 1

$$x_A < -F_N \quad \text{and} \quad x_F < -F_N .$$

By inductive assumption $x_F \geqq x_A$ and $Q_F = x_F + L(0) \geqq x_A + L(0) = Q_A$.

Case 2

$$x_A < -F_N \quad \text{and} \quad x_F \geqq -F_N .$$

Use the same argument as Case 2 above.

Case 3

$$x_A \geqq -F_N .$$

Now F_N might arrive while there are still items in the stockpile. In this case we must consider all possible policies A and not just those which issue F_N last. If no stockouts occur, then by Pierskalla [11] Theorem 1.1, FIFO is optimal. If a stockout did occur at some time, then we know that item S_n (the oldest initial item) could not be issued last. Let F denote the initial age of the last item issued $F \varepsilon \{F_1, \cdots, F_N\}$ but $F \neq S_n$ (in fact $F \neq S_i$, $i = 1, \cdots, n$). The remainder of the proof for this case directly parallels the proof given for Lemma 1 above, and FIFO is optimal for the dynamic model.

Now the proof of Corollary 1 follows directly from Theorem 3. It is only necessary to show that FIFO is optimal for any realization of the dynamic stochastic model, for then we just weight the realizations by their probability of occurrence and sum them. But FIFO is optimal for each realization by using the same proof as for Theorem 3 where we consider the special subcases as given in Lemma 1.

In the previous theorems and corollaries of this section the finite truncation point S_0 played a major role in the proofs. It is precisely this point S_0 which makes extensions of the results to other types of decreasing mean value functions very difficult. This problem can be avoided if it is assumed that regardless of whatever issuing policy is being followed, no item ever reaches zero field life prior to its issuance. Actually we will assume:

Assumption 1: If x_i is the total field life of the first i items issued under any policy then $S_{i+1} + x_i < S_0$ for all $i = 1, \cdots, n - 1$ where S_{i+1} is the initial age of the $i + 1^{st}$ item issued.

This assumption implies (i) the amount of inventory on hand is not large relative to its age in the stockpile and to its rate of deterioration and (ii) no item in inventory ever has an excessively long field life on issuance, i.e., there is a finite upper bound to the actual field life that an item can have. Thus, for example, we would not permit the gamma distribution to represent the distribution of $X(S)$; however, a truncated gamma distribution may be acceptable provided its upper bound does not conflict with the requirements of Assumption 1.

Now Assumption 1 is not a particularly restrictive assumption. In a practical situation we would not want to procure so much inventory that some of it would deteriorate to zero in the stockpile. Secondly, the types of items we are considering such as tires, batteries, industrial chemicals, etc., do not deteriorate at a rapid rate. For example, if we represent the distribution of $X(S)$ by the uniform distribution and if we assume a linear mean value function we obtain a deterioration graph (see Fig. 3). Then when Assumption 1 holds $E[X(S)] = aS + b$ where $X(S)$ has probability density $f(t) = 1/[2(b - b_0)]$ for $aS + b_0 \leqq t \leqq aS + 2b - b_0$ and $b > b_0 > 0$, $a > -1$, and $f(t) = 0$ otherwise.

We now demonstrate a theorem which gives the optimal policy for this case among others.

Theorem 4. Let $X(S)$ for $S \geqq 0$ be a nonnegative random variable. Let

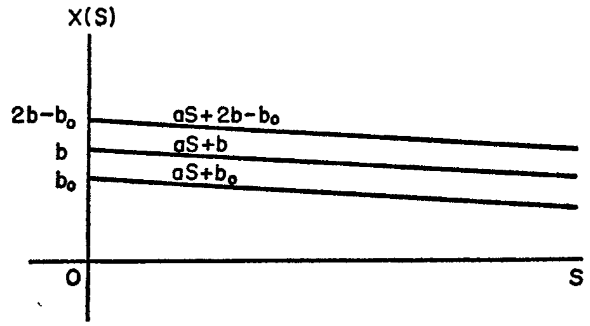

FIGURE 3

$E[X(S)] = aS + b$ where $b > 0$, $a > -1$. If Assumption 1 holds, then FIFO is optimal for $n \geqq 2$.

Proof. Let

$T_i \equiv$ the age of the i^{th} item issued when it is issued to the field,

$Y_i \equiv$ the field life of the i^{th} item issued after issuance to the field,

$Z_i \equiv$ total field life of the first i items issued.

Then we have

$$\begin{aligned} Y_1 &= Z_1 = X(T_1) \\ Y_i &= X(T_i + Z_{i-1}) \quad \text{for} \quad i > 1 \\ Z_i &= Y_i + Z_{i-1} \qquad \text{for} \quad i > 1. \end{aligned}$$

Let $E[X(T)] = g(T) = aT + b$.

Since

$$\begin{aligned} Z_i &= X(T_i + Z_{i-1}) + Z_{i-1} \quad \text{then} \\ E[Z_i] &= E[Z_{i-1}] + E[X(T_i + Z_{i-1})] \\ &= E[Z_{i-1}] + E[E[X(T_i + Z_{i-1}) \mid Z_{i-1}]] \\ &= E[Z_{i-1}] + \int E[X(T_i + Z_{i-1}) \mid Z_{i-1} = u]\, dF_{Z_{i-1}}(u) \\ &= E[Z_{i-1}] + \int [g(T_i) + au]\, dF_{Z_{i-1}}(u) \\ &= (1 + a)E[Z_{i-1}] + g(T_i) \quad \text{by Assumption 1} \\ &= \textstyle\sum_{j=0}^{i-1} (1 + a)^j g(T_{i-j}). \end{aligned}$$

Since $(1 + a)^j$ is increasing in j, $E[Z_i]$ is maximized by ordering the T_i's so that $g(T_1) \geqq g(T_2) \geqq \cdots \geqq g(T_i)$. But this ordering is just the FIFO ordering.

Thus, for the example of the uniform distribution given above, FIFO is optimal. Another interesting example would be the case $L_i(S) = a_i S + b_i$ where $a_i > -1$, $b_i > 0$ for all i and p_i is the probability of occurrence of $L_i(S)$ but without the restriction that S_0 is the same for all i. Then, if Assumption

1 holds, we have $E[X(S)] = \sum_i L_i(S)p_i = \sum_i (a_iS + b_i)p_i = \sum_i a_ip_iS + \sum_i b_ip_i = aS + b$ and $b > 0$, $a > -1$; we merely apply Theorem 4 and see that FIFO is optimal.

Again a further generalization of Theorem 4 can be stated.

Corollary 2. Let the hypotheses of Theorem 4 hold. If no stockouts occur, then FIFO is optimal for the dynamic model where N new items are added to inventory one each at the N different future times $t_1, \cdots, t_N$. The new items have age $S = 0$ when they arrive.

Proof. By Theorem 4 FIFO is optimal for the static model. The remainder of the proof is the same as for Theorem 1.1 of [10].

For the case of a linearly decreasing mean value function with slope $\geqq -1$ and Assumption 1, FIFO is the optimal policy. It is not known if FIFO is optimal when $0 > a > -1$ and Assumption 1 does not hold; however, in this latter case and for most real problems, if FIFO is not optimal, then it must be "close" to optimal. And since FIFO is easily implemented, in general, it would be reasonable to use FIFO whether or not Assumption 1 applies at all times.

It should be mentioned that the preceeding theorems, lemmas, and corollaries generalize to the case of batches of items of the same age in the stockpile. That is, at the start of the process a stockpile has M sets of indivisible identical items where the items in the i^{th} set all have the same initial age, S_i, for $i = 1, \cdots, M$. The initial age of the items in any set, say the i^{th} set, is different than the initial age of the items in any other set. Then if we assume $0 \leqq S_1 < S_2 < \cdots < S_M < S_0$ and that the i^{th} set contains n_i items for $i = 1, \cdots, M$, then $\sum_{i=1}^{M} n_i = n$. The same modification holds for adding batches of items of the same age to inventory at the future times $t_1, t_2, \cdots, t_n$. This extension of the preceding results follows directly from Theorem 2.1 of [10] [viz. Let $L(S)$ be a continuous function. If FIFO (LIFO) is optimal without batches of items, then FIFO (LIFO) is optimal when batches are allowed].

The author would like to express his gratitude to the referees for the helpful comments and suggestions they made.

References

1. Bomberger, E. E., "Optimal Inventory Depletion Policies," *Management Science*, Vol. 7, No. 3, April 1961, pp. 294–303.
2. Derman, C. and M. Klein, "Inventory Depletion Management," *Management Science*, Vol. 4, No. 4, July 1958, pp. 450–456.
3. —— and ——, "Discussion: A Note on the Optimal Depletion of Inventory," *Management Science*, Vol. 5, No. 2, January 1959, pp. 210–213.
4. Eilon, S., "Stock Depletion for Multi-Channel Outlets in Maintenance Work," *Proceedings of the 3rd International Conference on Operational Research*, Oslo, 1963, Dunod (Paris) and English Universities Press, Ltd., (London), 1964, pp. 238–246.
5. ——, "Obsolescence of Commodities Which Are Subject to Deterioration in Store," *Management Science*, Vol. 9, No. 4, pp. 623–642.
6. ——, "Problems of Stock Depletion," *The Engineer*, (United Kingdom), Vol. 214, No. 5570, October 1962, pp. 706–708.
7. ——, "FIFO and LIFO Policies in Inventory Management," *Management Science*, Vol. 7, No. 3, April 1961, pp. 304–316.

8. Greenwood, J. A., "Issue Priority," *Naval Research Logistics Quarterly*, Vol. 2, No. 4, pp. 251–268.
9. Lieberman, G. J., "LIFO vs. FIFO in Inventory Depletion Management," *Management Science*, Vol. 5, No. 1, October 1958, pp. 102–105.
10. Pierskalla, W. P., "Optimal Issuing Policies in Inventory Management—II," to be published in the *Proceedings of the 4th International Conference on Operations Research*, Cambridge, Massachusetts, 1966 (Also available as Tech. Memo No. 48, Operations Research Group, Case Institute of Technology, Cleveland, Ohio, March 1966).
11. ——, "Optimal Issuing Policies in Inventory Management—I," *Management Science*, Vol. 13, No. 5, January 1967, pp. 395–412, (Also available as Tech. Memo No. 44, Operations Research Group, Case Institute of Technology, Cleveland, Ohio, January 1966).
12. ——, "Optimal Issuing Policies in Inventory Management," *Technical Report No. 7*, Program in Operations Research, Stanford University, Stanford, California, August 28, 1965.
13. Veinott, A. F., Jr., *Optimal Ordering, Issuing, and Disposal of Inventory With Known Demand*, Unpublished PhD Dissertation, Columbia University, 1960.
14. Zehna, P. W., "Inventory Depletion Policies," Chapter 15 in Arrow, K. J., S. Karlin, and H. Scarf (eds.), *Studies in Applied Probability and Management Science*, Stanford, California, Stanford University Press, 1962.