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A BY-PRODUCT PRODUCTION SYSTEM WITH AN ALTERNATIVE*

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This paper considers the optimal control of a production system which is composed of two distinct production processes, types A and B, that produce two different products, 1 and 2, having distinct random demands. Production type A produces both products in amounts determined by a fixed set of production coefficients. Type B can only be used to make product 2. Costs consist of linear production costs and convex holding and shortage costs. Each period, the optimal production level of each type must be determined. The criterion is the minimum expected discounted total cost. Results show that the decision space of each period is partitioned into four regions by three monotone functions and a point. Extensions include capacitated production, nonstationary costs, lost sales, fixed lead times and the general m process–n product system.
(INVENTORY/PRODUCTION; INVENTORY/PRODUCTION–STOCHASTIC MODELS)

I. Introduction

In many production-inventory facilities it is necessary to schedule the manufacture of several products on different production processes. Often, the same quantity of finished products can be made using different combinations of production processes. In such cases two simultaneous decisions must be made. First, the inventory level of each product must be determined to best meet (random) demands. Second, the best (optimal) production level (e.g., number of runs) must be determined to achieve these inventories. Often these decisions cannot be made independently because production capacities or design characteristics of the production lines may prohibit arbitrary combinations. The system we consider is described in Figure 1. There are two inventory items, called products 1 and 2, and two different production processes called types A and B. Type A is capable of making both products, simultaneously, according to the production coefficients η_1 and η_2. When type A is operated at the unit level (e.g., a single run), η_1 and η_2 units of products 1 and 2 are obtained. We choose to call type A a by-product production process due to its multi-product capability. Type B is a single item production process and is capable of making only product 2. In this context, type B is the alternative method of obtaining product 2.

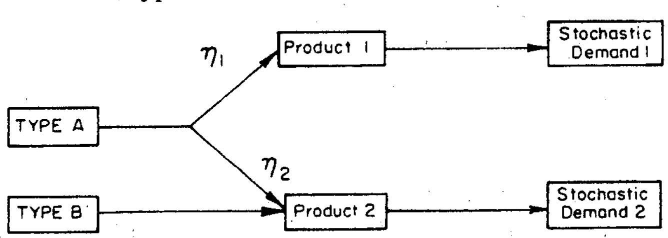

FIGURE 1. Block Diagram of the Production System.

A variety of real inventory-production systems found in industry can be described by diagrams similar to Figure 1, although many will be more complicated. By-product processes themselves occur quite naturally. Transistors are often classified into lots according to their quality. In this case, η_j would be the average percentage of transistors that are classified into lot j each run. The chemical industry and in particular the petroleum industry have production processes that are primarily of the

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by-product nature. For each barrel of crude oil processed through the cracking plant, η_j of hydrocarbon j are obtained. Various other blending problems such as the cutting stock problem (with random demands) can be viewed in this manner. In each of the above examples, any other means of producing any one single product would fit our type B.

This system is a specialization of what could be called an n-product m-production process inventory-production system, where the m processes have to be coordinated in an optimal fashion to prepare inventories of the n products. The m production processes would usually consist of a variety of different types of production processes.

In reviewing the relevant literature, one sees that a variety of papers have been written concerning the control of inventories for several products. The excellent survey paper written by Veinott [18] reviews the literature through 1965. Among the topics discussed by him are: substitute products, multilocation systems, ordering and repair and capital accumulation and production. Veinott [17] developed a very general multi-product model that allowed for several demand categories. Evans [7] considered the control of a by-product production system using two different production cost structures: under lost sales he used a linear joint production cost and under complete backlogging he allowed for a fixed set up cost and generalized K-convexity. Evans [6] considered a multi-product system with limited resources. Shah [15] explicitly treated different types of substitutability between products. Johnson [11] presented an infinite horizon model with fixed set up costs for an inventory system consisting of several different products. Silver [16] and Goyal [9], [10] considered the problem of joint replenishment when demands are known. Most of these papers considered multi-product inventories when the stock items are related in some fashion (resource limitations, by-products, etc.). A variety of deterministic multi-product models have been studied using linear programming. A number of these are discussed in the book by Johnson and Montgomery [12].

Apparently, the model described in Figure 1 has not yet been considered in the context of stochastic inventory theory.

The model is formulated in §2 and §3 and the analysis of the model is presented along with the characterization of the optimal policy. §4 provides some important extensions of the model.

2. Model Formulation and Assumptions

The model is of the periodic review type, where the planning horizon is N periods long. At the beginning of period n, $n = 1, 2, \ldots, N$, the initial inventories (before production) $x_n = (x_{n,1}, x_{n,2})$ are reviewed, where $x_{n,j}$ is the initial inventory of product j, $j = 1, 2$. Then, the starting inventories (stock levels after production but before demand) $y_n = (y_{n,1}, y_{n,2})$ and the production levels $p_n = (p_{n,A}, p_{n,B})$ are jointly determined. $y_{n,j}$ is the starting inventory of product j, $j = 1, 2$, in period n and $p_{n,A}$, $p_{n,B}$ the production levels of types A and B, respectively. Then, a random demand $D_n = (D_{n,1}, D_{n,2})$ is realized. We use a reverse recursion numbering scheme so that index N refers to the first period and 1 refers to the last period. When focusing attention to quantities within a period, we frequently drop the period index on the above quantities.

The following is a list of specific assumptions governing the development of our model. §4 provides a discussion for generalizing many of the assumptions stated below.

1. *Production Processes*

It is assumed that each production process is controlled by specifying the production level. For type A this level is p_A; for type B it is p_B. Thus, the amount of each product produced is $(\eta_1 p_A, p_B + \eta_2 p_A)$. We assume that η_1 and η_2 are constant; they are not decision variables. For the present, we assume that there are no lead times and that there are no upper bounds on production.

2. *Demands*

We assume that D_n, $n = 1, 2, \ldots, N$, form a sequence of nonnegative independent and identically distributed random vectors with a continuous joint density function $f(\cdot)$ and finite expected value. Although we allow dependencies among demands within a period, we do require that $D_{n,j}$ be satisfied only from inventories of product $j, j = 1, 2$. It is further assumed that unsatisfied demands are backlogged and are not lost.

3. *Production Costs*

(a) c_A (c_B) is the unit production cost on A (B), and it is assumed that $\eta_1^{-1}(c_A - \eta_2 c_B) > 0$.

(b) there are no fixed costs of production.

4. *Inventory Costs*

Let $g_i(\cdot)$ be the holding and shortage cost for product i, $i = 1, 2$, over any single period. We define the expected total holding and shortage cost over any period as

$$L(y) = \int_0^\infty \int_0^\infty [\, g_1(y_1 - t) + g_2(y_2 - u)] f(t, u)\, dt\, du.$$

We assume that

(1) $L(\cdot)$ is strictly convex over $\mathbf{R}_2$;

(2) $L(\cdot)$ has continuous second partial derivatives;

(3) $(c_A - \eta_2 c_B) y_1/\eta_1 + c_B y_2 + L(y) \to {}^{+}\infty$ whenever $\|y\| \to {}^{+}\infty$ (where $\|\cdot\|$ is the Euclidean norm). (3) assures the existence of an optimal policy.

5. *Discount Parameter*

Let $\alpha \in [0, 1]$. α denotes the parameter which relates costs in future periods to the present.

We use the following notation for derivatives and partial derivatives of functions. Let $h(\cdot)$ be a twice continuously differentiable scalar function, then $h'(\cdot)$ and $h''(\cdot)$ represent the first and second derivatives, respectively. Let $g(\cdot)$ be defined on $\mathbf{R}_n$ with continuous second partial derivatives. Then

$$g^{(i)}(x) = \frac{\partial}{\partial x_i} g(x), \qquad i = 1, 2, \ldots, n,$$

$$g^{(i,j)}(x) = \frac{\partial^2}{\partial x_j \partial x_i} g(x), \qquad i, j = 1, 2, \ldots, n.$$

The problem is to determine a policy that leads to the minimum expected discounted cost over the N period horizon. This leads to an optimization problem that is most easily conceptualized as a dynamic programming problem. Before formulating the dynamic program, we first describe the relationship between a feasible set of production levels $p = (p_A, p_B)$ and starting inventory levels $y = (y_1, y_2)$ given any initial inventory x. Given p and x, the starting inventories y must satisfy

$$y_1 = \eta_1 p_A + x_1, \tag{2.1a}$$

$$y_2 = \eta_2 p_A + p_B + x_2. \tag{2.1b}$$

Given y and x, the production levels p are determined by

$$p_A = \frac{1}{\eta_1}(y_1 - x_1), \tag{2.2a}$$

$$p_B = y_2 - \left(x_2 + \frac{\eta_2}{\eta_1}(y_1 - x_1)\right). \tag{2.2b}$$

Although the dynamic program is most easily defined in terms of p and x, we favor using y and x since they characterize the optimal policy.

The dynamic programming recursion is given by

$$C_n(x) = \inf\left\{ G_n(y) - \frac{1}{\eta_1}(c_A - \eta_2 c_B)x_1 - c_B x_2 \right\} \tag{2.3a}$$

subject to

$$y_1 \geqslant x_1, \tag{2.3b}$$

$$y_2 \geqslant x_2 + \frac{\eta_2}{\eta_1}(y_1 - x_1) \tag{2.3c}$$

for

$$G_n(y) = \frac{1}{\eta_1}(c_A - \eta_2 c_B)y_1 + c_B y_2 + L(y)$$

$$+\alpha \int_0^\infty \int_0^\infty C_{n-1}(y_1 - t, y_2 - u) f(t, u)\, dt\, du \tag{2.3d}$$

where $n = 1, 2, \ldots, N$, $y \in \mathbf{R}_2$ and $C_0(x) \equiv 0$ for all x.

3. Analysis of the Optimal Policy

The first result is of a purely technical nature, but provides the properties of $C_n(\cdot)$ and $G_n(\cdot)$ crucial to the characterization theorem.

THEOREM 3.1. *Under the assumptions made in* §2, *the following hold.*

1. $G_n(\cdot)$ *is continuous and strictly convex.*

2. *All sets of the form* $Q_n(\beta) = \{y \in \mathbf{R}_2;\ G_n(y) \leqslant \beta\}$ *are compact and convex for each bounded* $\beta \in \mathbf{R}$.

3. *The point* $y_n(x)$ *that solves* (2.3) *exists and is unique. Hence,* inf *can be replaced by* min *for each bounded* $x \in \mathbf{R}_2$ *and* $n = 1, 2, \ldots, N$.

4. $C_n(\cdot)$ *is convex and nonnegative on* $\mathbf{R}_2$.

The proof of Theorem 3.1 is provided in the Appendix.

The optimal policy, $y_n(x)$ is completely characterized by a point $S_n = (S_{n,1}, S_{n,2})$ and three scalar functions $r_n(\cdot)$, $w_n(\cdot)$, and $q_n(\cdot)$ as graphically described in Figure 2. Each arrow represents the path from an initial inventory (the tail) to its beginning inventory (the head) by some production combination. In Region I both A and B are used and notice that the starting inventory level will always be S_n. In region III only type B is used so the starting stock always lies on the graph of $q_n(\cdot)$. In region II only A is used. The path from x to $y_n(x)$ in this region is a line parallel to $w_n(\cdot)$. It is also important to point out that $r_n(\cdot)$, $w_n(\cdot)$ and $q_n(\cdot)$ are functions of x_1; this is a different interpretation than is found in [17].

We proceed to the characterization by making the necessary definitions and assumptions, region by region. Following this discussion is Theorem 3.2. Finally, we mention that existence and required properties of the partial derivatives of $G_n(\cdot)$ are summarized in Theorem A.1 in the Appendix.

For ease of reference, we now state the Kuhn-Tucker conditions for (2.3), which are necessary and sufficient for the policy $y_n(x)$ to be optimal. Theorem 3.1 assures the existence of a unique solution to these conditions.

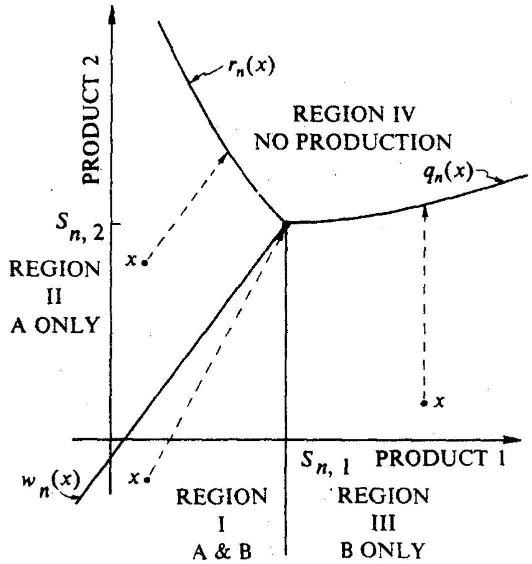

FIGURE 2. Characterization of the Optimal Policy in Period n.

$$G_n^{(1)}(y_n(x)) + \frac{\eta_2}{\eta_1} G_n^{(2)}(y_n(x)) \geqslant 0, \tag{K1}$$

$$G_n^{(2)}(y_n(x)) \geqslant 0, \tag{K2}$$

$$(y_{n,1}(x) - x_1) \cdot \left[G_n^{(1)}(y_n(x)) + \frac{\eta_2}{\eta_1} G_n^{(2)}(y_n(x)) \right] = 0, \tag{K3}$$

$$\left[y_{n,2}(x) - \left(x_2 + \frac{\eta_2}{\eta_1} (y_{n,1}(x) - x_1) \right) \right] \cdot G_n^{(2)}(y_n(x)) = 0, \tag{K4}$$

$$y_{n,2}(x) \geqslant x_2 + \frac{\eta_2}{\eta_1} (y_{n,1}(x) - x_1), \tag{K5}$$

$$y_{n,1}(x) \geqslant x_1. \tag{K6}$$

In the discussion which follows, we make use of the concept of attainability. To define this, let $y^\circ = (y_1^\circ, y_2^\circ)$ be an arbitrary inventory level after production and $x = (x_1, x_2)$ be the inventory prior to production. We say y° is attainable if (y°, x) satisfies (K5) and (K6).

Region I (Both A and B). Since both processes are used, the unique solution to (K1) through (K6) must satisfy (K5) and (K6) strictly. Define $S_n = (S_{n,1}, S_{n,2})$ as the global minimum of $G_n(\cdot)$. If S_n is attainable from x, it follows that $y_n(x) = S_n$. This is true iff x is in the set

$$R_I = \{ x \in \mathbf{R}_2 : x_1 < S_{n,1}, x_2 < w_n(x_1) \}$$

where

$$w_n(x_1) = S_{n,2} - \frac{\eta_2(S_{n,1} - x_1)}{\eta_1} \quad \text{for all } x_1.$$

The following lemma is required for the characterization of Region II.

LEMMA 3.1. *Let $G(\cdot)$ be a strictly convex function defined on $\mathbf{R}_2$ with continuous second partial derivatives. Assume that* (i) *$G(y) \to {}^{+}\infty$ whenever $\|y\| \to {}^{+}\infty$ and* (ii) *there is an $\eta > 0$ such that $G^{(1,j)}(y) + \eta G^{(2,j)}(y) > 0$ for all $y \in \mathbf{R}_2$, $j = 1, 2$. Then*

(a) *The global minimum $S = (S_1, S_2)$ of $G(\cdot)$ is unique.*

(b) *$r(x)$ that solves*

$$G^{(1)}(x, r(x)) + \eta G^{(2)}(x, r(x)) = 0$$

is continuously differentiable and nonincreasing, for all x. Furthermore, $r(S_1) = S_2$.

PROOF. (a) is a consequence of compact level sets and strict convexity. That $r(\cdot)$ exists and is differentiable is a consequence of the Implicit Function Theorem [13] which is applicable because of (ii). The nonincreasing property is evident upon differentiating the defining expression for $r(x)$ in (b). Finally, $r(S_1) = S_2$ is a consequence of the definition of S and the continuity of $r(\cdot)$.

Region II (Only A). When only A is used, the unique solution to (K1) through (K6) satisfies (K5) strictly and (K6) at equality. Thus the optimal policy is chosen to minimize $G_n(\cdot)$ along the line passing through x in the $\eta = (\eta_1, \eta_2)$ direction. Let $y_n^\circ \equiv y_n^\circ(x)$ solve

$$\min_{y \in \mathbf{R}_1} G_n\left(y, x_2 + \frac{\eta_2}{\eta_1}(y - x_1)\right).$$

If $y_n^\circ > x_1$, then $(y_n^\circ, x_2 + \eta_2(y_n^\circ - x_1)/\eta_1)$ is attainable from x and thus must solve (K1) through (K6). This is possible iff x is in the set

$$R_{\mathrm{II}} = \{x \in \mathbf{R}_2;\ x_1 < S_{n,1},\ w_n(x_1) \leqslant x_2 < r_n(x_1)\}$$

where $r_n(x_1)$ is defined as in Lemma 3.1 (b) with $G(\cdot) \equiv G_n(\cdot)$ and $\eta = \eta_2\eta_1^{-1}$. The lemma is applicable by Theorem A.1. Also, $y_n = (y_{n,1}, r_n(y_{n,1}))$.

LEMMA 3.2. *Let $G(\cdot)$ satisfy the hypotheses of Lemma* 3.1 *along with* (iii) $G^{(i,j)}(y) < 0$, *$y \in \mathbf{R}_2$, $i, j = 1, 2$, $i \neq j$. Then*

(a) *The global minimum $S = (S_1, S_2)$ is unique.*

(b) *$q(x)$ that solves*

$$G^{(2)}(x, q(x)) = 0$$

exists, is continuously differentiable and nondecreasing, for all x. In addition $q(S_1) = S_2$.

The proof is omitted due to its similarity to that of Lemma 3.1.

Region III (Only B). When only B is used, the solution to (K1) to (K6) must satisfy (K5) at equality and (K6) strictly. Let $q_n(x_1)$ be the global minimum of $G_n(x_1, \cdot)$. Then,

$$G_n^{(2)}(x_1, q_n(x_1)) = 0 \quad \text{for all } x_1.$$

If $q_n(x_1) > x_2$, the point $(x_1, q_n(x_1))$ is attainable and is the solution to (K1) through (K6). Therefore, Region III is operable iff x lies in the set

$$R_{\mathrm{III}} = \{x \in \mathbf{R}_2;\ x_1 \geqslant S_{n,1},\ x_2 < q_n(x_1)\}.$$

where $q_n(x_1)$ is defined as in Lemma 3.2(b) with $G(\cdot) \equiv G_n(\cdot)$ and $\eta = \eta_2\eta_1^{-1}$. Lemma 3.2 is applicable as a consequence of Theorem A.1.

Region IV (No production). Using the unique characterization above, it is easy to see that no production takes place iff x is in the set

$$R_{\mathrm{IV}} = \{x \in \mathbf{R}_2;\ x_1 < S_{n,1},\ x_2 \geqslant r_n(x_1)\} \cup \{x \in \mathbf{R}_2;\ x_1 \geqslant S_{n,1},\ x_2 \geqslant q_n(x_1)\}.$$

THEOREM 3.2. *Under the assumptions of Theorem* 3.1, *the characterization in period n is given by the four sets R_{I}, R_{II}, R_{III}, R_{IV} defined above and the corresponding policies.*

PROOF. The proof is by induction on n. The proof of the initiation step ($n = 1$) is proven in exactly the same manner as the general induction step. The method is to show, for each region, that the stated solution is attainable and satisfies the Kuhn-Tucker conditions. We omit the details. Q.E.D.

4. Extensions

Many of the assumptions made in §2 can be relaxed. We now discuss a variety of these extensions, and how they affect the model studied in §3.

4.1. *Nonstationary Demands and Costs*

It was assumed in §2 that all costs and demands remain stationary over time. This assumption was made simply for ease of exposition. Theorems 3.1 and 3.2 remain valid as long as the strict convexity of $G_n(\cdot)$ is maintained and $G_n(y) \to {}^{+}\infty$ whenever $\|y\| \to {}^{+}\infty$. This will be true, for example, if $G_n(\cdot) \geqslant G_{n-1}(\cdot) \geqslant \cdots \geqslant G_1(\cdot)$.

4.2. *Production Capacities*

Probably the most restrictive assumption made in §2 was that production on types A and B is unlimited, since this may be unrealistic in some cases. However, relaxing this assumption does not significantly change the form of the optimal policy as is demonstrated below.

Let the finite production limits on types A and B be M_{A} and M_{B}, respectively. In §2 it was assumed $M_{A} = M_{B} = \infty$. Then, the optimization problem is given by (2.3) with the following additional constraints.

$$y_1 \leqslant \eta_1 M_{A} + x_1, \tag{4.1a}$$

$$y_2 \leqslant M_{B} + x_2 + \frac{\eta_2(y_1 - x_1)}{\eta_1}. \tag{4.1b}$$

Following the procedure developed in §3 the optimal policy is characterized as in Figure 3. Numerals I–IV correspond to regions I–IV as defined by Theorem (3.2). The

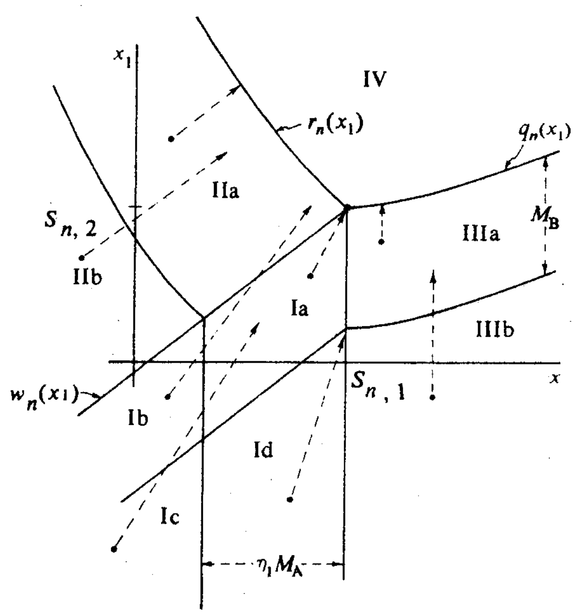

FIGURE 3. Characterization of the Optimal Policy when There is Finite Production Capacity.

subscripts refer to sub-regions created by the presence of production limitations. Notice that the characterization is still completely determined by $r_n(x_1)$, $q_n(x_1)$, $w_n(x_1)$ and S_n as in Theorem 3.2, but with the addition of displacements of these by factors involving the production limits $M = (M_A, M_B)$.

4.3. *Lost Sales*

In §2 it was assumed that all unsatisfied demands are backlogged and will therefore be satisfied eventually. In many situations this may not be a reasonable assumption. The model can be extended to the lost sales case without changing the characterization set forth in Theorem 3.2. However, since the transfer function is no longer linear, the convexity of $G_n(\cdot)$ must be established in a somewhat different fashion. The method of proof is virtually the same as the one used by Evans [6] and can be found in Deuermeyer [4].

4.4. *Fixed Lead Times*

In §2 it was assumed that all production was instantaneous. We now extend the model of §2 to the case where there are delivery lead times for each product. Specifically, let λ and μ be the constant lead times for products 1 and 2 respectively. We implicitly assume that the lead time for product 2 is the same regardless of how it is produced. Let w_j be the amount of product 1 to be delivered j periods into the future, $j = 1, 2, \ldots, \lambda - 1$. Similarly for product 2 we define v_i, $i = 1, 2, \ldots, \mu - 1$. Let $F_i^{*k}(t)$ be the k-fold convolution of $F_i(\cdot)$ with itself, where $F_i(\cdot)$ is the cumulative distribution of D_i. Furthermore, for ease of exposition assume that D_1 and D_2 are independent. Define

$$u_1 = x_1 + \sum_{j=1}^{\lambda-1} w_j \quad \text{and} \quad u_2 = x_2 + \sum_{i=1}^{\mu-1} v_i;$$

then, $u = (u_1, u_2)$ represents the vector of pipeline inventories. For ease of exposition, assume $n > \lambda \geqslant \mu$. Then, we replace (2.3) by

$$\tilde{C}_n(x; w_1, \ldots, w_{\lambda-1}; v_1, \ldots, v_{\mu-1})$$
$$= \min\left\{ \frac{1}{\eta_1}(c_A - \eta_2 c_B)z_1 + c_B z_2 + L(x) \right.$$
$$\left. + \alpha \int_0^\infty \int_0^\infty \tilde{C}_n(x_1 + w_1 - t_1, x_2 + v_1 - t_2; w_2, \ldots, z_1, v_2, \ldots, z_2) f_1(t_1) f_2(t_2) dt_1\, dt_2 \right\} \tag{4.2}$$

subject to

$$z_1 \geqslant 0,$$
$$z_2 \geqslant \eta_2 z_1$$

Following the procedure used by Scarf (Chapter 10 of [2]) we can rewrite the problem as:

$$\tilde{C}_n(x; w_1, \ldots, w_{\lambda-1}; v_1, \ldots, v_{\mu-1})$$
$$= \tilde{C}_\lambda(x; w_1, \ldots, w_{\lambda-1}; v_1, \ldots, v_{\mu-1}) + C_n(u) \tag{4.3}$$

where

$$C_n(u) = \min\{ \tilde{G}_n(y) - 1/\eta_1(c_A - \eta_2 c_B)u_1 - c_B u_2 \},$$
$$y_1 \geqslant u_1,$$
$$y_2 \geqslant u_2 + \frac{\eta_2}{\eta_1}(y_1 - u_1) \tag{4.4}$$

with

$$\tilde{G}_n(y) = \frac{1}{\eta_1}(c_A - \eta_2 c_B)y_1 + c_B y_2 + \alpha^\lambda \int_0^\infty \int_0^\infty L(y-t) f_1^{*\lambda}(t_1) f_2^{*\lambda}(t_2) dt_1\, dt_2 + \alpha \int_0^\infty \int_0^\infty C_n(y-t) f_1(t_1) f_2(t_2) dt_1\, dt_2.$$

If we let $\tilde{L}(y)$ be the third term in the above equation, we see that (4.4) is analogous to (2.3). Thus, the leadtime problem is characterized using functions defined in the same manner as $w_n(\cdot)$, $r_n(\cdot)$ and $q_n(\cdot)$ in §3. The important difference is that the decision in each period is a function of total pipeline inventories u rather than on hand inventories x.

4.5. *The General m-Process–n-Product Case*

Our results can be extended in a natural way to allow for any general number of m production processes and n products, so long as all production costs are linear. In addition, any by-product processes must have the property that the production coefficients do not change—that is, they cannot be decision variables. In many situations, such as in blending problems, part of the decision is to determine the optimum blend. Our model could handle this case by considering say m by-product processes, each on the same machine, but having different pre-determined production coefficients. In fact this approach is at the root of most linear programming solutions to blending problems.

In general, there will be as many decision regions as there are possible production process combinations, and the decision space will be of the same dimensions as there are products.

5. Summary and Suggestions for Further Research

This article has presented a new class of multi-product inventory systems, where decisions must be made in regular intervals of time (periods) concerning the optimal stock levels to maintain in order to anticipate stochastic demands. The novel feature of this new class is the notion of coordinating different types of production processes, operated in parallel (and independently), which are used to supply the stock items to the inventory system. Theoretical results were presented that characterized the optimal production policy in each period, in terms of these monotone functions $r_n(x)$, $q_n(x)$, and $w_n(x)$, and a point S_n. In addition, extensions of the model to allow for nonstationary costs and demands, lost sales, finite production capacities and lead times were considered. The general m-process, n-product analog of our model was also discussed. However, there are still some interesting problems still unsolved, such as allowing for fixed costs in production and developing efficient solution techniques for determining the optimal and near optimal policies.

Appendix

This appendix contains the proofs of the theorems and lemmas needed to establish Theorem 3.2 stated in §3. The following well known result (see Rockafellar [14]) is used to prove Theorem 3.1.

LEMMA A.1. *Let*

$$C(x) = \min_{y \in D(x)} B(x,y) = B(x, y^*(x))$$

where $y^*(x) \in D(x)$ and $x \in A$. Assume C and B are real valued. Then, if A is a convex set, B is jointly convex and if $X = \{(x, y);\ y \in D(x)\}$ is convex, then $C(x)$ is convex in x.

PROOF OF THEOREM 3.1. The proof is by induction on the period index, n. First, consider $n = 1$. (1) follows directly from properties assumed on $L(\cdot)$. By assumption, $G_1(y) \to \infty$ whenever $\|y\| \to {}^+\infty$. Hence, for any bounded β, $Q_1(\beta)$ is bounded and convex. The continuity of $G_1(\cdot)$ finally asserts (2). The optimization problem can be written with constraint set

$$y \in A(x) \cap Q_1(\hat{\beta})$$

where

$$A(x) = \left\{ y \in \mathbf{R}_2;\ y_1 \geq x_1, y_2 - \frac{\eta_2}{\eta_1} y_1 \geq x_2 - \frac{\eta_2}{\eta} x_1 \right\}$$

and $\hat{\beta} = G_1(x)$. But $A(x) \cap Q_1(\hat{\beta})$ is compact so that (3) follows. The first assertion in (4) follows from an application of Lemma A1. To prove the second notice that

$$C_1(x) = \frac{1}{\eta_1}(c_A - \eta_2 c_B)(y_{n,1}(x) - x_1) + c_B(y_{n,2}(x) - x_2) + L(y(x)) \geq 0.$$

Now, assume the Theorem is true for periods $1, 2, \ldots, n-1$. We prove it true for period n. (1) follows first since the functions $y - D$, $C_{n-1}(y)$, $C_{n-1}(y - D)$, $E[C_{n-1}(y - D)]$ are convex. The strict convexity of $L(\cdot)$ is sufficient for $G_n(\cdot)$ to be strictly convex. The continuity of $G_n(\cdot)$ follows since the effective domain (see [14]) of $G_n(\cdot)$ is all of $\mathbf{R}_2$ and convex functions are necessarily continuous on the interior of their effective domain. (2) follows immediately since $G_n(y) \geq G_1(y)$. Parts (3) and (4) are proven exactly the same as their counterparts in the case $n = 1$. Q.E.D.

THEOREM A.1. *Under the hypothesis of Theorem* 3.1, *the following hold.*
(i) $G_n(\cdot)$ *is twice continuously differentiable on* $\mathbf{R}_2$.
(ii) (a) $G_n^{(1,j)}(y) + \eta_2 G_n^{(2,j)}(y)/\eta_1 > 0$, $y \in \mathbf{R}_2$, $j = 1, 2$.
(b) $G_n^{(i,j)}(y) \leq 0$, $y \in \mathbf{R}_2$, $i, j = 1, 2$, $i \neq j$.
(iii) $C_n^{(i,j)}(\cdot)$ *is piecewise continuous on* $\mathbf{R}_2$, $i = 1, 2$, $j = 1, 2$.
(iv) (a) $C_n^{(1,j)}(x) + \eta_2 C_n^{(2,j)}(x)/\eta_1 \geq 0$, $j = 1, 2$.
(b) $C_n^{(1,2)}(x) = C_n^{(2,1)}(x) \leq 0$.
for every x in the interior of each of the four regions.

For the proof we refer to [5].[1]

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