

## ANALYSIS OF ORDERING AND ALLOCATION POLICIES FOR MULTI-ECHELON, AGE-DIFFERENTIATED INVENTORY SYSTEMS \*

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This paper considers the problem of determining optimal order and allocation policies for a multi-echelon inventory system in which the product is differentiated by its age. The cost structure includes standard inventory costs as well as a penalty cost for using the product beyond a certain age. Two age-allocation policies are considered: one where the quantity of product of a given age shipped from the central depot to the warehouse is proportional to each warehouse's order; and one where the quantity is based on a fixed fraction, with the possibility of re-allocation of excess supply. Optimal inventory levels at the central depot and at each satellite location are considered for both cases. In addition, conditions for the equivalence of the allocation policies are derived. Potential applications of the models' results include central and regional blood banks, food and chemical product distribution networks, and age-differentiated military logistic systems.

### 1. Introduction

A large and important fraction of inventory investment for many firms is to be found in multi-echelon distribution systems containing products differentiated by their age. Age-differentiated products occupy an intermediate position between pure services, which are not storable, and most manufactured items which have a virtually indefinite lifetime. Examples of age-differentiated products include food, blood, pharmaceuticals, photographic film, chemicals, and certain military ordnance. In some cases such products are perishable if they cannot be used to meet demand after reaching a specified maximum age (e.g., 21 days for whole blood).

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to total demand (summed over all warehouses) for the system.

- (2) A fixed fraction of the quantity of central depot stock of each age issued to meet total warehouse demand is initially allocated to each warehouse. In the event that a warehouse's allocation exceeds its total request from the depot, the excess is redistributed to another warehouse whose request exceeds its initial allocation.

The first age-allocation policy represents an equitable distribution of the product since each warehouse receives stock of each age category in direct proportion to its demand. By fixing the proportion initially allocated, the second policy establishes a priority rule for each warehouse. For example, if a warehouse's allocation fraction is zero, it only receives product after *all* other warehouses in the system are serviced by the FIFO issuing doctrine. Consequently, such a warehouse would receive a higher proportion of younger stock in its shipment, since oldest stock is issued first by FIFO and the central depot receives only fresh stock. Conversely, the warehouse whose fraction is one will have its order filled by the older stock first, and will receive fresh stock only if the entire supply at the central depot is less than that warehouse's order quantity. In a central blood bank system, where the warehouses are hospital blood banks, we often observe that hospitals doing specialized procedures (e.g., open heart surgery) receive a higher priority for fresh blood. Similarly, in a retail distribution system, it is conceivable that market areas served by certain warehouses may have higher priority for younger stock due to differences in profitability and corporate marketing strategy.

As noted above, the cost structure for this model consists of shortage, holding, and over-age, costs. Order costs were not included since order policy is restricted to the critical-number class wherein the expected quantity ordered per period is a constant (equal to mean demand).

Several approaches have been used to analyze single-echelon perishable inventories in which a penalty is paid for over-age units where such units are discarded from the inventory. In this paper a penalty cost for holding over-age stock is used but such units are not removed from the inventory. By setting the penalty cost sufficiently high, we can ensure that the expected quantity of over-age stock consumed can be made arbitrarily small, and thereby ensure that this approach provides a reasonable approximation to the multi-echelon perishable problem. Moreover, the complexity of the problem is reduced, since perishability need not be considered explicitly in the system transfer equations.

In the next section we relate this model to previous work in perishable and multi-echelon inventories. Section 3 presents the model notation and assumptions in detail. Section 4 contains the derivation of expected costs for both allocation policies. For the case of the random-fraction allocation policy a general condition to ensure convexity of expected costs is derived. This condition is somewhat unintuitive, but is shown to be always satisfied for the case where product lifetime is two or three periods and where critical inventory levels are set to equalize the pro-

bability of shortage at each warehouse or where each warehouse has the same critical inventory level. We consider the special case where the central depot target inventory level is either fixed or determined independently of the warehouse stocking and allocation decisions. Such a case allows us to focus on the differences between the two allocation policies. In this case, convexity of expected costs is demonstrated for both allocation policies. Comparison of optimal warehouse ordering policies between the two allocation systems is carried out for this case in section 5. It is shown that it is possible to compute a fixed-fraction allocation rule which yields an optimal order policy identical to the optimal order policy under the random-fraction allocation rule. This result is of importance because it demonstrates the existence of an *a priori* prioritization of warehouses which can achieve equitable (random fraction) age allocation. The paper concludes with a discussion of the key results and areas for further research.

## 2. Literature review

There has been considerable progress in both the areas of multi-echelon inventory and perishable product inventory systems.

Clark and Scarf [39] were the first to formulate and characterize the form of the optimal policy in a multi-period, multi-echelon inventory model subject to stochastic demands. Gross [87] considered the transshipment decision in a wheel system with stochastic demands. Zangwill [254] studied a deterministic demand model structured as an arbitrary acyclic network where the output of any facility cannot return to the facility either directly or indirectly. Tan [229] considered optimal allocation policies of a multi-echelon inventory system where the warehouse can allocate stock to the satellite facilities in each period, but can reorder from an exogenous source only after a fixed number of periods. The optimal quantity to be shipped from a warehouse to each satellite facility in each period is determined by the system configuration in the period.

Veinott [235] developed a general single-facility inventory model. Optimal ordering policies are given for a multi-product, dynamic, non-stationary inventory model, which takes the form of a single critical-number for each period and for each product. Bessler and Veinott [15] later interpreted the stocks of different products as the stocks of a common product carried at different facilities in a multi-echelon system, and hence optimal ordering policies at all facilities are also of the single critical-number type in each period.

Early work on perishable inventory models was concerned with issuing policy. Analysis focused on LIFO (last in, first out) and FIFO (first in, first out) policies. Models have been developed where the field life of the perishable product is deterministic (Eilon, [54]) or random (Pierskalla, [174]), and where there is either a single demand source withdrawing items from the stockpile (Lieberman, [136]; Pierskalla, [174]), or multiple demand sources (Eilon, [60]). FIFO issuing policies

are typically optimal under reasonable cost function conditions. Pierskalla and Roach [175] showed that FIFO was optimal for a perishable product model of a hospital or central blood bank where the utility of the product is non-increasing with age. They showed, in particular, that FIFO is optimal for three objective functions: maximize the utility of the product consumed; minimize the stock-out probability; and minimize the quantity of stock outdated.

More recently there has been significant progress in developing mathematical models of single-echelon perishable inventory systems where the objective is to develop optimal order policies: Nahmias and Pierskalla [165,166] Nahmias [164, 163], Fries [82], and Cohen [41]. Nahmias and Pierskalla [165] developed a model in which the amount to be outdated was expressed recursively in terms of previous outdates and demands. The cost function, including shortages and outdates, was then formulated as a dynamic program. The optimal ordering policy is shown to be dependent on the age distribution of the inventory, and is thus non-stationary. Since the optimal solution to the dynamic program is difficult to compute, approximations on the optimal ordering policies have been studied and evaluated in Nahmias [164,163] and Chazan and Gal [26].

Fries [82], independently of Nahmias and Pierskalla [165], also formulated the problem as a dynamic program. These two models differ in the way that outdates are counted in the objective function. The model of Nahmias and Pierskalla [165] counts the quantity of stock which outdates in the current period, whereas Fries [82] counts the quantity of the current order to be outdated in the  $m - 1$ st period in the future (where the product lifetime is  $m$ ). The Fries approach led to the dependence of the optimal ordering policy on the length of the planning horizon relative to the life of the product. However, the optimal ordering policy is still dependent on the age distribution of the inventory. (Nahmias [161] compared the two approaches directly.) Both Fries and Nahmias and Pierskalla also noted that as the life of the product  $m$  approaches infinity the single critical-number policy becomes optimal.

Since the optimal ordering policy in the dynamic program is difficult to compute even for moderate  $m$ , several authors have examined an ordering policy which is easy to implement, namely the stationary single critical-number policy (Van Zyl [232], Cohen [41], Nahmias [163]). Cohen [41] demonstrated the existence of an invariant distribution of the disposition process and developed closed-form solutions for this disposition process for the two-period lifetime case. Numerical procedures were also developed for longer product lifetimes. Chazan and Gal [26] demonstrated that expected outdates for the stationary critical-number policy were convex, and developed bounds for the determination of optimal policy for an arbitrary product lifetime. In a recent paper Cohen and Pekelman [44] examined the stochastic process of inventory systems governed by LIFO issuing.

The analysis of multi-echelon, age-differentiated systems is not nearly as well developed as is the work on single-echelon perishable systems. The model of this paper and a number of its results (theorem 1 and its corollaries) were developed by

Yen [245] in his doctoral dissertation. Prastacos [183,181] considered the allocation problem in a perishable product, two-echelon system in which inputs are random and not controlled. The perishability process was reduced to a two-period lifetime process by considering only fresh and old-age categories for the product. Recently Prastacos [182] considered the same problem where LIFO issuing is in force. Except for Yen's thesis [245], this paper, and Veinott's dissertation [234], which considered similar problems where demand is deterministic, we know of no results where ordering decisions, allocation, and product age differentiation are considered together.

There is also a large literature on the blood inventory management problem. Although these analyses have occasionally considered multi-echelon systems, they have, for the most part, been carried out as simulation studies. See Cohen and Pierskalla [46] for a recent example of this approach.

### 3. A two-echelon, age-differentiated model

#### 3.1. Notation and model structure

A two-echelon, two-warehouse, age-differentiated inventory system is shown in fig. 1. A facility, labeled as facility 0, supplies two other facilities, labeled facility 1 and facility 2, respectively. Facilities 1 and 2 are subject to non-negative, independent stochastic demand from external users in each time period. Facility 0 does not have any external users; however, it has internal users in the sense that orders from facilities 1 and 2 have to be satisfied. Let  $D_{ij}$  be the demand at facility  $j$  in period  $i$  with finite expectation and with the distribution function  $F_{ij}(\cdot)$ . Let  $D_i = (D_{ij})$ .

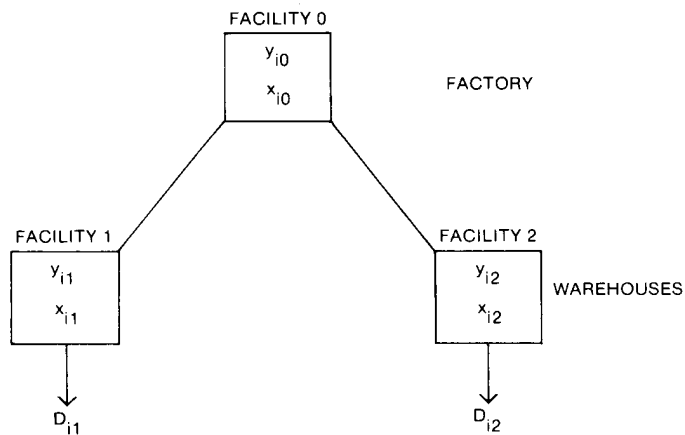


Figure 1

The generalization of this model to a wheel structure with an arbitrary number of destination facilities is notationally complicated but conceptually straightforward. An order for stock may be placed at the beginning of each period and delivery is immediate. Let  $y_{ijn}$  denote the quantity of inventory on hand of age  $n$  after orders have been placed in period  $i$  at facility  $j$  but before demands. Let  $y_{ij} = (y_{ijn})$  be the vector of such inventories and let  $y_{ij}^k = \sum_{n=k}^{\infty} y_{ijn}$  denote all inventory on hand which is of age older than or equal to  $k$  at facility  $j$  in period  $i$  after ordering but before demands and denote by  $y_i^k$  the vector  $(y_{ij}^k)$ . Consequently,  $y_{ij}^1$  will be the total inventory available for meeting demand in period  $i$  at facility  $j$ . Let  $y_i = (y_{ijk})$  be the matrix which represents the system inventory configuration of all ages at all facilities in period  $i$ . Since  $y_{ijk} = y_{ij}^k - y_{ij}^{k+1}$ , once the matrix  $(y_{ij}^k)$  is known,  $y_{ijk}$  can be calculated and vice versa. Hence,  $(y_{ij}^k)$  shall also be referred to as the inventory configuration of all ages at all facilities in period  $i$ .

After  $D_i$  is removed from  $y_i^1$ , the remainder is carried over to the next period and its age increased by 1. In the beginning of period  $i$  before ordering and after the ages are updated there will be a vector  $x_{ij} = (x_{ijn})$  of initial inventories on hand of age  $n$  at facility  $j$ . Let  $x_{ij}^k = \sum_{n=k}^{\infty} x_{ijn}$  denote the amount of inventory on hand which is of age  $k$  or older at the beginning of period  $i$  at facility  $j$ . Since no inventory can be of age 1, define  $x_{ij}^1$  as the backlogged demand from period  $i - 1$ . The total system inventory configuration of all ages at all facilities before ordering in period  $i$  will be denoted by the matrix  $x_i = (x_{ijk})$ .

There are specified rules by which demands are satisfied and orders are filled. It is assumed that there is an unlimited supply at an external source to fill orders from facility 0. After facility 0 has received its delivery in the period, the amount available there will be used to fill orders from facilities 1 and 2. In case there is insufficient stock at facility 0 to fill those orders, facilities 1 and 2 will still get what they ordered; the insufficient amount is assumed to be borrowed from an external source and to be returned in the next period. In this case the borrowed amount will be indicated by a negative inventory level at facility 0 and  $y_{ij}^1 - x_{ij}^1$  will be the amount facility  $j$  ordered and received in period  $i$  for  $j = 1$  and 2. Demands at facilities 1 and 2 are satisfied by their own stock and excess demands in the current period are backlogged there. The shortage incurred at facility  $j$  in period  $i$ ,  $v_{ij}(D - y)$ , may be expressed as

$$v_{ij}(D - y) = (D_{ij} - y_{ij}^1)^+, \quad j = 0, 1, 2, \tag{1}$$

where  $a^+ = \max(a, 0)$ .

It is assumed that the age of all incoming orders for facility 0 is 1. When demands arrive from facilities 1 and 2, the stock is then issued according to a FIFO policy, that is, the oldest stock will be issued first. This issued stock is placed in inventory at facility 1 or 2 waiting to be issued to its users on a FIFO basis. Because the FIFO issuing policy is used throughout the system,  $x_{i+1,j}^{k+1}$  may be expressed as

$$x_{i+1,j}^{k+1} = (y_{ij}^k - D_{ij})^+, \quad j = 0, 1, 2 \text{ and } k = 1, 2, \dots, \tag{2}$$

$$x_{i+1,j}^1 = y_{ij}^1 - D_{ij}, \quad j = 0, 1, 2. \quad (3)$$

The argument of the function  $x_{ij}^k$  has been suppressed for ease of writing. The quantity of stock remaining against which a holding cost is charged is  $h_{ij}(D - y) = (y_{ij}^1 - D_{ij})^+$ .

Once the units to be issued by facility 0 are determined by the FIFO issuing policy, it is necessary to decide how to allocate those units between facilities 1 and 2. Let  $z_{ij}^k(y - x)$  be the amount of stock of age older than or equal to  $k$  allocated from facility 0 to facility  $j$  in period  $i$ , when the vector  $y - x = (y_i^1 - x_i^1)$  is ordered by the system. The sum of allocations of age  $k$  or older at facilities 1 and 2 must not be larger than the total amount age  $k$  or older at facility 0 or the total amount ordered. This may be expressed mathematically:

$$\sum_{j=1}^2 z_{ij}^k(y - x) = \min\left(\sum_{j=1}^2 (y_{ij}^1 - x_{ij}^1), y_{i0}^k\right), \quad \text{for } k > 1 \quad (4)$$

and

$$\sum_{j=1}^2 z_{ij}^1(y - x) = \sum_{j=1}^2 (y_{ij}^1 - x_{ij}^1). \quad (5)$$

To this point we have defined all relevant variables for the model. In the next subsection the various components of the objective function are introduced. The final subsection of section 3 explicitly defines the class of order policies and allocation policies to be considered in the paper.

### 3.2. Objective function

We assume three types of costs in a period  $i$ : shortage, holding and over-age. It follows from the stationary critical-number policy that the ordering quantity becomes the demand of the previous period and therefore its expected value is a constant. The shortage cost  $p_j$  allocates a penalty for the system to be out of stock and is charged against expected value of shortage  $v_{ij}(D - y)$ . The holding cost  $c_j$  is charged against the expected value of total stock remaining,  $h_{ij}(D - y)$ . The over-age cost  $q_j$  is the penalty paid for having over-aged stock. However, this over-age cost is charged against  $x_{i+m-1,j}^m$  rather than  $x_{ij}^m$  because the decision variable associated with  $x_{i+m-1,j}^m$  is  $y_{ij}^1$  which is the inventory after ordering in the current period. By assuming that  $x_{0j}^k = 0$  for all  $j$  and  $k$ ,  $x_{ij}^k$  may be expressed recursively in terms of demand and  $y_{ij}^1$ . These expressions follow from the definition of variables and FIFO. Their formal proof can be found in Yen [245].

$$y_{i0}^k = x_{i0}^k = \left( y_{i-k+1,0}^1 - \sum_{n=1}^{k-1} D_{i-n,0} \right)^+, \quad \text{for } k > 1 \text{ and } i \geq k. \quad (6)$$

$$= 0 \quad \text{if } i < k.$$

Equation (6) indicates that stock of age  $k$  or more at the central depot before or after ordering (for  $k > 1$ ) is determined by the depletion of the sum of  $k - 1$  consecutive demands. (Recall that incoming stock at facility 0 is of age 1 and hence inventory before and after ordering are equal for  $k > 1$ .)

It follows from (6) that the quantity of stock of age  $m$  or greater at the depot in period  $i + m - 1$  is related to the depot target inventory level in period  $i$  by the following formula:

$$x_{i+m-1,0}^m = \left( y_{i0}^1 - \sum_{n=0}^{m-2} D_{i+n,0} \right)^+. \quad (7)$$

The corresponding formula for warehouse  $j$  is

$$x_{ij}^k = \left( y_{i-k+1,j}^1 + \sum_{n=1}^{k-2} z_{i-n,j}^{k-n} (y - x) - \sum_{n=1}^{k-1} D_{i-n,j} \right)^+,$$

$$\text{for } k > 1, i > k, \text{ and } j = 1, 2 \quad (8)$$

$$= 0 \quad \text{if } i < k.$$

where it was necessary to include the shipments of quantities of various ages to the warehouse from the depot. Equation (8) is established in Yen [245] by an inductive argument. The following equation is a direct consequence of (8):

$$x_{i+m-1,j}^m = \left( y_{ij}^1 + \sum_{n=1}^{m-2} z_{i+n,j}^{n+1} (y - x) - \sum_{n=0}^{m-2} D_{i+n,j} \right)^+,$$

$$\text{for } j = 1, 2. \quad (9)$$

Equations (7) and (9) define the random variable for over-age stock at the depot and warehouses respectively. These equations will be used in our derivation of expected costs.

We have not yet specified, however, the allocation policy in the system. Therefore, the value of the objective function will vary from one age-allocation policy to another. Let  $\alpha_{ij}$  be an age-allocation policy in period  $i$  which (initially) allocates an



amount of age  $k$  or greater to warehouse  $j$ , defined by

$$z_{ij}^k(y - x) = \alpha_{ij} \min\left(\sum_{j=1}^2 (y_{ij}^1 - x_{ij}^1), y_{i0}^k\right)$$

for  $0 < \alpha_{ij} < 1$ ,  $\alpha_{i1} + \alpha_{i2} = 1$  and  $k = 2, \dots, m$ . Note the age-allocation policy is defined only up to  $m$  because we shall charge an over-age cost for units older than or equal to  $m$ . Let  $\alpha_i = (\alpha_{i1}, \alpha_{i2})$ , and  $\alpha = (\alpha_i)_{i=1,2,\dots}$ .

Given the system will be following the FIFO issuing policy, and allocation policy  $(\alpha_i, \alpha_{i+1}, \dots, \alpha_{i+m-1})$  and an ordering policy  $(y_1^1, \dots, y_{i+m-1}^1)$  in the next  $m$  periods, a cost  $W_i(\cdot)$  will be incurred in period  $i$ .  $W_i(\cdot)$  can be expressed explicitly in terms of demands and inventory level after ordering by equations (1) and (9):

$$\begin{aligned} W_i(y_1^1, \dots, y_{i+m-1}^1, D_i, \dots, D_{i+m-1}, \alpha_i, \dots, \alpha_{i+m-1}) \\ = \sum_{j=0}^2 (p_j v_{ij}(D - y) + c_j h_{ij}(D - y) + q_j x_{i+m-1,j}^m). \end{aligned}$$

Because of the lengthy notation in expressing the dependence of  $x_{i+m-1,j}^m$  and  $W_i(\cdot)$  on  $(y_1^1, \dots, y_{i+m-1}^1, D_i, \dots, D_{i+m-1}, \alpha_i, \dots, \alpha_{i+m-1})$ , we shall suppress all of them and include only those relevant to the development of the various results as needed.

Let  $G_i(\cdot)$  be the expected value of  $W_i$ , then

$$G_i(\cdot) = \sum_{j=0}^2 E[p_j v_{ij}(D = y) + c_j h_{ij}(D - y) + q_j x_{i+m-1,j}^m].$$

Our objective is to find an ordering policy  $(y_1^1, y_2^1, \dots)$  to minimize the sum of expected, discounted costs,

$$\min_{(y_1^1, y_2^1, \dots)} G(y) = \sum_{i=1}^{\infty} r^i G_i(y_1^1, \dots, y_{i+m-1}^1) \tag{10}$$

where  $r$  is a discount factor with  $0 < r < 1$  and  $y_i^1 \in R_+^3$  for  $i = 1, 2, \dots$ , and where the allocation policy  $\alpha$  is specified.

### 3.3. Order policy and allocation policy restrictions

Additional assumptions will be made before proceeding to analyze optimal ordering policy. We shall restrict the ordering policy to the stationary single critical-

number policy, that is, the inventory on hand after ordering  $y_{ij}^1$  does not depend on  $i$ . Consequently, in each period an amount exactly equal to the demand in the previous period will be ordered so as to reach the desirable inventory level. Note this is true only if all excess demand is backlogged and the demand is non-negative so that the inventory level before ordering is always less than or equal to the desired inventory level after ordering. Furthermore, the demand in period  $i$  for facility 0 is the sum of the ordering amounts (the demands) from facilities 1 and 2 in the previous period, so  $D_{i0} = D_{i-1,1} + D_{i-1,2}$ , and  $D_{i0}$  will consequently be the ordering amount for facility 0 in period  $i + 1$ . The demands  $(D_{ij})_{i=1,2,\dots}$ , are independent random variables having a common distribution  $F_j(\cdot)$  with density  $f_j(\cdot)$  for  $j = 1, 2$ .

With regard to the allocation policy, two cases are considered. In case I we assume that each facility will receive an allocation of all ages proportional to its ordering amount with respect to the total ordering amount from facilities 1 and 2. Specifically, for this case, in each period  $i$ ,  $\alpha_{ij} = D_{i-1,j}/D_{i0}$ , and for case II  $\alpha_{ij}$  is fixed for all  $i$  and  $j$ , where values may be chosen to reflect relative costs for each warehouse or a qualitative priority ranking of each warehouse. After initial allocation to each warehouse, under case II, surplus at warehouse  $j$  (if it exists) is re-allocated to warehouse  $k \neq j$  (if there is a shortage at  $k$ ).

We now define the two allocation policies explicitly. Let

$$S_0 = 0, S_n = \sum_{k=0}^{n-1} D_{i+k,0},$$

for any  $n \leq m$  we define:

*Allocation policy I*

$$z_{i+n,j}^{n+1}(y-x) = \frac{D_{i+n-1,j}}{D_{i+n,0}} \min(D_{i+n,0}, (y_{i0}^1 - S_n)^+). \tag{11}$$

*Allocation policy II*

$$z_{i+n,j}^{n+1}(y-x) = \min \{ D_{i+n-1,j}, \alpha_{ij} A_{i+n} + [(1 - \alpha_{ij}) A_{i+n} - (D_{i+n,0} - D_{i+n-1,j})]^+ \}, \tag{12}$$

where

$$A_{i+n} = \min(D_{i+n,0}, (y_{i0}^1 - S_n)^+).$$

In both cases I and II, we note that, while each warehouse receives its entire order (e.g., a total of  $D_{i,0}$  is always shipped from facility 0 and hence  $z_{ij}^1(y-x) = D_{i-1,j}$ ;

the demand of last period), the age composition of that order may vary due to the allocation policy.

It is important to note that all costs in period  $i$  (shortage, holding, over-age) are only a function of the critical inventory vector  $y_i^1 = (y_{i0}^1, y_{i1}^1, y_{i2}^1)$  in period  $i$ . Moreover, since as noted above  $y_{ij}^1$  does not depend on  $i$ , by the critical number policy, it is sufficient to consider minimization of  $G_i(\cdot)$  for arbitrary  $i$ . Finally, since expected holding and shortage costs are convex, it will be sufficient to examine the convexity of  $E[x_{i+m-1}^m]$  to ensure convexity of  $G(y)$ . This particular issue is addressed in the next section.

#### 4. Optimal order policies

The following properties can be easily established (see Yen [245]), and are important for the determination of an optimal ordering policy.

##### Property 1

$v_{ij}(D - y)$ ,  $Ev_{ij}(D - y)$ ,  $h_{ij}(D - y)$  and  $Eh_{ij}(D - y)$  are convex functions of  $y_i^1$  over  $R_+^3$  for  $j = 0, 1, 2$ .

##### Property 2

$x_{i+m-1,j}^m$  is a continuous function of  $y_i^1$  for  $j = 0, 1, 2$ , for allocation policies I and II.

The following theorem, first derived in Yen [245], demonstrates that, for allocation policy I, it is possible to establish convexity of expected over-age stock in the system under certain conditions on the demand process (see appendix for all proofs). This condition cannot be easily demonstrated to be true for all values of  $(y_0^1, y_1^1, y_2^1)$ .

**Theorem 1.** For allocation policy I,

$$\begin{aligned} \text{If } & \sum_{j=2}^2 \sum_{l=2}^{m-2} E \left[ \frac{D_{ij}}{D_{i1} + D_{i2}} F_j^{M-l-1} (y_{ij}^1 - D_{i1} - D_{i2}) \right] f_0^l(y_{i0}^1) \\ & \geq \sum_{j=1}^2 \sum_{l=2}^{m-2} E \left[ \frac{D_{ij}}{D_{i1} + D_{i2}} F_0^{l-1} (y_{ij}^1 - D_{i1} - D_{i2}) \right] F_j^{m-l}(y_{ij}^1) \end{aligned}$$

then  $\sum_{j=0}^2 Ex_{i+m-1,j}^m$  is convex in  $y_i^1$ .

**Corollary 1.** If  $m = 2$  or  $3$ , then  $\sum_{j=0}^2 Ex_{i+m-1,j}^m$  is convex in  $y_i^1$ .

**Corollary 2.** If the events  $D_{ij} \leq y_{ij}^1$  are equivalent (equal probability) for  $j = 1, 2$ ,

then  $\sum_{j=0}^2 Ex_{i+m-1,j}^m$  is convex in  $y_i^1$ .

**Corollary 3.** If  $D_{ij}$  are independent and identically distributed and if  $y_{i1}^1 = y_{i2}^1$ , then  $\sum_{j=0}^2 Ex_{i+m-1,j}^m$  is convex in  $y_i^1$ .

**Lemma 1.** If  $\sum_{j=0}^2 Ex_{i+m-1,j}^m$  is convex, and if  $q_1 = q_2$ , then  $G(\cdot)$  is convex.

Theorem 1, its corollaries and lemma 1 indicate that when over-age costs are the same for the warehouses, and when demands at the warehouses are independent and identically distributed, then the over-age quantity function is convex in the critical-number decision variables. Yen [245] also derived a general condition for the optimality of allocation policy I and showed that this condition is satisfied for the same special cases noted above which were needed to ensure convexity.

The following theorem establishes convexity of expected over-age stock for both allocation policies when the level of stock at the central facility is fixed. As noted previously, this case may occur if depot stock ordering is controlled at a higher level or is uncontrollable. The restriction allows us to focus on the differences between warehouse ordering policies under each age-allocation policy.

**Theorem 2.** If  $y_{i0}^1$  is assumed to be fixed or determined independently of  $y_{i1}^1$  and  $y_{i2}^1$ , then  $Ex_{i+m-1,j}^m$  is convex for  $j = 1, 2$  and for allocation policies I and II.

Given the convexity of  $\sum_{j=0}^2 Ex_{i+m-1,j}^m$  in  $y_i^1$ , and since  $Ev_{ij}(D-y)$  and  $Eh_{ij}(D-y)$  are also convex in  $y_i^1$  for all  $j$ , the mathematical program given by (10) is a convex program. Consequently, optimal policies exist (level sets for the expected cost function are bounded since over-age costs increase as  $y_i^1$  increases and shortage costs increase as  $y_i^1$  decreases to zero), and nonlinear programming codes can be used to obtain the optimal  $y$ 's.

Analysis of the general case (where  $y_{i0}^1$  is variable) for allocation policy II was also carried out and led to results similar to that of theorem 1. Again, unconditioned convexity cannot be easily established. For the special case where  $y_{i0}^1$  is a constant, theorem 2 indicates that solution of the first-order condition equations for both cases I and II will yield unique optimal inventory levels for each warehouse. In the final section we examine the sensitivity of these optimal levels to parametric changes and we compare the relative magnitudes for each case under the assumption that  $y_{i0}^1$  is fixed.

## 5. Comparison of optimal policies

From the proof of theorem 1 (see appendix equation A4) and the definition of  $G_i(\cdot)$ , we note that for the allocation policy I, optimal inventory level at warehouse  $j$ , after ordering, for the case where central depot stock is fixed, is the solution to

the following equation:

$$\begin{aligned}
 & q_j \left\{ F_j^{m-1}(y_{ij}^1) \{1 - F_0(y_{i0}^1)\} + F_j(y_{ij}^1) F_0^{m-1}(y_{i0}^1) \right. \\
 & \quad \left. + \sum_{l=1}^{m-2} \int_0^{y_{i0}^1} \int_0^\infty \int_{\rho_j(y)}^\infty F_j^{m-l-1}(\beta_j(y)) F_j(dt_j) F_k(dt_k) F_0^l(dt_0) \right\} \\
 & \quad + (p_j + c_j) F_j(y_{ij}^1) = p_j. \tag{13}
 \end{aligned}$$

By theorem 2, when  $y_{i0}^1$  is fixed, (13) is also a sufficient condition for the optimal policy. Let  $y_I^*$  be the unique value of  $y_{ij}^1$  which solves (13) (where I refers to allocation policy I).

The solution to the following equation yields the optimal inventory level for warehouse  $j$  under allocation policy II (for the case where depot stock is fixed):

$$\begin{aligned}
 & q_j \left\{ F_j^{m-1}(y_{ij}^1) \{1 - F_0(y_{i0}^1)\} + F_j(y_{ij}^1) F_0^{m-1}(y_{i0}^1) \right. \\
 & \quad \left. + \sum_{l=1}^{m-2} \left[ \int_0^{y_{i0}^1} \int_{\gamma_k(y)}^\infty \int_{\gamma_j(y)}^\infty F_j^{m-l-1}(y_{ij}^1 + \gamma_j(y) - t_j) F_j(dt_j) \right. \right. \\
 & \quad \times F_k(dt_k) F_0^l(dt_0) + \int_0^{y_{i0}^1} \int_0^{\gamma_k(y)} \int_{\rho_j(y)}^\infty F_j^{m-l-1}(y_{ij}^1 + \rho_j(y) - t_j) F_j(dt_j) \\
 & \quad \times F_k(dt_k) F_0^l(dt_0) + \int_0^{y_{i0}^1} \int_{\gamma_k(y)}^\infty \int_{\rho_j(y)}^{\gamma_j(y)} F_j^{m-l-1}(y_{ij}^1) F_j(dt_j) \\
 & \quad \left. \left. \times F_k(dt_k) F_0^l(dt_0) \right] \right\} + (p_j + c_j) F_j(y_{ij}^1) \\
 & \quad = p_j. \tag{14}
 \end{aligned}$$

where

$$j \neq k, j = 1, 2;$$

$$\gamma_j(y) = \alpha_{ij}(y_{i0}^1 - t_0),$$

$$\gamma_k(y) = \alpha_{ik}(y_{i0}^1 - t_0),$$

$$\rho_j(y) = y_{i0}^1 - t_0 - t_k.$$

Again by theorem 2, the value of  $y_{ij}^1$ , to be denoted by  $y_{II}^*(\alpha_{ij})$  which solves (14) is the unique optimal solution for allocation policy II when  $y_{i0}^1$  is fixed. Note that we have explicitly included the dependence of the optimal inventory level under allocation policy II, on the value of  $\alpha_{ij}$ , the initial allocation fraction for warehouse  $j$ .

It is important to note that the solution of (13) and (14) is feasible since we have reduced the optimization problem to one of minimization of a one-dimensional convex function.

In the following theorem we demonstrate the relationship between the optimal inventory levels associated with each allocation policy.

**Theorem 3.** Assume  $y_{i0}^1$  is fixed:

Let 
$$y_{II}^*(0) = \lim_{\alpha_{ij} \rightarrow 0} y_{II}^*(\alpha_{ij})$$

and

$$y_{II}^*(1) = \lim_{\alpha_{ij} \rightarrow 1} y_{II}^*(\alpha_{ij}) ,$$

then

- (i)  $y_{II}^*(1) \leq y_I^* \leq y_{II}^*(0)$ ,
- (ii)  $y_{II}^*(\alpha_{ij})$  is continuous in  $\alpha_{ij}$ ,
- (iii) there exists an  $\hat{\alpha}_{ij} \in [0,1]$  such that  $y_I^* = y_{II}^*(\hat{\alpha}_{ij})$  and moreover  $\hat{\alpha}_{ij}$  is the solution to the following equation:

$$\begin{aligned} & \left[ \sum_{l=1}^{m-2} \int_0^{y_{i0}^1} \int_0^\infty \int_{\rho_j(y)}^\infty F_j^{m-l-1}(\beta_j(y_1^*)) F_j(dt_j) F_k(dt_k) F_0^l(dt_0) \right. \\ & - \int_0^{y_{i0}^1} \int_{\gamma_k(y)}^\infty \int_{\gamma_j(y)}^\infty F_j^{m-l-1}(y_1^* + \gamma_j(y) - t_j) F_j(dt_j) F_k(dt_k) F_0^l(dt_0) \\ & - \int_0^{y_{i0}^1} \int_0^{\gamma_k(y)} \int_{\rho_j(y)}^\infty F_j^{m-l-1}(y_1^* + \rho_j(y) - t_j) F_j(dt_j) F_k(dt_k) F_0^l(dt_0) \\ & \left. - \int_0^{y_{i0}^1} \int_{\gamma_k(y)}^\infty \int_{\rho_j(y)}^{\gamma_j(y)} F_j^{m-l-1}(y_1^*) F_j(dt_j) F_k(dt_k) F_0^l(dt_0) \right] = 0 , \quad (15) \end{aligned}$$

where

$$\beta_j(y^*) = y_1^* + \frac{t_j}{t_j + t_k} (y_{i0}^1 - t_0) - t_j.$$

Equation (15) indicates that it is possible to compute an allocation fraction (and hence an *a priori* warehouse prioritization) which yields the same expected costs and optimal warehouse order policies as that of allocation policy I. As noted previously, allocation policy I is equitable and under certain conditions is also optimal. Allocation policy II, on the other hand, may be easier to implement in those situations where consistent (rather than random) allocation rules are important. The equivalence of these two allocation policies has important managerial implications. Further research needs to be undertaken to investigate the sensitivity of the optimal order policy  $y_{II}^*(\alpha_{ij})$  to changes in  $\alpha_{ij}$  and to evaluate the magnitude of the difference between  $y_{II}^*(\alpha_{ij})$  and  $y_I^*$  as a function of  $\alpha_{ij}$ .

## 6. Conclusions

This paper has examined a specific class of multi-echelon inventory systems where the product can be differentiated by age. The interaction between optimal order policies and age-allocation policies was examined for the case of stationary critical number ordering and both random and fixed-fraction allocation rules. The random allocation rule can be viewed as equitable, and the fixed-fraction rule allows for prioritization of each warehouse. A general condition for convexity of expected costs was developed for the random fraction allocation rule model. Since this condition was shown to be satisfied for a number of special cases, we conjecture that the condition may, indeed, not be needed. Proof of this conjecture remains for future research.

Consideration of the cases where the central depot stocking level was assumed to be fixed or determined independently of warehouse stocking levels led to the derivation of closed form results for optimal warehouse order policy for each age allocation rule. It was also possible to prove the existence of a fixed allocation fraction value which makes the two age-allocation systems equivalent. Further research is needed to explore the general validity of convexity and to apply one-dimensional search algorithms to solve the necessary condition equations associated with the fixed depot stock model. Investigation of the equivalence between the age-allocation policies should also be carried out. Analysis of the more general problem where perishability (finite lifetime) is treated directly rather than by means of an over-age cost in an age-differentiated model, remains as a challenging research problem in the context of multi-echelon systems.

Appendix

Proof of theorem 1

It is sufficient to show that the Hessian for  $\sum_{j=0}^2 Ex_{i+m-1,j}^m$  is positive semi-definite. Denote by  $D_k Ex_{i+m-1,j}^m$  the partial derivative of  $Ex_{i+m-1,j}^m$  with respect to  $y_{ik}^1$  for  $j, k = 0, 1, 2$ . Let

$$S_0 = 0 \text{ and } S_n = \sum_{k=0}^{n-1} D_{i+k,0} .$$

Define

$$N(y) = \sup \{n: S_n < y_{i0}^1\} .$$

By conditioning  $N(y)$ ,  $Ex_{i+m-1,j}^m$  for  $j = 1, 2$  can be expressed as

$$\begin{aligned} Ex_{i+m-1,j}^m &= E \left\{ \left( y_{ij}^1 - \sum_{n=0}^{m-2} D_{i+n,j} \right)^+ ; N(y) = 0 \right\} + E \{ y_{ij}^1 - D_{i+m-2,j} \}^+ ; \\ N(y) \geq m-1 &+ \sum_{l=1}^{m-2} E \left\{ \left( y_{ij}^1 + \frac{D_{i+k-1,j}}{D_{i+k,0}} (y_{i0}^1 - S_l) \right. \right. \\ &\left. \left. - \sum_{n=l-1}^{m-2} D_{i+n,j} \right)^+ ; N(y) = l \right\} . \end{aligned} \tag{A1}$$

Because of the equivalence of the events  $(N(y) = l)$  and  $S_l < y_{i0}^1 \leq S_{l+1}$  for  $l = 0, 1, \dots, m-2$  and of  $(N(y) \geq m-1)$  and  $(S_{m-1} < y_{i0}^1)$  and because of the independence of random variables  $\sum_{n=0}^{m-2} D_{i+n,j}$  and  $D_{i0}$  and of  $D_{i+m-2,j}$  and  $S_{m-1}$  for  $j = 1, 2$ , (A1) can then be rewritten as

$$\begin{aligned} Ex_{i+m-1,j}^m &= \int_0^{y_{ij}^1} (y_{ij}^1 - t_j) F_j^{m-1}(dt_j) \{1 - F_0(y_{i0}^1)\} \\ &+ \int_0^{y_{ij}^1} (y_{ij}^1 - t_j) F_j(dt_j) \{F_0^{m-1}(y_{i0}^1)\} \end{aligned}$$



$$\begin{aligned}
& + \sum_{l=1}^{m-2} \int_0^{y_{i0}^1} \int_0^\infty \int_{\rho_j(y)}^\infty \int_0^{\beta_j(y)} \left\{ y_{ij}^1 + \frac{t_j}{t_j + t_k} (y_{i0}^1 - t_0) - t_j - u \right\} \\
& \times F_j^{m-l-1} (du) F_j(dt_j) F_k(dt_k) F_0^l(dt_0), \tag{A2}
\end{aligned}$$

where  $j \neq k$ ,  $j = 1, 2$ ;  $\rho_j(y) = y_{i0}^1 - t_0 - t_k$  and  $\beta_j(y) = y_{ij}^1 + [t_j/(t_j + t_k)] (y_{i0}^1 - t_0) - t_j$ .

Since the integrands in (A2) are continuous and differentiable between the limits of integration, the first-order partial derivatives can be calculated by the chain rule:

$$\begin{aligned}
D_0 Ex_{i+m-1,j}^m & = \int_0^{y_{ij}^1} (y_{ij}^1 - t_j) F_j^{m-1} (dt_j) \{-f_0(y_{i0}^1)\} \\
& + \int_0^{y_{ij}^1} (y_{ij}^1 - t_j) F_j(dt_j) \cdot \{f_0^{m-1}(y_{i0}^1)\} \\
& + \sum_{l=1}^{m-2} \int_0^\infty \int_0^\infty \int_0^{y_{ij}^1 - t_j} (y_{ij}^1 - t_j - u) F_j^{m-l-1} (du) F_j(dt_j) F_k(dt_k) f_0^l(y_{i0}^1) \\
& - \sum_{l=1}^{m-2} \int_0^{y_{i0}^1} \int_0^\infty \int_0^{y_{ij}^1} (y_{ij}^1 - u) F_j^{m-l-1} (du) f_j(y_{i0}^1 - t_k - t_0) \\
& \times F_k(dt_k) F_0^l(dt_0) + \sum_{l=1}^{m-2} \int_0^{y_{i0}^1} \int_0^\infty \int_{\rho_j(y)}^\infty \frac{t_j}{t_j + t_k} F_j^{m-l-1}(\beta_j(y)) \\
& \times F_j(dt_j) F_k(dt_k) F_0^l(dt_0). \tag{A3}
\end{aligned}$$

It is noteworthy that the first four terms can be cancelled out because the third and fourth terms are equivalent to

$$\sum_{l=1}^{m-2} \int_0^{y_{ij}^1} (y_{ij}^1 - t_j) F_j^{m-l-1} (dt_j) f_0^l(y_{i0}^1)$$

and

$$\sum_{i=1}^{m-2} \int_0^{y_{ij}^1} (y_{ij}^1 - u) F_j^{m-l-1}(du) f_0^{l+1}(y_{i0}^1),$$

respectively. The other first-order partial derivatives are calculated as follows:

$$D_j Ex_{i+m-1,j}^m = F_j^{m-1}(y_{ij}^1) \{1 - F_0(y_{i0}^1)\} + F_j(y_{ij}^1) F_0^{m-1}(y_{i0}^1) + \sum_{l=1}^{m-2} \int_0^{y_{i0}^1} \int_0^\infty \int_{\rho_j(y)}^\infty F_j^{m-l-1}(\beta_j(y)) F_j(dt_j) F_k(dt_k) F_0^l(dt_0), \quad (A4)$$

$$D_k Ex_{i+m-1,j}^m = 0, \quad \text{for } k \neq j \text{ and } j \neq 0,$$

$$D_0 Ex_{i+m-1,0}^m = F_0^{m-1}(y_{i0}^1),$$

and

$$D_k Ex_{i+m-1,0}^m = 0, \quad \text{for } k = 1, 2.$$

The second-order partial derivatives can be calculated by taking partial derivatives from the first-order partial derivatives. Again by the chain rule,

$$D_0 D_0 Ex_{i+m-1,j}^m = \sum_{l=1}^{m-2} \int_0^\infty \int_0^\infty \frac{t_j}{t_j + t_k} F_j^{m-l-1}(y_{ij}^1 - t_j) F_j(dt_j) F_k(dt_k) f_0^l(y_{i0}^1) - \sum_{l=1}^{m-2} \int_0^{y_{i0}^1} \int_0^\infty \frac{t_j}{t_j + t_k} f_0^l(y_{i0}^1 - t_j - t_k) F_j(dt_j) F_k(dt_k) F_j^{m-l-1}(y_{ij}^1) + \sum_{l=1}^{m-2} \int_0^{y_{i0}^1} \int_0^\infty \int_{\rho_j(y)}^\infty \left(\frac{t_j}{t_j + t_k}\right)^2 f_j^{m-l-1}(\beta_j(y)) F_j(dt_j) F_k(dt_k) F_0^l(dt_0), \quad (A5)$$

$$\begin{aligned}
D_j D_0 Ex_{i+m-1,j}^m &= D_0 D_j Ex_{i+m-1,j}^m \\
&= \sum_{l=1}^{m-2} \int_0^{y_{i0}^1} \int_0^\infty \int_{\rho_j(y)}^\infty \frac{t_j}{t_j + t_k} f_j^{m-l-1}(\beta_j(y)) F_j(dt_j) F_k(dt_k) F_0^l(dt_0)
\end{aligned} \tag{A6}$$

$$\begin{aligned}
D_j D_j Ex_{i+m-1,j}^m &= f_j^{m-1}(y_{ij}^1) \{1 - F_0(y_{i0}^1)\} + f_j(y_{ij}^1) F_0^m(y_{i0}^1) \\
&+ \sum_{l=1}^{m-2} \int_0^{y_{i0}^1} \int_0^\infty \int_{\rho_j(y)}^\infty f_j^{m-l-1}(\beta_j(y)) F_j(dt_j) F_k(dt_k) F_0^l(dt_0),
\end{aligned} \tag{A7}$$

$$D_k D_j Ex_{i+m-1,j}^m = 0, \quad \text{for } k \neq j \text{ and } j = 1, 2, \tag{A8}$$

and

$$D_0 D_0 Ex_{i+m-1,0}^m = f_0^{m-1}(y_{i0}^1). \tag{A9}$$

Now from the hypothesis and the following relationship:

$$\begin{aligned}
&\sum_{j=1}^2 \int_0^{y_{i0}^1} \int_0^\infty \frac{t_j}{t_j + t_k} f_0^{m-2}(y_{i0}^1 - t_j - t_k) F_j(dt_j) F_k(dt_k) F_j(y_{ij}^1) \\
&\leq \max_{j=1,2} \{F_j(y_{ij}^1)\} \\
&\times \int_0^{y_{i0}^1} \int_0^\infty \sum_{j=1}^2 \frac{t_j}{t_j + t_k} f_0^{m-2}(y_{i0}^1 - t_j - t_k) F_j(dt_j) F_k(dt_k) \\
&= \max_{j=1,2} \{F_j(y_{ij}^1)\} \cdot f_0^{m-1}(y_{i0}^1) \leq f_0^{m-1}(y_{i0}^1),
\end{aligned}$$

we have

$$\begin{aligned}
D_0 D_0 \sum_{j=0}^2 Ex_{i+m-1,j}^m &\geq \sum_{l=1}^{m-2} \int_0^{y_{i0}^1} \int_0^\infty \int_{\rho_j(y)}^\infty \left(\frac{t_j}{t_j + t_k}\right)^2 f_j^{m-l-1}(\beta_j(y)) \\
&\times F_j(dt_j) F_k(dt_k) F_0^l(dt_0).
\end{aligned}$$

The Hessian is then positive semi-definite by the Schwartz inequality, that is,

$$\begin{aligned}
 & \left[ \sum_{l=1}^{m-2} \int_0^{y_{i0}^1} \int_0^\infty \int_{\rho_j(y)}^\infty \frac{t_j}{t_j + t_k} f_j^{m-l-1}(\beta_j(y)) F_j(dt_j) F_k(dt_k) F_0^l(dt_0) \right]^2 \\
 &= \left[ \int_0^{y_{i0}^1} \int_0^\infty \int_{\rho_j(y)}^\infty \left( \frac{t_j}{t_j + t_k} \right) \sum_{l=1}^{m-2} f_j^{m-l-1}(\beta_j(y)) f_j(t_j) f_k(t_k) \right. \\
 & \quad \left. \times f_0^l(t_0) dt_j dt_k dt_0 \right]^2 \\
 &\leq \int_0^{y_{i0}^1} \int_0^\infty \int_{\rho_j(y)}^\infty \left( \frac{t_j}{t_j + t_k} \right)^2 \sum_{l=1}^{m-2} f_j^{m-l-1}(\beta_j(y)) f_j(t_j) f_k(t_k) \\
 & \quad \times f_0^l(t_0) dt_j dt_k dt_0 \\
 & \quad \int_0^{y_{i0}^1} \int_0^\infty \int_{\rho_j(y)}^\infty \sum_{l=1}^{m-2} f_j^{m-l-1}(\beta_j(y)) f_j(t_j) f_k(t_k) f_0^l(t_0) dt_j dt_k dt_0 \\
 &\leq \int_0^{y_{i0}^1} \int_0^\infty \int_{\rho_j(y)}^\infty \sum_{l=1}^{m-2} f_j^{m-l-1}(\beta_j(y)) f_j(t_j) f_k(t_k) f_0^l(t_0) dt_0 dt_k dt_j \quad \text{QED}
 \end{aligned}$$

*Proof of corollary 1*

When  $m = 2$  or  $3$  the hypothesis of theorem 1 is satisfied. Furthermore, when  $m = 2$ ,  $Ex_{i+m-1,j}^m$  is reduced to  $E(y_{ij}^1 - D_{ij})^+$  for  $j = 0, 1, 2$ , which is convex and represents the inventory on hand at the end of period  $i$  which will become age 2 at the beginning of period  $i + 1$ .

*Proof of corollary 2*

From the hypothesis we have  $F_1^{m-l}(y_{i1}^1) = F_2^{m-l}(y_{i2}^1)$  for  $l = 2, \dots, m-2$ . Since

$$E\left(\frac{D_{ij}}{D_{i1} + D_{i2}} F_j^{m-l-1}(y_{ij}^1 - D_{ij})\right) = E\left(\frac{D_{ij}}{D_{i1} + D_{i2}}; \sum_{n=0}^{m-l-1} D_{i+n,j} \leq y_{ij}^1\right)$$

we have

$$\sum_{j=1}^2 E\left(\frac{D_{ij}}{D_{i1} D_{i2}}; \sum_{n=0}^{m-l-1} D_{i+n,j} \leq y_{ij}^1\right) = F_j^{m-l}(y_{ij}^1).$$

Therefore the hypothesis in theorem 1 is satisfied because

$$\begin{aligned}
 & \sum_{j=1}^2 \left[ E\left(\frac{D_{ij}}{D_{i1} + D_{i2}} F_j^{m-l-1}(y_{ij}^1 - D_{i1})\right) f_0^l(y_{i0}^1) \right. \\
 & \quad \left. - E\left(\frac{D_{ij}}{D_{i1} + D_{i2}} f_0^{l-1}(y_{i0}^1 - D_{i1} - D_{i2})\right) F_j^{m-l}(y_{ij}^1) \right]
 \end{aligned}$$

$$\begin{aligned}
 &= \sum_{j=1}^2 [F_j^{m-l}(y_{ij}^1) f_0^l(y_{i0}^1) - f_0^l(y_{i0}^1) F_j^{m-l-1}(y_{ij}^1)] \\
 &= 0, \quad \text{for } l = 2, \dots, m - 2.
 \end{aligned}$$

*Proof of corollary 3*

Since the events  $\{D_{ij} \leq y_{ij}^1\}$  are equivalent from the hypotheses, therefore the assertion is true from corollary 2.

*Proof of lemma 1*

Since the sum of convex functions is convex, consequently  $G(\cdot)$  is convex in  $y_i^1$ .

*Proof of theorem 2*

*Allocation policy I*

By (A7) from the proof of theorem 1 we note that  $D_j D_j Ex_{i+m-1,j}^m \geq 0$ . By (A8) we note that  $D_k D_j Ex_{i+m-1,j}^m = 0$  for  $k \neq j$  and  $j = 1, 2$ . Convexity follows trivially.

*Allocation policy II*

By an argument similar to that in the proof of theorem 1 for case I, it can be demonstrated that

$$\begin{aligned}
 Ex_{i+m-1,j}^m &= \int_0^{y_{ij}^1} (y_{ij}^1 - t_j) F_j^{m-1}(dt_j) \cdot \{1 - F_0(y_{i0}^1)\} \\
 &+ \int_0^{y_{ij}^1} (y_{ij}^1 - t_j) F_j(dt_j) \cdot \{F_0^{m-1}(y_{i0}^1)\} \\
 &+ \sum_{l=1}^{m-2} \left\{ \int_0^{y_{i0}^1} \int_{\gamma_k(y)}^\infty \int_{\gamma_j(y)}^\infty \int_0^{y_{ij}^1 + \gamma_j(y) - t_j} [y_{ij}^1 + \gamma_j(y) - t_j - u] \right. \\
 &\times F_j^{m-l-1}(du) F_j(dt_j) F_k(dt_k) F_0^l(dt_0) \\
 &\left. + \int_0^{y_{i0}^1} \int_0^{\gamma_k(y)} \int_{\rho_j(y)}^\infty \int_0^{y_{ij}^1 + \rho_j(y) - t_j} [y_{ij}^1 + \rho_j(y) - t_j - u] F_j^{m-l-1}(du) \right.
 \end{aligned}$$

$$\begin{aligned} & \times F_j(dt_j)F_k(dt_k)F_0^l(dt_0) \\ & + \int_0^{y_{i0}^1} \int_{\gamma_k(y)}^\infty \int_{\rho_j(y)}^{y_{ij}^1} [y_{ij}^1 - u] F_j^{m-l-1}(du) F_j(dt_j) F_k(dt_k) F_0^l(dt_0) \Big\} , \end{aligned} \tag{A10}$$

where

$$\begin{aligned} \gamma_j(y) &= \alpha_{ij}(y_{i0}^1 - t_0) , \\ \gamma_k(y) &= \alpha_{ik}(y_{i0}^1 - t_0) ; \quad k \neq j , \\ \rho_j(y) &= y_{i0}^1 - t_0 - t_k . \end{aligned}$$

The first-order partial derivative with respect to  $y_{ij}^1$  is

$$\begin{aligned} D_j Ex_{i+m-1,j}^m &= F_j^{m-1}(y_{ij}^1) \{1 - F_0(y_{i0}^1)\} + F_j(y_{ij}^1) F_0^{m-1}(y_{i0}^1) \\ & + \sum_{l=1}^{m-2} \left\{ \int_0^{y_{i0}^1} \int_{\gamma_k(y)}^\infty \int_{\gamma_j(y)}^\infty F_j^{m-l-1}(y_{ij}^1 + \gamma_j(y) - t_j) \right. \\ & \times F_j(dt_j) F_k(dt_k) F_0^l(dt_0) \\ & + \int_0^{y_{i0}^1} \int_0^{\gamma_k(y)} \int_{\rho_j(y)}^\infty F_j^{m-l-1}(y_{ij}^1 + \rho_j(y) - t_j) F_j(dt_j) F_k(dt_k) F_0^l(dt_0) \\ & \left. + \int_0^{y_{i0}^1} \int_{\gamma_k(y)}^\infty \int_{\rho_j(y)}^{\gamma_j(y)} F_j^{m-l-1}(y_{ij}^1) F_j(dt_j) F_k(dt_k) F_0^l(dt_0) \right\} , \end{aligned}$$

and

$$D_k Ex_{i+m-1,j}^m = 0 , \quad k \neq j , j \neq 0 .$$

It follows then that

$$D_j D_j Ex_{i+m-1,j}^m = \sum_{l=1}^{m-2} \left\{ \alpha_{ij} f_0^l(y_{i0}^1) F_j^{m-l}(y_{ij}^1) \right\} ,$$

$$+ (1 - \alpha_{ij}) \alpha_{ik} \int_0^{y_{i0}^1} f_k(\alpha_{ik} t_0) F_0^l(dt_0) F_j^{m-l}(y_{ij}^1) \Big\} \geq 0$$

and hence convexity is established. QED

*Proof of theorem 3*

Let  $\alpha_{ij} = 0$ , which implies that  $\alpha_{ik} = 1$ ,  $\gamma_j(y) = 0$ ,  $\rho_j(y) \leq \gamma_k(y)$ . After eliminating common terms, we ignore the outer integral, with respect to  $t_0$ , in (13) and (14) and note that the three inner integrals in the summation in (14) become

$$\begin{aligned} & \int_{\gamma_k(y)}^{\infty} \int_0^{\infty} F_j^{m-l-1}(y_{ij}^1 - t_j) F_j(dt_j) F_k(dt_k) \\ & + \int_0^{\gamma_k(y)} \int_{\rho_j(y)}^{\infty} F_j^{m-l-1}(y_{ij}^1 + \rho_j(y) - t_j) F_j(dt_j) F_k(dt_k) \\ & - \int_{\gamma_k(y)}^{\infty} \int_0^{\rho_j(y)} F_j^{m-l-1}(y_{ij}^1) F_j(dt_j) F_k(dt_k). \end{aligned}$$

By decomposing the first integral above into two integrals, we get

$$\begin{aligned} & \int_{\gamma_k(y)}^{\infty} \int_0^{\rho_j(y)} F_j^{m-l-1}(y_{ij}^1 - t_j) F_j(dt_j) F_k(dt_k) \\ & + \int_{\gamma_k(y)}^{\infty} \int_{\rho_j(y)}^{\infty} F_j^{m-l-1}(y_{ij}^1 - t_j) F_j(dt_j) F_k(dt_k) \\ & + \int_0^{\gamma_k(y)} \int_{\rho_j(y)}^{\infty} F_j^{m-l-1}(y_{ij}^1 + \rho_j(y) - t_j) F_j(dt_j) F_k(dt_k) \\ & - \int_{\gamma_k(y)}^{\infty} \int_0^{\rho_j(y)} F_j^{m-l-1}(y_{ij}^1) F_j(dt_j) F_k(dt_k) \end{aligned}$$

$$\begin{aligned} &\leq \int_0^\infty \int_{\rho_j(y)}^\infty F_j^{m-l-1} (y_{ij}^1 + (\rho_j(y))^+ - t_j) F_j(dt_j) F_k(dt_k) \\ &\leq \int_0^\infty \int_{\rho_j(y)}^\infty F_j^{m-l-1} (\beta_j(y)) F_j(dt_j) F_k(dt_k), \end{aligned}$$

which is the inner integral in the summation in (13).

The final inequality follows from the monotonicity of  $F_j(\cdot)$  and,

$$\beta_j(y) \geq y_{ij}^1 + (\rho_j(y))^+ - t_j$$

since

$$\frac{t_j}{t_j + t_k} (y_0 - t_0) \geq (y_0 - t_0 - t_k)^+$$

if and only if

$$y_0 - t_0 - t_k = \rho_j(y) \leq t_j,$$

which is true by the region of integration.

Given the convexity of the expected over-age stock function for each allocation policy and the equality of the expected shortage function, the above relationship is sufficient to prove that  $y_{II}^*(0) \geq y_I^*$ .

We now let  $\alpha_{ij} = 1$  and show by a similar argument that  $y_{II}^*(0) \geq y_{II}^*(1)$ . Looking at the three inner integrals with  $\alpha_{ij} = 1$  we see that

$$\begin{aligned} &\int_0^\infty \int_{\gamma_j(y)}^\infty F_j^{m-l-1} (y_{ij}^1 + \gamma_j(y) - t_j) F_j(dt_j) F_k(dt_k) \\ &+ \int_0^\infty \int_{\rho_j(y)}^{\gamma_j(y)} F_j^{m-l-1} (y_{ij}^1) F_j(dt_j) F_k(dt_k) \\ &\geq \int_0^\infty \int_{\rho_j(y)}^\infty F_j^{m-l-1} (y_{ij}^1 + \rho_j(y) - t_j) F_k(dt_k) \end{aligned}$$

and hence the lower bound for  $\alpha_{ij} = 0$  is the upper bound for  $\alpha_{ij} = 1$ , which establishes that  $y_{II}^*(0) \geq y_{II}^*(1)$ .



Finally we note that

$$\begin{aligned}
& \int_0^{\infty} \int_{\gamma_j(y)}^{\infty} F_j^{m-l-1} (y_{ij}^1 + \gamma_j(y) - t_j) F_j(dt_j) F_k(dt_k) \\
& + \int_0^{\infty} \int_{\rho_j(y)}^{\gamma_j(y)} F_j^{m-l-1} (y_{ij}^1) F_j(dt_j) F_k(dt_k) \\
& \geq \int_0^{\infty} \int_{\rho_j(y)}^{\infty} F_j^{m-l-1} (y_{ij}^1) F_j(dt_j) F_k(dt_k) \\
& \geq \int_0^{\infty} \int_{\rho_j(y)}^{\infty} F_j^{m-l-1} (\beta_j(y)) F_j(dt_j) F_k(dt_k)
\end{aligned}$$

since

$$\frac{t_j}{t_j + t_k} (y_0 - t_0) \leq t_j$$

if and only if

$$y_0 - t_0 - t_k = \rho_j(y) \leq t_j,$$

which is true by the region of integration. Thus  $y_{\text{II}}^*(1) \leq y_{\text{I}}^* \leq y_{\text{II}}^*(0)$ .

We next observe that  $y_{\text{II}}(\alpha_{ij})$  is continuous in  $\alpha_{ij}$  since (14) is the sum of continuous functions of  $\alpha_{ij}$ . Hence there exists an  $\hat{\alpha}_{ij}$  such that  $y_{\text{I}}^* = y_{\text{II}}^*(\hat{\alpha}_{ij})$  and moreover  $\hat{\alpha}_{ij}$  is the root of (15). QED