Reprinted from BEHAVIORAL SCIENCE
Vol. 14, No. 5, September, 1969
Printed in U.S.A.

ANALYSIS OF A MULTISTAGE INVENTORY TASK: COMMENTS ON A PAPER BY AMNON RAPOPORT

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In an interesting paper in *Behavioral Science*, Volume 12 (1967), Amnon Rapoport reported on his tests of the dynamic decision-making capabilities of a group of students. He found that the students did not perform well relative to the normative results predicted by two quantitative inventory models he had chosen. It is the purpose of this paper to point out that the wrong normative models were chosen and that if the appropriate inventory model had been picked, the students may have performed reasonably well.

IN AN interesting paper (1967), Amnon Rapoport tested and analyzed the dynamic decision-making capabilities of a group of students. The subjects were sequentially given several single product multistage inventory problems and were required to determine in each stage the inventory level on hand after ordering but before the occurrence of an unknown random demand. Their responses to these problems were compared with the optimal amount on hand after ordering as predicted by two inventory models of Arrow, Karlin, and Scarf (1958). The result of the experiment was that, in general, the dynamic policies chosen by the subjects did not compare favorably with the optimal policies given by the two models.

It is the purpose of this paper to point out that neither of the two models chosen by Rapoport reflects the inventory situations perceived by the students in the experiment, and that indeed if the appropriate inventory model is chosen, it is believed that on the average the students may have performed remarkably well in making near-optimal dynamic decisions.

RAPOPORT'S RESULTS: SUMMARY

Rapoport used two inventory models. One, call it Model I, is a single stage inventory model which yields the constant-level optimal policy y given by the solution to

$$\int_y^\infty \psi(\xi)\, d\xi = \frac{c+h}{h+p+r} \tag{1}$$

where c, h and p are the per unit costs of ordering, holding, and shortage of an item in inventory, r is the per unit revenue of an item sold, and $\psi(\xi)$ is the probability density function of demand. The second model, call it Model II, is an infinite stage inventory model which yields the constant-level optimal policy y given by the solution to

$$\int_y^\infty \psi(\xi)\, d\xi = \frac{c(1-\alpha)+h}{h+p+r-\alpha c} \tag{2}$$

where c is a discount factor $0 \leqq \alpha < 1$. In his paper Rapoport sets $\alpha = 1$ which yields the number y as the solution to

$$\int_y^\infty \psi(\xi)\, d\xi = \frac{h}{h+p+r-c}. \tag{3}$$

Technically this value for y is not the only optimal policy because by setting $\alpha = 1$ we incur infinite costs over our infinite horizon and all other y's also yield the same infinite costs. Thus all y's are optimal in that they minimize costs over an infinite horizon.

In the experiment, the students were given the problem of choosing the number of items to produce in order to bring the inventory of the item up to some level. They were told that the number of decisions (that is, the number of stages) is unknown, and the demand in each stage for each problem is an unknown independent identically distributed random variable. Also they were informed that they would be given a sequence of such problems; they were in fact given six. For this work each received \$1.50 + \$1/5000 of the total profit obtained by his decisions.

Thus the inventory model primarily considered by Rapoport was the infinite stage model with critical number y, given in (3). However, the actual model facing the student was one where he knew that there would only be a *finite* number of stages for each problem due to the impossibility of conducting an infinite stage experiment. Furthermore, considering the fact he would play several games for \$1.50+, he obviously surmised that no game could last for many stages. Rapoport indicates that some of the students thought the number of stages would be around 15.

Some of the conclusions resulting from this experiment were:

(1) For each problem the two inventory models predict a constant level of inventory for all stages. These constants are denoted by $y^*(EV)$ for Model I and $y^*(DP)$ for Model II. $y^*(EV)$ and $y^*(DP)$ are derived from Equations (1) and (3) respectively. Let $\bar{y}_i = \sum_{k=1}^{N} y_{ik}/N$ where N = number of students tested and y_{ik} is the inventory level for student k in stage i. Thus $\bar{y}_i$ is the mean inventory level in period i for all students for a particular problem. Rapoport shows that for Problems Two, Three and Six, the average student decisions $\bar{y}_i$ show a significant linear and nonlinear decreasing trend over the stages of the problems. For Problem Five the nonlinear effect is also significant. He concludes that the students violate the two inventory models which predict that there should be no linear decreasing or nonlinear effects of the $\bar{y}_i$ on i. Indeed the violation of the models is apparent from his graphs

(2) There was significant correlation for each of Problems Three through Six between $\bar{y}_i$ and the demands ξ_j $j = i-1, i-2, \cdots, 2, 1$ in the preceding $i-1$ stages. Each of the inventory models predicts that there is zero correlation between the inventory levels and the preceding demands.

(3) In the experiment the students were divided into two groups—Group A consisted of students with some prior knowledge of the demand distribution $\Psi(\xi)$ and Group B consisted of students with no prior knowledge. The results showed that after a few trials (approximately 2 problems) there is no significant difference between Group A's decisions and Group B's decisions.

TWO ALTERNATIVE MODELS

The next two models to be presented, III and IV, are more representative of the actual problems faced by the students. Model III was developed in a paper by Bellman, Glicksberg, and Gross (1955). Since it is a special case of Model IV—to which more time is devoted—it will only be presented briefly.

Model III

Model III is the single-product n-stage inventory model with convex costs and independent identically distributed random demands in each stage. If the stages are numbered backward from $n, n-1, \cdots, 2, 1$ the minimum discounted expected costs for stage j given a starting inventory of x are

$$f_j(x) = min_{y \geqq x} \{c\cdot(y-x) + L(y) + \alpha f_{j-1}(0)\int_y^\infty \psi(\xi)\, d\xi + \alpha \cdot \int_0^y f_{j-1}(y-\xi)\psi(\xi)\, d\xi\} \tag{4}$$

where

$$L(y) = h\int_0^y (y-\xi)\psi(\xi)\, d\xi - r\int_0^\infty \xi\psi(\xi)\, d\xi + (p+r)\int_y^\infty (\xi-y)\psi(\xi)\, d\xi$$

and $f_0(\cdot) \equiv 0$.

PROBLEM 2

Period	20	19	18	17	16	15	14	13	12	11	10	9	8	7	6	5	4	3	2	1
$y^*(EV)$	254	254	254	254	254	254	254	254	254	254	254	254	254	254	254	254	254	254	254	254
$y^*(DP)$	304	304	304	304	304	304	304	304	304	304	304	304	304	304	204	304	304	304	304	304
y^*(Model III)	304	304	304	304	304	304	304	304	304	304	304	304	304	304	303	303	303	303	303	254
$y^*(a)$	304	304	304	304	304	304	298	298	297	296	296	295	294	293	291	289	286	281	273	254
$y^*(b)$	304	304	304	304	304	303	304	304	299	289	285	281	277	274	270	267	263	260	257	254
$y^*(c)$	304	304	304	304	304	304	304	304	304	304	303	303	303	303	302	301	299	293	281	254
$y^*(d)$	304	304	304	304	303	303	293	293	292	291	290	289	288	286	284	281	278	273	266	254

PROBLEM 3

Period	20	19	18	17	16	15	14	13	12	11	10	9	8	7	6	5	4	3	2	1
$y^*(EV)$	277	277	277	277	277	277	277	277	277	277	277	277	277	277	277	277	277	277	277	277
$y^*(DP)$	334	334	334	334	334	334	334	334	334	334	334	334	334	334	334	334	334	334	334	334
y^*(Model III)	333	333	333	333	333	333	333	333	333	333	333	333	333	333	333	333	333	333	333	277
$y^*(a)$	333	333	333	333	333	333	328	326	326	325	324	324	322	321	319	316	313	307	298	277
$y^*(b)$	333	333	333	333	333	333	333	333	328	315	311	307	303	299	295	291	287	284	281	277
$y^*(c)$	333	333	333	333	333	333	333	333	333	333	333	333	333	333	333	329	328	321	307	277
$y^*(d)$	333	333	333	333	333	333	321	321	320	319	318	316	315	313	310	308	303	298	290	277

PROBLEM 4

Period	20	19	18	17	16	15	14	13	12	11	10	9	8	7	6	5	4	3	2	1
$y^*(EV)$	296	296	296	296	296	296	296	296	296	296	296	296	296	296	296	296	296	296	296	296
$y^*(DP)$	388	388	388	388	388	388	388	388	388	388	388	388	388	388	388	388	388	388	388	388
y^*(Model III)	387	387	387	387	387	387	387	387	387	387	387	387	387	387	387	387	387	387	387	296
$y^*(a)$	387	387	387	387	387	387	370	369	369	366	364	363	360	360	354	351	344	336	323	296
$y^*(b)$	387	387	387	387	387	387	387	387	372	351	342	335	329	324	319	314	309	304	300	296
$y^*(c)$	387	387	387	387	387	387	387	387	387	387	387	387	387	387	381	378	370	360	336	296
$y^*(d)$	387	387	387	387	387	387	360	360	360	354	352	351	347	344	342	336	329	323	313	296

PROBLEM 5

Period	20	19	18	17	16	15	14	13	12	11	10	9	8	7	6	5	4	3	2	1
$y^*(EV)$	298	298	298	298	298	298	298	298	298	298	298	298	298	298	298	298	298	298	298	298
$y^*(DP)$	358	358	358	358	358	358	358	358	358	358	358	358	358	358	358	358	358	358	358	358
y^*(Model III)	360	360	360	360	360	360	360	360	360	360	360	360	360	360	360	360	360	360	360	299
$y^*(a)$	360	360	360	360	360	360	351	351	351	347	347	345	344	343	342	342	334	329	320	299
$y^*(b)$	360	360	360	360	360	360	360	360	351	337	333	329	325	320	316	312	309	305	302	299
$y^*(c)$	360	360	360	360	360	360	360	360	360	360	360	360	360	360	360	353	351	343	329	299
$y^*(d)$	360	360	360	360	360	360	343	343	342	342	342	342	336	334	333	329	325	320	311	299

PROBLEM 6

The results for this problem are the same as for problem #6 since the costs functions are identical.

The optimal policy for this model is a single critical number y_j^*, for each stage j. y_j^* is the solution to:

$$c + h - (h+p+r) \int_y^\infty \psi(\xi)\, d\xi + \alpha \int_0^y f'_{j-1}(y-\xi)\psi(\xi)\, d\xi = 0, \tag{5}$$

where $f'(z) = \dfrac{df(z)}{dz}$.

Actually the easiest way to obtain y_j^* is to use either Fibonacci or Lattice search on the expression in the brackets in (4) (see Wilde, 1964).

An interesting characteristic of this model is that the stage by stage critical numbers y_j^*, $j = 1, \cdots, n$ are nondecreasing. That is,

$$y^*(EV) = y_1^* \leqq y_2^* \leqq \cdots \leqq y_{n-1}^* \leqq y_n^* \leqq y^*(DP). \tag{6}$$

It is easy to see the implications of (6) in relation to the students' performance. As the experiment begins, y_n^* is the largest critical number chosen. As the experiment proceeds, nonincreasing numbers y_j^* will continue to be chosen until Stage 1 is reached —at which point y_1^* is chosen and the process terminates at the next step. In general this choice of critical numbers is consistent with the $\bar{y}_j$'s chosen by the students. However, the students did not know n! Because of this, Model IV is more representative of the situation they faced than either of the Models III, II or I.

Model IV

Model IV has the same assumptions as Model III except that it is now assumed that the length of the horizon n is not known but is a random variable with known (a priori) probability mass function Π_j, $j = 1, \cdots, M$. M represents the total number of stages and is the largest number chosen by the experimentee such that with probability one, the experimentee believes (or knows) that the problem will terminate. It is also assumed that the length of the horizon is independent of the distribution of demands, $\Psi(\xi)$. This model represents the problem facing the students. They were, in reality, faced with a finite-stage single product convex cost inventory problem where the experimenter had fixed the number of stages n in advance but to them this number n was an unknown random variable independent of the demands ξ_j in each stage.

In an early paper, Barankin and Denny (1960) presented the essential elements of Model IV; however, they did not develop its properties. Let w denote the period in which obsolescence occurs and let p_r be the conditional probability that the problem terminates in stage $r \leqq t \leqq M$, given that the problem has not terminated in stages $M, M-1, \cdots, t+1$. Then

$$P_r = P\{w = r \mid w \leqq t\}$$

$$(7) \qquad = \frac{P\{w = r \text{ and } w < t\}}{P\{w \leqq t\}}$$

$$= \frac{\Pi_r}{\sum_{k=1}^{t} \Pi_k}; \; r = 1, \cdots, t.$$

Of course $1 - p_r$ is the conditional probability that the problem will not terminate in stage r, given it has survived to stage t.

The optimal critical numbers y_j^* may be obtained by solving the minimum conditional expected cost function

$$\bar{f}_j(x) = min_{y \geqq x} \{c \cdot (y - x)$$

$$(8) \qquad + L(y) + \alpha(1 - p_j)$$

$$E[\bar{f}_{j-1}(max(0, y - \xi))]\},$$

where $L(y)$ is the same as before and

$$E[\bar{f}_{j-1}(max(0, y - \xi))] = \bar{f}_{j-1}(0)$$

$$\cdot \int_y^\infty \psi(\xi)\, d\xi + \int_0^y \bar{f}_{j-1}(y - \xi)\psi(\xi)\, d\xi.$$

The y_j^* is obtained by differentiating the term in brackets in (8) and setting the resulting expression to zero. Since the term in brackets is convex in y the resulting y_j^* yields the minimum. If it is assumed that the conditional probabilities $(1 - p_j)$ are an increasing function of j or equivalently that the Π_j is an increasing failure rate distribution then it is not difficult to show that

$$(9) \qquad y^*(EV) = y_1^* \leqq y_2^* \leqq \cdots \leqq y_n^* \leqq \cdots \leqq y^*(DP).$$

It should be noted that Model III is a special case of Model IV with $\Pi_1 = 1$ and $\Pi_j = 0, j = 2, \cdots, n$.

Is it reasonable to assume that $1 - p_j$ is an increasing function of j? To a student faced with the previously described experiment it could easily be reasonable since the following distributions all have the required property:

(*a*) the equally likely distribution

$$\Pi_j = \begin{cases} \dfrac{1}{T} & \text{for} \quad j = 1, \cdots, T \\ 0 & \text{otherwise.} \end{cases}$$

(*b*) the truncated Poisson distribution

$$\Pi_j = \begin{cases} \left(\dfrac{\lambda^{j-1}}{(j-1)!}\right) \cdot \left(\displaystyle\sum_{k=1}^{T} \dfrac{\lambda^{k-1}}{(k-1)!}\right)^{-1} & \\ \qquad \text{for} \quad j = 1, \cdots, T & \\ \qquad \lambda > 0 & \\ 0 \qquad \text{otherwise.} & \end{cases}$$

(*c*) the truncated geometric distribution

$$\Pi_j = \begin{cases} q^{j-1} \cdot \left(\displaystyle\sum_{k=1}^{T} q^{k-1}\right)^{-1} & \text{for } j = 1, \cdots, T \\ & 0 < q < 1 \\ 0 & \text{otherwise.} \end{cases}$$

(*d*) the discrete triangular distribution

$$\Pi_j = \begin{cases} j. \left(\displaystyle\sum_{k=1}^{T} k\right)^{-1} & \text{for } j = 1, \cdots, T \\ 0 & \text{otherwise.} \end{cases}$$

The number T is the number chosen by the student such that the probability of termination Π_T first becomes positive ($T \leqq M$).

Any of the probability mass functions (*a*), (*b*) or (*d*) might be a reasonable prior distribution for the student to choose. (*c*) would not be reasonable since it geometrically weights the final periods heavier than the earlier periods and would not coincide with some students' prediction of termination in about 15 periods. Probability mass function (*c*) was included in this study to give a more complete analysis of the behavior of the critical numbers under different distributions for the obsolescense random variable, n.

Whether or not any students picked (consciously or otherwise) any of these prior distributions is not and cannot be known. It is the author's opinion that these distributions represent the types of distributions which persons bring to uncertain situations of this type; however this opinion has not been statistically validated.

OPTIMAL POLICIES FOR MODELS III AND IV

In keeping with Rapoport's decision to draw his samples for a normal distribution with mean 251.51 and standard deviation 77.58 but to restrict the actual values to integers between 000 and 999, the following computations were obtained[1] using the same normal distribution but ignoring the probability of a sample in the range $-\infty$ to 0. That is $P\{X \leqq 0\} \doteq .00059$ was ignored. Furthermore the search for the optimal policies in each stage y_j^* was confined to the integers 000 to 400 and lattice search was used. For these two reasons the actual y_j^* obtained vary a slight amount from the optimal policies obtained by differentiating the expressions in brackets in (4) and (8) for Models III and IV respectively.

For Model IV the four distributions listed in section 3 above were used. These four distributions will be denoted by a, b, c and d corresponding to their lettering in section 3. The numbers M and T were chosen as $M = 20$ and $T = 14$; thus for the first six stages, numbered 20, 19, $\cdots$, 15, the probability the process will terminate is zero. This assumption that the problem will last at least six stages without terminating is completely arbitrary but tends to go along with some student comments that they expected the problem to last about 15 stages. $M = 20$ was chosen arbitrarily also as a maximum limit a student would believe the problem to run. Other assumptions on M and T could be made but with the types of distributions given by a, b, c and d the qualitative results would be about the same. That is, we will

[1] The computation weie made on the Case Institute of Technnlogy digital computer UNIVAC 1v07.

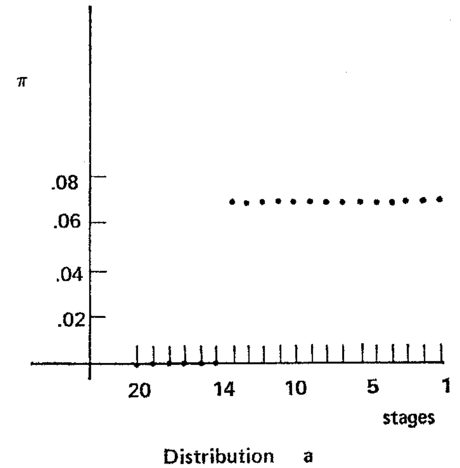

FIGURE 1

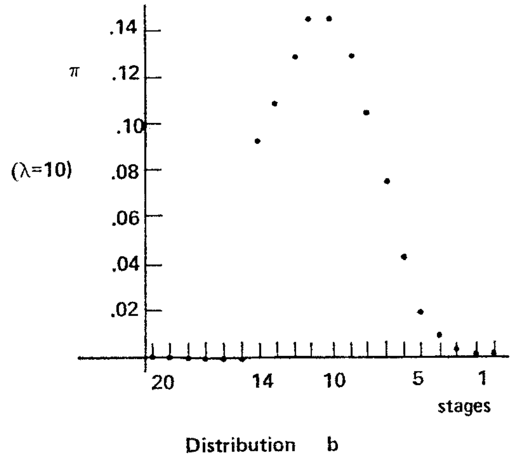

FIGURE 2

find that the optimal critical numbers decrease from y_{20}^* to y_1^* in an interesting manner. Obviously it would be interesting to check other values of $M > 20$; however if T is unchanged the additional values of y_j^* would be just the numbers $y^*(DP)$ since the steady state is reached prior to the 20th period (numbering backwards).

For distributions a, b, c and d Figures 1, 2, 3 and 4 indicate the choice of parameters and the behavior of the π distributions over the 20 stage horizon.

As mentioned earlier the critical numbers obtained using Model IV and distributions a, b, c and d are quite interesting. These results appear in Figures 5–8. (The actual critical numbers are tabulated at the end of this article.) An examination of the results shown in these figures, reveals that had the students known the actual termination stage n, they would have used Model III. The results of Model III essentially show that the best policy is $y^*(DP)$ up to the terminal stage and $y^*(EV)$ is optimal for the terminal stage. That is, the critical numbers given in (6) rapidly achieve the steady state number $y^*(DP)$.

But we know that the students did not know n. Furthermore, given either of the priors b or d (the author believes that b

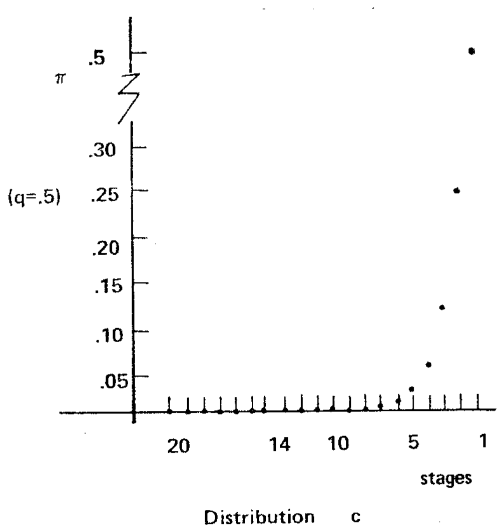

FIGURE 3

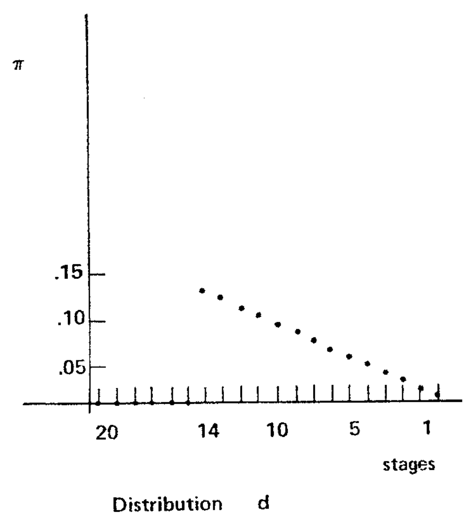

FIGURE 4

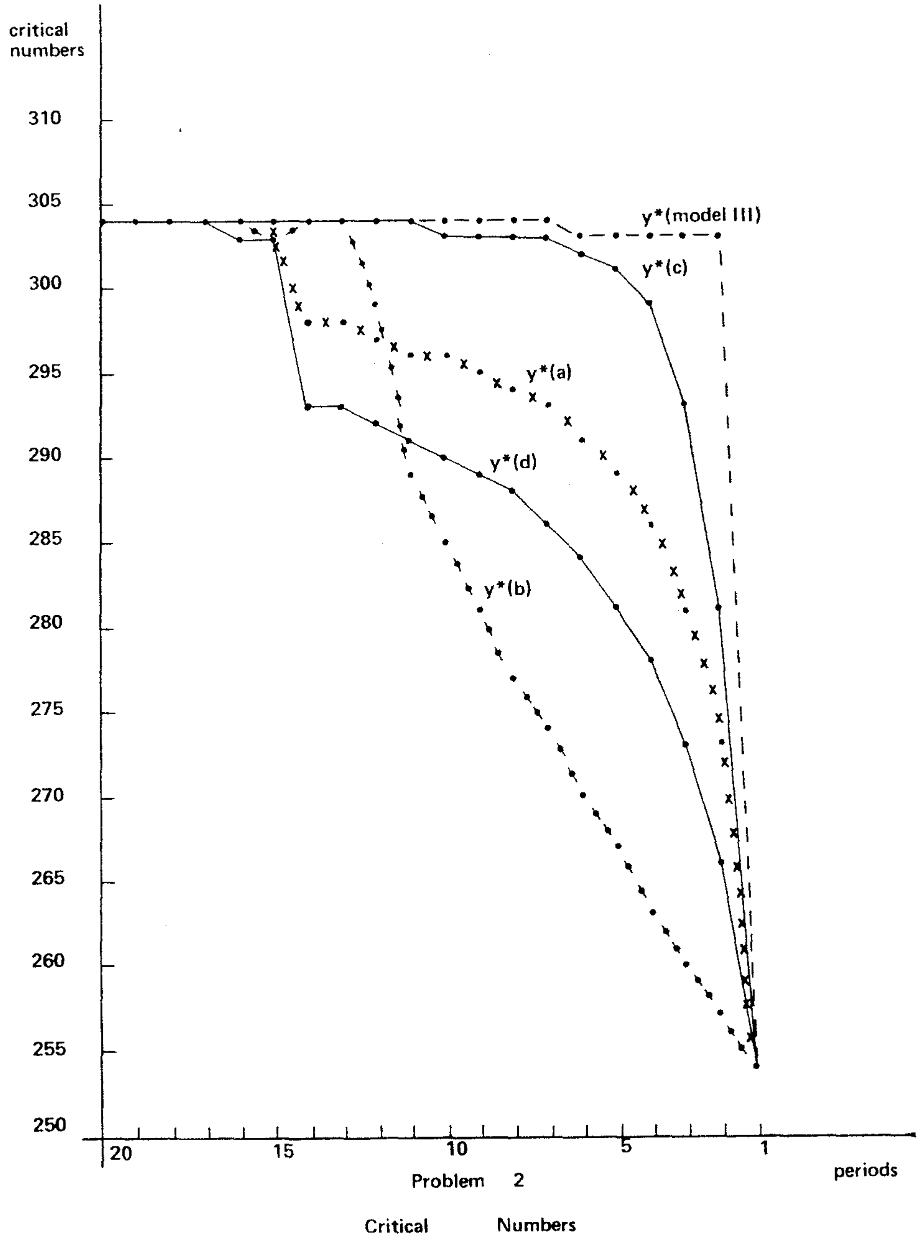

FIGURE 5

and d appear the most likely priors which would be chosen from the list of four priors given previously) the critical numbers $y^*(b)$ and $y^*(d)$ decrease rather rapidly after the probabilities Π_j are introduced at a positive level.

RESULTS

The results $y^*(b)$ and $y^*(d)$ have been superimposed on Rapoport's Figures 1, 2, 3, 4 and 5 of $\bar{y}_j$ for each problem. It should be pointed out again that perhaps none of the distributions a, b, c, and d reflect the distributions brought to the situation. Hence in this regard it may be misleading to plot $y^*(b)$ and $y^*(d)$ against the students' average choices since this would imply that the *average* prior was $y^*(b)$ or $y^*(d)$, whereas the average prior could conceivably be quite different. The plotting was done not to suggest that b or d was the average of the actual students' choices but to show that

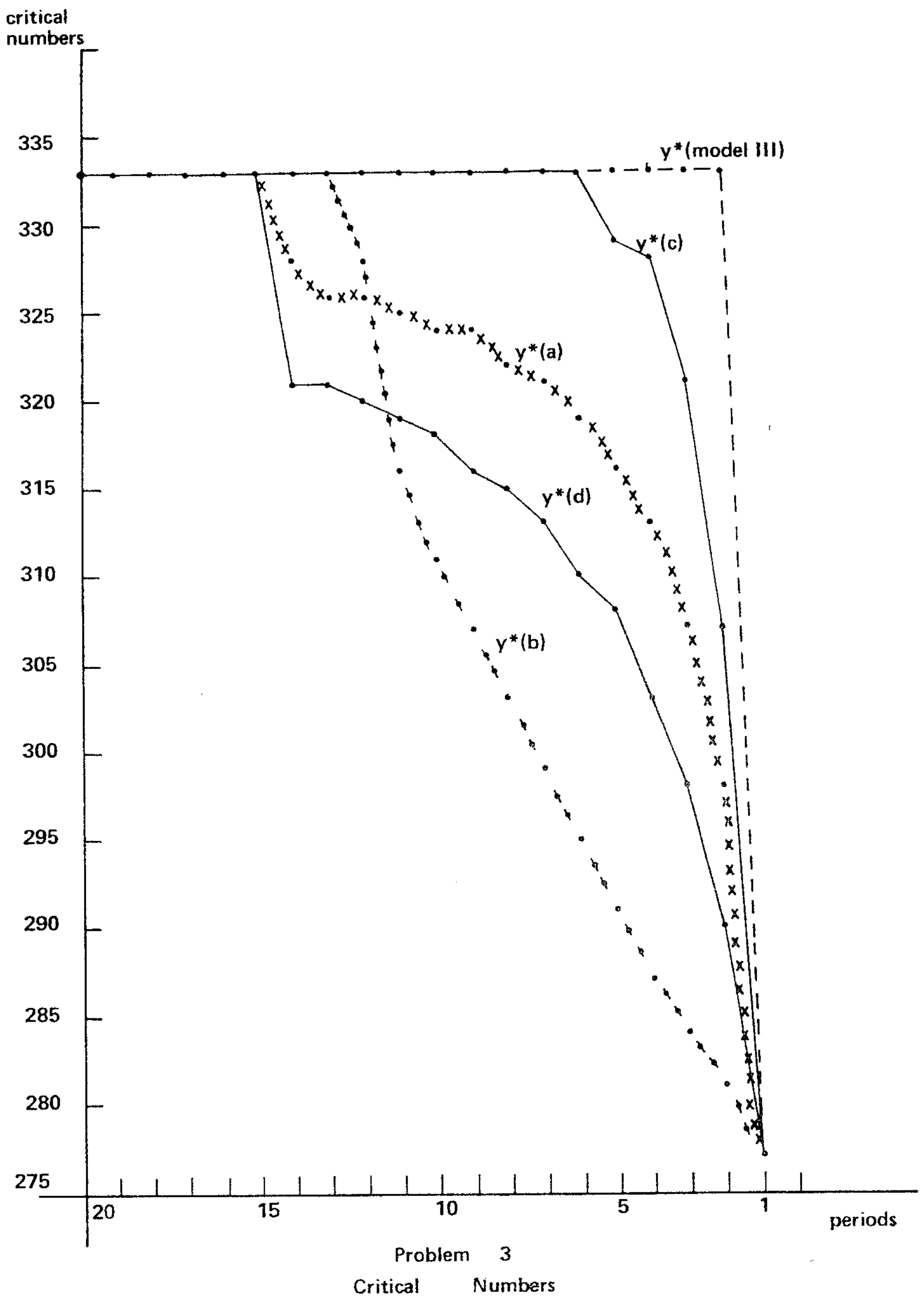

FIGURE 6

there are distributions available which could reasonably have been selected by the subjects and which yield a decreasing choice of y_j^*'s. Indeed, it is mathematically possible to find a prior distribution which would very closely fit the subjects' data but that would just be fitting the model to the data and is unjustified in a study of this nature.

Now note that the $\bar{y}_j$'s in general depict a convex decreasing trend, whereas the $y^*(b)$ and $y^*(d)$ show a concave decreasing trend. Thus the average student choice, although decreasing as appropriate, does not decrease at an appropriate rate for the choice of priors b and d. We can conclude that either the students failed to comprehend the underlying dynamic stochastic process thoroughly or the choice of priors b and d were

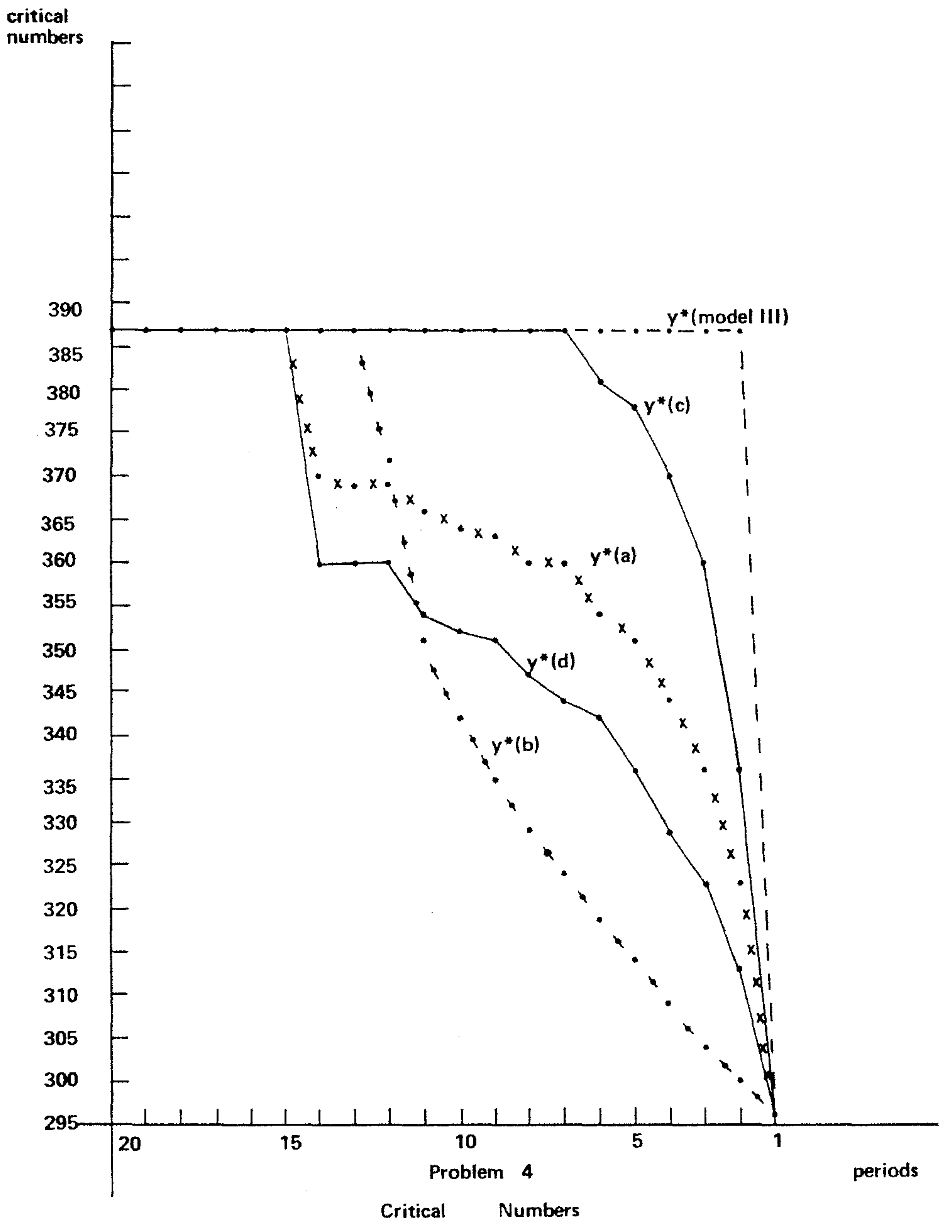

FIGURE 7

wrongly chosen to be representative of their priors, or both. From the available data it is difficult to say which conclusion is more accurate—most likely, it is some combination of the two. Furthermore, as noted by Rapoport the aggregate $\bar{y}_j$'s are too low. If the $\bar{y}_j$ curves were shifted upward, they would represent better choices.

Both Model III and Model IV predict that there should be zero correlation between $\bar{y}_i$ and the demands for the stages preceding stage i. Thus, Rapoport's conclusion that the students did not react appropriately here in relation to the problems they faced still holds for these models. In addition, the fact that Group B students adapted to their original lack of knowledge of $\Psi(\xi)$ rather rapidly is also independent of the choice of Models III or IV, and Rapoport's conclusions in this regard remain unchanged.

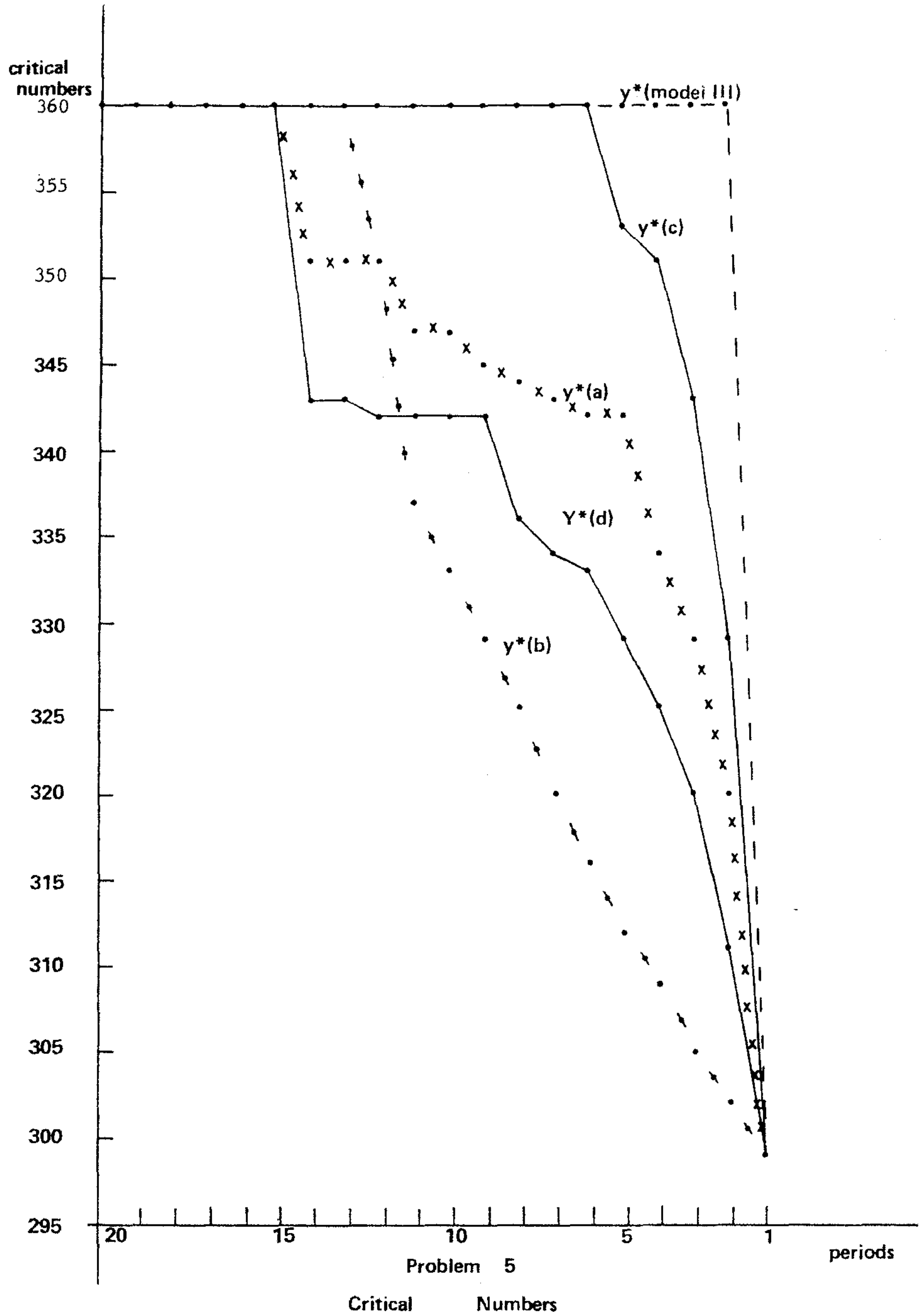

FIGURE 8

CONCLUSION

It appears that the students did not accept the infinite period model but rather that they substituted a model of the form of Model IV for their decision making. With regard to their aggregate choices $\bar{y}_j$, the author believes that they acted very dynamically and impressively due to the fact that they had little or no prior knowledge of inventory theory. The $\bar{y}_j$ curves decrease as the model predicts; however, they decrease at the wrong rate for some choices of prior distributions, and are in general deflated.

This paper suggests that work needs to be done on second-order conditions on the

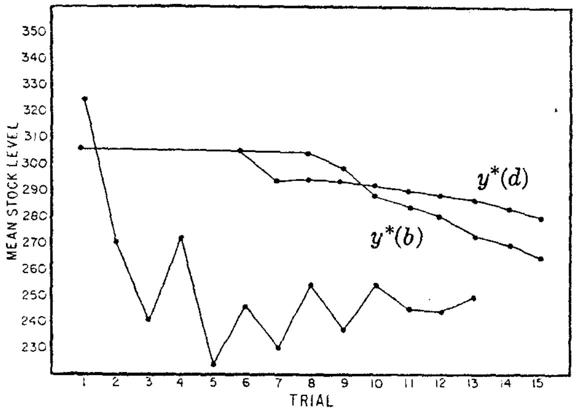

FIGURE 9
Mean Stock Level Plotted as a Function of Trial Number for Problem 2.

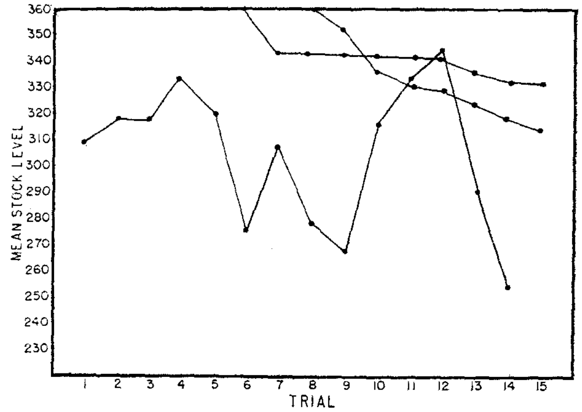

FIGURE 12
Mean Stock Level Plotted as a Function of Trial Number for Problem 5.

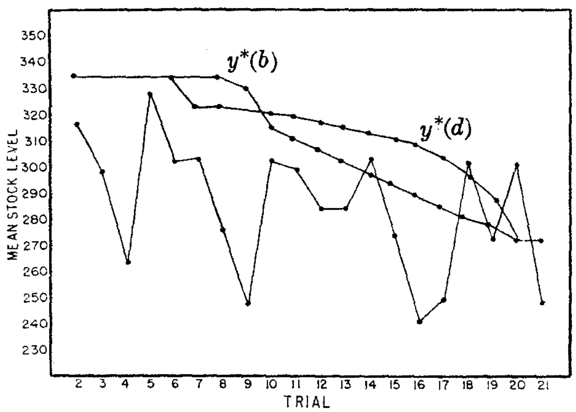

FIGURE 10
Mean Stock Level Plotted as a Function of Trial Number for Problem 3.

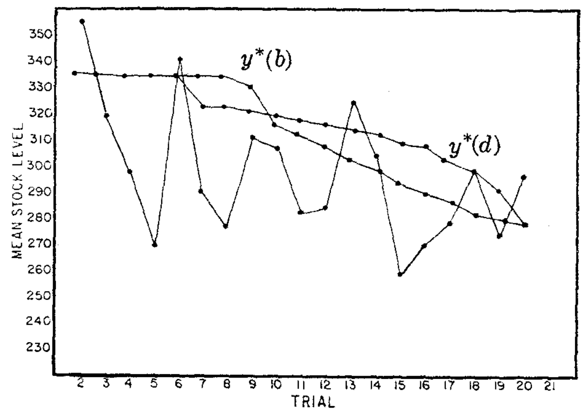

FIGURE 13
Mean Stock Level Plotted as a Function of Trial Number for Problem 6.

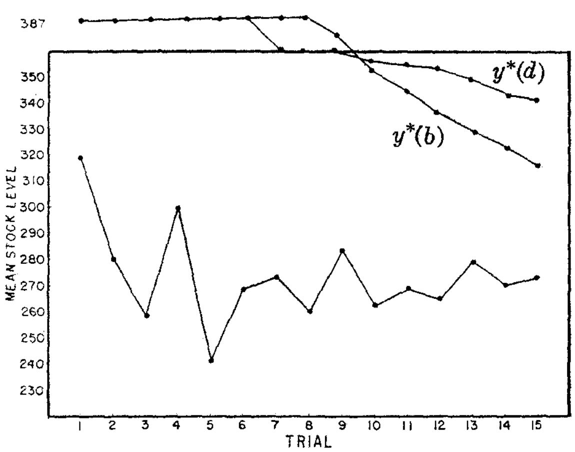

FIGURE 11
Mean Stock Level Plotted as a Function of Trial Number for Problem 4.

rate of change of the subjects response to problems, and on the prior distributions the subjects bring to problems. Another interesting question is why the $\bar{y}_j$'s are deflated in relation to the optimal y^*'s. Is it because the subjects have a very different and unusual prior distribution of the terminal point of the problem, or is it because the subjects are essentially conservative in a testing environment of this nature, or some other reasons entirely? The foregoing suggests that more research needs to be conducted on the topic of whether people can and do think dynamically.

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(Manuscript received December 16, 1968)

Certainly, style is not affectation. Conscious though it may be, when self-conscious it is an obstruction. Its purpose, to my way of thinking, is to give the reader pleasure by sparing him the work which the writer is duty-bound to have done for him. Writers, notwithstanding their hopes or ambitions, may or may not be artists. But there is no excuse for their not being artisans. The style is the man, we are told. True in the final and spiritual sense as this is, style is more than that. It is the writing man *in print*. It is, so to speak, his written voice and, if it is truly his voice, even in print it should be his and his alone. The closer it comes to the illusion of speech, perhaps the better. Yet the closeness of the written word to the spoken can, and in fact should, never be more than an illusion. For the point of the written word is planning, as surely as the charm of the spoken one is its lack of it.

JOHN MASON BROWN, *Still Seeing Things*