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OPTIMAL ORDERING POLICIES FOR A PRODUCT THAT PERISHES IN TWO PERIODS SUBJECT TO STOCHASTIC DEMAND*

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ABSTRACT

This paper considers the problem of computing optimal ordering policies for a product that has a life of exactly two periods when demand is random. Initially costs are charged against runouts (stockouts) and outdating (perishing). By charging outdating costs according to the expected amount of outdating one period into the future, a feasible one period model is constructed. The central theorem deals with the n-stage dynamic problem and demonstrates the appropriate cost functions are convex in the decision variable and also provides bounds on certain derivatives. The model is then generalized to include ordering and holding costs. The paper is concluded with a discussion of the infinite horizon problem.

INTRODUCTION

Although some researchers feel that the single product inventory problem has been completely solved, there is one very important aspect to the problem which has received little attention in the literature; specifically, optimal ordering policies for a perishable product. Veinott [8] considered a somewhat restricted class of deterministic models in his dissertation. Bulinskaya [2] considered the case where the product perished at the end of the period in which it was ordered so that he was able to utilize the standard single product model. The most salient analysis done on this problem was that of Van Zyl [7]. The approach taken by Van Zyl was to charge only ordering and runout costs. Because of outdating, the average inventory entering each period would be less so that orders would be larger, thus in some sense accounting for perishing costs. Our approach will be to charge directly a cost for outdating so that reasonable one period results can be obtained and the cost of outdating can be made explicit.

Most of the effort involving a perishable product has been directed towards optimal issuing policies; however, from an applications point of view, the ordering problem is just as important. The farmer with the facility for stocking unsold produce is essentially faced with an ordering problem when deciding how much to plant. Hospitals that can order blood from a central blood bank face a similar problem. In addition, one can find applications of this problem to chemicals, radioactive elements, other aspects of the food processing industry, drugs, etc.

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Our motivation in this paper is to develop a model for describing ordering policies for the product that perishes in two periods in the hope of introducing a structure that can be extended to solve the general problem. Wherever ordering occurs exactly twice during the product's useful life, this model will be applicable.

In the next section, the nature of the model is described and explicit statements of the assumptions are given. These assumptions are discussed in some detail and the probabilistic processes describing the perishability of the product are given.

In the third section, the existence of optimal ordering policies is established and many of its properties are demonstrated. Specialized results for the backlog case and the lost sales case are given as corollaries to the main theorems.

The fourth section is devoted to generalizing the results obtained to include ordering and holding costs as well. The introduction of a salvage value at the end of the horizon induces a type of stationarity on the model which is analogous to the single critical number obtained in Veinott [9].

In the last section, the results are extended to the discounted infinite horizon model.

THE NATURE OF THE MODEL—THE ONE PERIOD PROBLEM

Initially we will make the following assumptions. In the discussion which follows these assumptions, it will be pointed out which assumptions may be relaxed and which cannot be.

(1) All orders are placed at the start of the period and received instantaneously.

(2) All stock arrives new.

(3) Demands in each period are independent identically distributed nonnegative random variables with distribution function F and density f.

(4) Inventory is depleted according to a FIFO policy; that is, oldest first.

(5) Costs are charged linearly against

(a) unsatisfied demand (runouts) at r per unit

(b) deterioration (outdates) at θ per unit.

(6) If the product has not been depleted by the end of two periods then it deteriorates and must be discarded at a cost given in 5(b) above.

(7) Unsatisfied demand is backlogged.

Discussion of the Assumptions

(1) This assumption is standard for models which don't allow a delivery lag.

(2) This assumption is an important assumption for our model. As we will see, there is very little one can say when the stock that arrives is of mixed age.

(3) The fact that the demands are identically distributed is not an important assumption in the finite horizon problem. The assumption is made for notational convenience. It will be demonstrated that our results apply to nonstationary demand as well.

(4) This assumption is actually a consequence of the model and can easily be shown to be the optimal depletion policy. More general results of this type can be found in Pierskalla and Roach [5].

(5) We have assumed only two costs for the following reason: Consideration of a deterioration cost greatly complicates the model and for most perishable product problems deterioration and shortage costs are more important factors than holding and ordering costs. Again, though, this restriction is not crucial to the nature of the model and a more general cost structure will be considered.

(6) This assumption is the basic deterioration assumption.

(7) This assumption is not a crucial assumption. The model will also be applicable for the lost demand case, as will be indicated.

The state variable will be called x and will represent the amount of one period old product being brought into the period. The decision variable y represents the amount of new product being ordered. The first problem that must be considered is how to charge the costs. The obvious solution is charge costs against the expected cost incurred in the period so that

$$E\ [\text{Runouts}] = r\,E[D-(x+y)]^{+} = r\int_{x+y}^{\infty} (t-(x+y))dF(t)$$

$$E\ [\text{Outdates}] = \theta\,E[x-D]^{+} = \theta\int_{0}^{x} (x-t)dF(t),$$

where $f^{+} = \max\ (f, 0)$ and F is the demand distribution. Letting

$$L(x, y) = r\int_{x+y}^{\infty} [(t-(x+y)]\ dF(t) + \theta\int_{0}^{x} (x-t)\ dF(t),$$

we see that this method has the property that y does not appear in the outdating cost, so that the charge for out-dating is actually independent of our order. Hence the value of y that minimizes $L(x, y)$ for any value of x is $y = +\infty$.

The direct approach to charging outdates leads to somewhat undesirable first period results so that we must develop a different method for charging the outdating cost.

Although there are undoubtedly many ways to charge the outdate cost, e.g., we could start the process with two periods to go, instead we will charge this cost to what the expected outdates will be after one period into the future. In this way, we will actually be charging the outdating cost against the expected outdating of the present order y. This approach has several advantages: first, it is reasonable for a real problem since the future demands are not known with certainty; second, it is easily generalizable to the case where a product perishes in m periods (rather than just two); third, it leads to a mathematically sensible one period problem as opposed to an approach which charges the outdate cost on the expected outdates $E[x-D]^{+}$ at the end of one period.

Let D_1, D_2 represent random variables denoting demands in two successive periods. Then the total amount of outdating of our present order will be

$$Z = \{y - [D_2 + (D_1 - x)^{+}]\}^{+}.$$

This can be seen most easily by considering the two possibilities $D_1 \leqslant x$ and $D_1 > x$ separately. If $D_1 \leqslant x$, then the demand in the first period is satisfied totally by one period old stock (x) and $x - D_1$ outdates, so that only the demand D_2 depletes from y, and when $D_2 \geqslant y$ there will be no outdating, and when $D_2 < y$, $y - D_2$ will outdate. In the other case, when $D_1 > x$, the total demand depleting from y will be $D_1 + D_2 - x$.

THEOREM 1: The random variable Z has distribution function H given by

$$H_{x,y}(t) = 1 - \int_{0}^{y-t} F(u+x)\,f(y-t-u)du \qquad t \geqslant 0$$

$$= 0 \qquad t < 0$$

and

$$E(Z) = \int_0^y F(u+x)\, F(y-u)du.$$

PROOF: We wish to compute

$$P\{(y - [D_2 + (D_1 - x)^+])^+ \leq t\} = P\{y - (D_2 + (D_1 - x)^+) \leq t\}$$

for $t \geq 0$, $t < y$.

$$\begin{aligned}P\{y - [D_2 + (D_1 - x)^+] \leq t\} &= P\{y - (D_2 + (D_1 - x)^+) \leq t \cap D_1 \leq x\} \\ &\quad + P\{y - (D_2 + (D_1 - x)^+) \leq t \cap D_1 > x\} \\ &= P\{D_2 \geq y - t\} \cdot P\{D_1 \leq x\} + P\{D_1 + D_2 \geq y + x - t,\ D_1 > x\} \\ &= [1 - F(y-t)]F(x) + \int_x^\infty [1 - F(y - t + x - u)]f(u)du \\ &= F(x) - F(y-t)F(x) + (1 - F(x)) \\ &\quad - \int_x^{y-t+x} F(y - t + x - u)f(u)du,\end{aligned}$$

where

$$\begin{aligned}\int_x^{y-t+x} F(y-t+x-u)f(u)du &= \int_0^{y-t} F(y-t-v)f(v+x)dv \\ &= -F(y-t)F(x) + \int_0^{y-t} F(v+x)f(y-t-v)dv.\end{aligned}$$

The last equality results from a change of orders of integration. Hence, substituting above we get

$$P\{y - [D_2 + (D_1 - x)^+]^+ \leq t\} = \begin{cases} 1 & \text{if} \quad t > y \\ 1 - \int_0^{y-t} F(u+x)f(y-t-u)du & \quad 0 \leq t \leq y \\ 0 & \quad t < 0. \end{cases}$$

Since Z is a nonnegative random variable it follows that

$$\begin{aligned}E(Z) &= \int_0^\infty [1 - H_{x,y}(t)]dt = \int_0^y \int_0^{y-t} F(u+x)f(y-t-u)du\, dt \\ &= \int_0^y \int_0^v F(u+x)f(v-u)du\, dv\end{aligned}$$

$$= \int_0^y \int_v^y F(u+x) f(v-u)\,dv\,du = \int_0^y F(u+x) F(y-u)\,du. \qquad \text{Q.E.D.}$$

By utilizing this result, our one period expected cost function becomes

$$L(x, y) = r \int_{x+y}^{\infty} [t - (x+y)] f(t)\,dt + \theta \int_0^y F(t+x) F(y-t)\,dt.$$

Before examining the structure we've generated in detail, one can note a number of interesting properties immediately. Notice that $L(x, y)$ is not a function of $x+y$ as it is in the conventional models; the outdating cost differentiates between the inventory brought into the period and that ordered at the beginning of the period. Because of this property, the significant work on myopic policies by Veinott [8] and Bessler and Veinott [1] will not be applicable nor will single critical numbers or (s, S) policies be optimal.

THE DYNAMIC PROBLEM—FINITE HORIZON

The principle of optimality for this model takes the following form:

$$C_n(x) = \inf_{y \geq 0} \{L(x, y) + \alpha \int_0^{\infty} C_{n-1}(y - (t-x)^+) f(t)\,dt\},$$

where $C_n(x)$ has the usual interpretation of being the minimum expected cost given we have x on hand and there are n periods remaining in the horizon with $C_0(x) = 0$ for all x. Note that we are discounting costs $(0 < \alpha \leq 1)$. The form of the transfer function $s(x, y, t) = y - (t-x)^+$ is a consequence of assumptions that we deplete according to a FIFO policy and demand is backlogged.

Define $B_n(x, y) = L(x, y) + \alpha \int_0^{\infty} C_{n-1}(y - (t-x)^+) f(t)\,dt$ and let $y_n(x)$ be such that

$$C_n(x) = B_n(x, y_n(x)) = \min_{y \geq 0} \{B_n(x, y)\}.$$

Then the purpose of the next theorem will be to demonstrate the existence of $y_n(x)$ and to enumerate some of its properties. The results obtained will be applicable to the finite horizon problem where the length of the horizon is arbitrary. Apologies are in order for the length of the theorem, but the induction argument requires all nine steps to be proven simultaneously.

THEOREM 2: Assuming that (a) the demand density $f(x)$ is continuous for all $x > 0$, (b) $f(x) = 0$ for all $x < 0$, and (c) $f(x) > 0$ for $x > 0$. We then have

(1) $B_n(x, y)$ is convex in y for all fixed x and is strictly convex in y for a fixed x in a neighborhood of the global minimum.

(2) $\lim_{y \to 0} \frac{\partial B_n(x, y)}{\partial y} < 0$ and $\lim_{y \to +\infty} \frac{\partial B_n(x, y)}{\partial y} > 0$ for all x.

(3) There is a unique $y_n(x)$ given by the solution to $\frac{\partial B_n(x, y)}{\partial y} = 0$, and $y_n(x) \in (0, \infty)$. In addition $\frac{dy_n(x)}{dx} = y_n'(x)$ exists and is continuous for all x.

(4) $C_n'(x) = -\alpha C_{n-1}'(y_n(x)) F(x) - \theta F(y_n(x)) F(x)$.

(5) $-1 \leqslant y_n'(x) < 0$. In addition $y_n'(x) > -1$ if $x > 0$.

(6) $C_n(x)$ is twice continuously differentiable over $(-\infty, 0)$ and $(0, \infty)$. $C_n''(x)$ is continuous at $x = 0$ whenever $f(x)$ is continuous at $x = 0$.

(7) $\theta f(x) + \alpha C_n''(x) \geqslant 0$ for all x.

(8) $-\theta F(x) \leqslant C_n'(x) \leqslant 0$ for all x and $C_n'(x) = 0$ for all $x \leqslant 0$. In addition $C_n'(x) < 0$ for $x \in (0, +\infty)$ and $n \geqslant 1$.

(9) $\lim_{x \to +\infty} y_n(x) = 0$ and $\lim_{x \to +\infty} C_n'(x) = 0$.

Before proving this theorem, it is useful to describe what some of these results mean. Parts (8) and (9) tell us that the optimal cost function, $C_n(x)$, is constant for $x \leqslant 0$ and then decreases to zero as $x \to +\infty$. Similarly part (5) indicates the optimal ordering function, $y_n(x)$, is decreasing in x, but at a rate greater than -1 and by part (9), this function also asymptotically approaches zero as $x \to +\infty$.

PROOF: The proof is by induction.

Assume the theorem is true for $1, 2, \ldots, n-1$.

The case of $n = 1$ follows in the same manner as when n is arbitrary with $C_0 \equiv 0$, so that the logic need not be repeated.

(1)
$$B_n(x, y) = r\int_{x+y}^{\infty} [t - (x+y)]f(t)\,dt + \theta\int_0^y F(x+t)F(y-t)\,dt + \alpha\int_0^{\infty} C_{n-1}(y - (t-x)^+)f(t)\,dt$$

$$= r\int_{x+y}^{\infty} [t - (x+y)]f(t)\,dt + \theta\int_0^y F(x+t)F(y-t)\,dt + \alpha C_{n-1}(y)F(x) + \alpha\int_x^{\infty} C_{n-1}(x+y-t)f(t)\,dt$$

$$\frac{\partial B_n(x, y)}{\partial y} = -r[1 - F(x+y)] + \theta\int_0^y F(x+y-t)f(t)\,dt + \alpha C_{n-1}'(y)F(x) + \alpha\int_x^{x+y} C_{n-1}'(x+y-t)f(t)\,dt,$$

since by the inductive assumption on (6) and (8) C_{n-1}' exists and $C_{n-1}'(u) = 0$ for all $u \leqslant 0$.

(2)
$$\frac{\partial^2 B_n(x, y)}{\partial y^2} = rf(x+y) + \theta F(x)f(y) + \theta\int_0^y f(x+y-t)f(t)\,dt + \alpha C_{n-1}''(y)F(x) + \alpha\int_x^{x+y} C_{n-1}''(x+y-t)f(t)\,dt$$

$$= rf(x+y) + F(x)[\theta f(y) + \alpha C_{n-1}''(y)] + \int_x^{x+y} [\theta f(x+y-t) + \alpha C_{n-1}''(x+y-t)]f(t)\,dt$$

$$\geqslant 0,$$

since by the inductive assumptions on (6) and (7) we have C_{n-1}'' exists and $\theta f(u) + C_{n-1}''(u) \geqslant 0$ for all u.

Note that in the case where f has a jump at 0, the differentiation under the integral sign in the third and fifth terms above would normally not be permitted. However, since the jumps occur at the endpoints of the range of integration and f and C_n'' are bounded the operation is valid. (See Van Zyl [7] for a lemma which establishes this.)

$$\lim_{y\to 0^+}\frac{\partial B_n(x,y)}{\partial y}=\lim_{y\to 0^+}\Big\{-r[1-F(x+y)]+\theta\int_0^y F(x+y-t)f(t)\,dt$$
$$+\alpha C'_{n-1}(y)F(x)+\alpha\int_x^{x+y}C'_{n-1}(x+y-t)f(t)\,dt\Big\}$$
$$=-r[1-F(x)]<0 \qquad \text{since } F(x)<1,$$

and by the inductive assumptions on (6) and (8) C'_{n-1} is continuous and $C'_{n-1}(0)=0$.

$$\lim_{y\to+\infty}\frac{\partial B_n(x,y)}{\partial y}=\theta-\alpha\lim_{y\to+\infty}\int_x^{x+y}C'_{n-1}(x+y-t)f(t)\,dt$$

by the inductive assumption on (9)

$$\geqslant \theta+\alpha\lim_{y\to\infty}\int_x^{x+y}-\theta F(x+y-t)f(t)\,dt,$$

by the inductive assumption on (8)

$$\geqslant \theta-\alpha\theta\lim_{y\to\infty}\int_x^{x+y}f(t)\,dt=\theta-\alpha\theta(1-F(x))\geqslant 0.$$

Thus $B_n(x,y)$ is convex and $\lim_{y\to 0^+}\frac{\partial B_n(x,y)}{\partial y}<0$ and $\lim_{y\to\infty}\frac{\partial B_n(x,y)}{\partial y}>0$ imply $y_n(x)\epsilon(0,+\infty)$, where

$y_n(x)$ solves $\frac{\partial B_n(x,y)}{\partial y}=0$.

Now if $x<0$, then we must have $y_n(x)>|x|$ or else solving $\frac{\partial B_n(x,y)}{\partial y}=0$ we would obtain

$$-r[1-F(x+y_n(x))]+\theta\int_0^{y_n(x)}F(x+y_n(x)-t)f(t)\,dt+\alpha C_{n-1}(y_n(x))F(x)$$
$$+\alpha\int_x^{x+y_n(x)}C'_{n-1}(x+y_n(x)-t)f(t)\,dt=-r\neq 0,$$

which is impossible. Hence for all x, $f(x+y_n(x))>0$ and we see that in a neighborhood of $y_n(x)$

$$\frac{\partial^2 B_n(x,y)}{\partial y^2}>0$$

or $B_n(x,y)$ is strictly convex in y in a neighborhood of $y_n(x)$. Thus $y_n(x)$ is the unique global minimum of $B_n(x, y)$. Define $T_n(x, y) = \dfrac{\partial B_n(x, y)}{\partial y}$ for all $(x, y) \in E^2$. Then

$$(3) \quad T_n(x, y) = -r[1-F(x+y)] + \theta \int_0^y F(x+y-t)f(t)dt + \alpha C'_{n-1}(y)F(x) + \alpha \int_x^{x+y} C'_{n-1}(x+y-t)f(t)dt$$

is continuously differentiable on E^2 by the inductive assumption on (6). Now $(x, y_n(x))$ uniquely satisfies $T_n(x, y_n(x)) = 0$ and

$$\frac{\partial T_n(x, y_n(x))}{\partial y} = \frac{\partial^2 B_n(x, y_n(x))}{\partial y^2} > 0.$$

Hence by the implicit function theorem $y_n(x)$ is continuously differentiable in a neighborhood of x. But x was arbitrary in E^1 hence $y_n(x)$ is continuously differentiable for all x.

$$(4) \qquad C_n(x) = B_n(x, y_n(x))$$

$$\frac{dC_n(x)}{dx} = \left.\frac{\partial B_n(x, y)}{\partial x}\right|_{y=y_n(x)} + \left.\frac{\partial B_n(x, y)}{\partial y}\right|_{y=y_n(x)} \cdot y'_n(x)$$

$$= \left.\frac{\partial B_n(x, y)}{\partial x}\right|_{y=y_n(x)} \text{ since } \left.\frac{\partial B_n(x, y)}{\partial y}\right|_{y=y_n(x)} = 0.$$

Since $B_n(x, y_n(x)) = L(x, y_n(x)) + \alpha \int_0^\infty C_{n-1}[y_n(x) - (t-x)^+]f(t)dt$ it follows that

$$\frac{dC_n(x)}{dx} = \frac{\partial L(x, y)}{\partial x} + \alpha\, C_{n-1}(y)f(x) - \alpha\, C_{n-1}(y)f(x) + \alpha \int_x^\infty C'_{n-1}(x+y-t)f(t)dt \,\Big|_{y=y_n(x)}$$

$$= -r[1-F(x+y_n(x))] + \theta \int_0^{y_n(x)} f(x+y_n(x)-t)F(t)dt + \alpha \int_x^{x+y_n(x)} C'_{n-1}(x+y_n(x)-t)f(t)dt$$

by inductive assumption on (8).

But $y_n(x)$ satisfies

$$-r[1-F(x+y_n(x))] + \theta \int_0^{y_n(x)} F(x+y_n(x)-t)f(t)dt + \alpha C'_{n-1}(y_n(x))F(x)$$

$$+\alpha\int_x^{x+y_n(x)} C'_{n-1}(x+y_n(x)-t)f(t)\,dt=0.$$

Substituting, we obtain

$$\frac{dC_n(x)}{dx}=\theta\int_0^{y_n(x)}[f(x+y_n(x)-t)F(t)-F(x+y_n(x)-t)f(t)]\,dt-\alpha C'_{n-1}(y_n(x))F(x)$$

$$=-\theta F(y_n(x))F(x)-\alpha C'_{n-1}(y_n(x))F(x).$$

Since $y'_n(x)$ exists and since $T_n(x,y_n(x))=0$ for all x then

$$(5)\qquad \frac{\partial T_n(x,y_n(x))}{\partial x}=\frac{\partial^2 B_n(x,y_n(x))}{\partial x\,\partial y}$$

$$=rf(x+y_n(x))[1+y'_n(x)]+\theta y'_n(x)F(x)f(y_n(x))$$

$$-\theta\int_0^{y_n(x)} f(x+y_n(x)-t)[1+y'_n(x)]f(t)\,dt+\alpha C''_{n-1}(y_n(x))y'_n(x)F(x)$$

$$+\alpha\int_x^{x+y_n(x)} C''_{n-1}(x+y_n(x)-t)[1+y'_n(x)]f(t)\,dt,$$

by the inductive assumptions on (6) and (8) C''_{n-1} exists and $C'_{n-1}(0)=0$. Now since $y_n(x)>|x|$ if $x<0$ then $f(x+y_n(x))>0$ for all x; thus we can solve for $y'_n(x)$ to obtain

$$y'_n(x)=\frac{N(x,y_n(x))}{D(x,y_n(x))}\qquad\text{for all } x,$$

where

$$N(x,y_n(x))=-rf(x+y_n(x))+\int_x^{x+y_n(x)}[\theta f(x+y_n(x)-t)+\alpha C''_n(x+y_n(x)-t)]f(t)\,dt$$

<0 by inductive assumption on (7).

Also

$$D(x,y_n(x))=rf(x+y_n(x))+\theta F(x)f(y_n(x))+\int_x^{x+y_n(x)}[\theta f(x+y_n(x)-t)$$

$$+\alpha C''_{n-1}(x+y_n(x)-t)]f(t)\,dt+F(x)[\theta f(y_n(x))+\alpha C''_{n-1}(y_n(x))]$$

>0 by inductive assumption on (7).

Also since $-N(x,\ y_n(x))\leqslant D(x,\ y_n(x))$ it follows that $-1\leqslant y'_n(x)<0$. Notice that if $x>0$ the inequality is strict in both directions. We have that $y'_n(x)$ will exist and will be continuous by the inductive assumption on (6) (namely that C''_{n-1} exists and is continuous) so that

$$(6)\qquad C''_n(x)=\frac{d}{dx}[-\alpha C'_{n-1}(y_n(x))F(x)-\theta F(y_n(x))F(x)]$$

$$= -\alpha C''_{n-1}(y_n(x))y'_n(x)F(x) - \alpha C'_{n-1}(y_n(x))f(x)$$
$$-\theta f(y_n(x))y'_n(x)F(x) - \theta F(y_n(x))f(x)$$

is continuous.

$$\text{(7)} \qquad \theta f(x) + \alpha C''_n(x) = \theta f(x)[1 - \alpha F(y_n(x))] - \alpha^2 C'_{n-1}(y_n(x))f(x) - y'_n(x)\alpha F(x)[\theta f(y_n(x)) + \alpha C''_{n-1}(y_n(x))] \geqslant 0$$

since each term is nonnegative by the inductive assumptions on (7) and (8) and since $y'_n(x) < 0$.

$$\text{(8)} \qquad C'_n(x) = -\alpha C'_{n-1}(y_n(x))F(x) - \theta F(y_n(x))F(x) \geqslant -\theta F(y_n(x))F(x)$$

since $C'_{n-1}(x) \leqslant 0$, and

$$C'_n(x) \leqslant +\alpha\theta F(y_n(x))F(x) - \theta F(y_n(x))F(x) \leqslant 0$$

since $C'_{n-1}(x) \geqslant -\theta F(x)$.
Hence

$$-\theta F(x) \leqslant C'_n(x) \leqslant 0 \qquad \text{for all } x.$$

Assume to the contrary that $\lim_{x\to\infty} y_n(x) = w > 0$. (We know the limit exists since $y_n(x)$ is strictly decreasing and bounded below by zero.) Now we have for all x

$$\text{(9)} \qquad 0 = \lim_{x\to\infty}\Big\{-r[1 - F(x + y_n(x))] + \theta\int_0^{y_n(x)} F(x + y_n(x) - t)f(t)\,dt + \alpha C'_{n-1}(y_n(x))F(x) + \alpha\int_x^{x+y_n(x)} C'_{n-1}(x + y_n(x) - t)f(t)\,dt\Big\}$$

$$= \alpha C'_{n-1}(w) + \lim_{x\to\infty}\Big\{+\theta\int_0^{y_n(x)} F(x + y_n(x) - t)f(t)\,dt + \alpha\int_x^{x+y_n(x)} C'_{n-1}(x + y_n(x) - t)f(t)\,dt\Big\}$$

$$\geqslant \alpha C'_{n-1}(w) + \theta\lim_{x\to\infty}\int_0^{y_n(x)} F(x)f(t)\,dt + \alpha\lim_{x\to\infty}\inf_{u\in E} C'_{n-1}(u)\int_x^{x+y_n(x)} f(t)\,dt$$

since $F(x) \leqslant F(x + y_n(x) - t)$ for all $t \in [0, y_n(x)]$.

By inductive assumptions (6), (8), and (9), since C'_{n-1} is continuous on E^1 and $\lim C'_{n-1}(x) = 0$ and $C'_{n-1}(x) = 0$ for all $x \leqslant 0$ then $\inf_{x\in E^1} C'_{n-1}(x) \equiv K > -\infty$. Thus

$$\lim_{x\to\infty}\inf_{u} C'_{n-1}(u)\int_x^{x+y_n(x)} f(t)\,dt = K\lim_{x\to\infty}\{F(x + y_n(x)) - F(x)\} = 0.$$

We see then from above that

$$0 \geqslant \alpha C'_{n-1}(w) + \theta F(w) \geqslant -\alpha\theta F(y_{n-1}(w))F(w) + \theta F(w) = \theta F(w)[1 - \alpha F(y_{n-1}(w))] > 0$$

via the inductive assumption on (8), and the fact that $w > 0$. Hence we achieve an impossibility, so it must be the case that

$$\lim_{x\to\infty} y_n(x) = 0.$$

Now

$$\lim_{x\to\infty} C_n'(x) = \lim_{x\to\infty} [-\alpha C_{n-1}'(y_n(x))\, F(x) - \theta F(y_n(x))\, F(x)]$$

$$= -\alpha C_{n-1}'(0) - \theta F(0) = 0. \qquad \text{Q.E.D.}$$

There are some very novel features of this model that are apparent from the proof. In particular, in the standard multi-period dynamic model for a nonperishable product the method of demonstrating convexity is quite different. One shows inductively that C_n is convex in its argument, which implies that $\int_0^\infty C_n\, [\underset{\sim}{s}\,(z, t)]\, f(t)\, dt$ is convex so that $c(z-x) + L(z) + \int_0^\infty C_n\, (\underset{\sim}{s}\,(z, t))\, f(t)\, dt$ is convex, where $L(z)$ is the expected holding and shortage costs when z is the inventory on hand after ordering. In our model through, C_n, in general, will not be convex so that it was necessary to use different methods to demonstrate the convexity of G_n.

The following corollary shows that when demand is backlogged, the optimal order quantity is a function of only a single critical number.

COROLLARY 1: If $x < 0$, then $y_n(x) = y_n(0) + |x|$.

PROOF: We know $y_n(x)$ satisfies

$$-r[1-F(x+y_n(x))] + \theta \int_0^{y_n(x)} F(x+y_n(x)-t) f(t)\, dt + \alpha \int_0^\infty C_{n-1}'(y_n(x) - (t-x)^+) f(t)\, dt = 0.$$

For $x < 0$ this becomes

$$-r(1-F(x+y_n(x))) + \theta \int_0^{y_n(x)+x} F(x+y_n(x)-t) f(t)\, dt + \alpha \int_0^\infty C_{n-1}'(y_n(x)+x-t) f(t)\, dt = 0.$$

Assume the solution is $y_n(x) = y_n(0) + |x| = y_n(0) - x$. Then by substituting in the above proof—

$$-r[1-F(y_n(0))] + \theta \int_0^{y_n(0)} F(y_n(0)-t) f(t)\, dt + \alpha \int_0^\infty C_{n-1}'(y_n(0)-t) f(t)\, dt.$$

But this is precisely the defining relationship for $y_n(0)$. Hence the substitution must necessarily have been valid. Q.E.D.

In the following corollaries we will indicate what generalizations apply.

COROLLARY 2: If the demand is nonstationary (where the indexing corresponds to the indexing of the functions C_n) and each of the demand densities satisfies the assumptions of Theorem 2, then all of the results of Theorem 2 and Corollary 1 remain valid with the following alterations:

(a) The recursive equations take the form

$$C_n(x)=\min_{y\geqslant 0}\left\{L_n(x,y)+\alpha\int_0^\infty C_{n-1}(y-(t-x)^+)f_n(t)\,dt\right\},$$

where

$$L_n(x,y)=-r\int_{x+y}^\infty [t-(x+y)]f_n(t)\,dt+\theta\int_0^{y_n(x)} F_n(t+x)F_{n-1}(y-t)\,dt.$$

(b) Part (4) of the theorem becomes

$$C_n'(x)=-\alpha C_{n-1}'(y_n(x))F_n(x)-\theta F_{n-1}(y_n(x))F_n(x).$$

(c) Part (7) of the theorem becomes

$$\theta f_n(x)+\alpha C_n''(x)\geqslant 0 \qquad \text{for all } x.$$

(d) Part (8) of the theorem becomes

$$-\theta F_{n-1}(y_n(x))F_n(x)\leqslant C_n'(x)\leqslant 0.$$

PROOF: (a) The functional equation form is well known. The form for the one period cost $L_n(x,y)$ is actually a consequence of Theorem 1. If we let the random variables be indexed backwards in accordance with the functional equations then from Theorem 1:

$$\begin{aligned}H_{x,y}(t)&=P\{(y-[D_{n-1}+(D_n-x)^+])\leqslant t\}\\&=\begin{cases}1-\int_0^{y-t} F_n(t+x)f_{n-1}(y-t)\,dt & t>0\\ 0 & t\leqslant 0,\end{cases}\end{aligned}$$

so that $E\,[\text{outdates}]=\int_0^y F_n(t+x)F_{n-1}(y-t)\,dt.$

(b) Clearly (1)–(3) in Theorem 2 will not be affected by indexing the demands as long as each of the distributions, F_n, satisfies the assumptions of the theorem. To show (4) is true:

$$C_n(x)=L_n(x,y_n(x))+\alpha\int_0^\infty C_{n-1}(y_n(x)-(t-x)^+)f_n(t)\,dt,$$

$$\frac{dC_n(x)}{dx}=\frac{dL_n(x,y_n(x))}{dx}+\alpha\int_x^{x+y_n(x)} C_{n-1}'(x+y_n(x)-t)f_n(t)\,dt,$$

where $y_n(x)$ satisfies

$$-r[1-F_n(x+y_n(x))]+\theta\int_0^{y_n(x)}F_n(t+x)f_{n-1}(y_n(x)-t)\,dt+\alpha C'_{n-1}(y_n(x))F_n(x)$$
$$+\alpha\int_x^{x+y_n(x)}C'_{n-1}(x+y_n(x)-t)f_n(t)\,dt=0.$$

Since

$$\frac{dL_n(x,y_n(x))}{dx}=-r[1-F_n(x+y_n(x))]+\theta\int_0^{y_n(x)}f_n(t+x)F_{n-1}(y_n(x)-t)\,dt,$$

it follows that

$$\frac{dC_n(x)}{dx}=\theta\int_0^{y_n(x)}[f_n(t+x)F_{n-1}(y_n(x)-t)-F_n(t+x)f_{n-1}(y_n(x)-t)]\,dt-\alpha C'_n(y_n(x))F_n(x)$$
$$=-\alpha C'_n(y_n(x))F_n(x)-\theta F_{n-1}(y_n(x))F_n(x)$$

so that (4) holds; (5) and (6) will remain valid in spite of the nonstationary demands. For (7) we will get

$$\alpha C''_n(x)+\theta f_n(x)=\theta f_n(x)[1-\alpha F_{n-1}(y_n(x))]-\alpha^2C'_{n-1}[y_n(x)]f_n(x)$$
$$-\alpha y'_n(x)F_n(x)[\theta f_{n-1}(y_n(x))+\alpha C''_{n-1}(y_n(x))]\geqslant 0.$$

Again the first term is positive; $C'_{n-1}(u)\leqslant 0$ for all u so that the second term is positive, and the inductive assumption on (7) yields the positivity of the last term.

(d) The generalization follows directly from (b) above.

The remaining sections of Theorem 2 remain valid under nonstationary demand.

Q.E.D.

Because of our method of projecting the outdate costs into the future, we must assume that the demand distribution, F_0, corresponding to the period after the end of the horizon is known (recall that periods are numbered backwards).

COROLLARY 3: If we do not allow the demand to be backlogged (lost sales case), where the transfer function takes the form

$$s(x,y,t)=(y-(t-x)^+)^+,$$

then all of the results of Theorem 2 remain valid (where $x\geqslant 0$ always).

PROOF: For $n=1$ the two problems are identical, so all results will follow. Assume all the results hold for $n=1$. Then

$$B_n(x,y)=r\int_{x+y}^{\infty}[t-(x+y)]f(t)\,dt+\theta\int_0^y F(x+t)F(y-t)\,dt+\alpha\int_0^{\infty}C_{n-1}[(y-(t-x)^+)^+]f(t)\,dt$$
$$=r\int_{x+y}^{\infty}[t-(x+y)]f(t)\,dt+\theta\int_0^y F(x+y-t)F(t)\,dt+\alpha C_{n-1}(y)F(x)$$
$$+\alpha\int_x^{x+y}C_{n-1}(y+x-t)f(t)\,dt+\alpha C_{n-1}(0)[1-F(x+y)],$$

$$\frac{\partial B_n(x,y)}{\partial y} = -r\,[1-F(x+y)] + \theta \int_0^y F(x+y-t)f(t)\,dt + \alpha C'_{n-1}(y)F(x) + \alpha \int_x^{x+y} C'_{n-1}(y+x-t)f(t)\,dt,$$

which is precisely the same form as in the backlogging case, so that (1), (2), and (3) will follow in the same way. To show (4) holds as well:

$$C_n(x) = \min_{y\geq 0} \{L(x,y) + \alpha \int_0^\infty C_{n-1}[(y-(t-x)^+)^+]f(t)\,dt\}$$

$$= L(x, y_n(x)) + \alpha C_{n-1}(y_n(x))F(x) + \alpha \int_x^{x+y_n(x)} C_{n-1}(y_n(x)+x-t)f(t)\,dt + \alpha C_{n-1}(0)\,[1-F(x+y_n(x))],$$

$$C'_n(x) = -r\,[1-F(x+y_n(x))] + \theta \int_0^{y_n(x)} f(x+y_n(x)-t)F(t)\,dt + \alpha \int_x^{x+y_n(x)} C'_{n-1}(y_n(x)+x-t)f(t)\,dt,$$

since all terms involving $y'_n(x)$ drop out. But $y_n(x)$ satisfies:

$$-r\,[1-F(x+y_n(x))] + \theta \int_0^{y_n(x)} F(x+y_n(x)-t)f(t)\,dt + \alpha C'_{n-1}(y_n(x))F(x) + \alpha \int_x^{x+y_n(x)} C'_{n-1}(x+y_n(x)-t)f(t)\,dt = 0.$$

Hence

$$C'_n(x) = \theta \int_0^{y_n(x)} [f(x+y_n(x)-t)F(t) - F(x+y_n(x)-t)f(t)]\,dt - \alpha C'_{n-1}(y_n(x))F(x)$$

$$= -\theta F(y_n(x))F(x) - \alpha C'_{n-1}(y_n(x))F(x)$$

as in Theorem 2.

Since the defining relationship for $y_n(x)$ is precisely the same in both cases, $y'_n(x)$ is also equivalent and (5) will hold. The results (6), (7), and (8) follow from (4), and (9) and (10) hold independent of backlogging assumptions. Note that all the results for this corollary only hold for $x \geqslant 0$. Q.E.D.

ADDITION OF HOLDING AND ORDERING COSTS

An important consideration in our model is whether or not the results can be extended to a more general cost structure including holding and ordering costs. The answer is that it can, but the form of the policy changes; in particular, it will not always be advantageous to place an order.

Assume that there is a charge of $c > 0$ per unit for each unit ordered and a charge of $h > 0$ per unit (charged at the end of the period before outdating, but after demands occur) for each unit carried. Hence the one period expected cost function now becomes

$$L(x,y)=cy+h\int_0^{x+y}(x+y-t)f(t)dt+r\int_{x+y}^{\infty}(t-(x+y))f(t)dt+\theta\int_0^{y}F(v+x)F(y-v)dv.$$

We will also make the assumption that there is a salvage value at the end of the horizon which is equivalent to the ordering cost so that

$$C_1(x)=\inf_{y\geq 0}\left\{L(x,y)-\alpha c\int_0^{\infty}\{y-(t-x)^+\}f(t)dt\right\}.$$

It turns out that this assumption allows us to obtain a certain type of stationarity in our results. Notice that if we define

$$C_0(x)=-cx \qquad \text{for all } x,$$

then the functional equations are still consistently defined.

We will use the notation C_n and B_n as in Theorem 2. Also the same assumptions are made on the demand distribution. We also assume that $r > (1-\alpha)\,c$ (otherwise we would never order to a positive level).

We then have the following analogous result to Theorem 2:

THEOREM 3: Assume F has the same properties as in Theorem 2 and that $r > c(1-\alpha)$. Define the number $\bar{x}$ as the unique positive solution to

$$(1-\alpha)c+hF(\bar{x})-r[1-F(\bar{x})]=0,\ \text{i.e.,}\left[\ \bar{x}=F^{-1}\left[\frac{r-(1-\alpha)c}{r+h}\right]\right].$$

Then we have

(1) $B_n(x, y)$ is convex in y for all x, and is strictly convex in the neighborhood of the global minimum for $x \leq \bar{x}$.

(2) $\lim_{y\to 0}\dfrac{\partial B_n(x,y)}{\partial y} < 0$ if and only if $x < \bar{x}$, $\lim_{y\to+\infty}\dfrac{\partial B_n(x,y)}{\partial y} > 0$ for all x.

(3) If $x < \bar{x}$ then there exists a unique nonnegative solution $y_n(x)$ which solves

$$\left.\frac{\partial B_n(x,y)}{\partial y}\right|_{y=y_n(x)}=0.$$

In addition $y_n'(x)$ exists and is continuous for $x < \bar{x}$.

If $x \geq \bar{x}$, then the optimal policy is not to order, i.e., $y_n(x)=0$.

(4) If $x < \bar{x}$ then

$$C_n'(x)=-c-\alpha C_{n-1}'[y_n(x)]F(x)-\theta F(y_n(x))F(x).$$

(5), (6), and (7) of Theorem 2 hold for $x < \bar{x}$.

(8) If $0 < x < \bar{x}$, then $-c - \theta F(x) \leq C_n'(x) \leq 0$ and $C_n'(x) = -c$ if $x \leq 0$.

(9) If $x > \bar{x}$ then $C_n''(x) > 0$. $C_n'(x) > -c$. In addition C_n and C_n' are continuous at $\bar{x}$.

PROOF: Let $n = 1$.

(1), (2), and (3): By assumption we obtain

$$B_1(x, y) = L(x, y) - \alpha c \int_0^\infty (y - (t-x)^+) f(t)\,dt$$

$$= cy + h\int_0^{x+y} (x+y-t) f(t)\,dt + r\int_{x+y}^\infty (t - (x+y)) f(t)\,dt + \theta\int_0^y F(v+x) F(y-v)\,dv$$

$$- \alpha c\left[yF(x) + \int_x^\infty (y+x-t) f(t)\,dt\right],$$

$$\frac{\partial B_1(x, y)}{\partial y} = c - \alpha c + hF(x+y) - r[1 - F(x+y)] + \theta\int_0^y F(v+x) f(y-v)\,dv,$$

and

$$\frac{\partial^2 B_1(x, y)}{\partial y^2} = (h+r) f(x+y) + \theta F(x) f(y) + \theta\int_0^y f(v+x) f(y-v)\,dv \geq 0.$$

Hence

$$\left.\frac{\partial B_1(x, y)}{\partial y}\right|_{y=0} = c\,(1-\alpha) + hF(x) - r\,[1 - F(x)] = c\,(1-\alpha) - r + (h+r)\,F(x).$$

Now since $\bar{x}$ is defined where $c\,(1-\alpha) + hF(\bar{x}) - r\,[1 - F(\bar{x})] = 0$ and $\left.\frac{\partial B_1(x, y)}{\partial y}\right|_{y=0}$ is a monotone nondecreasing function of x (and is strictly increasing in a neighborhood of $\bar{x}$), it follows that

$$\left.\frac{\partial B_1(x, y)}{\partial y}\right|_{y=0} < 0 \qquad \text{if } x < \bar{x},$$

$$\left.\frac{\partial B_1(x, y)}{\partial y}\right|_{y=0} > 0 \qquad \text{if } x > \bar{x},$$

and

$$\lim_{y\to\infty} \frac{\partial B_1(x, y)}{\partial y} \geq (1-\alpha)\,c + h + \theta > 0 \text{ for all } x.$$

Hence it follows that if $x < \bar{x}$, the optimal policy is to order to $y_1(x)$ where $y_1(x)$ solves

$$\left.\frac{\partial B_1(x, y)}{\partial y}\right|_{y=y_1(x)} = 0,$$

and if $x > \bar{x}$ we don't order. Notice if $x = \bar{x}$ then $y_1(\bar{x}) = 0$, so the policy is continuous at $\bar{x}$.

The differentiability of $y_1(x)$ follows in precisely the same manner as in Theorem 2 when $x < \bar{x}$.

(4) If $x < \bar{x}$ then

$$C_1(x) = B_1(x, y_1(x))$$

$$C_1'(x) = \left.\frac{\partial B_1(x, y)}{\partial x}\right|_{y=y_1(x)} + \left.\frac{\partial B_1(x, y)}{\partial y}\right|_{y=y_1(x)} \cdot y_1'(x) = \left.\frac{\partial B_1(x, y)}{\partial x}\right|_{y=y_1(x)}$$

$$= hF\,(x+y_1(x)) - r\,[1-F\,(x+y_1(x))] + \theta\int_0^{y_1(x)} f\,(v+x)\,F\,(y_1(x)-v)\,dv - \alpha c\,(1-F(x))$$

$$= \{c\,(1-\alpha) + h\,[F\,(x+y_1(x))] - r\,[1-F\,(x+y_1(x))]$$

$$+\,\theta\int_0^{y_1(x)} F\,(v+x)\,f\,(y_1(x)-v)\,dv\Big\} - c - \theta F\,(x)\,F\,(y_1(x)) + \alpha cF(x)$$

$$= -c - \theta F(x) F(y_1(x)) + \alpha c F(x).$$

The term in brackets is

$$\text{exactly } \left.\frac{\partial B_1(x, y)}{\partial y}\right|_{y=y_1(x)} = 0.$$

Also notice that $C_0'(x) = -c$, so this conforms with our notation.

(5) Solving for $y_1'(x)$, we obtain (as long as $x < \bar{x}$) precisely the same expression as in Theorem 2 with r replaced by $r+h$. All terms involving c drop out.

$$\lim_{x \uparrow \bar{x}} y_1'(x) = \frac{-(r+h)f(\bar{x})}{(r+h)f(\bar{x}) + \theta F(\bar{x})f(0)} < 0$$

so that there is a discontinuity of y_1' at $\bar{x}$, since $y_1'(x) = 0$, for $x > \bar{x}$. Notice if $f(0) = 0$, then $\lim_{x \uparrow \bar{x}} y_1'(x) = -1$

(6) and (7) follow since C_1'' is identical. Notice that C_1'' will not be continuous at $\bar{x}$ by the fact that $y_1'(x)$ is not continuous at $\bar{x}$.

(8) follows directly from (4).

(9) If $x > \bar{x}$, $C_1(x) = B_1(x, 0)$.

$$C_1'(x) = hF(x) - r[1-F(x)] - \alpha c[1-F(x)]$$

$$= (1-\alpha)c + hF(x) - r[1-F(x)] - c + \alpha cF(x) > -c$$

$$C_1''(x) = (h+r+\alpha c)f(x) > 0 \text{ since } x > \bar{x} > 0.$$

Notice $\lim_{x \downarrow \bar{x}} C_1(x) = B_1(\bar{x}, 0) = \lim_{x \uparrow \bar{x}} C_1(x)$ and $\lim_{x \downarrow \bar{x}} C_1'(x) = \lim_{x \uparrow \bar{x}} C_1'(x) = -c + \alpha cF(\bar{x})$.

Assume that the theorem holds for $1, 2, \ldots, n-1$.

(1), (2), and (3):

$$B_n(x, y) = L(x, y) + \alpha \int_0^\infty C_{n-1}(y-(t-x)^+) f(t)\,dt.$$

Let $y < \bar{x}$.

$$\frac{\partial B_n(x, y)}{\partial y} = c + hF(x+y) - r[1-F(x+y)] + \theta \int_0^y F(v+x) f(y-v)\,dv$$

$$+ \alpha C'_{n-1}(y) F(x) + \alpha \int_x^{x+y} C'_{n-1}(y+x-t) f(t)\,dt - \alpha c[1-F(x-y)].$$

$$\frac{\partial^2 B_n(x, y)}{\partial y^2} = (h+r) f(x+y) + \theta F(x) f(y) + \theta \int_0^y f(v+x) f(y-v)\,dv + \alpha C''_n(y) F(x)$$

$$+ \int_x^{x+y} C''_{n-1}(y+x-t) f(t)\,dt + \alpha c f(x+y) - \alpha c f(x+y)$$

$$\geqslant (h+r) f(x+y) + \theta(1-\alpha) f(y) + \theta(1-\alpha) \int_0^y f(v+x) f(y-v)\,dv$$

$\geqslant 0$ by the inductive assumption on (7).

If $y > \bar{x}$, then

$$B_n(x, y) = L(x, y) + \alpha C_{n-1}(y) F(x) + \alpha \int_x^{y+x-\bar{x}} C_{n-1}(y+x-t) f(t)\,dt + \alpha \int_{y+x-\bar{x}}^\infty C_{n-1}(y+x-t) f(t)\,dt.$$

$$\frac{\partial B_n(x, y)}{\partial y} = c + hF(x+y) - r[1-F(x+y)] + \theta \int_0^y F(v+x) f(y-v)$$

$$+ \alpha C'_{n-1}(y) F(x) + \alpha \int_x^{y+x-\bar{x}} C'_{n-1}(y+x-t) f(t)\,dt + \alpha C_{n-1}(\bar{x}) f(y+x-\bar{x}) - \alpha C_{n-1}(\bar{x}) f(y+x-\bar{x})$$

$$+ \alpha \int_{y+x-\bar{x}}^{y+x} C'_{n-1}(y+x-t) f(t)\,dt - \alpha c[1-F(y+x)].$$

Notice that the upper limit on the last integral became $y+x$ by the inductive assumption on (8).

$$\frac{\partial^2 B_n(x, y)}{\partial y^2} = (h+r) f(x+y) + \theta F(x) f(y) + \theta \int_0^y f(v+x) f(y-v)\,dv$$

$$+ \alpha C''_{n-1}(y) F(x) + \alpha \int_x^{y+x-\bar{x}} C''_{n-1}(y+x-t) f(t)\,dt + \alpha C'_{n-1}(\bar{x}) f(y+x-\bar{x})$$

$$- \alpha C'_{n-1}(\bar{x}) f(y+x-\bar{x}) + \alpha \int_{y+x-\bar{x}}^{y+x} C''_{n-1}(y+x-t) f(t)\,dt - \alpha c f(y+x) + \alpha c f(y+x).$$

Although C''_{n-1} may have a jump at $\bar{x}$ and at 0 the differentiation under the integral sign is justified by Van Zyl's lemma.

Now since $y>\bar{x}$, $C''_{n-1}(y)>0$ by the inductive assumption on (9),

$$\int_x^{y+x-\bar{x}} C''_{n-1}(y+x-t)\,f(t)\,dt=\int_{\bar{x}}^{y} C''_{n-1}(u)\,f(y+x-u)\;du>0$$

and in the same way,

$$\int_{y+x-\bar{x}}^{y+x} C''_{n-1}(y+x-t)\,f(t)\,dt=\int_0^{\bar{x}} C''_{n-1}(u)\,f(y+x-u)\;du$$

$$\geqslant -\theta\int_0^{\bar{x}} f(u)\,f(y+x-u)\;du$$

$$\geqslant -\theta\int_0^{y} f(u)\,f(y+x-u)\;du$$

by the inductive assumption on (7).

Hence

$$\frac{\partial^2 B_n(x,y)}{\partial y^2}>(h+r)\,f(x+y)+\theta F(x)\,f(y)+\theta\,(1-\alpha)\int_0^y f(v+x)\,f(y-v)\;dv\geqslant 0$$

and convexity is established over $(\bar{x},\infty)$.

Since $B_n(x,y)$ and $\dfrac{\partial B_n(x,y)}{\partial y}$ are continuous at $\bar{x}$ the convexity follows.

$$\lim_{y\to 0}\frac{\partial B_n(x,y)}{\partial y}=c+hF(x)-r[1-F(x)]+\alpha\lim_{y\to 0} C'_{n-1}(y)F(x)+\alpha\int_x^\infty \lim_{y\to 0} C'_{n-1}(y+x-t)\,f(t)\,dt$$

$$=c(1-\alpha)+hF(x)-r[1-F(x)]$$

by the inductive assumption on (8) so that

$$\left.\frac{\partial B_n(\bar{x},y)}{\partial y}\right|_{y=0}=0$$

and it follows that if $x<\bar{x}$, $\left.\dfrac{\partial B_n(x,y)}{\partial y}\right|_{y=0}<0$ and the optimal policy is to order $y_n(x)$ where $y_n(x)$ is the solution to

$$\left.\frac{\partial B_n(x,y)}{\partial y}\right|_{y=y_n(x)}=0$$

and if $x > \bar{x}, \left.\dfrac{\partial B_n(x, y)}{\partial y}\right|_{y=0} > 0$, so it is advantageous not to order. Notice $y_n(\bar{x}) = 0$.

(4) If $x < \bar{x}$, then

$$C_n'(x) = hF(x+y_n(x)) - r[1-F(x+y_n(x)] + \theta \int_0^{y_n(x)} f(v+x)F(y_n(x)-v)dv + \alpha f(x)C_{n-1}(y_n(x))$$

$$- \alpha f(x)C_{n-1}(y_n(x)) + \alpha \int_x^\infty C_{n-1}'(y_n(x)+x-t)f(t)dt$$

$$= \Big\{c + hF(x+y_n(x)) - r[1-F(x+y_n(x)] + \theta \int_0^{y_n(x)} F(v+x)f(y_n(x)-v)dv$$

$$+ \alpha F(x)C_{n-1}'(y_n(x)) + \alpha \int_x^\infty C_{n-1}'(y_n(x)+x-t)f(t)dt\Big\} - c - \alpha F(x)C_{n-1}(y_n(x))$$

$$- \theta F(y_n(x))F(x)$$

$$= -c - \alpha F(x)C_{n-1}(y_n(x)) - \theta F(y_n(x))F(x)$$

by the definition of $y_n(x)$.

(5), (6), and (7) follow in precisely the same way as in Theorem 2 since the expressions for $y_n'(x)$ and $C_n''(x)$ are the same for $x < \bar{x}$.

Notice that $\lim_{x \uparrow \bar{x}} y_n'(x) < 0$ and $= -1$ if $f(0) = 0$, as in the $n = 1$ case.

(8) From (4) and the inductive assumption on (8), we obtain for $x < \bar{x}$

$$C_n'(x) \leq -c - \alpha F(x)[-c - \theta F(y_n(x)] - \theta F(y_n(x)F(x)$$

$$\leq -c(1-\alpha) - \theta(1-\alpha)F(y_n(x))F(x) \leq 0,$$

$$C_n'(x) \geq -c - \theta F(y_n(x))F(x) \geq -c - \theta F(x).$$

From (4) $C_n'(x) = -c$ if $x \leq 0$.

(9) If $x > \bar{x}$ then

$$C_n(x) = L(x, 0) + \alpha \int_0^\infty C_{n-1}(-(t-x)^+)f(t)dt$$

$$= L(x, 0) + \alpha C_{n-1}(0)F(x) + \alpha \int_x^\infty C_{n-1}(x-t)f(t)dt.$$

Hence

$$C_n'(x) = hF(x) - r[1-F(x)] - \alpha c[1-F(x)]$$

$$> (1-\alpha)c + hF(\bar{x}) - r[1-F(\bar{x})] - c + \alpha c F(x) > -c$$

$$C''_n(x) = (h+r+\alpha c) f(x) > 0.$$

The continuity of $C_n(x)$ and $C'_n(x)$ at $\bar{x}$ follows easily since $y_n(\bar{x}) = 0$ by (1), (2), and (3), and $C'_{n-1}(0) = -c$ by (8). Thus $C'_n(\bar{x}) = -c + \alpha c F(\bar{x})$ independent of n. Q.E.D.

Notice that the introduction of a salvage value at the end of the horizon insures that the numbers $\bar{x}$ don't depend on the period n. If we exclude this assumption then $\bar{x}_2 > \bar{x}_1$ and $\bar{x}_n = \bar{x}_{n-1}$, $2 < n < N$, provided we number the periods backwards.

We remark that the results of Corollaries 1, 2, and 3 remain valid with the addition of ordering and holding costs.

The model considered by Van Zyl is precisely this one with $h = \theta = 0$. Note that in the case where $\theta = 0$ the results of the theorem remain valid with the added property that C_n is convex everywhere.

We have the following result:

COROLLARY 4: For $0 < x < \bar{x}$ and all n,

$$0 < y_n(0) < y_n(x) + x < \bar{x}.$$

PROOF: Since $y'_n(x) > -1$ and $y_n(x)$ is continuous,

$$-1 < \frac{y_n(x) - y_n(0)}{x} \text{ from the mean value theorem}$$

$$=> y_n(x) + x > y_n(0).$$

Also since $y'_n(x) > -1$ for $0 < x < \bar{x}$, $\lim_{x \uparrow \bar{x}} y'_n(x) \geqslant -1$ and $y_n(x)$ is continuous,

$$-1 < \frac{y_n(\bar{x}) - y_n(x)}{\bar{x} - x}$$

$$=> x + y_n(x) < y_n(\bar{x}) + \bar{x} = \bar{x}. \qquad \text{Q.E.D.}$$

This is a particularly significant result. It says that if the initial inventory is less than $\bar{x}$ then we would order in every period. If $x > \bar{x}$, then we would not order until the inventory on hand fell below $\bar{x}$ and then we would order in each subsequent period.

In the case where $y_n(0)$ and $\bar{x}$ are relatively close, one could approximate an optimal ordering policy by assuming the inventory on hand after ordering is some number interpolated between $y_n(0)$ and $\bar{x}$.

THE DYNAMIC PROBLEM—INFINITE HORIZON

The objective in analyzing the infinite horizon problem is to demonstrate the existence of a stationary policy that minimizes the long run expected discounted cost and to determine methods for computing that policy. The formalism we will employ is that of Denardo [3].

The state space of our problem is $R = (-\infty, \infty)$.

Define:

$W = \{w : w$ is a real valued Borel measurable function on $R\}$. The metric on W is defined in the usual way by

$$\rho(u, v) = \sup_{x \in R} | u(x) - v(x) |.$$

Also let

$$V = \{u, v \in W : \rho(u, v) < \infty \text{ and } u, v \text{ are continuous}\}.$$

The subset V is introduced to insure the property of completeness. We note that, in general, V will not be uniquely defined depending on our choice of r (per unit runout cost) and θ (per unit outdate cost). We deal with one specific subset V of W.

Denardo assumed that his space V was the space of bounded functions on R. It was recognized by Porteus [6] that this assumption was really not necessary and that the results of the theory apply to the more general classes of functions indicated above.

For any ordering policy define the mapping H_y on V as

$$H_y v(x) = L(x, y(x)) + \alpha \int_0^\infty v(y(x) - (t-x)^+) f(t)\, dt$$

and

$$Av(x) = \inf_{y \geq 0} \left\{ L(x, y) + \alpha \int_0^\infty v(y - (t-x)^+) f(t)\, dt \right\}.$$

For the infinite horizon case we assume $0 < \alpha < 1$. We then have the following result:

THEOREM 3: The mappings H_y and A are both positive contractions on V. In addition both of these mappings satisfy the monotonicity property, that is, if $u(x) \geqslant v(x)$ (for all $x \epsilon X$) then $H_y u(x) \geqslant H_y v(x)$, and similarly for A. Finally there exists an optimal stationary policy $y^*(x)$ which is the solution to the equation

$$u^*(x) = L(x, y^*(x)) + \alpha \int_0^\infty u^*(y^*(x) - (t-x)^+) f(t)\, dt,$$

where u^* is the unique fixed point of A. The proof is obvious from Denardo's and Porteus' work.

The term $u^*(x)$ has the interpretation of being the minimum expected cost, given we are in state x and infinitely many periods remain in the horizon. To see that the infinite horizon problem arises in a natural way as the limit of the finite horizon problem as the length of the horizon gets large, let $v_0 \equiv 0$ $v_n = A^n v_0$. Then it follows that v_n is exactly C_n of Theorem 2 and the function

$$\left\{ L(x, y) + \alpha \int_0^\infty v_n(y - (t-x)^+) f(t)\, dt \right\}$$

will be convex in y. Hence the sequence of functions $y_n \to y^*$.

The computational schemes developed employ precisely this idea that the infinite horizon problem is in a sense the limit of the finite horizon problem. Algorithms for successive approximations are considered in Denardo and Porteus.

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