OPTIMAL ORDERING POLICIES FOR PERISHABLE INVENTORY - I*

Steven Nahmias

Department of Industrial Engineering
University of Pittsburgh
Pittsburgh, Pennsylvania 15213

and

William P. Pierskalla

Department of Industrial Engineering
and Management Sciences
Northwestern University
Evanston, Illinois 60201

Optimal ordering policies for perishable inventories has received scant attention in the literature. The goal of this paper is to generalize some recent work of Nahmias and Pierskalla [11] in this area. We consider the case of a single product at a single installation where the demands in successive periods are random variables. The case where demands are deterministic turns out, in the simple case, to have a trivial solution.

The model we consider is a periodic review model. That is, ordering takes place at the start of a period and costs are incurred during the period. The period length is arbitrary but fixed.

We will make the following assumptions:
(1) All orders are placed at the start of the period and received instantaneously. (2) All stock arrives new. (3) Demands in successive periods are independent random variables with known distributions and densities. (4) Inventory is depleted according to a FIFO policy; that is, oldest first. (5) Costs are charged linearly against (a) unsatisfied demand (runouts) (b) deterioration (outdates). (6) If the product has not been depleted by demand by the time it reaches age m periods then it deteriorates and must be discarded at a specified per-unit cost. (7) Unsatisfied demand is backlogged.

The first problem to consider in developing a model is describing the state variable of the system. Since it is necessary to keep track of the amount of product on hand at each age level, we define x_i = amount of product on hand which will outdate (perish) exactly i periods into the future and the vector $\underline{x} = (x_{m-1}, x_{m-2}, \ldots, x_1)$ represents the total initial inventory at each age level. The decision variable, y, represents the amount of new product being ordered. For convenience, define

$$x = \Sigma_{i=1}^{m-1} x_i.$$

The most effective way to describe the dynamics of the problem is to look at it as a special type of multi-echelon problem. Let the i^{th} echelon correspond to the term x_i and contain the amount of product which will outdate exactly i periods into the future. Then in transferring from one period to the next, stock is transmitted to the next lower echelon; new stock enters only at the highest echelon and all remaining stock in lowest echelon must be discarded at the end of the period. The FIFO assumption is interpreted as follows: demand originates only at echelon 1 and is transmitted up to the next higher echelon until either the demand is satisfied or unsatisfied demand is backlogged at echelon m.

From these considerations we arrive at the following transfer function: If x is the amount of stock on hand at the beginning of any period, y is amount of the order placed at the start of the period and t represents a realization of the demand in the period, then the amount of stock on hand at the start of the next period is

$$(\underline{s}(y,\underline{x},t)) = (s_1(y,\underline{x},t),\ldots,s_{m-1}(y,\underline{x},t))$$

$$s_i(y,\underline{x},t) = [x_{i+1} - [t - \sum_{j=1}^{i} x_m]^+]^+ \qquad 1 \le i \le m-2$$

$$s_{m-1}(y,\underline{x},t) = \begin{cases} y - [t - \sum_{j=1}^{m-1} x_j]^+ & \text{if demand is backlogged} \\ [y - [t - \sum_{j=1}^{m-1} x_j]^+]^+ & \text{if demand is not backlogged} \end{cases}$$

where $f^+ = \max(f,0)$.

Let $D_1, D_2, \ldots$ be the demands in successive periods each of which has specified distributions $F_1, F_2, \ldots$ and densities $f_1, f_2, \ldots$. Let y be the amount of the order. Then the total amount of stock on hand after ordering will be $y + \sum_{i=1}^{m-1} x_i$ (= y + x). We define the random variables $R_1 = y + \sum_{i=1}^{m-1} x_i$,

$$R_2 = [y + \sum_{i=2}^{m-1} x_i - (D_1 - x_1)^+]^+,$$

$$R_3 = [y + \sum_{i=3}^{m-1} x_i - \{D_2 + (D_1 - x_1)^+ - x_2\}^+]^+$$

etc. Thus R_j represents the total amount of

$y + \sum_{i=1}^{m-1} x_i$ on hand exactly j - 1 periods into the future. Notice that we need not worry about future orders since our FIFO assumption assures us that all of the inventory y will be depleted before any future order.

In order to simplify the notation, we define recursively $B_0 = 0,\ldots,B_j[D_j + B_{j-1}-x_j]^+$ $1 \le j \le m - 1$.

The random variable B_j can be interpreted as the total unsatisfied demand in period j after depleting all of the product that would have outdated in the period. Notice that R_m represents the amount of y on hand at the start of the m^{th} period so that $R_m^* = [y - (D_m + B_{m-1})]^+$ represents the amount of y that perishes.

In order to determine the distribution of R_m^* it is convenient to define

$$G_n(t;\underset{\sim}{x}(n-1)) = P\{D_n + B_{n-1} \le t\},\ 1 \le n \le m$$

where the vector $\underset{\sim}{x}(n) = (x_n,\ldots,x_1)$.

We have the following

THEOREM 1.1:

$$G_n(t;\underset{\sim}{x}(n-1)) = \int_0^t G_{n-1}(v + x_{n-1};\underset{\sim}{x}(n-2))f_n(t-v)dv,\quad 1 \le n \le m$$

where $G_0(t) = 1$.

For consciseness most of the proofs have been omitted. The interested reader may supply them himself or refer to [12].

We will use the notation $G_n^{(i)}(\underset{\sim}{x}(n))$ to refer to the first partial derivative of G_n with respect to the i^{th} argument. We then have the following results:

COROLLARY 1.1: Assume that the demand distributions F_k possess densities f_k that are continuous everywhere. Then the functions $G_n^{(i)}(t;x(n-1))$ are continuous over $R_n (1 \le i \le m - 1)$. If f_n has a jump at 0 then $G_n^{(i)}(t;\underset{\sim}{x}(n-1))$ are all continuous over R_n for $i > 2$ and $G_n^{(1)}(t;\underset{\sim}{x}(n-1))$ is continuous in all its arguments but possesses a jump at t = 0.

COROLLARY 1.2: $\lim_{x_{n-i}\to\infty} G_n(\underset{\sim}{x}(n)) = G_i(x_n,\ldots,x_{n-i+1})$, $2 \le n \le m$

where G_i is a function of $F_{n-i+1},\ldots,F_n$ respectively.

The motivation behind defining the functions G_n will become aparent in the next lemma.

LEMMA 1.2: If $x = (x_{m-1},\ldots,x_1)$ is the inventory on hand before ordering, and y is the amount of the order, then the expected number of outdates of y, m periods into the future, is $\int_0^y G_m(u;\underset{\sim}{x})du$.

The important results regarding the one period model are summarized in the following theorem:

THEOREM 1.2: If the charge for runouts is r per unit and the charge for outdates is θ per unit (r > 0, θ > 0), then the total expected one period cost given that we have $\underset{\sim}{x}$ on hand and order y is

$$L(\underset{\sim}{x},y) = r\int_{x+y}^{\infty} [t - (x + y)]f_1(t)dt + \theta\int_0^y G_m(u:\underset{\sim}{x})du.$$

The function $L(\underset{\sim}{x},y)$ is convex in y and the function $y^*(\underset{\sim}{x})$ given by $L(\underset{\sim}{x},y^*(\underset{\sim}{x})) = \min_{y\ge 0}[L(\underset{\sim}{x},y)]$ exists and its range is contained in $[0,\infty)$. The constrained minimization is equivalent to the unconstrained minimization so that $L(\underset{\sim}{x},y^*(\underset{\sim}{x})) = \min_y[L(\underset{\sim}{x},y)]$ where $y^*(x)$ will be uniquely defined whenever f_1 is such that $f_1(t) > 0$ for all t > 0.

THE FINITE HORIZON DYNAMIC PROBLEM

The dynamic problem has the dynamics (that is, the transfer function) built into the problem on two levels. This is one of the features that makes this model unique and somewhat more involved analytically than most stochastic inventory models.

From section I the expected outdating charge is $\theta\int_0^y G_m(u;x)du$ where G_m is defined recursively in THEOREM 1.1. It turns out that another recursion obtained from averaging G_m composed with the transfer function is also valid. This result will be a key one for analyzing the dynamic model and is described in the following

THEOREM 2.1.

$$G_n(y;\underset{\sim}{x}(n-1)) = \int_0^{\infty} G_{n-1}[\underset{\sim}{s}(y,\underset{\sim}{x}(n-1),t)]f(t)dt$$

$2 \le n \le m$.

As a direct consequence of the theorem we have the following result.

COROLLARY 2.1. Define the vector valued functions

$$\underset{\sim}{z}_j(t) = (y,x_{m-1},\ldots,x_{j+1},\sum_{i=1}^{j} x_i - t,0,\ldots,0)\quad 2 \le j \le m$$

$\underset{\sim}{z}_1(t) = (y,x_{m-1},\ldots,x_2)$ and $x_m = y$.

$$\text{Then } G_m(y;\underset{\sim}{x}) = \sum_{j=1}^{m}\int_{w_{j-1}}^{w_j} G_{m-1}[\underset{\sim}{z}_j(t)]f(t)dt$$

for $m \ge 2$ where $w_j = \sum_{i=1}^{j-1} x_i$.

Also if we define $z_j(t;k)$ as the last k component of $z_j(t)$ we have that

$$G_{k+1}(\underset{\sim}{x}(k+1)) = \sum_{j=1}^{k+1} \int_{w_{j-1}}^{w_j} G_k(\underset{\sim}{z}_j(t;k))f(t)dt \qquad 1 \le k \le m-1.$$

Throughout this chapter we will assume the following notation. If A is a function on R^k, then $A^{(i)}$ denotes the first partial derivative of A with respect to the i^{th} variable and $A^{(i,j)}$ the second cross partial derivative with respect to the ith and jth variables respectively for i, j < k. If either i > k or j > k then $A(i,j) \equiv 0$.

There are a number of results which will follow directly from COROLLARY 2.1.

For example, $G_{m-1}(\underset{\sim}{x}) = \sum_{j=1}^{m-1} \int_{w_{j-1}}^{w_j} G_{m-1}(\underset{\sim}{z}_j(t;m-2))f(t)dt$

so that

$$G_{m-1}^{(i)}(\underset{\sim}{x}) = \sum_{j=1}^{m-i} \int_{w_{j-1}}^{w_j} G_{m-1}^{(i)}(\underset{\sim}{z}_j(t;m-2))f(t)dt + \sum_{j=m-i+1}^{m-1} \int_{w_{j-1}}^{w_j} G_{m-1}^{(m-j+1)}(\underset{\sim}{z}_j(t;m-2))f(t)dt \qquad 1 \le i \le m-2.$$

Notice that all terms involving differentiation of the limits drop out. Also

$$G_{m-1}^{(m-1)}(\underset{\sim}{x}) = \sum_{j=2}^{m-1} \int_{w_{j-1}}^{w_j} G_{m-1}^{(m-j+1)}(\underset{\sim}{z}_j(t;m-2))f(t)dt.$$

This results from differentiation of the limit on the first integral. Hence for $2 \le i \le m-1$ we obtain

$$G_{m-1}^{(i-1)}(\underset{\sim}{x}) - G_{m-1}^{(i)}(\underset{\sim}{x}) = \sum_{j=1}^{m-1} \int_{w_{j-1}}^{w_j} \{G_{m-2}^{(i-1)}[\underset{\sim}{z}_j(t;m-2)] - G_{m-2}^{(i)}[\underset{\sim}{z}_j(t;m-2)]\}f(t)dt.$$

This and similar results will be used in the proof of THEOREM 2.2.

Since our methods rely upon differentiation we will need a number of lemmas which consider various relationships between the functions G_k and its derivatives.

<u>LEMMA 2.1</u>: For any real number a > 0 we have

$$\frac{\partial}{\partial x_{m-i}}\left[\int_0^a G_m(u;\underset{\sim}{x})du\right] = G_m(a;\underset{\sim}{x})$$

$$\sum_{j=1}^{i} G_{m-j}(\underset{\sim}{x}(m-j))H_j(a;\bar{\underset{\sim}{x}}(m-j))$$

where $H_1(a) = F(a)$

$$H_j(a;\bar{\underset{\sim}{x}}(m-j)) = \int_0^a F(a-v_{m-1}) \int_0^{v_{m-1}+x_{m-1}} f(v_{m-1}+x_{m-1}-v_{m-2})\cdots \int_0^{v_{m-j+2}+x_{m-j+2}} f(v_{m-j+2}+x_{m-j+2}-v_{m-j+1})f(v_{m-j+1}+x_{m-j+1}) dv_{m-j+1}\cdots dv_{m-1}$$

$$\underset{\sim}{x} = (x_{m-1},\ldots,x_1) = (\underset{\sim}{x}(m-j),\underset{\sim}{x}(m-j))$$

$$\underset{\sim}{x}(m-j) = (x_{m-j},\ldots,x_1)$$

$$\bar{\underset{\sim}{x}}(m-j) = (x_{m-1},\ldots,x_{m-j+1}).$$

Notice that if we differentiate both sides of the equation with respect to a we obtain the identity

$$G_m^{(i+1)}(a;\underset{\sim}{x}) = G_m^{(1)}(a;\underset{\sim}{x}) - \sum_{j=1}^{i} G_{m-j}(\underset{\sim}{x}(m-j))H_j^{(1)}(a;\bar{\underset{\sim}{x}}(m-j))$$

which implies

$$G_m^{(i)}(a;\underset{\sim}{x}) - G_m^{(i+1)}(a;\underset{\sim}{x}) = G_{m-i}(\underset{\sim}{x}(m-i))H_i^{(1)}(a;\bar{\underset{\sim}{x}}(m-i))$$

Notice that the interchanging of orders of differentiation is valid by COROLLARY 1.1.

<u>LEMMA 2.2</u> The function H_j defined in Lemma 2.1 satisfy the following recursion:

$$H_j(a;\bar{\underset{\sim}{x}}(m-j)) = G_j(a;\bar{\underset{\sim}{x}}(m-j)) - \sum_{i=1}^{j-1} G_{j-i}(x_{m-i},\ldots,x_{m-j+1})H_i(a;\bar{\underset{\sim}{x}}(m-i)),$$

Clearly, relationships between the H_j and G_j functions will play a major role in the analysis of the dynamic problem so that we also list the following lemmas.

<u>LEMMA 2.3</u>:

$$H_j^{(1)}(a;\bar{\underset{\sim}{x}}(m-j)) = G_j^{(j)}(a;\bar{\underset{\sim}{x}}(m-j)).$$

<u>LEMMA 2.4</u>:

$$H_j(a;\bar{\underset{\sim}{x}}(m-j)) = \int_0^a f(a-v)H_{j-1}(v+x_{m-1};x_{m-2},\ldots,x_{m-j+1})\,dv \quad -F(a)H_{j-1}(\bar{\underset{\sim}{x}}(m-j)).$$

<u>LEMMA 2.5</u>:

$$H_j(a;\bar{\underset{\sim}{x}}(m-j)) \le 1 - \sum_{k=1}^{j-1} H_k(x_{m-j+k},\ldots,x_{m-j+1}).$$

<u>LEMMA 2.6</u>:

$$\sum_{j=i}^{m-1} G_{m-j}(\underset{\sim}{x}(m-j))[1 - \sum_{k=1}^{j-i} H_k(x_{m-j+k},\ldots,x_{m-j+1})] = \sum_{k=1}^{m-i} H_k(\underset{\sim}{x}(k)) \qquad \text{for } 1 \le i \le m-1.$$

The following orderings are valid:

<u>LEMMA 2.7</u>:

(i) $G_k(\underset{\sim}{x}(k)) \le G_{k-1}(x_k,\ldots,x_2)$

(ii) $H_k(\underset{\sim}{x}(k)) \le H_{k-1}(x_k,\ldots,x_2)$

(iii) $G_k(\underset{\sim}{x}(k)) \ge H_k(\underset{\sim}{x}(k))$

for $1 \le k \le m$.

At this point we are ready to introduce the central theorem of the paper. The notation that will be used will be consistent with that introduced in the lemmas. We note particularly that in previous results we had

$$\underset{\sim}{z}_j(t) = (y,x_{n-1},\ldots,x_{j+1},\sum_{i=1}^{j} x_i - t,0,\ldots,0)$$

which depends on $(\underset{\sim}{x},y)$ as well as j and t. From now on we will assume that the first component of z_j is the function $y_n(\underset{\sim}{x})$ which will represent the optimal quantity to be ordered given we have $\underset{\sim}{x}$ on hand and n periods remain in the horizon (as in the previous section). Often the dependence on $\underset{\sim}{x}$ will be suppressed and we'll write y_n.

The assumptions we will make for the theorem are virtually equivalent to those made in Nahmias and Pierskalla [11] for the m = 2 case. We consider the finite horizon problem; that is, the length of the horizon is some fixed positive integer N.

The functional equations take the form

$$C_n(\underset{\sim}{x}) = \inf_{y\geq 0} \{L(\underset{\sim}{x},y) + \alpha\int_0^\infty C_{n-1}(\underset{\sim}{s}(y,\underset{\sim}{x},t))f(t)dt\}$$

$$= \inf_{y\geq 0} \{B_n(\underset{\sim}{x},y)\} \quad \text{where } C_0(\underset{\sim}{x}) = 0 \text{ for all } \underset{\sim}{x},$$

where $L(\underset{\sim}{x},y)$ is the one period expected cost function introduced earlier, and $\underset{\sim}{s}(y,\underset{\sim}{x},t)$ is the (vector valued) transfer function. Recall $x = \sum_{i=1}^{m-1} x_i$ and $w_j = \sum_{i=1}^{j} x_i$ so that x and w_{m-1} are identical. Hence x has the usual interpretation of being the total amount of inventory on hand before ordering.

The proof for the case of m an arbitrary positive integer greater than or equal to 2 is set up to conform closely with the m = 2 case. As in that case $C_n(\underset{\sim}{x})$ will turn out not to be a convex function, but $B_n(\underset{\sim}{x},y)$ will be convex in y. Hence it becomes necessary to use the calculus to obtain convexity. The analysis becomes significantly more involved when developing bounds for the derivatives and sections (7) and (8) of the theorem represent the major analytical effort of the proof. The key inequality in establishing convexity for the m = 2 case was

$$C_n''(x) \geq -\theta f(x).$$

In the general case we need

$$C_n^{(1,1)}(x) \geq -\theta\, G_{m-1}^{(1)}(\underset{\sim}{x}),$$

but to show this holds in period n requires the development of a network of inequalities.

<u>THEOREM 2.2</u>: Assume that

(i) The demand distribution F possesses a bounded continuous density f such that

$$f(t) > 0 \text{ if } t > 0,\ f(t) = 0 \text{ if } t < 0.$$

(ii) The discount factor is a $\varepsilon(0,1)$.

Then (1) $B_n(\underset{\sim}{x},y)$ is a convex in y for all $\underset{\sim}{x} \in R^{m-1}$ and is strictly convex in a neighborhood of the global minimum

(2)

(3) The is a unique $y_n(\underset{\sim}{x})$ given as the solution to $\left.\frac{\partial B_n(\underset{\sim}{x},y)}{\partial y}\right|_{y=y_n(\underset{\sim}{x})} = 0$ and $y_n(\underset{\sim}{x}) \in (0,\infty)$. In addition $y_n^{(i)}(\underset{\sim}{x})$ exists and is continuous for all $\underset{\sim}{x}$, $1 \leq i \leq m-1$.

(4)

$$C_n^{(i)}(\underset{\sim}{x}) = -\theta \sum_{j=1}^{i} G_{m-j}(\underset{\sim}{x}(m-j))H_j(y_n(\underset{\sim}{x});\bar{\underset{\sim}{x}}(m-j))$$

$$+ \alpha \sum_{j=1}^{m-i} \int_{w_{j-1}}^{w_j} \{C_{n-1}^{(i+1)}[\underset{\sim}{z}_j(t)] - C_{n-1}^{(1)}[\underset{\sim}{z}_j(t)]\}f(t)dt$$

$$+ \alpha \sum_{j=m-i+1}^{m-1} \int_{w_{j-1}}^{w_j} \{C_{n-1}^{(m-j+1)}[\underset{\sim}{z}_j(t)] - C_{n-1}^{(1)}[\underset{\sim}{z}_j(t)]\}f(t)dt$$

where $\underset{\sim}{z}_j(t)$ and w_j are as previopsly defined and $C_n^{(m)}(\underset{\sim}{x}) \equiv 0$. The result holds for $1 \leq i \leq m-1$.

(5) $-1 \leq y_n^{(1)}(\underset{\sim}{x}) \leq y_n^{(2)}(\underset{\sim}{x}) \leq \cdots \leq y_n^{(m-1)}(\underset{\sim}{x}) < 0.$

(6) a) $C_n^{(i,k)}(\underset{\sim}{x})$ exists and is continuous for all $x \in \mathbb{R}^{m-1}$ and $i \leq k$, $1 \leq m-1$. However, $C_n^{(1,1)}(t;\underset{\sim}{x}(m-2))$ will be discontinuous at t = 0 whenever f(t) is discontinuous at t = 0.

b) $C_n^{(i)}[\bar{\underset{\sim}{x}}(m-i),\underset{\sim}{0}] - C_n^{(i-1)}[\bar{\underset{\sim}{x}}(m-i),\underset{\sim}{0}] = 0$ for $2 \leq i \leq m-1$. The notation $\underset{\sim}{0}$ is meant to be interpreted as the last m − 1 components being zeros.

(7) a) $C_n^{(1,j)}(\underset{\sim}{x}) \geq -\theta\, G_{m-1}^{(j)}(\underset{\sim}{x}) \qquad 1 \leq j \leq m-1$

b) $C_n^{(i,j)}(\underset{\sim}{x}) - C_n^{(i-1,j)}(\underset{\sim}{x}) \geq -\theta G_{m-i}^{(j-i+1)}(\underset{\sim}{x}(m-i))\cdot[1 - \sum_{k=1}^{i-1} H_k(x_{m-i+k},\ldots,x_{m-i+1})] \quad m-1 \geq j \geq i \geq 1$

c) $C_n^{(1,i)}(\underset{\sim}{x}) - C_n^{(1,i-1)}(\underset{\sim}{x}) \leq \theta[G_{m-1}^{(i-1)}(\underset{\sim}{x}) - C_{m-1}^{(i)}(\underset{\sim}{x})] \quad m-1 \geq i \geq 2$

d) $[C_n^{(i,j)}(\underset{\sim}{x}) - C_n^{(i-1,j)}(\underset{\sim}{x})] - [C_n^{(i,j-1)}(\underset{\sim}{x}) - C_n^{(i-1,j-1)}(\underset{\sim}{x})]$

$$\leq \theta[G_{m-i}^{(j-i)}(\underset{\sim}{x}(m-i)) - G_{m-i}^{(j-i+1)}(\underset{\sim}{x}(m-i))]\cdot[1 - \sum_{k=1}^{i-1} H_k(x_{m-i+k},\ldots,x_{m-i+1})] \quad \text{for } m-1 \geq j > i \geq 2$$

(8) a) $-\theta \sum_{j=1}^{i} G_{m-j}(\underset{\sim}{x}(m-j))[1 - \sum_{k=1}^{j-1} H_k (x_{m-j+k},\ldots,x_{m-j+1}] \le C_n^{(i)}(\underset{\sim}{x}) \le 0$

for $1 \le i \le m-1$ and for all $\underset{\sim}{x}$

b) $C_n^{(i)}(\underset{\sim}{x}) - C_n^{(j)}(\underset{\sim}{x}) \le \theta \sum_{k=j+1}^{i} G_{m-k}(\underset{\sim}{x}(m-k)) [\sum_{q=k-j}^{k-1} H_q(x_{m-k+q},\ldots,x_{m-k+1})]$

for $1 \le j < i \le m-1$ and all $\underset{\sim}{x}$

c) $C_n^{(i)}(\underset{\sim}{x}) - C_n^{(j)}(\underset{\sim}{x}) \ge -\theta \sum_{k=j+1}^{i} G_{m-k}(\underset{\sim}{x}(m-k)) \cdot [1 - \sum_{q=1}^{k-1} H_q(x_{m-k+q},\ldots,x_{m-k+1})]$

for $1 \le j < i \le m-1$ and for all $\underset{\sim}{x}$

d) $C_n^{(i)}(\underset{\sim}{x}) = 0$ for $\underset{\sim}{x} = (x_{m-1},\underset{\sim}{0})$ and $x_{m-1} \le 0$.

(9) a) $\lim_{x_i \to +\infty} y_n(\underset{\sim}{x}) = 0 \qquad 1 \le i \le m-1$

b) $\lim_{x_j \to +\infty} C_n^{(i)}(\underset{\sim}{x}) = 0 \qquad 1 \le i,j \le m-1.$

Proof:

Assume the result holds for $1,2,\ldots,n-1$. (The $n = 1$ case follows in a similar manner with $C_0(\underset{\sim}{x}) = 0$ for all $\underset{\sim}{x}$).

(1) and (2)

$$B_n(\underset{\sim}{x},y) = r \int_{x+y}^{\infty} [t - (x+y)]f(t)dt + \theta \int_0^y G_m(u;\underset{\sim}{x})du + \alpha \int_0^{\infty} C_{n-1}[\underset{\sim}{s}(y,\underset{\sim}{x},t)]f(t)dt$$

$$\frac{\partial B_n(\underset{\sim}{x},y)}{\partial y} = -r[1 - F(x+y)] + G_m(y;\underset{\sim}{x}) + \alpha \int_{w_{m-1}}^{w_{m-1}+y} C_{n-1}^{(1)}[y + x - t, 0\ldots0]f(t)dt + \alpha \sum_{j=1}^{m-1} \int_{w_{j-1}}^{w_j} C_{n-1}^{(1)}[\underset{\sim}{z}_j(t)]f(t)dt$$

by the inductive assumption on (8,d) $\le -r[1 - F(x+y)] + \theta G_m(y;\underset{\sim}{x})$ by the Inductive assumption on (8,a).

Hence $\left.\frac{\partial B_n(x,y)}{\partial y}\right|_{y=0} \le -r[1 - F(x)] < 0$

$$\frac{\partial B_n(\underset{\sim}{x},y)}{\partial y} \ge -r[1 - F(x+y)] + \theta G_m(y;\underset{\sim}{x}) - \alpha\theta \sum_{j=1}^{m} \int_{w_{j-1}}^{w_j} G_{m-1}(\underset{\sim}{z}_j(t))f(t)dt$$

(where $w_m = x + y$) by the inductive assumption on (8,a)

$= -r[1 - F(x+y)] + \theta G_m(y;\underset{\sim}{x})[1 - \alpha]$ by COROLLARY 2.1.

Hence $\lim_{y \to +\infty} \frac{\partial B_n(\underset{\sim}{x},y)}{\partial y} \ge 1 - \alpha > 0$. Now

$$\frac{\partial^2 B_n(\underset{\sim}{x},y)}{\partial y^2} = rf(x+y) + \theta G_m^{(1)}(y;\underset{\sim}{x}) + \alpha \sum_{j=1}^{m} \int_{w_{j-1}}^{w_j} C_{n-1}^{(1,1)}(\underset{\sim}{z}_j(t))f(t)dt.$$

Note that when f has a discontinuity at zero $C_{n-1}^{(1,1)}$ will also have a discontinuity at zero in its first argument via the inductive assumption on (6,a). The differentiation under the integral sign is a problem only when $j = m$ and is justified by a lemma appearing in Van Zyl [15]. The boundedness of the first and second partials follows as long as f is bounded.

$$\frac{\partial^2 B_n(\underset{\sim}{x},y)}{\partial y^2} \ge rf(x+y) + \theta G_m^{(1)}(y;\underset{\sim}{x}) - \alpha\theta \sum_{j=1}^{m} \int_{w_{j-1}}^{w_j} G_{m-1}^{(1)}[\underset{\sim}{z}_j(t)]f(t)dt$$

by the inductive assumption on (7)a) with $j = 1$
$j = 1$

$= rf(x+y) + \theta G_m^{(1)}(y;\underset{\sim}{x})[1 - a] \ge 0,$

by COROLLARY 2.1.

Hence $y_n(\underset{\sim}{x})$ satisfies $\left.\frac{\partial B_n(x,y)}{\partial y}\right|_{y=y_n(\underset{\sim}{x})} = 0.$

The fact that $B_n(\underset{\sim}{x},y)$ is strictly convex in a neighborhood of $y_n(\underset{\sim}{x})$ follows from the fact that $x + y_n(\underset{\sim}{x}) > 0$. This is obviously true when $x \ge 0$. If $x < 0$ then $x_{m-1} < 0$ and $x_i = 0$, $1 \le i \le m-2$. Assume $y_n + x < 0$, $(y_n + x_{m-1} < 0)$.

$$\left.\frac{\partial B_n(\underset{\sim}{x},y)}{\partial y}\right|_{y=y_n(\underset{\sim}{x})} = -r + \theta \int_0^{y_n} G_{m-1}(v + x_{m-1};\underset{\sim}{0})f(y_n - v)dv + \alpha \sum_{j=1}^{m} \int_{w_{j-1}}^{w_j} C_{n-1}^{(1)}[\underset{\sim}{z}_j(t)]f(t)dt$$

$$\le -r + \theta \int_0^{y_n} G_{m-1}(v + x_{m-1};0)f(y_n - v)dv = -r < 0.$$

Notice the second term is identically zero since $v + x_{m-1} \le 0$ for all $v \in [0,y_n]$. Hence this contradicts the definition of y_n so that $x + y_n > 0$, and $B_n(\underset{\sim}{x},y)$ is strictly convex in a neighborhood of y_n. (3) The existence and uniqueness of y_n follows.

Again define $T_n(\underset{\sim}{x},y) = \frac{\partial B_n(\underset{\sim}{x},y)}{\partial y}$

$T_n(\underset{\sim}{x},y)$ is continuously differentiable over R^m via COROLLARY 1.1 and the inductive assumption on (6). Since y_n solves $T_n(\underset{\sim}{x},y_n)=0$ for all $\underset{\sim}{x}$ and

$$\left.\frac{\partial T_n(\underset{\sim}{x},y)}{\partial y}\right|_{y=y_n} = \left.\frac{\partial^2 B_n(\underset{\sim}{x},y)}{\partial y^2}\right|_{y=y_n} > 0$$

it follows from the Implicit Function Theorem that $y_n^{(i)}$ exists and is continuous for all $\underset{\sim}{x}$ and for $1 \le i \le m-1$.

(4) From the above it follows that

$$C_n(\underset{\sim}{x}) = \inf_{y \geq 0} \{B_n(\underset{\sim}{x},y)\} = \min_y\{B_n(\underset{\sim}{x},y)\} = B_n(\underset{\sim}{x},y_n) \text{ so that}$$

$$C_n^{(i)}(\underset{\sim}{x}) = -r[1 - F(x + y_n)] + \theta G_m(y_n;\underset{\sim}{x}) - \theta \sum_{j=1}^{i} G_{m-j}(\underset{\sim}{x}(m-j))H_j(y_n;\bar{\underset{\sim}{x}}(m-j))$$

$$+ \frac{\partial}{\partial x_{m-i}}\left[\alpha \int_0^\infty C_{n-1}[\underset{\sim}{s}(y,\underset{\sim}{x},t)]f(t)dt\right] \text{ by LEMMA 2.1.}$$

If $1 \leq i \leq m - 2$, then

$$\frac{\partial}{\partial x_{m-i}}\left[\int_0^\infty C_{n-1}[\underset{\sim}{s}(y,\underset{\sim}{x},t)]f(t)dt\right] = \sum_{j=1}^{m-i} \int_{w_{j-1}}^{w_j} C_{n-1}^{(i+1)}[\underset{\sim}{z}_j(t)]f(t)dt$$

$$+ \sum_{j=m-i+1}^{m-1} \int_{w_{j-1}}^{w_j} C_{n-1}^{(m-j+1)}[\underset{\sim}{z}_j(t)]f(t)dt + \int_{w_{m-1}}^{\infty} C_{n-1}^{(1)}[\underset{\sim}{z}_m(t)]f(t)dt$$

$$+ C_{n-1}[\underset{\sim}{z}_{m-i}(w_{m-i})]f(w_{m-i}) + \sum_{j=m-i+1}^{m-1}\{C_{n-1}[\underset{\sim}{z}_j(w_j)]f(w_j) - C_{n-1}[\underset{\sim}{z}_{j-1}(w_{j-1})]$$

$$\cdot f(w_{j-1})\} - C_{n-1}[\underset{\sim}{z}_{m-1}(w_{m-1})]f(w_{m-1})$$

which is a consequence of the definition of $\underset{\sim}{s}(y,\underset{\sim}{x},t)$. Notice that all terms involving differentiation of limits drop out.

If we agree to interpret $C_{n-1}^{(m)}(\underset{\sim}{x}) = 0$, then for $1 \leq i \leq m - 1$ we can write

$$C_n^{(i)}(\underset{\sim}{x}) = -r[1 - F(x + y_n)] + \theta G_m(y_n;\underset{\sim}{x}) - \theta \sum_{j=1}^{i} G_{m-j}(\underset{\sim}{x}(m-j))H_j(y_n;\bar{\underset{\sim}{x}}(m-j))$$

$$+ \alpha \sum_{j=1}^{m-i} \int_{w_{j-1}}^{w_j} C_{n-1}^{(i+1)}[\underset{\sim}{z}_j(t)]f(t)dt + \alpha \sum_{j=m-i+1}^{m-1} \int_{w_{j-1}}^{w_j} C_{n-1}^{(m-j+1)}[\underset{\sim}{z}_j(t)]$$

$$\cdot f(t)dt + \alpha \int_{w_{m-1}}^{\infty} C_{n-1}^{(1)}[\underset{\sim}{z}_m(t)]f(t)dt$$

$$= \left\{-r[1 - F(x + y_n)] + \theta G_m(y_n;\underset{\sim}{x}) + \alpha \sum_{j=1}^{m-1} \int_{w_{j-1}}^{w_j} C_{n-1}^{(1)}[\underset{\sim}{z}_j(t)]f(t)dt\right.$$

$$\left. + \alpha \int_{w_{m-1}}^{\infty} C_{n-1}^{(1)}[\underset{\sim}{z}_m(t)]f(t)dt\right\} - \theta \sum_{j=1}^{i} G_{m-j}(\underset{\sim}{x}(m-j))H_j(y_n;\bar{\underset{\sim}{x}}(m-j))$$

$$+ \alpha \sum_{j=1}^{m-i} \int_{w_{j-1}}^{w_j} \left(C_{n-1}^{(i+1)}[\underset{\sim}{z}_j(t)] - C_{n-1}^{(1)}[\underset{\sim}{z}_j(t)]\right)f(t)dt$$

$$+ \alpha \sum_{j=m-i+1}^{m-1} \int_{w_{j-1}}^{w_j} \left(C_{n-1}^{(m-j+1)}[\underset{\sim}{z}_j(t)] - C_{n-1}^{(1)}[\underset{\sim}{z}_j(t)]\right)f(t)dt.$$

The proof is completed by recognizing that the term in brackets is

$$\left.\frac{\partial B_n(\underset{\sim}{x},y)}{\partial y}\right|_{y=y_n} = 0.$$

(5) From (1) and (2) we have that y_n satisfies

$$-r[1 - F(x + y_n)] + \theta G_m(y_n;\underset{\sim}{x}) + \alpha \sum_{j=1}^{m} \int_{w_{j-1}}^{w_j} C_{n-1}^{(1)}[\underset{\sim}{z}_j(t)]f(t)dt = 0$$

where $x_m = y_n$. Notice that the upper limit becomes $x + y_n$ on the last sum via the inductive assumption on (8,d).

Differentiating implicitly with respect to x_{m-1} we obtain

$$rf(x + y_n)[1 + y_n^{(i)}] + \theta G_m^{(1)}(y_n;\underset{\sim}{x})y_n^{(i)} + \theta G_m^{(i+1)}(y_n;\underset{\sim}{x})$$

$$+ \alpha \sum_{j=1}^{m} \int_{w_{j-1}}^{w_j} C_{n-1}^{(1,1)}(\underset{\sim}{z}_j(t))f(t)dt \cdot y_n^{(i)} + \alpha \sum_{j=1}^{m-i} \int_{w_{j-1}}^{w_j} C_{n-1}^{(1,i+1)}(\underset{\sim}{z}_j(t))f(t)dt$$

$$+ \alpha \sum_{j=m-i+1}^{m} \int_{w_{j-1}}^{w_j} C_{n-1}^{(1,m-j+1)}(\underset{\sim}{z}_j(t))f(t)dt + \alpha C_{n-1}^{(1)}[\underset{\sim}{z}_{m-i}(w_{m-i})]f(w_{m-i})$$

$$+ \alpha \sum_{j=m-i+1}^{m} \{C_{n-1}^{(1)}[\underset{\sim}{z}_j(w_j)]f(w_j) - C_{n-1}^{(1)}(\underset{\sim}{z}_{j-1}(w_{j-1}))f(w_{j-1})\}$$

(where $C_{n-1}^{(1,m)} \equiv 0$). The terms involving differentiation of the limits (i.e. the last line above) reduce to $C_{n-1}^{(1)}[\underset{\sim}{z}_m(w_m)]f(w_m)$, where $\underset{\sim}{z}_m(w_m) = \underset{\sim}{0}$ so that this term is zero by the inductive assumption on (8,d). Hence

$$y_n^{(i)}(\underset{\sim}{x}) = -\frac{N_i(y_n;\underset{\sim}{x})}{D(y_n;\underset{\sim}{x})} \quad \text{where}$$

$$N_i(y_n;\underset{\sim}{x}) = [rf(x + y_n) + \theta G_m^{(i+1)}(y_n;\underset{\sim}{x}) + \alpha \sum_{j=1}^{m-i} \int_{w_{j-1}}^{w_j} C_{n-1}^{(1,i+1)}(\underset{\sim}{z}_j(t))f(t)dt + \alpha \sum_{j=m-i+1}^{m} \int_{w_{j-1}}^{w_j} C_{n-1}^{(1,m-j+1)}$$

$$(\underset{\sim}{z}_j(t))f(t)dt] \quad \text{for } i \leq m - 1$$

(with $C_n^{(1,m)}(\underset{\sim}{x}) \equiv 0$ and

$$D(y_n;\underset{\sim}{x}) = rf(x + y_n) + \theta G_m^{(1)}(y_n;\underset{\sim}{x}) + \alpha \sum_{j=1}^{m} \int_{w_{j-1}}^{w_j} C_{n-1}^{(1,1)}(\underset{\sim}{z}_j(t))f(t)dt.$$

Now

$$N_i(y_n;x) \geq rf(x + y_n) + \theta G_m^{(i+1)}(y_n;\underset{\sim}{x})$$

$$- \alpha\theta \sum_{j=1}^{m-i} \int_{w_{j-1}}^{w_j} G_{m-1}^{(i+1)}[\underset{\sim}{z}_j(t)]f(t)dt - \alpha\theta \sum_{j=m-i+1}^{m} \int_{w_{j-1}}^{w_j} G_{m-1}^{(m-j+1)}$$

$$\cdot[\underset{\sim}{z}_j(t)]f(t)dt \quad \text{by the inductive assumption on (7,a)}$$

$$= rf(x + y_n) + \theta(1 - \alpha)G_m^{(i+1)}(y_n;\underset{\sim}{x}) > 0$$

which results by differentiating both sides of the equality of COROLLARY 2.1 with respect to x_{m-i-1} (see the comments following the Corollary). Hence $-N_i(y_n;\underset{\sim}{x}) < 0$.

Similarly, $D(y_n;\underset{\sim}{x}) \geq rf(x + y_n) + \theta(1 - \alpha)G_m^{(1)}(y_n;\underset{\sim}{x}) > 0$

so that $y_n^{(i)}(\underset{\sim}{x}) < 0$.

One can show that if we define $D(y_n;x) = N_0(y_n;x)$ then

$N_{i-1}(y_n;x) - N_i(y_n;x) \ge 0$ for $1 \le i \le m - 1$.

From this the result follows.

(6) a) The result follows directly by computing the second cross partial derivatives using the expression of part (4) COROLLARY 1.1, and the inductive assumption on 6(a).

(6)b) $$c_n^{(i)}(x) - c_n^{(i-1)}(x) = -\theta G_{m-i}(x(m-i))H_i(y_n;\bar{x}(m-i)) + \alpha \sum_{j=1}^{m-i} \int_{w_{j-1}}^{w_j} [c_{n-1}^{(i+1)}(z_j(t)) - c_{n-1}^{(i)}(z_j(t))]f(t)dt.$$

If the last $(m-1)$ components of x are 0 then it follows that $w_j = w_{j-1} = 0$ for $2 \le j \le m - 1$ so the result follows easily.

(7) The inequalities of sections (7) and (8) represent the major analytical effort of the proof. The methods used for each section are quite similar and require the application of the appropriate inductive assumptions and THEOREM 2.1. Detailed proofs of each section can be found in Nahmias [12]. The following chart indicates the logical implications

to prove	use the induction assumption on
7(a)	7(c) with i = 2
7(b)	7(b) and 7(c) if i < m - 1. 7(a) if i = j = m - 1.
7(c)	7(c) and 7(d) if i < m - 1. 7(b) if i - m - 1.
7(d)	7(c) and 7(d)
8(a)	8(c) with j = 1 if i < m - 1. 8(b) and 8(c) if i = m - 1.
8(b)	8(b) if i < m - 1. 8(a) and 8(b) if i = m - 1.
8(c)	8(c)

In addition, the various lemmas are used where needed.

(9) a) y_n is defined implicitly by the relationship $-r[1 - F(x + y_n)] + \theta G_m(y_n;x)$

$$+ \alpha \sum_{j=1}^{m} \int_{w_{j-1}}^{w_j} c_{n-1}^{(1)}[z_j(t)]f(t)dt = 0.$$

Assume that $\lim_{x_{m-i}\to+\infty} y_n(x) = w > 0$. Notice that

$$\lim_{x_{m-i}\to+\infty} \sum_{j=1}^{m} \int_{w_{j-1}}^{w_j} c_{n-1}^{(1)}(z_j(t))f(t)dt$$

$$\ge \lim_{x_{m-i}\to+\infty} \sum_{j=1}^{m} \int_{w_{j-1}}^{w_j} G_{m-1}[z_j(t)]f(t)dt$$

$$= \lim_{x_{m-i}\to+\infty} G_m(w;x_{m-1},\dots,x_1)$$

where $\bar{z}_j(t)$ is identical to $z_j(t)$ except that w is substituted for y_n. The first equality follows from the inductive assumption on (8,a) and the last equality from COROLLARY 2.1.

$= G_{m-i}(w;\bar{x}(m-i))$ from COROLLARY 1.2.

Hence letting $x_{m-i}\to+\infty$ in the defining relationship for y_n we get

$$0 = \lim_{x_{m-i}\to+\infty} \Big[-r[1 - F(x + y_n)] + \theta G_m(y_n;x) + \alpha \sum_{j=1}^{m} \int_{w_{j-1}}^{w_j} c_{n-1}^{(1)}[z_j(t)]f(t)dt\Big]$$

$$\ge 0 + \theta G_{m-1}(w;\bar{x}(m-i)) - \alpha\theta G_{m-i}(w;\bar{x}(m-i)) > 0,$$

which follows from the arguments above and a reapplication of COROLLARY 1.2. But this contradicts the definition of y_n. Hence

$$\lim_{x_{m-i}\to+\infty} y_n(x) = 0.$$

(9) b) $$c_n^{(i)}(x) - c_n^{(i-1)}(x) = -\theta G_{m-i}(x(m-i))H_i(y_n;\bar{x}(m-i)) + \alpha \sum_{j=1}^{m-i} \int_{w_{j-1}}^{w_j} \{c_{n-1}^{(i+1)}[z_j(t)] - c_{n-1}^{(i)}[z_j(t)]\}f(t)dt$$

so that $\lim_{x_j\to+\infty} \{c_n^{(1)}(x) - c_n^{(i-1)}(x)\} = 0$ by (9,a) and the inductive assumption on (9,b).

Since $\lim_{x_j\to+\infty} c_n^{(1)}(x) = 0$ by a similar argument it follows that $\lim_{x_j\to+\infty} c_n^{(i)}(x) = 0$.

We have assumed that excess demand is backlogged. The mechanism for accounting for the backlog is to let $x = (x_{m-1}, 0)$ where $x_{m-1} < 0$. The following result is a key one regarding backlogged demand.

COROLLARY 2.2. If demand is backlogged (that is, $x = (x_{m-1}, 0)$ and $x_{m-1} < 0$), then

$$y_n(x) = y_n(0) + |x_{m-1}|.$$

Proof: We know that y_n satisfies

$$-r[1 - F(x_{m-1} + y_n)] + \theta G_m(y_n;x) + \alpha \int_0^\infty c_{n-1}^{(1)}[s(y,x,t)]f(t)dt = 0.$$

Now if $x = (x_{m-1}, 0)$ where $x_{m-1} < 0$ it follows that

$$G_m(y_n;x) = \int_0^{y_n} G_{m-1}(x_{m-1} + y_n - v;0)f(v)dv$$

$$= \int_0^{y_n+x_{m-1}} F^{(m-1)*}(x_{m-1} + y_n - v)f(v)dv$$

where $F^{(m-1)*} = \underbrace{F * F * \cdots * F}_{m-1 \text{ times}}$ (F convoluted with itself m - 1 times) and

$$\alpha \int_0^\infty C_{n-1}^{(1)}[\underset{\sim}{s}(y,\underset{\sim}{x},t)]f(t)dt = \alpha \int_0^{x_{m-1}+y_n} C_{n-1}^{(1)}[y_n + x_{m-1} - t, 0 \ldots 0]f(t)dt.$$

Assume a solution of the form $y_n(\underset{\sim}{x}) = y_n(\underset{\sim}{0}) + |x_{m-1}|$. Then

$$-r[1 - F(y_n(\underset{\sim}{0}))] + \theta \int_0^{y_n(\underset{\sim}{0})} F^{(m-1)*}(y_n(0) - v)f(v)dv + \int_0^{y_n(\underset{\sim}{0})} C_{n-1}^{(1)}[y_n(\underset{\sim}{0}) - t, 0 \ldots 0]f(t)dt = 0.$$

Since this is precisely the defining relationship for $y_n(\underset{\sim}{0})$ the assumed solution must have been valid.

COROLLARY 2.3. If we assume that demand is not backlogged (lost sales case) where the transfer function now has the form

$$s_i(y,\underset{\sim}{x},t) = [x_{i+1} - (t - \sum_{j=1}^{i} x_j)^+]^+ \qquad 1 \le i \le m-1$$

(where $x_m = y_n$) then all of the results of THEOREM 2.2 remain valid.

Throughout the development of this paper we have assumed that demand is stationary. It turns out that this assumption is not a crucial one for our analysis.

COROLLARY 2.4. If we assume demands are non-stationary then the results of THEOREM 2.2 remain valid (with suitable additional indices to account for demands).

In conclusion, we note that existence of optimal policies for the infinite horizon problem is a direct consequence of recent work in dynamic programming (see for example Denardo [6]). The methods used for the m = 2 case are reported in Nahmias and Pierskalla [11] and generalize directly to the case where m is an arbitrary positive integer.

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