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OPTIMAL ISSUING POLICIES FOR PERISHABLE INVENTORY*†

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Optimal issuing policies for some particular classes of perishable inventory problems are derived under several possible objective functions. The inventory considered is one in which stock is grouped into categories according to shelf age. Demand occurs for each of the categories, and may be satisfied by inventory units from that category or from any "younger" category. It is shown for most of the objective functions considered that the optimal policy is to issue the oldest unit which will satisfy the demand. A prime example of the classes of inventory problems considered in this paper is the issuance of whole blood from a hospital or central blood bank.

1. Introduction

Optimal issuing policies for some particular classes of perishable inventory problems are derived under several possible objective functions. The inventory considered is one in which the stock is grouped into categories according to age. Demand occurs for each of the categories, and may be satisfied by inventory units from that category or from any "younger" category. Demands are initially considered to be deterministic, and the resupply of the inventory is also considered to be known. Later these two assumptions are relaxed to permit stochastic demands and resupply. It is shown that for most of the objective functions considered, the optimal issuing policy is FIFO (First In, First Out). In this case the FIFO policy issues the oldest unit which will satisfy the demand feasibly.

The importance of studying these particular classes of inventory models can be seen from looking at the following example: Consider a hospital blood bank which must determine the optimal issuing and reorder policies for whole blood in order to meet the demands of the patients at the hospital. Many factors affect the ordering and issuing policies for whole blood. Some of these factors are

(a) the supply of blood is random, and comes from many different sources such as volunteers, corporate blood plans, paid donors, etc.,

(b) the demand for blood is random,

(c) much of the blood demanded is not used, and is returned to the inventory,

(d) blood is a perishable commodity and, for practical purposes, it is assumed to deteriorate to lower freshness categories on a step function basis over a 21- or 28-day horizon,

(e) the demand for blood of a particular category may be satisfied from a fresher category but not an older category.

It would be misleading to say that whole blood is the only important commodity with these perishability characteristics. Inventories of food products at the retail or wholesale store and at the home or commissary as well as many agricultural products and certain types of chemicals and drugs have some or all of these characteristics. Almost uniformly, previous papers dealing with perishable inventories have focused

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on the optimal ordering and issuing policies when demand and supply are assumed known. The early works of Greenwood [10], C. Derman and M. Klein [2], [3], G. J. Lieberman [15], P. Zehna [28], S. Eilon [4], [5], [6], [7] and W. Pierskalla [21], [22] were of this form. Their primary objective was to give general conditions when one or the other of two issuing policies, LIFO (Last In, First Out) or FIFO, was optimal. In his thesis, A. F. Veinott, Jr. [27] investigated optimal issuing, ordering and disposal policies under the assumptions of known demands and deterministic supply. Although some work was done treating the utility of the commodity at the time of issue as a random variable [2], [20], [23], [28], none of the studies of perishable inventory problems has entertained the interesting and relevant properties listed above for whole blood inventories.

Furthermore, some of the important work on the blood inventory problem done by Jennings [11], [12], Keeney [14], Elston [8], Elston and Pickrel [9] and Pegels [16] has not taken into account the previous work on perishable inventory control.

Thus, it is with the foregoing research in mind, and the fact that an important problem has not been resolved, that we provided the following results in partial answer to an interesting and important class of perishable inventory problems.

The next section gives the specific assumptions and the mathematical model underlying the relevant classes of inventory problems studied in this paper; also each objective function considered is discussed in some detail, and its relevance to the blood inventory example is brought out.

In §3 the results for the finite horizon discrete demand and discrete supply cases are stated with the proofs given in the appendix. Again, the example of the blood inventory problem is used to illustrate the relevance of these results and a numerical example is provided.

Finally, in the fourth section, generalization of the preceding models to other types of cost and utility functions is considered, and suggestions for future research are delineated.

2. The Model

The mathematical model which is developed and analyzed in the subsequent sections is based on the following assumptions:

(1) The items deteriorate over time on a step function basis; that is, the "freshness" of the items is a nonincreasing step function (in the case of blood inventory the "freshness" would be a function of the survival rate of red cells and chemicals and organic compounds in the blood).

(2) The demands for the items occur periodically and initially will be assumed to be known.

(3) Replenishment of the inventory may be made by items of any age; that is, not all items entering the inventory will be new but, indeed, may be several periods old, in which case their freshness depends on what age they are in regard to the step function freshness curve. (In the blood inventory this assumption is very important since much of the blood withdrawn from the bank is not used for the specific withdrawal purpose and after one or two days is returned to the bank. Obviously, the blood continues to deteriorate while it is held, hence it cannot be considered "new" when it reenters the bank.)

(4) The quantity of items added to the inventory is assumed to be known, initially. This assumption will be relaxed later.

(5) The demand for an item of a given freshness level may be satisfied from the given level or any higher level, *i.e.*, any "younger" item in the stock.

(6) The model is dynamic in the sense that there is a time horizon of n periods considered (n is any positive integer).

In a completely deterministic model we can easily require that no stockouts occur provided there are no limitations on the supply of items. On the other hand, stockouts are an important factor in any real problem. For this reason, it is necessary to make some assumptions about the demand process. The standard assumptions are (i) all demand is lost if there is a stockout, (ii) all demand is backlogged and filled from the next supply of items. We will show that in the case of backlogging, FIFO is the optimal policy for all objective functions considered in §2, and in the case of lost demand FIFO is optimal for two of the three objective functions.

In his thesis, A. F. Veinott, Jr. [27] investigated a model similar to the above model and determined the optimal issuing and disposal policies for his model. This model assumed that the utility function had the form of a nonincreasing concave-convex curve. A special case of this model then would be a nonincreasing step function consisting of only one step; that is, the utility function would be equal to a constant for a certain selected number of periods then it would drop down to be equal to another constant for all remaining periods. In this latter case he showed that in order to maximize the total utility the optimal issuing policy must be of the form issue all items lying on the upper utility level by means of FIFO after which one may use any policy for issuing the items in the lower level. Our problem is somewhat more complicated than the model analyzed by Veinott in that we permit many levels for the nonincreasing step function and, indeed, in the blood inventory case we must consider at least three such levels. Furthermore, we are interested in restocking the model with items of any age level, and only items of the given demanded age or younger may be used to fulfill the demand.

In order to state the problem mathematically, it is convenient to use the following notation:

(1) $n \equiv$ number of periods the process operates,

(2) $M \equiv$ number of age categories for the deteriorating item,

(3) $p_j - p_{j-1} \equiv$ length of age category j in periods for $j = 1, \cdots, M$ and $p_0 = 1$ (i.e., category j consists of all items of age less then p_j periods and greater than or equal to p_{j-1} periods),

(4) $V_j \equiv$ nonnegative value of one unit of stock in category j (by assumption (1), $V_1 \geqq V_2 \geqq \cdots \geqq V_M$),

(5) $R_i \equiv$ cumulative value in period i of all filled demands (insofar as possible) plus the value of the stock on hand at the end of period i

$$R_i = R_{i-1} + \textstyle\sum_{j=1}^{M} V_j D_{ij} + \sum_{j=1}^{M} V_j I_{ij} - \sum_{j=1}^{M} V_j I_{i-1,j}, \qquad i = 1, \cdots, n,$$

$D_{ij} \equiv$ total demand filled for items of category j in period i,

$I_{ij} \equiv$ the nonnegative inventory of items of category j remaining after demands are filled in period i,

$I_{0j} \equiv$ initial inventory and $R_0 \equiv \sum_{j=1}^{M} V_j I_{0j}$,

(6) $R = (R_1, R_2, \cdots, R_n)$,

(7) $S_{ij} \equiv$ nonnegative stockout of items of category j at the end of period i,

(8) P is any policy which states which items are to be used to fill the demands. P is a feasible issuing policy if all demands are satisfied whenever there are items in stock to satisfy the demands.

FIFO is the feasible policy which issues the oldest item satisfying assumption (5).

LIFO is the feasible policy which issues the youngest item satisfying assumption (5).

The question of an appropriate objective function in the blood inventory example or in the perishable inventory problem in general is an important consideration. Three different objective functions are described below, and arguments can be presented for their use and appropriateness. Fortunately, in the case of backlogged demand there is no need to weigh their relative merits to make a choice among them, since the optimal issuing policy is the same for each. However, in the lost demand case FIFO is optimal for only the last two objective functions, and, as is shown by the example, FIFO is not optimal for the first objective function.

The first objective function considered is based on a utility approach. The total current utility of the system is defined as the value of all demands satisfied in the past plus the value of items in stock at present. The value of demand filled is linear, and depends on the category of the demand. It is assumed that satisfaction of a demand for a "younger" category has a higher value. In the blood bank problem this would indicate the more critical uses of the fresher blood. The value for filling a particular demand is then assumed to be the same whether the freshest or the oldest item (which can be used to fill the demand) is used, since the value is derived from the use of the item which the demand specifies. The value of the stock in inventory is defined as the value of the highest demand which it can fill, i.e., the value of the stock is its "potential" value in satisfying demands. We then have a series of values per item, $V_1, V_2, \cdots, V_M$, where V_1 is value of the freshest category per item, V_2 is the value of the second freshest category, etc., with the property $V_1 \geqq V_2 \geqq V_3 \geqq \cdots \geqq V_M \geqq 0$. It is shown that regardless of the actual values of the V_i's, as long as the above inequalities are maintained, FIFO maximizes the utility of the system provided all excess demand is backlogged.

The second objective function considered is to minimize the total number of units backlogged at any given time or, in the lost demand cases, minimize the total number of lost demands. It is shown that FIFO again optimizes this objective over all feasible issuing policies.

The third objective function is to minimize the total number of items reaching the last age category. This last category could be construed as an overage category where items in the category are too old to be used. Once again, FIFO is shown to be the optimal policy.

3. FIFO Optimality

In this section the notation $D_{ij}^F, I_{ij}^F, S_{ij}^F$ denote the demand, inventory and stockout for items of category j in period i when a FIFO policy is followed. If the superscript F is not used then an arbitrary feasible policy is being followed.

THEOREM 1. *If all excess demand is backlogged, then FIFO maximizes R_i for all $i = 1, \cdots, n$ and all feasible issuing policies.*

Another way of stating Theorem 1 is for any $V_1 \geqq V_2 \geqq \cdots \geqq V_M \geqq 0$ and if excess demand is backlogged, then $R_i^F \geqq R_i$ for all $i = 1, \cdots, n$. Now

$$\begin{aligned} R_i^F &= R_{i-1}^F + \textstyle\sum_{j=1}^M V_j(D_{ij}^F + I_{ij}^F - I_{i-1,j}^F) \\ &= \textstyle\sum_{t=1}^i \sum_{j=1}^M V_j D_{tj}^F + \sum_{j=1}^M V_j I_{ij}^F . \end{aligned}$$

Thus

$$R_i^F = \sum_{j=1}^M V_j \left(\sum_{t=1}^i D_{tj}^F + I_{ij}^F\right) \geqq \sum_{j=1}^M V_j \left(\sum_{t=1}^i D_{tj} + I_{ij}\right) = R_i .$$

Letting $V_1 = V_2 = \cdots = V_k = 1$ and $V_{k+1} = \cdots = V_M = 0$ we see for all k

LEMMA 1. *If all excess demand is backlogged, then*

$$\sum_{j=1}^k \sum_{t=1}^i (D_{tj}^F - D_{tj}) \geqq \sum_{j=1}^k (I_{ij} - I_{ij}^F)$$

for all $i = 1, \cdots, n$ *and all* $k = 1, \cdots, M$.

Note that $\sum_{j=1}^k \sum_{t=1}^i D_{tj}$ is the total demand *filled* for items of category k or less through period i for an arbitrary policy. Now the total demand through period i for items of category k or less is:

(i) when backlogging is allowed

$$\sum_{t=1}^i \sum_{j=1}^k D_{tj} + \sum_{j=1}^k S_{ij}$$

regardless of what feasible policy is followed, *i.e.*,

$$\sum_{j=1}^k \left(\sum_{t=1}^i D_{tj} + S_{ij}\right) = \sum_{j=1}^k \left(\sum_{t=1}^i D_{tj}^F + S_{ij}^F\right);$$

(ii) when excess demand is lost

$$\sum_{t=1}^i \sum_{j=1}^k (D_{tj} + S_{tj}),$$

regardless of what feasible policy is followed, *i.e.*,

$$\sum_{t=1}^i \sum_{j=1}^k (D_{tj} + S_{tj}) = \sum_{t=1}^i \sum_{j=1}^k (D_{tj}^F + S_{tj}^F).$$

LEMMA 2. *When backlogging is allowed,*

$$\sum_{j=1}^k (S_{ij} - S_{ij}^F) \geqq \sum_{j=1}^k (I_{ij} - I_{ij}^F)$$

for all $i = 1, \cdots, n$ *and* $k = 1, \cdots, M$.

Lemma 2 reveals some interesting properties about the cumulative differences in stockouts between FIFO and an arbitrary policy, say P. In the backlog case, the cumulative difference of the backlog amounts for items of category k or less *within* a period i is at least as great as the cumulative difference in their inventories. Roughly the lemma says that whenever a policy P has more inventory than FIFO then it has even more stockouts.

We now proceed to the next main result, namely, FIFO minimizes the total stockouts over all feasible issuing policies. Indeed we will see an even stronger result is true.

THEOREM 2. *For all feasible issuing policies,*

(i) *when backlogging is allowed,*

$$\sum_{j=1}^k S_{ij} \geqq \sum_{j=1}^k S_{ij}^F$$

for all $i = 1, \cdots, n$ *and* $k = 1, \cdots, M$,

(ii) *when excess demand is lost,*

$$\sum_{t=1}^i \sum_{j=1}^M S_{tj} \geqq \sum_{t=1}^i \sum_{j=1}^M S_{tj}^F$$

for all $i = 1, \cdots, n$.

Thus not only does FIFO minimize the total stockout $\sum_{j=1}^M S_{ij}$ and the total lost demand $\sum_{t=1}^i \sum_{j=1}^M S_{tj}$ through period i, but in the case of backlogging FIFO minimizes the cumulative stockout for all items of category k or less for all $k = 1, \cdots, M$.

LEMMA 3. *For all feasible issuing policies*

$$\sum_{j=1}^{M} I_{ij} \geqq \sum_{j=1}^{M} I_{ij}^{F} \quad \text{for all} \quad i = 1, \cdots, n.$$

Thus for both backlogging and lost demand FIFO minimizes the total inventory in each period. But perhaps a more important result is that FIFO minimizes the total inventory reaching the last age category M. Presumably the items in this last category are not useable.

THEOREM 3. *For all feasible issuing policies,*

$$I_{iM} \geqq I_{iM}^{F} \quad \text{for all} \quad i = 1, \cdots, n.$$

Thus when backlogging is allowed FIFO is optimal for each of the three objective functions considered. Because of this result, one does not need to choose among these objective functions nor does one have to consider weighted combinations of them. Unfortunately in the lost demand case, FIFO is only optimal when one wishes to minimize the total lost demand or minimize the number of items reaching age category M. The following example illustrates the conclusions reached in the above theorems and lemmas. In addition it points out that Theorem 1 and Lemmas 1 and 2 do not hold in the case of lost demands.

$$M = 4, \quad n = 5, \qquad V_1 = 3, \quad V_2 = 2, \quad V_3 = 1, \quad V_4 = 0,$$

$$p_0 = 1, \quad p_1 = 4, \quad p_2 = 7, \quad p_3 = 13, \quad p_4 = +\infty.$$

The inventory status at the end of each period ($i = 1, 2, 3, 4, 5$), the backlog by category ($j =$ I, II, III, IV) and the lost demand by category are given for FIFO and for a policy P where P issues the youngest item in the category for which the demand occurs until that category is exhausted, then proceeds to the next younger category and does the same.

This example was chosen to illustrate Theorems 1, 2 and 3 as well as to point out

TABLE 1

Initial Inventory and Additions to Inventory

Age of Items in Stock / Units	1	2	3	4	5	6	7	8	9	10	11	12
Initial Inventory	5	3	0	0	2	1	1	3	0	0	1	2
Additions to Inventory at beginning of period 2	4	0	0	0	0	0	0	0	0	1	0	0
3	7	0	0	0	0	0	0	0	0	0	1	5
4	2	3	0	0	4	0	2	0	0	2	0	3
5	6	1	1	1	0	0	0	0	1	0	0	0

TABLE 2

Demands (in units)

Period / Category	1	2	3	4	5
I	5	7	6	6	3
II	2	1	0	6	4
III	1	3	3	1	1
IV	0	0	8	0	1

that in the lost demand case Theorem 1 and Lemmas 1 and 2 do not hold. Note in Tables 3 and 4 that $R_i^F \geqq R_i^P$ for $i = 1, \cdots, 5$, but in Tables 5 and 6, $R_3^F < R_3^P$ and $R_4^F < R_4^P$. Similarly from Tables 3 and 4, it is easy to see that the results of Lemmas 1 and 2 are satisfied by the example; however, in Tables 5 and 6, if one sets $i = 4$ and $k = 1$ we see that

$$\sum_{j=1}^{1} \sum_{t=1}^{4} (D_{tj}^F - D_{tj}^P) = -1$$

and

$$\sum_{j=1}^{1} (I_{4j}^P - I_{4j}^F) = 0$$

show that Lemma 1 does not hold for lost demands, and

$$\sum_{j=1}^{1} \sum_{t=1}^{4} (S_{tj}^P - S_{tj}^F) = -1$$

shows that the result corresponding to Lemma 2 for lost demands does not hold.

TABLE 3

Backlog Inventory Status Using FIFO

Age of Items / Number of Items End of Period	I			II			III						IV		R_i	$\Sigma_{j=1}^{M} S_{ij}$	I_{iM}	FIFO Backlog by Category			
	1	2	3	4	5	6	7	8	9	10	11	12	13	14···				I	II	III	IV
1	3	0	0	0	1	0	1	3	0	0	1	1	0	0	37	0	0	0	0	0	0
2	0	0	0	0	0	0	0	1	2	0	0	0	0	1	49	0	1	0	0	0	0
3	0	0	0	0	0	0	0	0	0	0	0	0	0	0	67	0	0	0	0	0	0
4	0	0	0	0	0	0	2	0	0	2	0	2	0	0	97	3	0	1	2	0	0
5	0	0	0	0	0	0	0	2	1	0	1	0	1	0	118	1	1	0	1	0	0

TABLE 4

Backlog Inventory Status Using a Policy P

Age of Items / Number of Items End of Period	I			II			III						IV		R_i	$\Sigma_{j=1}^{M} S_{ij}$	I_{iM}	P Backlog by Category			
	1	2	3	4	5	6	7	8	9	10	11	12	13	14···				I	II	III	IV
1	0	3	0	0	0	1	0	3	0	0	1	2	0	0	37	0	0	0	0	0	0
2	0	0	0	0	0	0	0	0	1	1	0	1	2	0	47	1	2	0	1	0	0
3	0	0	0	0	0	0	0	0	0	0	0	0	0	0	67	0	0	0	0	0	0
4	0	0	0	0	0	0	1	0	0	2	0	3	0	0	97	3	0	1	2	0	0
5	0	0	0	0	0	0	0	0	1	0	2	0	2	0	117	1	2	0	1	0	0

TABLE 5

Lost Demand Inventory Status Using FIFO

Age of Items / Number of Items End of Period	I			II			III						IV		R_i	$\Sigma_{t=1}^{i} \Sigma_{j=1}^{M} S_{tj}$	I_{iM}	FIFO Lost Demand by Category			
	1	2	3	4	5	6	7	8	9	10	11	12	13	14···				I	II	III	IV
1	3	0	0	0	1	0	1	3	0	0	1	1	0	0	37	0	0	0	0	0	0
2	0	0	0	0	0	0	0	1	2	0	0	0	1	0	49	0	1	0	0	0	0
3	0	0	0	0	0	0	0	0	0	0	0	0	0	0	67	0	0	0	0	0	0
4	0	0	0	0	0	0	2	0	0	2	0	2	0	0	97	3	0	1	2	0	0
5	2	0	0	0	0	0	0	2	1	0	1	0	1	0	119	3	1	0	0	0	0

TABLE 6

Lost Demand Inventory Status Using a Policy P

Age of Items / Number of Items End of Period	I			II			III						IV		R_i	$\Sigma_{t=1}^{i} \Sigma_{j=1}^{M} S_{tj}$	I_{iM}	P Lost Demand by Category			
	1	2	3	4	5	6	7	8	9	10	11	12	13	14···				I	II	III	IV
1	0	3	0	0	0	1	0	3	0	0	1	2	0	0	37	0	0	0	0	0	0
2	0	0	0	0	0	0	0	0	1	1	0	1	2	0	47	1	2	0	1	0	0
3	1	0	0	0	0	0	0	0	0	0	0	0	0	0	68	1	0	0	0	0	0
4	0	0	0	0	0	0	1	0	0	2	0	3	0	0	98	3	0	0	2	0	0
5	0	1	1	0	0	0	0	0	1	0	2	0	2	0	116	3	2	0	0	0	0

Finally we note in Tables 3 and 4 that

$$\textstyle\sum_{j=1}^{M} S_{ij}^{F} \leqq \sum_{j=1}^{M} S_{ij}^{P} \quad \text{for all} \quad i = 1, \cdots, 5,$$

in Tables 5 and 6 that

$$\textstyle\sum_{t=1}^{i} \sum_{j=1}^{M} S_{tj}^{F} \leqq \sum_{t=1}^{i} \sum_{j=1}^{M} S_{ij}^{P} \quad \text{for} \quad i = 1, \cdots, 5$$

and in all cases

$$I_{iM}^{P} \geqq I_{iM}^{F} \quad \text{for} \quad i = 1, \cdots, 5.$$

4. Generalizations and Extensions

In the preceding sections it was assumed that the utility function was a decreasing step function with values $V_1 \geqq \cdots \geqq V_M \geqq 0$. If one examines the proofs of the earlier results, it becomes apparent that any nonnegative decreasing function $f(x)$ for $x \in [0, +\infty)$ will do provided there are only a finite number of demand categories ($j = 1, \cdots, M$). Indeed if $f(x)$ is strictly decreasing for all $x \in [0, +\infty)$, then FIFO is the only optimal policy (in continuous time).

A more important extension of the earlier results is to consider the demand and the additions to stock as stochastic processes. Here it is necessary to assume each process is independent of the issuing policy. Under this assumption the results of §3 still hold. Furthermore this assumption also implies that the amount of stock received and demands originating in period $i + 1$ are independent of the inventory on hand at the end of period i. In the case of the whole blood inventory it would be reasonable to assume that the demand for blood occurs independently of the issuing policy or the stock on hand. However the replenishment policy cannot be independent of the stock on hand. Indeed the less stock on hand means a larger amount must be ordered to meet the anticipated demands. If one holds the backlog to a constant regardless of the issuing policy, then from Lemma 2 it is apparent that

$$\textstyle\sum_{j=1}^{k} I_{ij}^{F} \geqq \sum_{j=1}^{k} I_{ij} \quad \text{for} \quad k = 1, \cdots, M \quad \text{and} \quad i = 1, \cdots, n.$$

That is, FIFO has a "younger" and "larger" stock on hand, and FIFO would therefore minimize the amount needed to meet the anticipated future demand.

Appendix

Proof of Theorem 1. The proof is given by construction. We will show that for any policy which is not FIFO through period i, we can construct another policy which is one unit closer to FIFO through period i and at least as good as the first policy. FIFO optimality follows through recursive application.

Consider any policy P which fills a demand for category k_1 ($k_1 = 1, \cdots, M$) in period i ($i = 1, \cdots, n$) with a unit q and there exists a unit q' such that

$$\text{age } (q') > \text{age } (q) \tag{1}$$

and q' can be used to fill the demand. Then we will show there is a feasible policy P' which fills the demand with q' and $R' \geqq R$ (we say the vector $R' \geqq R$ if and only if $R'_j \geqq R_j$ for $j = 1, \cdots, n$).

Define

$V_i^q \equiv$ value of unit q in period i,

$j \equiv$ period policy P issues item q' ($j = i + 1, \cdots, n + 1$),

$t \equiv$ first period P is stockout for a demand category k_2 (for convenience we call this demand k_2) which P' can fill with item q ($t = i + 1, \cdots, n + 1$),

$r \equiv$ the period in which P fills the demand k_2 with an item $\bar{q}$ ($r = t + 1, \cdots, n + 1$).

We use the convention of $j = n + 1, t = n + 1$ or $r = n + 1$, then the event describing j, t or r never occurs.

Note that $V_i^q = V_{k_1}$ if $p_{k_1} \leqq$ age $(q) \leqq p_{k_1+1} - 1$ and $V_i^q \geqq V_{k_1}$ in any case.

Case 1. $j < t$, *i.e.*, P issues q' and there are no intervening stockouts under P which P' can fill with item q. For all periods $u < i$, $R'_u = R_u$. For all periods $i \leqq u < j$, $R'_u - R_u = V_u^q - V_u^{q'} \geqq 0$ *i.e.*, P' still has q in stock and P has q' in stock, (1) holds and both policies have filled the same demand. On day j let P' issue q to satisfy the demand for which P issues q' each receiving value V_{k_1+L} ($L = 0, 1, 2, \cdots$) and henceforth the two policies issue the same items. Thus $R'_j - R_j = V_{k_1+L} = 0$. Since each policy has satisfied the same demands and has the same inventories remaining, $R'_u - R_u = 0$ for all $u > j$.

Case 2. $j \geqq t$, *i.e.*, before or at the same time P issues q' there is a stockout for demand k_2 in P, and P' issues q to satisfy k_2.

We have

$$V_t^q \geqq V_{k_2} \geqq V_t^{q'}, \tag{2}$$

since q can be used for k_2 and q' cannot.

Consider u such that $i \leqq u < t$. We still have $R'_u - R_u = V_u^q - V_u^{q'} \geqq 0$, since (1) holds.

Consider u such that $t \leqq u$. We have the situation that P' has issued q to meet demand k_2 and receives value V_{k_2}.

Subcase (a). $u < j$ *and* $u < r$. Then P has not issued q' and has not satisfied demand k_2 hence

$$R'_u - R_u = V_{k_2} - V_u^{q'} \geqq 0.$$

Subcase (b). $u < j$ *and* $u \geqq r$. Then P has not issued q' but P has satisfied demand k_2 with item $\bar{q}$. Then each policy has satisfied the same demands but P' has $\bar{q}$ in stock and P has q' in stock. Hence

$$R'_u - R_u = V_u^{\bar{q}} - V_u^{q'} \geqq 0$$

since $V_r^{\bar{q}} \geqq V_{k_2} \geqq V_r^{q'}$ implies $V_u^{\bar{q}} \geqq V_u^{q'}$ for all $u \geqq r$.

Subcase (c). $u \geqq j$ *and* $u < r$. Then P has issued q' for some demand k_3 and has received value V_{k_3} but P has not satisfied demand k_2. Now either

(i) P' satisfies demand k_3 with some item $\hat{q}$, then

$$R'_u - R_u = V_{k_2} + V_{k_3} - V_{k_3} - V_u^{\hat{q}} \geqq 0$$

since the age $(\hat{q})$ at time u is too old to satisfy demand k_2, or

(ii) P' cannot satisfy demand k_3, then

$$R'_u - R_u = V_{k_2} - V_{k_3} \geqq 0$$

since each policy has the same inventory.

Subcase (d). $u \geqq j$ *and* $u \geqq r$. Then P has satisfied demand k_2 with $\bar{q}$ and demand k_3 with q'. Let P' use $\bar{q}$ to satisfy demand k_3. Thus each policy has satisfied the same demands and has the same inventory so $R'_u - R_u = 0$.

PROOF OF LEMMA 1. By Theorem 1

$$R_i^F - R_i = \sum_{j=1}^{M} V_j[\sum_{t=1}^{i} (D_{tj}^F - D_{tj}) + I_{ij}^F - I_{ij}] \geqq 0, \qquad i = 1, \cdots, n,$$

for all V_j such that $V_1 \geqq V_2 \geqq \cdots \geqq V_M \geqq 0$.

Letting $V_1 = V_2 = \cdots = V_k = 1$ and $V_{k+1} = \cdots = V_M = 0$ we obtain

$$R_i^F - R_i = \sum_{j=1}^{k} [\sum_{t=1}^{i} (D_{tj}^F - D_{tj}) + I_{ij}^F - I_{ij}] \geqq 0.$$

PROOF OF LEMMA 2. In the case of backlogging

$$\sum_{t=1}^{i} \sum_{j=1}^{k} (D_{tj}^F - D_{tj}) = \sum_{j=1}^{k} (S_{ij} - S_{ij}^F) \quad \text{for} \quad k = 1, \cdots, M \quad \text{and} \quad i = 1, \cdots, n.$$

Thus by Lemma 1

$$\sum_{j=1}^{k} (S_{ij} - S_{ij}^F) \geqq \sum_{j=1}^{k} (I_{ij} - I_{ij}^F) \quad \text{for} \quad k = 1, \cdots, M \quad \text{and} \quad i = 1, \cdots, n.$$

PROOF OF THEOREM 2. Consider the backlog case. Assume to the contrary that

$$\sum_{j=1}^{k} (S_{ij} - S_{ij}^F) < 0 \tag{3}$$

for some i and k.

Let $k^* = \max \{k \mid (3) \text{ holds and } S_{ik}^F > 0\}$. Then

$$d_{k^*} = \sum_{j=1}^{k^*} (S_{ij}^F - S_{ij}) > 0$$

is the number of demands filled by P for items of category k^* or less that could not be filled by FIFO. Furthermore $I_{ij}^F = 0$ for $j = 1, \cdots, k^*$. Hence

$$\sum_{j=1}^{k^*} (I_{ij} - I_{ij}^F) = \sum_{j=1}^{k^*} I_{ij} \geqq 0$$

which by Lemma 2 contradicts (3) above.

The lost demand case is proved in a manner similar to that of Theorem 1. Let P and P' be two policies such that P issues item q and P' issues item q' to fill a demand k_1 where age $(q) <$ age (q'). Define State 1 to be the state such that each policy P and P' has the same number of stockouts. Since

$$\sum_{j=1}^{M} S_{0j} = \sum_{j=1}^{M} S'_{0j},$$

the process starts in State 1. Define State 2 to be the state such that policy P has one more stockout than policy P' and policy P has one more inventory item than policy P'. We will show that for $i = 1, \cdots, n$ the process must be in one of these two states. Assume the process is in State 1 in period i and P has issued q for k_1 and P' has issued q' for k_1.

Case 1. P issues q' for k. Let P' issue q for k. The process is still in State 1.

Case 2. P stocks out for k and P' stocks out for k'. The process is still in State 1.

Case 3. P stocks out for k and P' issues q for k.

Subcase (a). P still has q' in inventory. The process is in State 2 and P has one more stockout than P'.

Subcase (b). P issues q' for k' and P' stocks out for k'. The process is in State 1.

Subcase (c). P issues q' for k' and P' issues $\bar{q}$ for k'. Then P has $\bar{q}$ in inventory and is stockout for k. Thus the process is in State 2. Replacing q' by $\bar{q}$ in the preceding cases shows that the process is in either State 1 or 2 at all times. Hence

$$\sum_{t=1}^{i} \sum_{j=1}^{M} S_{tj} \geqq \sum_{t=1}^{i} \sum_{j=1}^{M} S'_{tj} \quad \text{for all} \quad i = 1, \cdots, n.$$

FIFO optimality follows by the recursive application of policies P'.

Proof of Lemma 3. Letting $p_M = +\infty$ then the total stock in the system through period i $(i = 1, \cdots, n)$ is:

$$\sum_{t=1}^{i} \sum_{j=1}^{M} D_{tj} + \sum_{j=1}^{M} I_{ij}$$

regardless of the issuing policy, *i.e.*

$$\sum_{t=1}^{i} \sum_{j=1}^{M} D_{tj}^{F} + \sum_{j=1}^{M} I_{ij}^{F} = \sum_{t=1}^{i} \sum_{j=1}^{M} D_{tj} + \sum_{j=1}^{M} I_{ij}$$

or

$$\sum_{t=1}^{i} \sum_{j=1}^{M} (D_{tj}^{F} - D_{tj}) = \sum_{j=1}^{M} (I_{ij} - I_{ij}^{F}).$$

But in the backlog case and Theorem 2

$$\sum_{t=1}^{i} \sum_{j=1}^{M} (D_{tj}^{F} - D_{tj}) = \sum_{j=1}^{M} (S_{ij} - S_{ij}^{F}) \geqq 0$$

and in the lost demand case and Theorem 2

$$\sum_{t=1}^{i} \sum_{j=1}^{M} (D_{tj}^{F} - D_{tj}) = \sum_{t=1}^{i} \sum_{j=1}^{M} (S_{ij} - S_{ij}^{F}) \geqq 0.$$

Hence $\sum_{j=1}^{M} (I_{ij} - I_{ij}^{F}) \geqq 0$.

Proof of Theorem 3. Consider the backlog case. Now

$$\sum_{t=1}^{i} \sum_{j=1}^{M} D_{tj}^{F} + \sum_{j=1}^{M} I_{ij}^{F} = \sum_{t=1}^{i} \sum_{j=1}^{M} D_{tj} + \sum_{j=1}^{M} I_{ij}. \tag{4}$$

By Lemma 1

$$\sum_{t=1}^{i} \sum_{j=1}^{M-1} D_{tj}^{F} + \sum_{j=1}^{M-1} I_{ij}^{F} \geqq \sum_{t=1}^{i} \sum_{j=1}^{M-1} D_{tj} + \sum_{j=1}^{M-1} I_{ij}. \tag{5}$$

Assuming there is no demand for category M items *i.e.* $D_{tM}^{F} = D_{tM} = 0$ for all $t = 1, \cdots, i$ then by (4) and (5) $I_{iM} \geqq I_{iM}^{F}$.

For the lost demand case, note in the proof of Theorem 2 (lost demand case) the two policies P and P' either have the same inventory or P' has q and P has q' or P has one more item than P'. If the inventory is the same, $I_{iM} = I'_{iM}$. If P' has q and P has q' and since age $(q) <$ age (q'), $I_{iM} \geqq I'_{iM}$. If P has one more item than P', $I_{iM} \geqq I'_{iM}$.

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