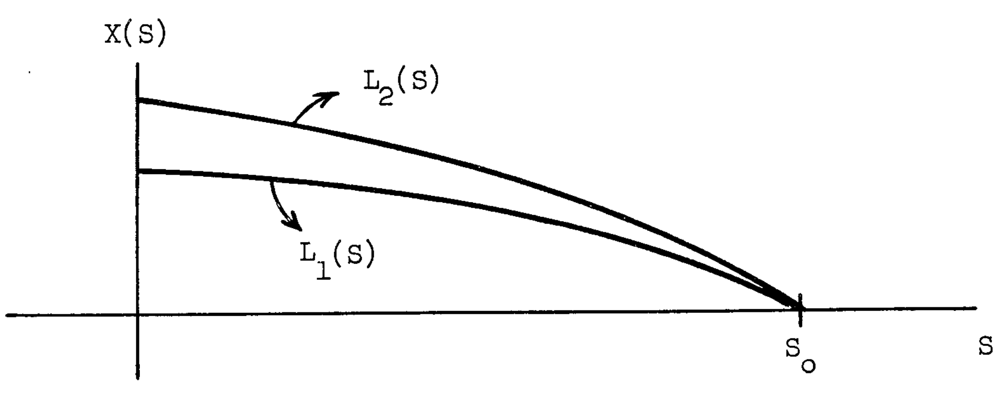
Chapter 6

A Stochastic Field Life Function

All of the previous results depend upon $L(S)$ being a known function. In practice, this assumption is often not satisfied. However, it may be the case that although the specific field life function for a given item may not be known, it may be known that any given item must obey one of a finite family of field life functions. More specifically we will assume that the field life of an item of age S on issuance is a nonnegative random variable, $X(S)$. $X(S)$ takes on a value $L_i(S)$ with probability p_i $(i = 1, \ldots, M, \sum_{i=1}^{M} p_i = 1, p_i \geq 0)$. Furthermore it will be assumed that $0 < L_1(S) < L_2(S) < \cdots < L_M(S)$ for all $S \epsilon [0, S_o)$ and $L_i(S) = L_{i+1}(S) = 0$ for all $S \epsilon [S_o, \infty)$ and for all $i = 1, \ldots, M - 1$.

Diagrammatically for $M = 2$ and $L_i(S)$ concave for $S \epsilon [0, S_o]$, we have



Thus with the other assumptions of the model (cf. Chapter 1) holding, the total field life, Q_{A_n}, for any given realization of the process and any policy A is a function of n dependent variables. A realization of the process is e.g. S_1 lies on L_2 when issued, S_2 lies on L_2 when issued, S_3 lies on L_1 when issued, etc. for S_4 through S_n.

The total expected return from policy A, U_{A_n}, is given by $U_{A_n} = E[Q_{A_n}]$. The objective of this stochastic process is to find the issue policy which maximizes U_n, the total expected field life.

The interpretation of this stochastic model follows easily. We have n items in inventory each of which deteriorates in accordance with one of M field life functions. It is not known which specific function an item will follow; however, the probability of it following a given function is fixed and independent of the age of the item. When an item is young, its range of possible values is large with respect to its actual field life, but as the items age those with lower quality initially, do not deteriorate as fast. The process of deterioration continues until all of the field life functions approach the same truncation point. When $p_i = 1$ for some i, this model reduces to the previous deterministic model.

Theorem 6.1: Let $X(S)$ be a random variable which takes on any one value of the M values $L_i(S)$ with probability p_i $(i = 1, 2, \ldots, M,$ $\sum_{i=1}^{M} p_i = 1,$ $p_i \geq 0)$. Let $\nu = 1$. Let $L_i(S)$ have the following properties:

(i) $L_i(S) = a_iS + b_i$ for all $S \epsilon [0, S_o]$ and $L_i(S) = 0$ for all $S \geq S_o$ and all $i = 1, \ldots, M$.

(ii) $b_M > b_{M-1} > \cdots > b_1 > 0 > a_1 > a_2 > \cdots > a_M > -1$

Then FIFO maximizes the total expected return for all $n \geq 2$.

Proof of Theorem 6.1: For any fixed policy there are M^n possible realizations of the total field life from the n items in the stockpile. Let $Q^j_{F_n}$ and $Q^j_{A_n}$ be the j^{th} such possible realization by FIFO and by an arbitrary policy A respectively, where the realizations are ordered from 1 to M^n on the basis of their probability of occurrence. Thus the probability of occurrence of $Q^j_{F_n}$, denoted by $P^{(j)}$ is identical to the probability of occurrence of $Q^j_{A_n}$. Thus

$$U_{F_n} - U_{A_n} = \sum_{j=1}^{M^n} (Q^j_{F_n} - Q^j_{A_n})P^{(j)} = E[Q_{F_n} - Q_{A_n}] . \qquad (6.1.1)$$

We must prove $U_{F_n} - U_{A_n} \geq 0$. It is sufficient to show

$$Q^j_{F_n} - Q^j_{A_n} \geq 0 \quad \text{for all } j = 1, \ldots, M^n . \qquad (6.1.2)$$

Let $n = 2$. Then for two items each realization must take one of these three cases:

(1) S_1 lies on $a_jS + b_j$ and S_2 lies on $a_jS + b_j$ for some $j = 1, \ldots, M$.

(2) S_1 lies on $a_jS + b_j$ and S_2 lies on $a_kS + b_k$ for some $j < k$.

(3) S_1 lies on $a_k S + b_k$ and S_2 lies on $a_j S + b_j$ for some $j < k$.

Note that since $L_j(S_o) = L_k(S_o) = 0$

$$\Rightarrow S_o = -\frac{b_j}{a_j} = -\frac{b_k}{a_k}$$

$$\Rightarrow a_j b_k = a_k b_j \quad \text{for all } j, k = 1, \ldots, M . \tag{6.1.3}$$

In addition, by lemma 2.1

$$L_k(S_1 + L_j(S_2)) > 0 \tag{6.1.4}$$

for all $j, k = 1, \ldots, M$.

<u>Case (1)</u>

(i) If $L_j(S_2 + L_j(S_1)) = a_j(S_2 + a_j S_1 + b_j) + b_j = 0$ then

$$Q_{F_2}^{(1)} - Q_{L_2}^{(1)} = a_j S_2 + b_j + a_j(S_1 + a_j S_2 + b_j) + b_j - a_j S_1 - b_j$$

$$= a_j S_2 + b_j + a_j^2 S_2 + a_j b_j$$

$$= (1 + a_j)(a_j S_2 + b_j) > 0$$

since $1 + a_j > 0$ and $a_j S_2 + b_j > 0$ by assumption (7) of the model.

(ii) If $L_j(S_2 + L_j(S_1)) > 0$ then

$$Q_{F_2}^{(1)} - Q_{L_2}^{(1)} = a_jS_2 + b_j + a_j(S_1 + a_jS_2 + b_j) + b_j$$
$$- [a_jS_1 + b_j + a_j(S_2 + a_jS_1 + b_j) + b_j]$$
$$= a_j^2(S_2 - S_1) > 0 \quad \text{since} \quad S_2 > S_1 .$$

Therefore $Q_{F_2}^{(1)} > Q_{L_2}^{(1)}$ in both subcases.

Case (2)

(i) If $L_k(S_2 + L_j(S_1)) = a_k(S_2 + a_jS_1 + b_j) + b_k = 0$ then

$$Q_{F_2}^{(2)} - Q_{L_2}^{(2)} = a_kS_2 + b_k + a_j(S_1 + a_kS_2 + b_k) + b_j - a_jS_1 - b_j$$
$$= (1 + a_j)(a_kS_2 + b_k) > 0$$

since $1 + a_j > 0$ and $a_kS_2 + b_k > 0$ by assumption (7).

(ii) If $L_k(S_2 + L_j(S_1)) > 0$ then using (6.1.3)

$$Q_{F_2}^{(2)} - Q_{L_2}^{(2)} = a_kS_2 + b_k + a_j(S_1 + a_kS_2 + b_k) + b_j$$
$$- [a_jS_1 + b_j + a_k(S_2 + a_jS_1 + b_j) + b_k]$$
$$= a_ja_k(S_2 - S_1) + a_jb_k - a_kb_j$$
$$= a_ja_k(S_2 - S_1) > 0$$

since $a_ja_k > 0$ and $S_2 - S_1 > 0$.

Therefore $Q_{F_2}^{(2)} > Q_{L_2}^{(2)}$ in both subcases.

<u>Case (3)</u>

(i) If $L_j(S_2 + L_k(S_1)) = 0$ then

$$Q_{F_2}^{(3)} - Q_{L_2}^{(3)} = a_jS_2 + b_j + a_k(S_1 + a_jS_2 + b_j) + b_k - a_kS_1 - b_k$$

$$= (1 + a_k)(a_jS_2 + b_j) > 0 .$$

(ii) If $L_j(S_2 + L_k(S_1)) > 0$ then using (6.1.3)

$$Q_{F_2}^{(3)} - Q_{L_2}^{(3)} = a_jS_2 + b_j + a_k(S_1 + a_jS_2 + b_j) + b_k$$

$$- [a_kS_1 + b_k + a_j(S_2 + a_kS_1 + b_k) + b_j]$$

$$= a_ja_k(S_2 - S_1) + a_kb_j - a_jb_k$$

$$= a_ja_k(S_2 - S_1) > 0 .$$

Therefore $Q_{F_2}^{(3)} > Q_{L_2}^{(3)}$ in both subcases.

Hence in all three cases (6.1.2) holds and

$$U_{F_2} > U_{L_2} .$$

Assume (6.1.2) holds for n, it will be shown that (6.1.2) holds for $n + 1$. That is we must show

$$Q_{F_{n+1}}^k - Q_{A_{n+1}}^k \geq 0 \quad \text{for all} \quad k = 1, \ldots , M^{n+1}.$$

Let S^* denote the initial age of the last item issued under policy A and let S^* have realization $L_r(S) = a_rS + b_r$ under outcome k.

Moreover, let x denote the total field life from the first n items issued under outcome k . Then

$$Q^k_{A_{n+1}}(x) = x + a_r(S^* + x) + b_r$$

and
$$\frac{dQ^k_{A_{n+1}}(x)}{dx} = 1 + a_r > 0 \; .$$

Hence $Q^k_{A_{n+1}}(x)$ is an increasing function of x , and $Q^k_{A_{n+1}}$ is maximized by making x as large as possible. But by the inductive assumption x is maximized by FIFO; therefore $Q^k_{A_{n+1}}$ is maximized by using FIFO on the first n items issued.

Now if $S^* \neq S_1$ then S_1 is issued next to last and since $Q^j_{F_2} - Q^j_{L_2} > 0$ for all $j = 1, \ldots , M^2$ then the total field life of $Q^{k^2}_{A_{n+1}}$ can be improved by interchanging the order of issue of the last two items. Hence S_1 is issued last and the first n items are issued by FIFO by the application, again, of the inductive assumption. Since k was arbitrary

$$Q^k_{F_{n+1}} - Q^k_{A_{n+1}} \geq 0 \quad \text{for all} \quad k = 1, \ldots , M^{n+1} .$$

Therefore $U_{F_{n+1}} - U_{A_{n+1}} \geq 0$ since A was any arbitrary policy. Thus FIFO maximizes the total expected return.

q.e.d.

Theorem 6.1 can be generalized to a countably infinite family of $L_i(S) = a_iS + b_i$ where $X(S)$ takes on any $L_i(S)$ with probability p_i

and $\sum_{i=1}^{\infty} p_i = 1$, $p_i \geq 0$ for all i . This generalization follows from the fact that $0 < Q^i_{A_n} \leq K < \infty$ for any policy A and all i , where K is a constant upper bound. Thus,

$$\begin{aligned} U_{A_n} &= \sum_{i=1}^{\infty} Q_{A_i} P^{(i)} \\ &= \sum_{k_1=1}^{\infty} \sum_{k_2=1}^{\infty} \cdots \sum_{k_n=1}^{\infty} Q_{A_{k_1 k_2 \cdots k_n}} p_{k_1} p_{k_2} \cdots p_{k_n} \\ &\leq K \sum_{k_1=1}^{\infty} \cdots \sum_{k_n=1}^{\infty} p_{k_1} \cdots p_{k_n} = K < \infty \end{aligned}$$

and

$$U_{F_n} - U_{A_n} = \sum_{i=1}^{\infty} (Q^i_{F_n} - Q^i_{A_n}) P^{(i)} < \infty$$

since also

$$0 < Q^i_{F_n} - Q^i_{A_n} < K \qquad \text{for all } i .$$

It was hoped that Theorem 6.1 could be extended to the general case for $L_i(S)$ concave nonincreasing. Repeated attempts to do this have not been successful. For one thing $Q^j_{F_n} - Q^j_{A_n} \not\geq 0$ for all j . However it is possible to establish a form of Theorem 6.1 for $L_i(S)$ concave nonincreasing when there are but two items in the stockpile.

Theorem 6.2: Let $X(S)$ be a random variable which takes on any one value of the M values $L_i(S)$ with probability p_i

$(i = 1, 2, \ldots, M, \ \sum_{i=1}^{M} p_i = 1, \ p_i \geq 0)$. Let $\nu = 1$. Let $n = 2$ items in the stockpile. Let $L_i(S)$ have the following properties:

(i) $L_i(S)$ is concave for all $S \epsilon [0, S_o]$

(ii) $L_i(S) = 0$ for all $S \epsilon [S_o, \infty)$ and $L_i(S) > 0$ for all $S \epsilon [0, S_o)$

(iii) $0 \geq L_1'(S) > L_2'(S) > \cdots > L_M'(S) \geq -1$ for $S \epsilon [0, S_o]$.

Then FIFO maximizes the total expected return.

Proof of Theorem 6.2: We must show $U_{F_2} - U_{L_2} \geq 0$. Define

$$Q_{F_{jk}} - Q_{L_{kj}} = L_j(S_2) + L_k(S_1 + L_j(S_2)) - [L_k(S_1) + L_j(S_2 + L_k(S_1))]$$

Then

$$U_{F_2} - U_{L_2} = \sum_{i=1}^{M^2} (Q^i_{F_2} - Q^i_{L_2}) P^{(i)} = E[Q_{F_2} - Q_{L_2}]$$

$$= \sum_{k=1}^{M} \sum_{j=1}^{M} (Q_{F_{jk}} - Q_{L_{kj}}) p_j p_k$$

$$= \sum_{k=1}^{M} (Q_{F_{kk}} - Q_{L_{kk}}) p_k^2 + \sum_{k=2}^{M} \sum_{j=1}^{k-1} [Q_{F_{jk}} - Q_{L_{kj}} + Q_{F_{kj}} - Q_{L_{jk}}] p_j p_k . \tag{6.1.5}$$

We will show

$$Q_{F_{kk}} - Q_{L_{kk}} \geq 0 \tag{6.1.6}$$

and

$$Q_{F_{jk}} - Q_{L_{kj}} + Q_{F_{kj}} - Q_{L_{jk}} \geq 0 \quad \text{for all } j, k; \ j \neq k \ . \tag{6.1.7}$$

Then (6.1.5) has $U_{F_2} - U_{L_2} \geq 0$ since $p_i \geq 0$ for all $i = 1, \ldots, M$.

Now (6.1.6) follows immediately from Lieberman [9] Theorem 3. Hence it is only necessary to prove (6.1.7). Let any j, k be given such that $j \neq k$, then

$$\begin{aligned} Q_{F_{jk}} - Q_{L_{kj}} + Q_{F_{kj}} - Q_{L_{jk}} &= L_j(S_2) + L_k(S_1 + L_j(S_2)) - L_k(S_1) \\ &\quad - L_j(S_2 + L_k(S_1)) + L_k(S_2) + L_j(S_1 + L_k(S_2)) \\ &\quad - L_j(S_1) - L_k(S_2 + L_j(S_1)) \ . \end{aligned} \tag{6.1.8}$$

To prove (6.1.8) is nonnegative, we first establish two relationships:

(1) By lemma 2.1, $S_1 + L_j(S_2) < S_o$ and $S_1 + L_k(S_2) < S_o$.

(2) $\dfrac{L_j(S_o) - L_j(S_2)}{S_o - S_2} \geq -1$

$\Rightarrow \ S_o \geq S_2 + L_j(S_2)$. Similarly

$S_o \geq S_2 + L_k(S_2)$.

<u>Case 1a</u>: $\qquad S_2 + L_k(S_1) < S_o$

Then since $L_i(\cdot)$ is concave nonincreasing

$$\frac{L_j(S_2) - L_j(S_1)}{S_2 - S_1} \geq \frac{L_j(S_2 + L_k(S_1)) - L_j(S_1 + L_k(S_1))}{S_2 - S_1}$$

$$\Rightarrow L_j(S_2) - L_j(S_1) \geq L_j(S_2 + L_k(S_1)) - L_j(S_1 + L_k(S_1))$$
$$\geq L_j(S_2 + L_k(S_1)) - L_j(S_1 + L_k(S_2))$$

since $S_2 - S_1 > 0$ and $L_j(S_1 + L_k(S_2)) \geq L_j(S_1 + L_k(S_1))$.
Hence we obtain

$$L_j(S_2) + L_j(S_1 + L_k(S_2)) - L_j(S_1) - L_j(S_2 + L_k(S_1)) \geq 0 . \quad (6.1.9)$$

<u>Case 1b</u>: $S_2 + L_k(S_1) \geq S_o$

Then

$$\frac{L_j(S_2) - L_j(S_1)}{S_2 - S_1} \geq \frac{L(S_o) - L_j(S_1 + L_k(S_2))}{S_o - (S_1 + L_k(S_2))}$$
$$\geq \frac{L(S_o) - L_j(S_1 + L_k(S_2))}{S_2 - S_1}$$

since by (2) above $S_o - L_k(S_2) \geq S_2$; hence $S_o - L_k(S_2) - S_1 \geq S_2 - S_1 > 0$ and since $L(S_o) - L_j(S_1 + L_k(S_2)) < 0$. But then we have

$$L_j(S_2) + L_j(S_1 + L_k(S_2)) - L_j(S_1) \geq 0 . \quad (6.1.10)$$

<u>Case 2a</u>: $S_2 + L_j(S_1) < S_o$

Then just as in case 1a with j and k interchanged, we obtain

$$L_k(S_2) + L_k(S_1 + L_j(S_2)) - L_k(S_1) - L_k(S_2 + L_j(S_1)) \geq 0 . \quad (6.1.11)$$

Case 2b: $S_2 + L_j(S_1) \geq S_o$

Then just as in case 1b with j and k interchanged, we obtain

$$L_k(S_2) + L_k(S_1 + L_j(S_2)) - L_k(S_1) \geq 0 \ . \qquad (6.1.12)$$

Then adding (6.1.9) to (6.1.11) or (6.1.12) we see that (6.1.7) holds. Or adding (6.1.10) to (6.1.11) or (6.1.12), again (6.1.7) holds. But we have exhausted all possibilities for (6.1.8). Hence (6.1.7) holds for all j, k, $j \neq k$ since j and k were arbitrary. Therefore

$$U_{F_2} - U_{L_2} \geq 0 \ .$$

q.e.d.

This theorem (6.2) also generalizes to the case where M is countably infinite.

A further generalization of Theorem 6.1 can be stated immediately.

Corollary 6.1: Let the hypotheses of theorem 6.1 hold. If no stockouts occur, then FIFO is optimal for the dynamic inventory depletion model.

Proof of Corollary 6.1: As shown in the proof of theorem 6.1, FIFO is optimal for any realization in the extended problem. Since no stockouts occur then FIFO for any realization in the original problem equals FIFO for the same realization in the extended problem. Now assume there exists some policy A which has a greater field life than FIFO for some realization of the original problem. But then policy A yields a greater total field life than FIFO in the extended problem since all policies of the original problem are contained in the set of all policies for the extended problem. Since FIFO is optimal for the extended problem we obtain a contradiction hence A cannot be optimal for the original problem. But A was arbitrary for any realization hence FIFO is optimal for the dynamic model.

q.e.d.

Chapter 7

Batches of Items of the Same Age in the Stockpile

It has always been assumed that the n items in inventory all have different initial ages. In general, this assumption is not necessary. With minor modifications, the theorems, lemmas and corollaries of the preceding chapters as well as most of the results of the papers referenced in the Bibliography can be stated for batches of items of the same age. More specifically we modify assumption (1) of the model as follows:

Assumption (1)': At the start of the process a stockpile has N sets of indivisible identical items where the items in the i^{th} set all have the same initial age, S_i, for $i = 1, \ldots, N$. The initial age of the items in any set, say the i^{th} set, is different than the initial age of the items in any other set. Assume $0 \leq S_1 < S_2 < \cdots < S_N < S_0$ and that the i^{th} set contains n_i items for $i = 1, \ldots, N$. Then $\sum_{i=1}^{N} n_i = n$.

For ease of adapting the previous results to the batch problem we make the following ordering of the n items.

Let the first n_1 items be numbered from 1 to n_1, the next n_2 items from $n_1 + 1$ to $n_1 + n_2$, the next n_3 items from $n_1 + n_2 + 1$ to $n_1 + n_2 + n_3$, etc. until all the items possess a number from 1 to n and such that $S_i \leq S_{i+1}$ for all $i = 1, \ldots, n-1$. We note that there are $\prod_{i=1}^{N} (n_i)!$ ways to complete the above ordering; hence choosing any one of these ways is somewhat arbitrary. However the total field life realized from the n items by any policy under any one of the $\prod (n_i)!$ ways is the same. Thus we choose an ordering and define FIFO (LIFO) as the policy which issues the highest (lowest) indexed item in the stockpile each time an item is issued.

We now prove a general theorem which applies to most of the previous results in inventory depletion theory.

<u>Theorem 7.1</u>: Let $L(S)$ be a continuous function. If FIFO (LIFO) is optimal when assumption (1) of the model holds, i.e., when there are no batches in the inventory, then FIFO (LIFO) is optimal when assumption (1)' holds, i.e., when batches are allowed.

<u>Proof of Theorem 7.1</u>: We will prove the theorem for FIFO; the theorem for LIFO follows <u>mutatis</u> <u>mutandis</u>.

Let $\epsilon_o = \min\limits_{1 \leq i \leq N-1} [S_{i+1} - S_i \, , \, S_o - S_N] > 0$. ϵ_o exists since $0 \leq S_1 < S_2 < \cdots < S_N < \infty$. Consider any ϵ such that

$$\epsilon_o > \epsilon > 0 \tag{7.1.1}$$

and consider the n items defined by

$$T_{ij} = S_i + \frac{\epsilon}{n_i - j + 1} \tag{7.1.2}$$

for all $j = 1, \ldots, n_i$ and $i = 1, \ldots, N$. Then from (7.1.2) we have

$$0 < T_{11} < T_{12} < \cdots < T_{1n_1} < T_{21} < \cdots < T_{N1} < \cdots < T_{Nn_N} < S_o .$$

(7.1.3)

Denote by $Q_F(\epsilon)$ and $Q_A(\epsilon)$ the total field life from the issuance of the n items of (7.1.3) by FIFO and by an arbitrary policy A, respectively. Denote by Q_{F_B} and Q_{A_B} the total field life from the issuance of the n items in batches by FIFO and by an arbitrary policy A, respectively.

Since $L(\cdot)$ is a continuous function, then Q_A is also a continuous function for any policy A.

Hence for any $\delta > 0$ and δ sufficiently small there is an $\epsilon > 0$ such that ϵ satisfies (7.1.1) and such that

$$|Q_{F_B} - Q_F(\epsilon)| < \delta$$

and

$$|Q_{A_B} - Q_A(\epsilon)| < \delta .$$

Then

$$Q_{F_B} + \delta > Q_F(\epsilon) \tag{7.1.4}$$

and

$$Q_A(\epsilon) > Q_{A_B} - \delta . \tag{7.1.5}$$

By hypothesis however

$$Q_F(\epsilon) \geq Q_A(\epsilon) \tag{7.1.6}$$

for all ϵ satisfying (7.1.1). Thus

$$Q_{F_B} > Q_{A_B} - 2\delta \; . \tag{7.1.7}$$

Now (7.1.4), (7.1.5), and (7.1.6) hold for all $\delta > 0$ where $Q_F(\epsilon)$ and $Q_A(\epsilon)$ are defined by the issuance of the items in (7.1.3). And since $\delta > 0$ can be made arbitrarily small, we have

$$Q_{F_B} \geq Q_{A_B}$$

for any arbitrary policy A.

q.e.d.

The foregoing proof also holds when we consider the stochastic case since U_A is the sum of continuous functions, Q_{A_i} ; hence U_A is also continuous. In the case of Chapter 6, the continuity of U_A in the countably infinite case follows from the dominated convergence theorem, i.e.

$$|U_{A_B} - U_A(\epsilon)| \leq \sum_{i=1}^{\infty} \left| Q_{A_{B,i}} - Q_{A_i}(\epsilon) \right| P^{(i)}$$

but since $\sum_{i=1}^{\infty} \left| Q_{A_{B,i}} - Q_{A_i}(\epsilon) \right| P^{(i)} \leq 2K$ where K is an upper bound for all Q_A and any A , then there exists an N such that

$\sum_{i=N+1}^{\infty} P^{(i)} < \frac{\epsilon}{4K}$ and we have

$$|U_{A_B} - U_A(\epsilon)| \leq \sum_{i=1}^{\infty} |Q_{A_{B,i}} - Q_{A_i}(\epsilon)| P^{(i)}$$

$$\leq \sum_{i=1}^{N} |Q_{A_{B,i}} - Q_{A_i}(\epsilon)| P^{(i)} + 2K\left(\frac{\epsilon}{4K}\right) .$$

But Q_A is continuous hence we can choose

$$|Q_{A_{B,i}} - Q_{A_i}(\epsilon)| < \frac{\epsilon}{2N}$$

then

$$|U_{A_B} - U_A(\epsilon)| \leq \sum_{i=1}^{N} |Q_{A_{B,i}} - Q_{A_i}(\epsilon)| P^{(i)} + \frac{\epsilon}{2}$$

$$\leq \sum_{i=1}^{N} P^{(i)} \frac{\epsilon}{2N} + \frac{\epsilon}{2}$$

$$< N\left(\frac{\epsilon}{2N}\right) + \frac{\epsilon}{2} = \epsilon .$$

We will now consider the batch problem relative to the results of each of the preceding chapters. As mentioned previously certain modifications to the hypotheses are in order. In the theorems, lemmas and corollaries where it was assumed that $L^-(S) \geq -1$ for all $S \epsilon (0, S_o]$ we will now assume

$$\frac{L(S) - L(S_o)}{S - S_o} > -1 \qquad \text{for all } S \epsilon [0, S_o) . \tag{7.1.8}$$

This assumption guarantees that none of the items under FIFO deteriorate to zero field life prior to issuance. Thus lemma 2.1 holds for the batch problem. Rather than restating and proving all of the results of the preceding chapters, we will just prove Lemma 2.1 under assumption (7.1.8) and for the remainder of the theorems and lemmas we will merely point out certain changes which are required in order to complete their proofs for the batch case. Some of the changes will not be listed since they are quite obvious when going through the proofs.

Proof of Lemma 2.1 assuming batches and (7.1.8): The proof will be by induction. Assume $N = 1$. Thus there are $n_1 = n$ items of the same age S_1, $0 \leq S_1 < S_o$. Now $L(S_1) > 0$ for the first item issued. Let x denote the total field life from the first $k - 1$ items issued by FIFO. Assume the lemma is true for the first k items issued, then $L(S_1 + x) > 0$ by the inductive assumption. By (7.1.8) we have

$$\frac{L(S_o) - L(S_1 + x)}{S_o - S_1 - x} > -1$$

which implies $S_1 + x + L(S_1 + x) < S_o$. But $S_1 + x + L(S_1 + x)$ is the age of the $k + 1^{st}$ item upon issuance; hence $L(S_1 + x + L(S_1 + x)) > 0$.

Now assume the lemma is true for $N - 1$. It will be proved true for N batches. Let y denote the total field life from the first $N - 1$ batches issued by FIFO but not including the last item issued in the last batch, i.e., y is the total field life from the FIFO issuance of the items indexed from n down through item $n_1 + 2$.

Then by the inductive assumption

$$L(S_2 + y) > 0$$

for item $n_1 + 1$ in the n_2^{nd} batch. By (7.1.8)

$$\frac{L(S_o) - L(S_2 + y)}{S_o - S_2 + y} > -1$$

implies

$$\begin{aligned} S_o &> S_2 + y + L(S_2 + y) \\ &> S_1 + y + L(S_2 + y) . \end{aligned} \qquad (7.1.9)$$

But $S_1 + y + L(S_2 + y)$ is the age of the first item in the n_1^{st} set issued by FIFO; hence $L(S_1 + y + L(S_2 + y)) > 0$. In addition (7.1.9) says that at the time of issue of the first item in the n_1^{st} set all of the items in the n_1^{st} set have age strictly less than S_o. Hence by the same argument given in the first paragraph of this proof all of the items in the n_1^{st} set have positive field life on issuance by FIFO.

q.e.d.

All of the theorems, lemmas, and corollaries of the preceding chapters can be adapted to the batch case using (7.1.8) when appropriate. If a theorem, lemma, or corollary is not mentioned in the following paragraphs it is because either the result needs no modification or else the modification is very slight and quite obvious. We will proceed chapter by chapter.

In lemmas 2.4 and 2.5 it must be assumed that the number of items in each batch for set I is the same as the number of items in the corresponding batch for set II. In theorem 2.5, it is necessary to consider several more cases since $S_{j_1} \leq S_{k_2}$. And in this theorem as well as elsewhere whenever the fact $S_i < S_j$ is used in

$$\frac{L(S_j + x) - L(S_i + x)}{S_j - S_i} \geq -1 \qquad (7.1.10)$$

to obtain $L(S_j + x) + S_j \geq L(S_i + x) + S_i$, then this latter inequality will hold even when $S_i = S_j$ and it is not necessary to use (7.1.10).

Most of the results in Chapter 3 are valid only with the addition of (7.1.8). However in lemmas 3.3, 3.4 and 3.5, it is necessary to allow $S_{j_k} \leq S_{j_{k+1}}$ for all $k = 1, \ldots, i - 1$. For theorem 3.6 the augmented set will now be $0 \leq S_1 \leq S_2 \leq \cdots \leq S_n < S_{n+1} < \cdots < S_{n+\nu}$, etc. In lemma 3.6 we can permit $S_1 \leq S_2$ and the conclusion $L^+(S_2 + L(S_1)) \geq -1$ is still valid. Finally, as in lemmas 2.4 and 2.5, we must assume for lemma 3.7 that the number of items in each batch for set I is the same as the number of items in the corresponding batch for set II.

For Chapter 4 we make the following changes. In lemmas 4.5 and 4.7 and in theorem 4.3 assume $S_i < \dfrac{c - b(1 + a)^{i-1}}{a(1 + a)^{i-1}}$. Furthermore in lemma 4.7 and part (c) of theorem 4.3 let $S_{i+1} \geq \dfrac{c - b(1 + a)^i}{a(1 + a)^i}$ and in part (a) of theorem 4.3 let $S_J < \dfrac{c - b(1 + a)^{[\frac{n-j}{\nu}]}}{a(1 + a)^{[\frac{n-j}{\nu}]}}$.

In Chapter 5 we modify (7.1.8) to

$$\frac{L(S) - L(S_o)}{S - S_o} > -1 \qquad \text{for all } S\epsilon(-\infty, S_o) . \tag{7.1.11}$$

Also batch additions to the inventory are permitted where each item in a given batch has field life $L(0)$ upon arrival at the stockpile. In the proof of lemma 5.1 note that if $S_i = S_j$ then $\frac{L(S_i) - L(S_o)}{S_i - S_o} = -1$ implies $L(S_i) + S_i = L(S_i) + S_j = S_o$; hence all items older than or the same age as S_i and issued after S_i has been issued can be omitted from consideration. The proof then goes through. In corollary 5.4 assume $S_{i_k} \geq S_{i_{k+1}}$ for all $k = 1, \ldots , j - 1$. Theorems 5.3 and 5.4 need a slight change to the definition of generalized modified policies. GMA now becomes "when a batch of items of size n_i arrives at the stockpile, then immediately issue all n_i items to those demand sources which have the least useful life remaining in their items currently in use; if $n_i > \nu$ then immediately issue ν of the new items, one to each demand source."

For Chapter 6, it is only necessary to assume (7.1.8). The two theorems and their extensions to the countably infinite case then follow.

Chapter 8

Summary and Conclusions

In the preceding chapters we have presented a considerably more general model than was originally formulated in earlier papers.

In Chapter 2 we have carried on the modification of assumption (6) which was started by Zehna [11] and Eilon [4]. For $L(S)$ concave nonincreasing and $L^{-}(S) \geq -1$ for $S \leq S_o$, common upper and lower bounds for FIFO and for the optimal policy were obtained when there are several demand sources. And if $[\frac{1}{2}(n+1)] \leq \nu \leq n$, then FIFO is optimal.

In Chapter 3 the assumption (4) of no penalty costs was removed and optimal policies were presented. For $L(S)$ convex nonincreasing, if LIFO maximizes the total field life for any i items in inventory then the optimal policy must be one of the n policies $L_{(i,\nu)*}$ $i = 1, \ldots, n$ which says issue the i youngest items by LIFO and discard the remaining $n - i$ items. For $L(S)$ concave nonincreasing, if $L^{-}(S) \geq -1$ for $S \leq S_o$ and if FIFO is optimal for i items in inventory then the optimal policy must be one of the n policies $F_{(i,\nu)*}$. When more restrictions were placed on $L^{-}(S)$ and on $L^{+}(S)$ we were able to reduce the search even further. In particular, if $L(S) = aS + b$ where $b > 0 > a > -1$ for $S \leq S_o$ then the specific optimal policy was obtained.

In Chapter 4 we considered an S-shaped field life function, $L(S)$ concave nonincreasing for $S \leq t$, $L(S) = L(t) = c > 0$ for $S \geq t$, and $L^{-}(S) \geq -1$ for $S \leq t$. In this case the optimal policy is one of the n policies F_iL, $i = 1, \ldots, n$, where F_iL says to issue the i youngest items by FIFO and the remaining $n - i$ items by LIFO. If $L(S)$ for $S \leq t$ is linear, then the specific optimal policy was obtained. Also we showed, if $L(S)$ is convex or concave decreasing, $L^{-}(S) < -1$ for all $S\epsilon(0, t]$ and $L(S) = L(t) = c > 0$ for all $S \geq t$, then LIFO is optimal.

In Chapter 5 the assumption (5) of a static inventory depletion model was removed and a dynamic model was introduced. It was seen that if $L(S)$ is concave nonincreasing, $L^{-}(S) \geq -1$ for $S \leq S_o$, if stock-outs did not occur then if FIFO was optimal in the static model, FIFO was optimal in the dynamic model. And if $L(S)$ is concave or convex decreasing and $L^{-}(S) < -1$ for $S \leq S_o$, then GML (generalized-modified-LIFO) is optimal.

In Chapter 6 we again looked at the field life function but now from the viewpoint of a stochastic model. It was proved that if $X(S)$ is a nonnegative random variable which takes on any one of a countable number of nonincreasing $L_i(S)$ with probability p_i and if $0 \geq L_i(S) > L_{i+1}(S) \geq -1$ for all i and any $S < S_o$, then (i) if $L_i(S) = a_iS + b_i$, FIFO is optimal for $n \geq 2$ items and (ii) if $L_i(S)$ is concave nonincreasing, FIFO is optimal for $n = 2$ items.

In Chapter 7 the batch problem (where one or more of the n items have the same initial age) was considered and the general result was presented that if $L(S)$ is a continuous function and if FIFO (LIFO) is optimal without batches, then FIFO (LIFO) is optimal with batches.

Consideration of the batch case led to the modification of assumption (1) of the model.

This work has presented changes to assumptions (1), (2), (4), (5) and (6). We did not consider field life functions which become negative since they do not seem to have a nice interpretation in the inventory context of the model. However it may be possible that negative $L(S)$ would be an interesting representation of some other model based on costs or profits.

Also we have not sought to change assumption (3). Assumption (3) in conjunction with assumption (6) defines the demand function on the inventory and this assumption is what gives this particular model its interest and its problems. It makes any current decision as to the optimal issuing policy entirely dependent upon all past decisions. Other authors have removed assumption (3) and assumption (6) to make the demands on the stockpile independent of the ages of the items in the stockpile. They have obtained some very interesting results (ref. [2], [3], [5], [10], and [11] in the Bibliography).

It should be noted that the theorems presented in the preceding chapters do not exhaust the possible changes to the model nor do they exhaust the possible variations on the changes already suggested. For example, the stochastic models presented in Chapter 5 of Zehna [11] and in Chapter 6 of this work are stated only for the case $\nu = 1$ and for special probability density functions and $L_i(S)$. Since the stochastic case is very important from a practical point of view, extensions to $\nu \geq 1$ and other field life functions would be desirable. It should be mentioned that these extensions appear to be difficult since the truncation point S_o makes most proofs rather complicated.

Some other areas of future research should be

(i) for the case $\nu > 1$ find the optimal policy for various $L(S)$'s and/or find more sufficient conditions for LIFO or FIFO optimality,

(ii) investigate other types of S-shaped functions,

(iii) consider the dynamic depletion model in relation to removing other assumptions of the model,

(iv) investigate appreciating field life functions, and

(v) look at a fixed (or minimum) spacing, δ , between the initial ages of the items, i.e., $S_{i+1} \geq S_i + \delta$.

This last area would correspond to the case when inventory items arrived at the stockpile in some fixed pattern as would be the case, e.g., if the production facility was turning out one item every time period.

As a final consideration some research should be devoted to different types of objective functions. An important case of this is pointed out in Eilon [6] and Eilon [5]. He suggested that if we are a seller (rather than consumer) of the items in the stockpile and if the holding costs per item are high, then we may wish to minimize the time the items spend in inventory. This type of objective function is, in a sense, the polar image of the objective function considered in the preceding chapters since we now wish to minimize the total field life of the stockpile (at least of the first $n - \nu$ items issued from the stockpile). However it is not, in general, true that if FIFO (LIFO) is optimal for the one objective function that LIFO (FIFO) is optimal for the other.

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