Chapter 5

The Dynamic Inventory-Depletion Model

As mentioned in the Introduction, we have always been assuming that new items are never added to the inventory after the process has started. In many instances this is an essential assumption of the model. However, a static model of this nature is not representative of actual inventory systems and it is interesting to ask "What sufficient conditions can be formulated when the model is dynamic, i.e., new items are continually added to the inventory, with the result that LIFO or FIFO is the optimal issuing policy?" The following sections give some answers to this question.

For the results of the following sections, we remove the assumption that new items are never added and replace it with this new assumption:

"If a new item is added to inventory, it has age $S = 0$ and initial field life $L(0)$ immediately upon entry to the inventory."

All of the other assumptions of the model, given in the introductory chapter, remain unchanged.

In addition to the new assumption we will assume that only a finite number, N, of new items are ever added to the inventory. The ages of the new items are all assumed to be different and we will denote the ages by $F_1, F_2, \ldots, F_N$ where $F_i > F_{i+1}$ means that item F_i arrives at the inventory before item F_{i+1}.

The assumption that N is finite is not necessarily restrictive since N can be chosen so large as to encompass the "going life" of any business concern.

We now construct two different problems called (i) the "original" problem and (ii) the "extended" problem. The original problem is the dynamic inventory problem of finding the optimal issuing policy for the n items $S_1 < S_2 < \cdots < S_n < S_o$ which are originally in the stockpile and the N items $F_1 > F_2 > \cdots > F_N$ which are added at arbitrary times in the future. The time of arrival of item F_N will be denoted by T (we are presently at time zero). The extended problem is the static inventory problem of finding the optimal issuing policy for the N + n items $F_N < F_{N-1} < \cdots < F_1 < S_1 < \cdots < S_n$ where all n + N items are originally in the stockpile and no new items are ever added. If we consider $S < 0$ as future time in the original problem, then the extended problem can be thought of as the original problem under the transformation $L(\hat{S}) = L(S + T)$ i.e., shift the ordinate axis left to the point $-T$ of the original problem. Reference figures 1 and 2 below.

"Original Problem"

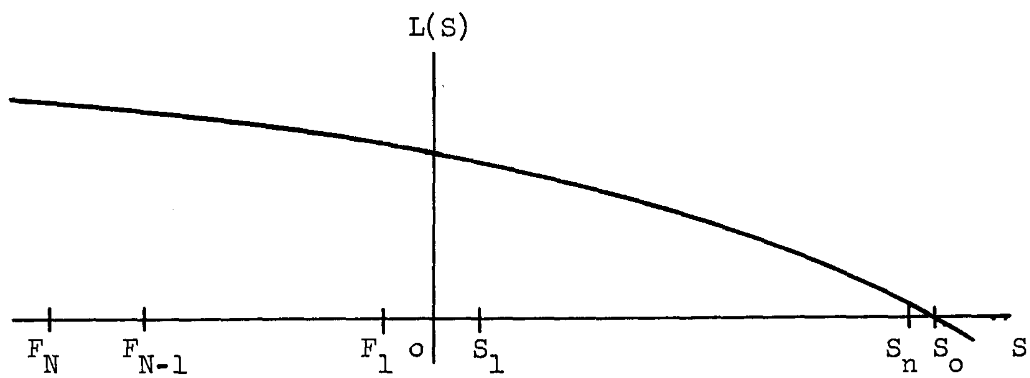


Figure 1

"Extended Problem"

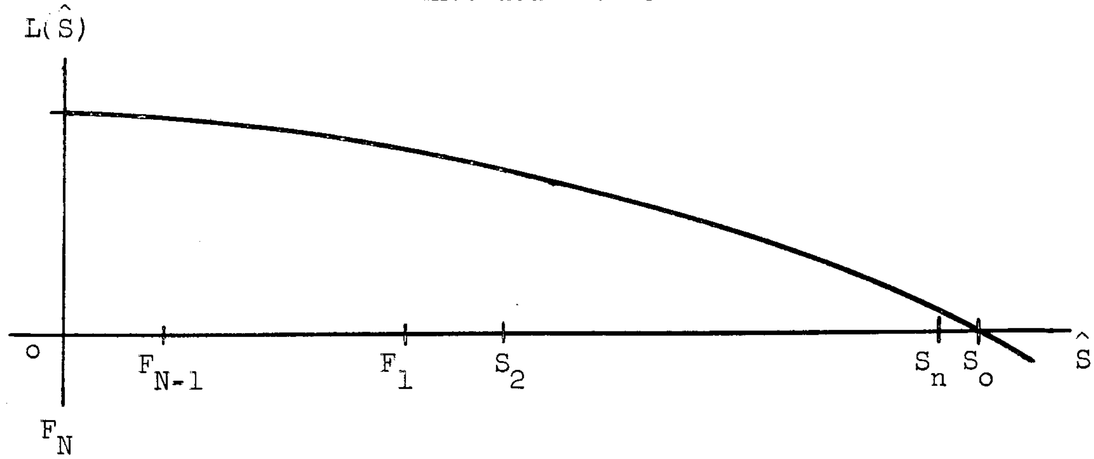


Figure 2

5.1 $L(S)$ Concave with $0 \geq L'(S) \geq -1$

Theorem 5.1: Let $L(S)$ be a concave nonincreasing function with $L'(S) \geq -1$ for all $S \leq S_o$. Let $\nu \geq 1$. If

(i) FIFO is optimal in the extended problem and

(ii) the arrival of items $F_1, \ldots, F_N$ in the original problem are timed so that no stockouts occur

then FIFO is the optimal issuing policy in the original problem, i.e. in the dynamic inventory problem.

Proof of Theorem 5.1: Since no stockouts occur, then the FIFO issuance of $S_n, \ldots, S_1$ and $F_1, \ldots, F_N$ in the original problem results in the same total field life as FIFO in the extended problem.

Now let us assume that there exists a policy A in the original problem which gives a greater total field life than FIFO. Then policy A must give a greater total field life than FIFO in the extended problem. This last statement follows from the fact that the set of all possible policies in the extended problem includes all policies of the original problem. But FIFO is optimal in the extended problem hence we have a contradiction. Therefore there cannot exist a policy A in the original problem which has a greater total field life than FIFO. Hence FIFO is optimal for the dynamic inventory problem.

q.e.d.

Corollary 5.1: Let $L(S) = aS + b$ for all $S \leq S_o$ and with $b > 0 > a > -1$. Let $\nu \geq 1$. If no stockouts occur in the original problem, then FIFO is optimal for this dynamic inventory model (the original problem).

Proof of Corollary 5.1: By Zehna [11] Theorems 4.1 and 4.3, FIFO is optimal for the extended problem; hence, by Theorem 5.1 above FIFO is optimal for the original problem.

q.e.d.

Corollary 5.2: Let $L(S)$ be a concave nonincreasing function with $L'(S) \geq -1$ for all $S \leq S_o$. Let $\nu = 1$.

If no stockouts occur in the original problem, then FIFO is optimal for this dynamic inventory model (the original problem).

Proof of Corollary 5.2: By Lieberman [9] Theorem 3, FIFO is optimal for the extended problem; hence by Theorem 5.1 above, FIFO is optimal for the original problem.

q.e.d.

Corollary 5.3: Let $L(S)$ be concave nonincreasing with $L^-(S) \geq -1$ for all $S \leq S_o$. Denote by $[x]$ the largest integer $\leq x$ where x

is a real number. Then if the number of demand sources ν has $[\frac{1}{2}(N + n + 1)] \leq \nu \leq n$ and if no stockouts occur in the original problem, then FIFO is optimal for the original problem.

<u>Proof of Corollary 5.3</u>: By Theorem 2.6 FIFO is optimal for the extended problem; hence, by Theorem 5.1 above, FIFO is optimal for the original problem.

q.e.d.

The preceding theorem and corollaries were concerned with $L(S)$ concave with slope ≥ -1 and in the linear case with $L'(S) > -1$. We now consider the linear case for $L'(S) = -1$, and show that FIFO is optimal for this case also. It is only necessary to prove that FIFO is optimal for the extended problem and then apply Theorem 5.1.

As was done in all of the preceding work, we assume that the stockpile has n items of initial ages $S_1 < \cdots < S_n < S_0$ at the start. And for the time being we do not consider adding any items to the stockpile.

<u>Lemma 5.1</u>: Let $L(S)$ be linear for all $S \leq S_0$ and $L'(S) = -1$ for all $S < S_0$. Let $\nu = 1$. Then any issuing policy is optimal and the total field life of the stockpile for any issuing policy is

$$Q^* = S_0 - S_1 = L(S_1) .$$

<u>Proof of Lemma 5.1</u>: Remember that we are only interested in the static model with n items $S_1 < \cdots < S_n < S_0$.

We first note that for any two items in inventory with current age $S_i < S_j$ $(< S_0)$ that

If S_i is chosen to be issued first, then at the expiration of the field life of S_i , S_j will have no field life remaining: (5.1.1)

$$\frac{L(S_o) - L(S_i)}{S_o - S_i} = -1 \qquad \text{where} \quad L(S_o) = 0$$

$$\Rightarrow \ -L(S_i) = -S_o + S_i$$

$$\Rightarrow \ S_o = S_i + L(S_i) < S_j + L(S_i)$$

$$\Rightarrow \ L(S_j + L(S_i)) = 0 \quad \text{since for all} \quad S > S_o \ , \quad L(S) = 0 \ ,$$

and

if S_j is chosen to be issued first, then at the expiration of the field life of S_j , S_i will still have positive field life remaining: (5.1.2)

$$\frac{L(S_o) - L(S_j)}{S_o - S_j} = -1$$

$$\Rightarrow \ S_o = S_j + L(S_j) > S_i + L(S_j)$$

$$\Rightarrow \ L(S_i + L(S_j)) > 0 \quad \text{since for all} \quad S < S_o \ , \quad L(S) > 0 \ .$$

We now use the above two properties to show: in any issue policy $A = [S_{i_1} , \ldots , S_{i_n}]$ we can omit any items S_{i_j} for which there is some $S_{i_{j-k}}$ $(k = 1, \ldots, j-1)$ such that $S_{i_{j-k}} < S_{i_j}$, since for these S_{i_j}, they will have no field life remaining when they are ready to be issued.

By statement (5.1.1) above when $S_{i_{j-k}}$ is issued it has current age, say S_t, and S_{i_j} has current age S_u (i.e., if the total field life of the items up to $S_{i_{j-k}}$ is Q, then $S_t = S_{i_{j-k}} + Q$ and $S_u = S_{i_j} + Q$) but $S_t < S_u$ $(< S_o)$ hence S_u has no field life remaining after S_t is issued.

Thus of all possible policies, we only have to consider policies where each succeeding item is younger than the previously issued item since any other policy will have total field life equivalent to one of these oldest to youngest ordered policies (where all items with field life of zero have been discarded).

Now by statement (5.1.2) since $S_1 < S_i$ for all $i = 2, \ldots, n$, we must have that upon issue at any time, S_1 will have positive field life. But as shown above any item issued after S_1 has field life of zero and can be discarded without issuance, hence S_1 is the last item to be issued under all policies which we need to consider.

It now remains to be shown that for any policy $B = [S_{i_1}, \ldots, S_1]$ where $S_{i_j} > S_{i_{j+1}}$, for all j in the policy, that B has a total field life of $S_o - S_1 = L(S_1)$.

Let policy B contain the issuance of k items $(k = 1, \ldots, n)$. Obviously if $k = 1$, then B is LIFO and

$$\frac{L(S_o) - L(S_1)}{S_o - S_1} = -1$$

$$\Rightarrow \quad L(S_1) = S_o - S_1 = Q_{LIFO} = Q^* \,. \qquad (5.1.3)$$

Let $k > 1$ and let the total field life of the $k - 1$ items up to but not including S_1 be denoted by x, then $S_1 + x < S_o$ and

$$\frac{L(S_1 + x) - L(S_o)}{S_1 + x - S_o} = -1$$

$$\Rightarrow \quad L(S_1 + x) = S_o - S_1 - x \,. \qquad (5.1.4)$$

But the total field life from policy B is

$$Q_B = L(S_1 + x) + x$$

hence

$$Q_B = L(S_1 + x) + x = S_o - S_1 = L(S_1)$$

by (5.1.3) and (5.1.4). Now policy B was arbitrary; thus any issue policy has total field life $S_o - S_1$; hence all issue policies are optimal.

<u>NOTE</u>: This result means

$$Q_{FIFO} = Q_{LIFO} = S_o - S_1 = L(S_1) \,.$$

q.e.d.

<u>Corollary 5.4</u>: Let $L(S)$ be linear, with $L'(S) = -1$ for all $S \leq S_o$. Let $\nu = 1$. Then any issue policy

$$A = [S_{i_1}, \ldots, S_{i_j}] \quad \text{where} \quad S_{i_k} > S_{i_{k+1}}$$

$$(k = 1, \ldots, j - 1)$$

has a total field life of

$$Q_A = L(S_{i_j}) = S_o - S_{i_j} .$$

<u>Proof of Corollary 5.4</u>: Let x denote the total field life up to but not including the issue of item S_{i_j}. Then

$$Q_A = L(S_{i_j} + x) + x$$

and by lemma 2.1 $S_{i_j} + x < S_o$.

Hence

$$\frac{L(S_{i_j} + x) - L(S_o)}{S_{i_j} + x - S_o} = -1$$

$$\Rightarrow Q_A = L(S_{i_j} + x) + x = S_o - S_{i_j}$$

but

$$\frac{L(S_{i_j}) - L(S_o)}{S_{i_j} - S_o} = -1$$

$$\Rightarrow L(S_{i_j}) = S_o - S_{i_j}$$

hence

$$Q_A = S_o - S_{i_j} = L(S_{i_j})$$

q.e.d.

Lemma 5.2: Let $L(S)$ be linear with $L'(S) = -1$ for all $S \leq S_o$. Let $\nu \geq 1$. Then any issuing policy which issues items $S_1, S_2, \ldots, S_\nu$ (i.e. the ν youngest items) each to a different demand source is optimal and the total field life from an optimal policy, Q^*, is given by

$$Q^* = \sum_{i=1}^{\nu} L(S_i) = \nu S_o - \sum_{i=1}^{\nu} S_i . \qquad (5.1.5)$$

Furthermore

$$Q_{FIFO,\nu} = Q_{LIFO,\nu} = Q^* .$$

Proof of Lemma 5.2: We will first show that any policy which issues items $S_1, S_2, \ldots, S_\nu$ each to different demand sources has total field life given by (5.1.5). We will then show that any other policy not of this form has field life less than (5.1.5). Finally we will show $Q_{FIFO,\nu} = Q_{LIFO,\nu} = Q^*$.

Consider any policy which issues $S_1, \ldots, S_\nu$ each to different demand sources say $M_1, \ldots, M_\nu$ respectively. Hence if demand source M_j receives the c items $[S_{j_{i_1}}, \ldots, S_{j_{i_c}}]$ then by the same argument as given in (5.1.1) and (5.1.2) of lemma 5.1 we only need to consider the ordering

$$A_{M_j} = [S_{j_{i_1}}, \dots , S_j] \text{ where } S_{j_{i_k}} > S_{j_{i_{k+1}}} .$$

Now by corollary 5.4, the total field life obtained from policy A_{M_j} is $Q_{A_{M_j}} = L(S_j) = S_o - S_j$. Since M_j was picked arbitrarily, then for all $j = 1, \dots , \nu$

$$Q_{A_{M_j}} = L(S_j) = S_o - S_j$$

and

$$Q = \sum_{j=1}^{\nu} Q_{A_{M_j}} = \sum_{j=1}^{\nu} L(S_j) = \sum_{j=1}^{\nu} (S_o - S_j) = \nu S_o - \sum_{j=1}^{\nu} S_j , \tag{5.1.6}$$

which is (5.1.5) as required.

Now let B be any policy which does not issue $S_1, \dots , S_\nu$ each to different $M_1, \dots , M_\nu$. Hence B must issue at least two of the items $S_1, \dots , S_\nu$ to the same demand source, say S_i and S_j are issued to M_k where $S_1 \leq S_i < S_j \leq S_\nu$. Now by (5.1.1), (5.1.2) and corollary 5.4 we have

$$Q_{B_{M_k}} = L(S_i) = S_i - S_o .$$

And since S_i and S_j are issued to M_k then there is at least one M_t such that the youngest item issued to M_t has initial age $S_t > S_\nu$.

Hence

$$Q_{B_{M_t}} = L(S_t) = S_o - S_t$$

by corollary 5.4 and (5.1.1) and (5.1.2).

Thus the total field life for policy B is at most

$$Q_B \leq \sum_{\substack{i=1 \\ i \neq j}}^{\nu} L(S_i) + L(S_t) = \nu S_o - S_t - \sum_{\substack{i=1 \\ i \neq j}}^{\nu} S_i$$

$$< \nu S_o - \sum_{i=1}^{\nu} S_i \quad \text{since} \quad S_\nu < S_t$$

$$= Q$$

in (5.1.6). Thus $Q = Q^*$ since B was any arbitrary policy. Now $Q_{LIFO,\nu}$ issues only $S_1, \ldots, S_\nu$ and each to different demand sources since for all $k > \nu$

$$S_k + L(S_\nu) > S_\nu + L(S_\nu) = S_o$$

$$\Rightarrow L(S_k + L(S_\nu)) = 0 \text{ .}$$

Hence

$$Q_{LIFO,\nu} = \sum_{i=1}^{\nu} L(S_i) = Q^* \text{ .}$$

Furthermore by lemma 2.3, we note that FIFO belongs to the class of policies such that items $S_1, \ldots, S_\nu$ are each issued to different $M_1, \ldots, M_\nu$; hence

$$Q_{FIFO,\nu} = \sum_{i=1}^{\nu} L(S_i) = Q^* .$$

q.e.d.

We are now able to state:

Theorem 5.2: Let $L(S)$ be linear with $L'(S) = -1$ for all $S \leq S_o$. Consider the original problem and the extended problem given in pages 141-142. Let $\nu \geq 1$. If no stockouts occur in the original problem, then FIFO is optimal for the original problem.

Proof of Theorem 5.2: By lemma 5.2, FIFO is optimal for the extended problem; hence by Theorem 5.1 FIFO is optimal for the original problem, the dynamic inventory model.

q.e.d.

Note that by lemma 5.2 if the F_i are known for all $i = N - \nu + 1, \ldots, N$ then the total field life for the model of Theorem 5.2 is

$$Q^* = Q_{FIFO,\nu} = \sum_{i=0}^{\nu-1} L(F_{N-i}) .$$

5.2 L(S) Concave or Convex with Slope < -1

In this section we seek the optimal issuing policy for the dynamic inventory model when $L(S)$ is concave or convex and has slope < -1 for all $0 \leq S \leq S_o$. The optimal policy is found for the case $\nu \geq 1$

demand sources; however it will be instructive to state the case $\nu = 1$ first and subsequently to state and prove the case for all ν $(1 \leq \nu \leq n)$. It is interesting to note that we no longer need the assumption that no stockouts occur; the reason for this will be discussed later.

We define a modified-LIFO policy (ML) for the case $\nu = 1$ in the following way: Use LIFO until a new item arrives, then discard the item currently in use and use the new item immediately.

In addition, it will be assumed that there is no penalty cost for the installation or removal of an item in the field.

Theorem 5.3: Let $L(S)$ be a convex or concave differentiable function on $[0, S_o]$ with $L'(S) < -1$ on $[0, S_o]$. Let $\nu = 1$. Then modified-LIFO is the optimal issuing policy for the original problem i.e. the dynamic inventory model.

Since this theorem is a special case $(\nu = 1)$ of Theorem 5.4, it will not be proved here. It was presented here in order to introduce the concept of a modified-LIFO policy and some sufficient conditions under which ML is optimal. For Theorem 5.4 it will be necessary to generalize the ML concept. But before so doing we present the following useful lemma.

Lemma 5.3: Let $L(S)$ be a convex or concave differentiable function on $[0, S_o]$ with $L'(S) < -1$ on $[0, S_o]$. Let there be n items $0 < S_1 < S_2 < \cdots < S_n < S_o$ in inventory and no new items are ever added to the inventory. Let $\nu \geq 1$. If x_i is the total field life contributed by demand source M_i under any arbitrary policy A and if the x_i are ordered $x_1 \geq x_2 \geq \cdots \geq x_\nu$, then

$$x_i \leq L(S_i) \qquad \text{for all} \quad i = 1, \ldots, \nu .$$

By Zehna [11] Theorems 4.2 and 4.3 we know that LIFO maximizes the total field life. This lemma states that not only is that fact true but also each demand source under LIFO receives more field life than from any other policy.

Proof of Lemma 5.3: Assume to the contrary that $x_i > L(S_i)$ for some $i = 1, \ldots, \nu$. Then x_i must contain one or more items $S_j < S_i$ for if all items S_k assigned to M_i under policy A are such that $S_k \geq S_i$, then since LIFO is optimal for $\nu = 1$ (cf. Zehna [11] Theorems 2.4 and 2.6) we would have $L(S_i) \geq x_i$ contrary to the assumption $x_i > L(S_i)$.

But if $S_j < S_i$ is assigned to M_i then there are at most $i - 2$ S's which have $S_k < S_i$, $k \neq j$, available for assignment to the $i - 1$ M_t's viz. $M_1, \ldots, M_{i-1}$. Hence some M_t, $t = 1, \ldots, i - 1$, does not receive any $S_k < S_i$. Therefore as stated in the preceding paragraph we must have $x_t \leq L(S_i)$ and

$$x_i > L(S_i) \geq x_t \qquad \text{where} \quad t < i .$$

But $x_i > x_t$ for $t < i$ contradicts the hypothesis of the lemma. Therefore

$$x_i \leq L(S_i) \qquad \text{for all} \quad i = 1, \ldots, \nu .$$

q.e.d.

Let A be any arbitrary policy for issuing the n items originally in the inventory and the N items added to the inventory in the future. We define a generalized-modified-A policy, GMA, for issuing items to the $\nu \geq 1$ demand sources in the following way: Use policy A until a new item arrives, then discard the oldest item currently in use in the field and immediately replace it with the new item. When $A \equiv$ LIFO we denote GMA by GML.

Theorem 5.4: Let $L(S)$ be a convex or concave differentiable function on $[0, S_o]$ with $L'(S) < -1$ on $[0, S_o]$. Let there be no penalty costs for the removal or the installation of an item in the field. Let $\nu \geq 1$. Then GML is the optimal issuing policy for the original problem, i.e., the dynamic inventory model.

Proof of Theorem 5.4: The proof will be by induction on N. Let $N = 1$. And let the time of arrival of the new item be denoted by t. (We are initially at time zero.)

We first show that under any policy A it is always better to discard some item currently in use and use the new item immediately.

Let T be the field life remaining to demand source M_i when the new item arrives. There are three cases:

Case (i): $0 < T < S_o$ then $\frac{L(0) - L(T)}{-T} < -1$ implies $L(0) > L(T) + T$ and it is better to use the new item immediately. For $j \neq i$ the field lives of the other M_j's are not affected by this change.

Case (ii): $S_o \leq T$ which implies $L(T) = 0$. Then $L(0) \geq L(S_1) \geq T = T + L(T)$ and again it is better to use the new item immediately.

Case (iii): $T = 0$ then $L(0) = T + L(T)$ and the new item should be installed immediately on arrival.

In the above we have implicitly assumed for $j \neq i$ that all M_j's have items currently in use. If some M_j did not have any items left and if $T > 0$ for M_i the new item would be assigned to M_j. This last remark is contained in the next two paragraphs.

We will show that the policy of assigning the new item to the demand source M_i which loses the least life by discarding the items currently assigned to it (they all have life zero after time $L(0)$) is better than assigning the new item to some other M_j, $j \neq i$.

Let M_i be the demand source with the least field life remaining at time t. Denote this remaining field life to M_i by T_{min}. $T_{min} \geq 0$. For any $j \neq i$, let T_j be the field life remaining to M_j. Then $T_j \geq T_{min}$. Let Q be the total field life obtained by all the M_k's, $k = 1, \ldots, \nu$, if the new item is not issued until the current items issued to M_i and M_j expire. Then

$$Q + L(0) - T_{min} \geq Q + L(0) - T_j \,, \quad \text{for any } j \neq i \,.$$

Hence under any policy A we obtain

Statement (1): The new item should be issued immediately upon its arrival to the demand source which must discard the least field life.

Thus statement (1) says that the optimal policy for the case $N = 1$ must belong to the class of generalized-modified policies.

We now show GML is optimal for $N = 1$. Consider any policy GMA with GMA $\neq$ GML. Let $x_1 \geq x_2 \geq \cdots \geq x_\nu$ be the field life contributed by $M_1, \ldots, M_\nu$ under policy A when the new item is not considered. By GMA the new item will be assigned to M_ν since x_ν is the smallest field life. We consider five mutually exclusive and exhaustive cases. Recall by lemma 5.3 that $x_i \leq L(S_i)$ for all i.

<u>Case 1</u> $x_\nu = L(S_\nu)$ and $t \leq L(S_\nu)$ t is the arrival time of the new item

Then

$$Q_{GML} = \sum_{i=1}^{\nu-1} L(S_i) + t + L(0) \geq \sum_{i=1}^{\nu-1} x_i + t + L(0) = Q_{GMA}$$

<u>Case 2</u> $x_\nu = L(S_\nu)$ and $t > L(S_\nu)$

Then

$$Q_{GML} = \sum_{i=1}^{\nu} L(S_i) + L(0) \geq \sum_{i=1}^{\nu} x_i + L(0) = Q_{GMA}$$

<u>Case 3</u> $x_\nu \leq L(S_\nu)$ and $t \leq x_\nu$

Then

$$Q_{GML} = \sum_{i=1}^{\nu-1} L(S_i) + t + L(0) \geq \sum_{i=1}^{\nu-1} x_i + t + L(0) = Q_{GMA}$$

<u>Case 4</u> $x_\nu < L(S_\nu)$ and $x_\nu < t \leq L(S_\nu)$

Then

$$Q_{GML} = \sum_{i=1}^{\nu-1} L(S_i) + t + L(0) > \sum_{i=1}^{\nu} x_i + L(0) = Q_{GMA}$$

<u>Case 5</u> $x_\nu < L(S_\nu)$ and $t > L(S_\nu)$

Then

$$Q_{GML} = \sum_{i=1}^{\nu} L(S_i) + L(0) > \sum_{i=1}^{\nu} x_i + L(0) = Q_{GMA}$$

In all cases $Q_{GML} \geq Q_{GMA}$ and since A was any arbitrary policy GML is optimal for $N = 1$.

Assume GML is optimal for adding $N = k$ items to inventory and it will be proved that GML is optimal for adding $N = k + 1$ items.

We will first establish that the optimal policy must belong to the class of generalized-modified policies. Let T be the field life remaining to M_i when the $k + 1^{st}$ item arrives. Let t_k and t_{k+1} denote the time of arrival of the k^{th} and $k + 1^{st}$ items respectively. Since arrivals are distinct events $t_k < t_{k+1}$ and all items in use or in the stockpile, except the $k + 1^{st}$ item, have age greater than zero at time t_{k+1}, then $L(0) > T \geq 0$, and we have the same three cases as before:

<u>Case (i)</u> $0 < T < S_o$

Then $$\frac{L(0) - L(T)}{-T} < -1$$

implies

$$L(0) > L(T) + T$$

<u>Case (ii)</u> $S_o \leq T$

Then

$$L(T) = 0$$

and

$$L(0) > T + L(T)$$

<u>Case (iii)</u> $0 = T$

Then

$$L(0) = T + L(T)$$

Hence the new item should always be installed immediately on arrival. Now by the same argument given in $N = 1$, the new item should be assigned to the demand source which loses the least field life. Thus Statement (1) applies also to the case of $N = k + 1$ and the optimal policy belongs to the class of generalized-modified policies.

In order to proceed further it is necessary to develop some additional notation. Let $Q_{M_i,N}$ and $x_{i,N}$ denote the total field life for M_i under GML and GMA respectively. In addition relabel the M_i's in GML and in GMA such that $Q_{M_i,N} \geq Q_{M_{i+1},N}$ and $x_{i,N} \geq x_{i+1,N}$ for all $i = 1, \ldots, \nu - 1$. It is possible that M_1 under GML is not the same M_1 as under GMA but this fact is of no importance in the following.

In the case $N = 1$ we showed $Q_{GML} \geq Q_{GMA}$ but also in conjunction with lemma 5.3 we showed that the total field life for each M_i under GML is greater than under GMA i.e., using the notation above

$$Q_{M_{i,1}} \geq x_{i,1} \qquad \text{for all} \quad i = 1, \ldots, \nu \tag{5.2.1}$$

It will now be proved that

$$Q_{M_{i,k+1}} \geq x_{i,k+1} \qquad \text{for all} \quad i = 1, \ldots, \nu \tag{5.2.2}$$

where we inductively assume

$$Q_{M_{i,k}} \geq x_{i,k} \qquad \text{for all} \quad i = 1, \ldots, \nu \,. \tag{5.2.3}$$

Now (5.2.3) and Statement (1) inform us that the $k + 1^{st}$ arrival is immediately assigned to M_ν. We only need to show $Q_{M_{\nu,k+1}} \geq x_{\nu,k+1}$. Let T_{GML} and T_{GMA} be the total field life remaining to M_ν at time t_{k+1} when GML and GMA are being followed respectively. By (5.2.3) $T_{GML} \geq T_{GMA} \geq 0$. We again consider the five mutually exclusive and exhaustive cases:

<u>Case 1</u> $\qquad x_{\nu,k} = Q_{M_{\nu,k}}$ and $t_{k+1} \leq Q_{M_{\nu,k}}$

Then $x_{\nu,k} = Q_{M_{\nu,k}}$ implies $T_{GML} = T_{GMA}$ and

$$Q_{M_{\nu,k+1}} = Q_{M_{\nu,k}} - T_{GML} + L(0) = x_{\nu,k} - T_{GMA} + L(0) = x_{\nu,k+1}$$

<u>Case 2</u> $\qquad x_{\nu,k} = Q_{M_{\nu,k}}$ and $t_{k+1} > Q_{M_{\nu,k}}$

Then $\qquad Q_{M_{\nu,k+1}} = Q_{M_{\nu,k}} + L(0) = x_{\nu,k} + L(0) = x_{\nu,k+1}$

<u>Case 3</u> $x_{\nu,k} \le Q_{M_{\nu,k}}$ and $t_{k+1} \le x_{\nu,k}$

Then $$Q_{M_{\nu,k}} - x_{\nu,k} = T_{GML} - T_{GMA}$$

and $$Q_{M_{\nu,k+1}} = Q_{M_{\nu,k}} - T_{GML} + L(0) = x_{\nu,k} - T_{GMA} + L(0) = x_{\nu,k+1}$$

<u>Case 4</u> $x_{\nu,k} < Q_{M_{\nu,k}}$ and $x_{\nu,k} < t_{k+1} \le Q_{M_{\nu,k}}$

Then $T_{GMA} = 0$ and $Q_{M_{\nu,k}} - x_{\nu,k} > T_{GML} - T_{GMA} = T_{GML}$

$$Q_{M_{\nu,k+1}} = Q_{M_{\nu,k}} - T_{GML} + L(0) > x_{\nu,k} + L(0) = x_{\nu,k+1}$$

<u>Case 5</u> $x_{\nu,k} < Q_{M_{\nu,k}}$ and $t_{k+1} > Q_{M_{\nu,k}}$

Then $$Q_{M_{\nu,k+1}} = Q_{M_{\nu,k}} + L(0) > x_{\nu,k} + L(0) = x_{\nu,k+1} .$$

Hence in all cases

$$Q_{M_{\nu,k+1}} \ge x_{\nu,k+1} \tag{5.2.4}$$

Now since the field life for the other M_i $i = 1, \ldots, \nu - 1$ are unchanged then by (5.2.3)

$$Q_{M_{i,k}} = Q_{M_{i,k+1}} \ge x_{i,k+1} = x_{i,k} \tag{5.2.5}$$

for all $i = 1, \ldots, \nu - 1.$

Combining (5.2.4) and (5.2.5) we see that (5.2.2) holds for all $i = 1, \ldots, \nu$.

But GML for $N = k + 1$ yields

$$Q_{GML} = \sum_{i=1}^{\nu-1} Q_{M_{i,k}} + Q_{M_{\nu,k+1}}$$

and GMA for $N = k + 1$ yields

$$Q_{GMA} = \sum_{i=1}^{\nu-1} x_{i,k} + x_{\nu,k+1} .$$

Hence by (5.2.4) and (5.2.5) $Q_{GML} \geq Q_{GMA}$ where A was any arbitrary policy. Therefore by induction GML is optimal for all N.

q.e.d.

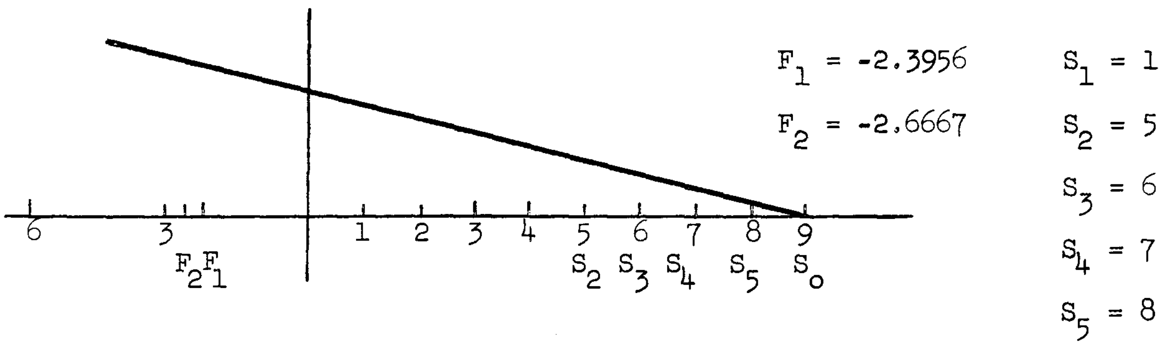
5.3 The Problem of Stockouts

In the results of section 5.1, it was assumed that the ordering schedule for new items was arranged so that stockouts did not occur. This assumption was essential for FIFO optimality as the following example shows:

$$L(S) = -\frac{1}{3} S + 3 \quad \text{for} \quad S \epsilon [0, 9]$$

$$= 0 \quad \text{for} \quad S \epsilon [9, \infty)$$

$$\nu = 2$$



Then FIFO = $[S_5, S_3, S_1, F_2 ; S_4, S_2, F_1]$ which yields $Q_F = 10.9883$ as compared to $A = [S_5, S_4, S_3, S_2, F_1 ; S_1, F_2]$ which yields $Q_A = 11.0623$ is definitely not optimal.

In the case of FIFO, however, a stockout occurred because the total field life for $[S_4, S_2]$ is 1.7781 whereas item F_1 does not arrive until $t_1 = 2.3956$.

It is interesting to note, however, that the results of section 5.2 do not require the assumption of no stockouts. The reason for this is essentially contained in lemma 5.3 which states that each demand source receives more field life under LIFO than from any other policy. Thus if we followed a non-LIFO policy, say GMA, we could expect stockouts to be more frequent and of a much longer duration. But, the new arriving item under any generalized modified policy is used to its fullest extent. Therefore the policy which minimizes the total stockout duration will maximize the total field life; and as shown in section 5.2 this optimal policy is GML for the dynamic depletion model.

As the concluding statement in this chapter, it should be noted that results were not presented for the case $L(S)$ convex decreasing with slope $L'(S) \geq -1$. Even if we assumed that LIFO was optimal for the static depletion model, there are numerous counterexamples for $\nu = 1$, $n = 2$, and $N = 1$ in the dynamic model where neither LIFO nor modified LIFO nor any of the other possible policies is optimal in all cases. If we desire to find the conditions in this simple case where LIFO or ML is optimal, it is necessary to make very restrictive assumptions on S_1, S_2 and F_1. We have not done this because the transition to general n and N even keeping $\nu = 1$ does not appear to be interesting from a practical point of view.